Modified Denavit-Hartenberg Parameters for Better Location of Joint Axis Systems in Robot Arms

L. Keith Barker
Modified Denavit-Hartenberg Parameters for Better Location of Joint Axis Systems in Robot Arms

L. Keith Barker

Langley Research Center
Hampton, Virginia
SUMMARY

The Denavit-Hartenberg parameters define the relative location of joint axis systems in a robot arm. A recent criticism is that one of these parameters approaches infinity as the rotational axes of two successive joints approach a parallel condition. This causes an ill-conditioned transformation matrix and locates a joint axis system far away from the joint itself. This paper introduces a simple modification of these parameters to easily overcome this criticism. This modification (which entails the constraint that a transverse vector between successive joint rotational axes be normal to one of the rotational axes, instead of both) leads to modified Denavit-Hartenberg parameters that more favorably locate successive joint axis systems.

An example is given with respect to the rotational axes of the elbow and shoulder joints in a robot arm. The regular and modified Denavit-Hartenberg parameters that locate the axis system of the elbow joint relative to the axis system of the shoulder joint are extracted in the example by an algebraic method with simulated measurements of three different locations of a point on a robot arm. For small misalignments of the elbow- and shoulder-joint rotational axes from a parallel condition, the regular Denavit-Hartenberg parameters, unlike the modified parameters, were found to be extremely sensitive to measurement accuracy.

INTRODUCTION

The motion of a robot hand is the result of joint movements in the robot arm. To transform operator commands to the robot hand into joint movements and to pass sensor information along the arm, the relative location of successive joint axis systems must be known. By far, the most popular way to describe the relative location and orientation of one joint axis system with respect to another is to use the well-known Denavit-Hartenberg parameters (ref. 1), which define the elements in transformation matrices. However, these parameters have been criticized recently (refs. 2 and 3) as unsuitable in the case where successive rotational axes approach a parallel condition. Specifically, several of the elements in the transformation matrix approach infinity, and a joint axis system is located far away from the joint itself.

The purpose of the present paper is to modify the Denavit-Hartenberg parameters to overcome the aforementioned criticism. The algebraic method in reference 4 is extended to extract the modified parameters.

SYMBOLS

\( \mathbf{\hat{A}}, \mathbf{\hat{B}}, \mathbf{\hat{C}} \) measurement vectors from world coordinate system to a point on robot arm

\( \mathbf{A}_{i-1} \) homogeneous transformation matrix from coordinate system \( i \) to \( i - 1 \)

\( a_i \) length of \( \mathbf{\hat{a}}_i \)

\( \mathbf{\hat{a}}_i \) common normal vector between \( Z_{i-1} \) and \( Z_i \); intersects \( Z_i \) at \( O_i \)
\( \vec{c}_i \) vector from world coordinate system to center of circular trajectory of a point on robot arm about line of rotation of joint \( i \)

\( \vec{d}_i \) constrained transverse vector from \( Z_{i-1} \) to \( Z_i \)

\( \mathbf{E} \) elbow of robot arm

\( \vec{e}_i \) unit vector normal to both \( Z_{i-1} \) and \( Z_i \) that, by definition, points along \( X_i \)

\( \mathbf{H} \) hand of robot arm

\( \vec{h}_i \) location of \( O_i \) from \( O_{i-1} \) in coordinate system \( i - 1 \)

\( i \) integer

\( \vec{z}_i \) vector from world coordinate system to a point on line of rotation for joint \( i \)

\( \mathbf{N} \) neck of robot arm

\( \vec{N} \) vector normal to circular trajectory plane of point on robot arm

\( O \) base of robot arm

\( O_i, O_w \) origin of coordinate system \( i \) for joint \( i + 1 \) and of world coordinate system, respectively

\( Q, Q_1, Q_2, Q(x, y, z) \) points in three-dimensional space

\( \vec{r}_i \) vector from \( O_w \) to \( O_i \)

\( R_{i-1} \) rotational transformation matrix from coordinate system \( i \) to \( i - 1 \)

\( r_i \) coordinate along \( Z_{i-1} \) of \( O_i \)

\( \vec{r}_i \) vector from \( O_{i-1} \) to coordinate along \( Z_{i-1} \) of \( O_i \)

\( \mathbf{S} \) shoulder of robot arm

\( \vec{u}_i \) unit vector along positive rotational axis of joint \( i \)

\( \vec{v}_2 \) normal vector between lines of rotation for joints 2 and 3

\( \mathbf{W} \) wrist of robot arm

\( X_i \) axis directed along common normal between \( Z_{i-1} \) and \( Z_i \)

\( X_w, Y_w, Z_w \) world coordinate axes

\( Y_i \) axis directed to complete right-hand axis system with \( X_i \) and \( Z_i \)

\( Z_i \) axis of positive rotation of joint \( i + 1 \)

\( a_i \) angle between \( Z_{i-1} \) and \( Z_i \), measured positive about \( X_i \)

\( \beta_i \) constant bias angle, which yields joint angle \( \theta'_i \) when summed with joint angle \( \theta_i \)
\[ \theta_A, \theta_B, \theta_C \] joint angle corresponding to measurement vectors \( \hat{A}, \hat{B}, \) and \( \hat{C}, \) respectively

\[ \theta_i \] joint angle with initial value (0°) corresponding to initial position of robot arm

\[ \theta_i' \] joint angle between \( X_{i-1} \) and \( X_i \), measured positively about \( Z_{i-1} \)

\[ \Delta \theta_i \] incremental positive change in \( \theta_i \)

\[ \lambda_i \] real variable in vector line equation for joint \( i \)

\[ \rho_i \] radius of circular trajectory of point on robot arm about line of rotation of joint \( i \)

\[ \xi_i, \eta_i, \zeta_i \] components of \( \hat{D}_i \) with respect to coordinate system \( i - 1 \) when \( \theta_i = 0^\circ \)

Mathematical notations:

- \( \cdot \) dot or scalar product
- \( \times \) cross or vector product
- \( T \) superscript to indicate transpose
- \( | \cdot | \) length of a vector or absolute value

**ANALYSIS**

The regular Denavit-Hartenberg parameters are first described and then the difficulties encountered with their use in practical robotic applications are discussed. The basic set of parameters is then modified such that the criticism is no longer valid. Next, an algebraic approach, adapted from reference 4, is extended to extract the modified set of parameters. Parameters are then extracted for near-parallel rotational axes in a simulated robot arm.

**Denavit-Hartenberg Parameters**

In figure 1, \( a_i \) (component of \( \hat{a}_i \) along \( X_i \)), \( r_i \) (component of \( \hat{r}_i \) along \( Z_{i-1} \)), \( a_i \), and \( \theta_i' \) referred to as the Denavit-Hartenberg parameters, completely characterize the relative location of successive joint axis systems.

**Joint axis systems.**—Figure 1 illustrates the axis systems associated with joints \( i \) and \( i + 1 \). By convention, joint \( i \) is associated with the coordinate system \( i - 1 \). Hence, joint \( i \) corresponds to the axis system with origin at \( O_{i-1} \), whereas joint \( i + 1 \) corresponds to the adjoining axis system with origin \( O_i \). By definition, the axis of rotation for joint \( i \) always lies along the associated \( Z_{i-1} \). The vector \( \hat{a}_i \), directed toward \( Z_i \), is the normal vector between \( Z_{i-1} \) and \( Z_i \). The intersection point of \( \hat{a}_i \) with \( Z_i \) locates the origin \( O_i \). The axis \( X_i \) originates from \( O_i \) in the same direction as \( \hat{a}_i \). In the event that \( Z_{i-1} \) and \( Z_i \) intersect (fig. 1(b)), \( \hat{a}_i \) is the zero vector, and \( X_i \) is then directed from this intersection in the direction of the cross product obtained by multiplying a unit vector along \( Z_{i-1} \) by a unit vector along \( Z_i \). The vector \( \hat{r}_i \) is from the
origin $O_{i-1}$ to the intersection of $\hat{a}_i$ with $Z_{i-1}$ (fig. 1(a)); for intersecting lines of rotation, $\hat{r}_i$ is a vector along $Z_{i-1}$ from $O_{i-1}$ to $O_i$ (fig. 1(b)). The angle $\alpha_i$ is the angle between a line parallel to $Z_{i-1}$ through the origin $O_i$ and $Z_i$, being measured positive about $X_i$ (fig. 1). Finally, the joint angle $\theta_i'$ is the angle between $X_{i-1}$ and a line parallel to $X_i$ through $O_{i-1}$ and is measured positive about $Z_{i-1}$ (fig. 1). For clarity, the axes $Y_i$ and $Y_{i-1}$, which simply complete right-hand axis systems, are omitted.

**Relationship between joint angles $\theta_i$ and $\theta_i'$.** In general, the joint angle $\theta_i'$ is not equal to the joint angle $\theta_i$, which is referenced to the initial position of the robot arm (e.g., the position of the robot arm in fig. 2). Corresponding values of $\theta_i$ and $\theta_i'$ are assumed to be related by the linear equation

$$\theta_i' = \theta_i + \beta_i$$

where $\beta_i$ is a constant bias, reflecting an initial offset in $\theta_i'$, because $\theta_i'$ and $\theta_i$ are generally measured from different starting positions. The joint angle $\theta_i$ is measurable prior to establishing the joint axis systems, but $\theta_i'$ is not yet measurable.

**Basic coordinate transformation.** The relative joint geometry dictates the basic transformation equations between adjacent joints. The coordinates of an arbitrary point $Q(x,y,z)$ with respect to the coordinate system $i$ can be transformed to coordinates of $Q$ with respect to the coordinate system $i-1$ by using the relation

$$\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}_{i-1} = A_{i-1}^{-1}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}_i$$

with

$$A_{i-1}^{-1} = \begin{bmatrix}
  i & i & i & i & i & i \\
  R_{i-1} & h_i \\
  0 & 0 & 0 & 1
\end{bmatrix}$$

where

$$R_{i-1} = \begin{bmatrix}
  \cos \theta_i' & -\cos \alpha_i \sin \theta_i' & \sin \alpha_i \sin \theta_i' \\
  \sin \theta_i' & \cos \alpha_i \cos \theta_i' & -\sin \alpha_i \cos \theta_i' \\
  0 & \sin \alpha_i & \cos \alpha_i
\end{bmatrix}$$
is the rotational transformation matrix that relates coordinate system \( i \) to \( i - 1 \) with respect to orientation, and where

\[
\begin{bmatrix}
  a_i 
  
  a_i \sin \theta_i 
  
  r_i 
\end{bmatrix}
\]

is the location of \( O_i \) from \( O_{i-1} \) in coordinate system \( i - 1 \). As can be seen, the elements in the basic transformation matrix \( A_{i-1}^i \) are expressed in terms of the Denavit-Hartenberg parameters.

Criticism of Denavit-Hartenberg Parameters

A criticism of the Denavit-Hartenberg parameters is that \( r_i \) as \( \theta_i \to 0^\circ \). Indeed, this criticism is justified in that this behavior causes: (1) Sensitivity of the parameter \( r_i \) to errors in misalignment from a parallel condition (\( \alpha_i = 0^\circ \)); (2) Ill-conditioned transformation matrices (\( r_i \) terms and matrix products approach infinity as \( \alpha_i \to 0^\circ \)); and (3) Excessive displacement of the axis system from the robot arm. The intent in this paper is to eliminate these weaknesses by modifying the Denavit-Hartenberg parameters.

Modified Denavit-Hartenberg Parameters

In this paper, the regular Denavit-Hartenberg parameters are modified by only insisting that a transverse vector between successive joint rotational axes be normal to one of the axes instead of to both axes. This simple modification leads to a more favorable location of successive joint axis systems.

As with the regular Denavit-Hartenberg parameters, let \( Z_i \) be the axis of rotation for joint \( i + 1 \), let \( X_i \) point in a direction that is normal to both \( Z_{i-1} \) and \( Z_i \), and let \( Y_i \) complete the right-hand coordinate system. The angle between \( Z_{i-1} \) and \( Z_i \) is \( \alpha_i \), measured positive for rotation about \( X_i \). The difference between the regular and modified parameters is in how \( O_i \) is located from \( O_{i-1} \).

In figure 3, \( \mathbf{\hat{u}}_i \) and \( \mathbf{\hat{u}}_{i+1} \) are unit vectors along the lines of rotation for joints \( i \) and \( i + 1 \), respectively. Also in figure 3, \( \mathbf{\hat{c}}_{i+1} \) (determined later) is a vector from \( O_w \) to a point on \( Z_i \); \( \mathbf{\hat{r}}_{i-1} \) and \( \mathbf{\hat{r}}_i \) locate \( O_{i-1} \) and \( O_i \) from \( O_w \), respectively. The vector \( \mathbf{\hat{d}}_i \) is from a point on \( Z_{i-1} \) to a point on \( Z_i \). The origin \( O_1 \) is removed from \( O_{i-1} \) by

\[
\mathbf{\hat{h}}_i = r_i \mathbf{\hat{u}}_i + \mathbf{\hat{d}}_i
\]
where

$$\hat{D}_i = \hat{R}_i - \hat{R}_{i-1} - r_i \hat{u}_i$$  \hspace{1cm}(7)$$

and where \( r_i \) is the coordinate of \( O_i \) along \( Z_{i-1} \).

When used in equation (3), \( \hat{R}_i \) is expressed in the coordinate system \( i - 1 \). When \( \theta_i = 0^\circ \),

$$\hat{R}_i = \begin{pmatrix} \xi_i \\ \eta_i \\ \zeta_i + r_i \end{pmatrix}$$  \hspace{1cm}(8)$$

where (using \( \hat{D}_i \) and \( \hat{e}_{i-1} \) for the condition \( \theta_i = 0^\circ \))

$$\xi_i = \hat{D}_i \cdot \hat{e}_{i-1}$$  \hspace{1cm}(9)$$

$$\eta_i = \hat{D}_i \cdot (\hat{u}_i \times \hat{e}_{i-1})$$  \hspace{1cm}(10)$$

and

$$\zeta_i = \hat{D}_i \cdot \hat{u}_i$$  \hspace{1cm}(11)$$

denote the projections of \( \hat{D}_i \) along \( X_{i-1}, Y_{i-1}, \) and \( Z_{i-1} \), respectively, and \( \hat{e}_{i-1} \) is a unit vector along \( X_{i-1} \). The parameters \( \xi_i, \eta_i, \zeta_i \), and \( r_i \) now take the place of the regular Denavit-Hartenberg parameters \( a_i \) and \( r_i \). As joint \( i \) rotates, the location of \( O_i \) from \( O_{i-1} \) changes with \( \theta_i \) as

$$\hat{R}_i = \begin{pmatrix} \xi_i \cos \theta_i - \eta_i \sin \theta_i \\ \xi_i \sin \theta_i + \eta_i \cos \theta_i \\ \zeta_i + r_i \end{pmatrix}$$  \hspace{1cm}(12)$$

which is used in equation (3). Equation (12) reduces to equation (5) if \( \xi_i = a_i \), \( \eta_i = \zeta_i = 0 \), and \( \theta_i = \theta_i' \).
Transverse vector normal to both $Z_i$ and $Z_{i-1}$ for regular Denavit-Hartenberg parameters. When constrained to be normal to both $Z_{i-1}$ and $Z_i$, $\hat{D}_i$ is located from $Q_1$ to $Q_2$ in figure 3. Then, $r_i$ becomes the location of the point $Q_1$ along $Z_{i-1}$. The direction of $\hat{D}_i$ is along $X_i$, and its component is $a_i$. Since $X_i$ is simply $X_{i-1}$ rotated by $\theta'_i$ about $Z_{i-1}$, $\hat{D}_i$ with components in coordinate system $i-1$ is

$$\hat{D}_i = \begin{pmatrix} a_i \cos \theta'_i \\ a_i \sin \theta'_i \\ 0 \end{pmatrix}$$

where $\theta'_i$ is $\theta_i$ measured from an initial zero position where $X_{i-1}$ and $X_i$ are parallel. Then, from equation (6), $\hat{h}_i$ is the same as equation (5) for the regular Denavit-Hartenberg parameters.

The problem with constraining $\hat{D}_i$ to be normal to both $Z_{i-1}$ and $Z_i$ is that $r_i$ can shift $O_i$ far away from its associated joint on the robot arm. Consequently, two alternate constraints are imposed on $\hat{D}_i$ in this paper. They are (1) Only constraining $\hat{D}_i$ to be normal to $Z_{i-1}$ and (2) Only constraining $\hat{D}_i$ to be normal to $Z_i$.

Transverse vector normal to $Z_{i-1}$. The transverse vector $\hat{D}_i$ is given by equation (7), with

$$\hat{R}_i = \hat{c}_{i+1} + \lambda_{i+1} \hat{u}_{i+1}$$

(14)

where $\hat{c}_{i+1}$ (discussed subsequently) is a point on $Z_i$. The value of $\lambda_{i+1}$, which simply moves the terminal point of $\hat{R}_i$ along $Z_i$, is chosen to satisfy the constraint

$$\hat{D}_i \cdot \hat{u}_i = 0$$

(15)

That is, $\hat{D}_i$ is normal to $Z_{i-1}$. Equations (7), (14), and (15) show that

$$\lambda_{i+1} = \frac{(\hat{R}_{i-1} + r_i \hat{u}_i - \hat{c}_{i+1}) \cdot \hat{u}_i}{\hat{u}_{i+1} \cdot \hat{u}_i}$$

(16)

Equations (9) to (12) then yield $\hat{R}_i$. Since $\hat{D}_i$ is normal to $Z_{i-1}$, $\zeta_i = 0$. The parameter $r_i$ is now arbitrary and is chosen to shift $O_i$ to a favorable location.
Transverse vector normal to $Z_i$.– The transverse vector $\hat{D}_i$ is given by equation (7), with $\hat{R}_i$ given by equation (14), where $\lambda_{i+1}$ is chosen to satisfy the constraint

$$\hat{D}_i \cdot \hat{u}_{i+1} = 0$$  \hfill (17)

That is, $\hat{D}_i$ is normal to $Z_i$. Equations (7), (14), and (17) yield

$$\lambda_{i+1} = (\hat{R}_{i-1} + r_i \hat{u}_i - \hat{C}_{i+1}) \cdot \hat{u}_{i+1}$$  \hfill (18)

Equations (9) to (12) then yield $h_i$, and $r_i$ is a free parameter that is chosen to shift $O_i$ to a favorable location.

Algebraic Method To Extract Modified Denavit-Hartenberg Parameters

Measurement data needed to extract the relative joint parameters is generated by individually varying the joint angles in the robot arm and measuring the location of a point on the robot hand or other extension. (See ref. 4.) Each joint angle is measured relative to a selected zero position; for example, see the position of the robot arm in figure 2.

Any point on the robot arm (except a point on the line of rotation) generates a circular trajectory as the joint angle $\theta_i$ is varied, and the location of a point on the robot arm is measured relative to a world reference axis system. As subsequently shown, three locations of the point on the circular trajectory, along with the corresponding values of $\theta_i$, are enough to determine the unit vector $\hat{u}_i$ along the line of rotation for joint $i$ and the vector $\hat{C}_i$ from the world axis system to the center of the circular trajectory of the point. With $\hat{u}_i$ and $\hat{C}_i$ for successive joints, the relative joint parameters are computed. First, joint $i$ is rotated to obtain $\hat{u}_i$ and $\hat{C}_i$. Then, with $\theta_i = 0^\circ$, joint $i + 1$ is rotated to obtain $\hat{u}_{i+1}$ and $\hat{C}_{i+1}$. The process is initiated by assuming a fictitious joint with a rotational axis along $Z_w$.

Circular trajectory center.– Let $\hat{A}$, $\hat{B}$, and $\hat{C}$ denote three positions of a point on a robot arm as the point moves in a circular trajectory about the line of rotation for joint $i$ (fig. 4), and let $\theta_A$, $\theta_B$, and $\theta_C$ be the corresponding values of $\theta_i$. The objective is to compute the vector $\hat{C}_i$ from the world axis system to the center of the circular trajectory. Let the radius of the circle be $\rho_i$, then

$$\left(\hat{A} - \hat{C}_i \right) \cdot \left(\hat{A} - \hat{C}_i \right) = \rho_i^2$$  \hfill (19)

$$\left(\hat{B} - \hat{C}_i \right) \cdot \left(\hat{B} - \hat{C}_i \right) = \rho_i^2$$  \hfill (20)
\[(\mathbf{c} - \mathbf{c}_1) \cdot (\mathbf{c} - \mathbf{c}_1) = \rho^2_1 \quad (21)\]

That is, all points are located at a distance \(\rho_1\) from the center of the circle. Equations (19) to (21) constitute three equations in four unknowns (\(\rho_1\) and the three components of \(\mathbf{c}_1\)). A fourth equation (mistakenly neglected in ref. 4) results from the constraint that the tip of \(\mathbf{c}_i\) lies in the plane of the data points; that is, one of the equations

\[(\mathbf{c}_i - \mathbf{a}) \cdot \mathbf{N} = 0 \quad (22)\]

\[(\mathbf{c}_i - \mathbf{b}) \cdot \mathbf{N} = 0 \quad (23)\]

\[(\mathbf{c}_i - \mathbf{c}) \cdot \mathbf{N} = 0 \quad (24)\]

where

\[
\mathbf{N} = (\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})
\]

is a normal vector to the circular trajectory plane.

Eliminating \(\rho_1\) from equations (19) and (20) by using equation (21) results in the following equations:

\[\mathbf{A} - \mathbf{a} \cdot (\mathbf{A} - \mathbf{c}_i) = (\mathbf{c} - \mathbf{c}_1) \cdot (\mathbf{c} - \mathbf{c}_1) \quad (26)\]

\[\mathbf{B} - \mathbf{c}_1 \cdot (\mathbf{B} - \mathbf{c}_1) = (\mathbf{c} - \mathbf{c}_1) \cdot (\mathbf{c} - \mathbf{c}_1) \quad (27)\]

Equations (22), (26), and (27) are three linear equations in three unknowns (the components of \(\mathbf{c}_i\)). An equivalent matrix equation for computing the components of \(\mathbf{c}_i\) is

\[
\begin{bmatrix}
N(1) & N(2) & N(3) \\
C(1) - A(1) & C(2) - A(2) & C(3) - A(3) \\
C(1) - B(1) & C(2) - B(2) & C(3) - B(3)
\end{bmatrix}
\begin{bmatrix}
C_1(1) \\
C_1(2) \\
C_1(3)
\end{bmatrix}
= \begin{bmatrix}
\mathbf{N} \cdot \mathbf{A} \\
\frac{1}{2}(\mathbf{c} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{a}) \\
\frac{1}{2}(\mathbf{c} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{b})
\end{bmatrix}
\]

Once \(\mathbf{c}_i\) is known, any one of equations (19) to (21) yields \(\rho^2_1\).
Unit vector $\hat{u}_i$.— Figure 5 shows the circular trajectory of a point on the robot arm and two position vectors $\hat{A}$ and $\hat{B}$, along with the incremental joint angle

$$\Delta \theta_i = \theta_B - \theta_A$$

between these position vectors. A unit vector normal to the plane of the circular trajectory and passing through point $C_i$ (whose coordinates are the components of the vector $\hat{C}_i$) is

$$\hat{u}_i = \frac{\hat{A} - \hat{C}_i \times \hat{B} - \hat{C}_i}{\rho_i^2 \sin \Delta \theta_i}$$

with $0^\circ < \Delta \theta_i < \pi$, $\hat{u}_i$ in equation (30) is in the same direction as the rotational vector of joint $i$. An average $\hat{u}_i$ should be used to reduce the effects of measurement errors.

Axis $X_i$.— The direction of $X_i$ is defined by a unit vector $\hat{e}_i$ (fig. 3) that is normal to both the line of rotation for joint $i$ and the line of rotation for joint $i + 1$. Such a unit vector is given by

$$\hat{e}_i = \frac{(\hat{u}_i \times \hat{u}_{i+1})/|\hat{u}_i \times \hat{u}_{i+1}|}{|\hat{u}_i \times \hat{u}_{i+1}|}$$

if $\hat{u}_i \times \hat{u}_{i+1} \neq 0$, or by

$$\hat{e}_i = \hat{u}_i/|\hat{u}_i|$$

if $\hat{u}_{i+1} \times \hat{u}_i = 0$. The vector cross product in equation (31) gives a vector which is normal to both $\hat{u}_i$ and $\hat{u}_{i+1}$ if the lines of rotation for joints $i$ and $i + 1$ are not parallel. If the rotational lines are parallel, equation (32) is used.

Alternatively, $\hat{e}_i$ is any nonzero solution to the simultaneous linear homogeneous equations

$$\hat{u}_i \cdot \hat{e}_i = 0$$

and

$$\hat{u}_{i+1} \cdot \hat{e}_i = 0$$

whether or not $\hat{u}_i$ and $\hat{u}_{i+1}$ are parallel.
Axes $Z_i$ and $Y_i$.- By convention, $Z_i$ points in the direction of positive rotation for joint $i+1$, that is, in the direction of $\hat{u}_{i+1}$. $Y_i$ completes a right-hand axis system with $X_i$ and $Z_i$.

Equation for $\tan \theta'_i$.- The equation for $\tan \theta'_i$ is

$$\tan \theta'_i = \frac{(\hat{e}_{i-1} \times \hat{e}_i) \cdot \hat{u}_i}{\hat{e}_{i-1} \cdot \hat{e}_i} \tag{35}$$

The numerator in equation (35) shows the cross product of a vector $\hat{e}_{i-1}$ along $X_{i-1}$ and a vector $\hat{e}_i$ along $X_i$. Forming the dot product of this result and a unit vector $\hat{u}_i$ along $Z_{i-1}$ (or equivalently $\hat{u}_1$) produces $\sin \theta'_i$ with the correct sign for a positive rotation about $Z_{i-1}$ (or equivalently $Z_1$). The denominator is equivalent to $\cos \theta'_i$. Hence, the fraction represents $\tan \theta'_i$, where $0^\circ < \theta'_i < 2\pi$. The joint angle $\theta'_i$ in equation (35) corresponds to the fixed position of joint $i$ after $\hat{u}_i$ has been determined and when joint $i+1$ is being varied to obtain $\hat{u}_{i+1}$. The bias $\beta_i$ in equation (1) is just the calculated value of $\theta'_i$ when $\theta'_i = 0^\circ$.

$\tan \alpha_i$ with $X_i$ in direction of $\hat{e}_i$.- The appropriate equation is

$$\tan \alpha_i = \frac{[(\hat{u}_i \times \hat{u}_{i+1}) \cdot \hat{e}_i] / \hat{u}_i \cdot \hat{u}_{i+1}} \tag{36}$$

The right-hand side of this equation shows the cross product of a vector along $Z_{i-1}$ (or $\hat{u}_i$) and a vector along $Z_i$ (or $\hat{u}_{i+1}$), and then the dot product of this result and a unit vector along $X_i$ (or $\hat{e}_i$) gives $\sin \alpha_i$. The dot product in the denominator yields $\cos \alpha_i$.

Parameters $\xi_i$, $\eta_i$, $\zeta_i$, and $\rho_i$.- At $\theta_i = 0^\circ$, the components of $\hat{D}_i$ in coordinate system $i-1$ are $\xi_i$, $\eta_i$, and $\zeta_i$ in equations (9) to (11), respectively, and $\rho_i$ is a free parameter chosen for favorable location of $\theta_i$. The transverse vector $\hat{D}_i$ is computed in equation (7) with $\hat{R}_i$ from equation (14), where $\lambda_{i+1}$ depends on how $\hat{D}_i$ is constrained. If $\hat{D}_i$ is constrained normal to $Z_{i-1}$, equation (16) is used for $\lambda_{i+1}$, but if $\hat{D}_i$ is constrained normal to $Z_{i-1}$, equation (18) is used. In use, the constant parameters $\xi_i$, $\eta_i$, $\zeta_i$, and $\rho_i$ appear in equation (12) for $\hat{h}_i$, which only varies with the joint angle $\theta_i$ (or $\theta'_i - \beta_i$, eq. (1)). The vector $\hat{h}_i$ is then used in equation (3).

**EXAMPLE**

For the robot arm depicted in figure 2, this example focuses on determining the relative location of the elbow-joint axis system with respect to the shoulder-joint axis system. A transverse vector between the rotational axis $Z_2$ of the elbow joint and the rotational axis $Z_1$ of the shoulder joint is used in locating the elbow-joint axis system.
Transverse-Vector Constraints

Three ways to locate the axis system result from three separate constraints on the transverse vector, which is constrained to be either (1) Normal to both rotational axes $Z_1$ and $Z_2$, (2) Normal to the elbow-joint rotational axis $Z_2$, or (3) Normal to the shoulder-joint rotational axis $Z_1$. The first constraint leads to the well-known Denavit-Hartenberg parameters.

Transverse vector normal to both $Z_1$ and $Z_2$. The rotational axes of the shoulder and elbow joints in figure 2 are parallel. The Denavit-Hartenberg parameters are $a_2 = 0^\circ$, $r_2 = SN$ along $Z_1$, $a_2 = ES$ in the $X_2$ direction, which is normal to both $Z_1$ and $Z_2$, and $\theta_2 = \theta_2 + \beta_2$, where $\beta_2 = 90^\circ$ is the angle between $X_1$ and $X_2$. These parameters locate the elbow-joint axis system as shown at the point E. Such is not the case, however, when the axes are not parallel. For example, suppose $Z_2$ is rotated counterclockwise by $0.1^\circ$ about $Y_2$. The Denavit-Hartenberg parameters then become $a_2 = 0.1^\circ$ (angle between $Z_1$ and $Z_2$), $r_2 = -9734$ in. (intersection of $Z_2$ with $Z_1$ along $Z_1$), and $a_2 = 0$ (since $Z_2$ intersects $Z_1$). Thus, the elbow-joint axis system is moved away (by $r_2$) from the robot arm by about 811 ft. This does not happen with the modified parameters introduced in this paper. Indeed, the elbow-joint axis system remains located at the desired point E.

Transverse vector normal to $Z_1$. When the transverse vector is normal to $Z_1$ (fig. 6(a)), the point E is located by distances $\xi_2$ along $X_1$, $\eta_2$ along $Y_1$, and $r_2 + \xi_2$ along $Z_1$, where $r_2$ is now the desired location of the elbow-joint axis system. For this example, in which $a_2 = 0.1^\circ$, these parameters are as follows: $\xi_2 = 0$, $\eta_2 = ES = 17$ in., $\xi_2 = 0$ (by the constraint on the transverse vector, $\xi_2$ is always zero and need not be determined), and $r_2 = SN = 6$ in. (by choice).

Transverse vector normal to $Z_2$. When the transverse vector is normal to $Z_2$ (fig. 6(b)), the parameters that locate the elbow-joint axis system are as follows: $\xi_2 = 0$, $\eta_2 = ES = 17$ in., $\xi_2 = -ES/\tan a_2 = -0.030$, and $r_2 = SN - \xi_2 = 6.060$ in. Hence, $r_2 + \xi_2 = SN = 6$ in.

Simulated Measurements

In figure 7, assume that the waist- and shoulder-joint axis systems have been determined. Now the problem is to determine the elbow-joint axis system by using simulated locations of a point on the robot arm for three distinct elbow-joint angles. Specifically, locations of the point W are assumed to be measured with respect to the waist-joint axis system (which, in this example, coincides with the world reference axis system) for three distinct values of $\theta_3$. The simulated measurements are based on the following dimensions (in inches): NO = 26, SN = 6, and ES = WE = 17. Examples of simulated measurements are as follows:

$$
\begin{bmatrix}
\theta_A = 0 \\
\hat{A} = (6.000) \\
60.000
\end{bmatrix}
$$
\[ \theta_B = 45^\circ, \quad \hat{B} = \begin{pmatrix} 12.021 \\ 6.009 \\ 55.021 \end{pmatrix} \]

\[ \theta_C = 90^\circ, \quad \hat{C} = \begin{pmatrix} 17.000 \\ 6.030 \\ 43.000 \end{pmatrix} \]

These measurements give the location of point \( W \) to the nearest one-thousandth of an inch as \( \theta_3 \) takes on the values of \( \theta_A \), \( \theta_B \), and \( \theta_C \). In computing the simulated vector positions \( \hat{A} \), \( \hat{B} \), and \( \hat{C} \) (see appendix), the fact that \( a_2 = 0.1^\circ \) is assumed to be unknown. The following vectors are known:

\[ \hat{R}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ \hat{C}_2 = \hat{R}_1 \]

\[ \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]

\[ \hat{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]

These vectors are expressed in the waist (or world) axis system. The vector from 0 to \( N \) is \( \hat{R}_1 \); \( \hat{C}_2 \) is the vector from the center of the circular trajectory of point \( W \) as the shoulder joint is rotated; \( \hat{e}_1 \) is a unit vector normal to \( z_0 \) and \( z_1 \) that defines \( X_1 \); and \( \hat{u}_2 \) is a unit vector in the direction of rotation of joint 2 (shoulder) in base coordinates.
Also, based on the simulated measurement data,

\[
\hat{c}_3 = \begin{pmatrix} 0.000 \\ 6.030 \\ 43.000 \end{pmatrix}
\]

and

\[
\hat{u}_3 = \begin{pmatrix} 0.000 \\ 1.000 \\ 0.002 \end{pmatrix}
\]

where \( \hat{c}_2 \) is a vector to the center of the circular trajectory from the waist coordinate system and \( \hat{u}_3 \) is a unit vector along \( Z_2 \) (the line of rotation of the elbow joint) in the waist coordinate system.

**Location of Origin of Elbow-Joint Axis System by Denavit-Hartenberg Parameters**

For the regular Denavit-Hartenberg parameters, the transverse vector, which is \( \hat{D}_2 \) in the notation of the text, is normal to both of the lines of rotation of the elbow and shoulder joints. Specifically (from ref. 4, pages 9 to 11, with \( i = 2 \) and \( \hat{D}_2 = \hat{v}_2^* \)),

\[
\hat{D}_2 = \hat{t}_3 - \hat{t}_2
\]

where

\[
\hat{t}_2 = \hat{c}_3 + \lambda_3 \hat{u}_3
\]

and

\[
\hat{t}_1 = \hat{c}_2 + \lambda_2 \hat{u}_2
\]

with

\[
\lambda_1 = \frac{[\hat{c}_3 - \hat{c}_2] \cdot [\hat{u}_2 - (\hat{u}_3 \cdot \hat{u}_2) \hat{u}_3]}{1 - (\hat{u}_3 \cdot \hat{u}_2)^2}
\]
\[ \lambda_2 = \frac{-(\mathbf{c}_3 - \mathbf{c}_2) \cdot (\mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{u}_2)\mathbf{u}_2)}{1 - (\mathbf{u}_3 \cdot \mathbf{u}_2)^2} \]  

That is, \( \lambda_1 \) and \( \lambda_2 \) are such that \( \mathbf{d}_2 \cdot \mathbf{u}_2 = 0 \) and \( \mathbf{d}_2 \cdot \mathbf{u}_3 = 0 \).

The regular Denavit-Hartenberg parameter \( r_2 \) is the component of

\[ \mathbf{r}_2 = \mathbf{r}_2 \cdot \mathbf{u}_2 \]  

along \( Z_1 \); that is,

\[ r_2 = r_2 \cdot \mathbf{u}_2 \]  

The regular Denavit-Hartenberg parameter \( a_2 \) is the component of

\[ \mathbf{a}_2 = \mathbf{a}_2 \cdot \mathbf{e}_1 \]  

along \( X_2 \); that is,

\[ a_2 = a_2 \cdot \mathbf{e}_1 \]

Based on the simulated measurement data, \( r_2 = -9789.719 \) in. and \( a_2 = 0.298 \) in. The true values are \( r = -9734.273 \) in. and \( a_2 = 0 \) in. Hence, the calculated values of \( r_2 \) and \( a_2 \) are in absolute error by 55.446 in. and 0.298 in., respectively.

Different values of \( a_2 \) are shown in the first column of table I, and the Denavit-Hartenberg parameters \( r_2 \) and \( a_2 \) that locate the origin of the elbow-joint axis system are shown in the second and third columns, respectively. Note that \( a_2 = 0 \) because the lines of rotation of the elbow and shoulder joints intersect. For measurements (components of \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \)) rounded off to the nearest ten-thousandth, one-thousandth, and one-hundredth of an inch, table I shows the absolute errors in the computed values of the parameters \( r_2 \) and \( a_2 \). As the elbow and shoulder joints approach a parallel condition \( (a_2 \neq 0^\circ) \), two important observations from table I are as follows: (1) The elbow-joint axis system is located far off the robot arm (large value of \( r_2 \)), and (2) The error in the computed value of \( r_2 \) is excessively large, even for very accurate measurements.
Location of Origin of Elbow-Joint Axis System by Modified Parameters

In this example for the modified parameters, the transverse vector from the elbow-joint axis of rotation to the shoulder-joint axis of rotation is normal to the elbow-joint axis of rotation. By choice, \( r_2 = SN = 6 \text{ in.} \) in this example. The true modified parameters are shown in table II for different values of \( \alpha_2 \) (angle between rotational axes of the elbow and shoulder joints) to locate the elbow-joint axis system at point E in figure 7. Errors in the calculated parameters using measurement data rounded off to the nearest ten-thousandth, one-thousandth, and one-hundredth of an inch are also shown. In addition to locating the elbow-joint axis system at the desired point E in figure 7, the modified parameters can be accurately calculated even as the lines of rotation of the elbow and shoulder joints approach a parallel condition (\( \alpha_2 \approx 0^\circ \)).

CONCLUDING REMARKS

At present, the most popular way to describe the relative location of successive joint axis systems in a robot arm is to use the Denavit-Hartenberg parameters. However, a recent justifiable criticism is that one of these parameters approaches infinity when two successive joints have nearly parallel rotational axes. Geometrically, this parameter removes the joint axis an excessive distance from the robot arm; computationally, this large parameter leads to an ill-conditioned transformation matrix. In this paper, a simple modification in the location of this axis system easily overcomes this criticism. This modification results by insisting that a transverse vector between successive joint rotational axes be normal to one of the rotational axes instead of to both axes. This simple modification leads to modified Denavit-Hartenberg parameters that favorably locate successive joint axis systems.

An example is given with respect to the rotational axes of the elbow and shoulder joints in a robot arm. The regular and modified Denavit-Hartenberg parameters (that locate the elbow-joint axis system relative to the shoulder-joint axis system) are extracted by an algebraic method via simulated measurement data.

A point near the robot hand (off the line of rotation) generates a circular trajectory as the elbow joint is rotated. Three position vectors to this point and the corresponding three elbow joint angles (referenced to an initial position) are simulated for the extraction process.

For small misalignments of the shoulder and elbow joints away from a parallel condition (i.e., parallel rotational axes), the Denavit-Hartenberg parameters locate the elbow-joint axis system far away from the robot arm; the modified parameters locate the axis system at the desired place on the robot arm. In addition, for a given accuracy of the measurements used in the parameter-extraction process, the extracted values for the Denavit-Hartenberg parameters yielded considerably larger errors than did the extracted values for the modified parameters. It appears that the modified parameters provide a more natural location of successive joint axis systems and are useful in the industrial calibration of robot arms.

NASA Langley Research Center
Hampton, VA 23665-5225
March 13, 1986
APPENDIX

EQUATIONS FOR SIMULATED MEASUREMENTS

By Euler’s theorem (ref. 5), a vector $\mathbf{r}$ rotated by an angle $\theta$ about a line (or unit vector $\mathbf{u}$) becomes

$$\mathbf{r}' = \cos \theta \mathbf{r} + \mathbf{r} \cdot \mathbf{u} (1 - \cos \theta) \mathbf{u} - (\mathbf{r} \times \mathbf{u}) \sin \theta$$  \hspace{1cm} (A1)

This equation is applied to obtain an expression for the location of point $W$ on the robot arm in figure A1 with respect to the waist axis system as the elbow joint $\theta_3$ is rotated for a specified value of $\alpha_2$.

For the position of the robot arm in figure A1, and with respect to the $X$, $Y$, and $Z$ axes, let $\mathbf{r} = (0, WE, 0)^T$ be a vector from $E$ to $W$, and let $\mathbf{u}_3 = (\sin \alpha_2, 0, \cos \alpha_2)^T$ be a vector along the rotational axis of the elbow joint. Moreover, let $\theta = \theta_3$ be the elbow-joint angle. Then $\mathbf{r}'$ gives the new location of $W$ as a function of $\theta_3$ and $\alpha_2$. The location of $W$ with respect to the waist axis system is then simply

$$\mathbf{W} = \begin{pmatrix} 0 \\ \mathbf{SN} \\ \mathbf{NO} + \mathbf{ES} \end{pmatrix} + \begin{pmatrix} r'_X \\ r'_Y \\ r'_Z \end{pmatrix}$$  \hspace{1cm} (A2)

where $r'_X$, $r'_Y$, and $r'_Z$ are the components of $\mathbf{r}'$ (or coordinates of $W$) along the $X$, $Y$, and $Z'$ axes.

Simulated measurement vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are the same as $\mathbf{W}$, except $\theta_3$ is replaced by $\theta_A$, $\theta_B$, and $\theta_C$, respectively. For the example in the text of this report, $\theta_A = 0^\circ$, $\theta_B = 45^\circ$, and $\theta_C = 90^\circ$. 

17
Figure A1. - Elbow-joint rotational axis, with respect to an auxiliary axis system, for simulating measurement data.
REFERENCES


TABLE I.- ERRORS IN CALCULATED DENAVIT-HARTENBERG PARAMETERS THAT LOCATE ORIGIN OF ELBOW-JOINT AXIS SYSTEM FOR ROBOT ARM

Simulated measurements to point on robot arm correspond to elbow-joint angles of 0°, 45°, and 90°

<table>
<thead>
<tr>
<th>True simulated value of -</th>
<th>Simulated point measurements rounded off to nearest -</th>
<th>Absolute error in computed value of -</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_2$, deg</td>
<td>$r_2$, in.</td>
<td>$a_2$, in.</td>
</tr>
<tr>
<td>---------------------------</td>
<td>-------------------------------------------------------</td>
<td>--------------------------------------</td>
</tr>
<tr>
<td>0.01</td>
<td>-97396.82</td>
<td>0.00</td>
</tr>
<tr>
<td>.01</td>
<td>-97396.82</td>
<td>0.00</td>
</tr>
<tr>
<td>.01</td>
<td>-97396.82</td>
<td>0.00</td>
</tr>
<tr>
<td>0.10</td>
<td>-9734.27</td>
<td>0.00</td>
</tr>
<tr>
<td>.10</td>
<td>-9734.27</td>
<td>0.00</td>
</tr>
<tr>
<td>.10</td>
<td>-9734.27</td>
<td>0.00</td>
</tr>
<tr>
<td>1.00</td>
<td>-967.93</td>
<td>0.00</td>
</tr>
<tr>
<td>1.00</td>
<td>-967.93</td>
<td>0.00</td>
</tr>
<tr>
<td>1.00</td>
<td>-967.93</td>
<td>0.00</td>
</tr>
<tr>
<td>10.00</td>
<td>-90.41</td>
<td>0.00</td>
</tr>
<tr>
<td>10.00</td>
<td>-90.41</td>
<td>0.00</td>
</tr>
<tr>
<td>10.00</td>
<td>-90.41</td>
<td>0.00</td>
</tr>
</tbody>
</table>

$10^{-4}$ $10^{-3}$ $10^{-2}$ $10^{-1}$
TABLE II.- ERRORS IN CALCULATED MODIFIED PARAMETERS THAT LOCATE ORIGIN OF ELBOW-JOINT AXIS SYSTEM FOR ROBOT ARM WHEN TRANSVERSE VECTOR IS CONSTRAINED NORMAL TO ELBOW-JOINT ROTATIONAL AXIS

Simulated measurements to point on robot arm correspond to elbow-joint angles of 0°, 45°, and 90°

<table>
<thead>
<tr>
<th>$a_2$, deg</th>
<th>$e_2$, in.</th>
<th>$n_2$, in.</th>
<th>$r_2$, in.</th>
<th>Simulated point measurements rounded off to nearest -</th>
<th>Absolute error in computed value of -</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$10^{-4}$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>0.01</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.01</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.01</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.10</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.10</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.00</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.00</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.00</td>
<td>0.00</td>
<td>17.00</td>
<td>6.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 1. - Denavit-Hartenberg parameters.
Figure 2. Initial position of robot arm and joint axis systems.
Figure 3.- Lines of rotation for successive joints.
Figure 4.— Three locations on a circular trajectory of a point on robot arm about line of rotation for joint 1.

Figure 5.— Unit vector in same direction as joint rotational vector.
Figure 6. Location of point $E$ from $N$ by $h_2$.

(a) $D_2$ normal to shoulder rotational axis $Z_1$.

(b) $D_2$ normal to elbow rotational axis $Z_2$.

26
Figure 7.- Geometry used in locating elbow-joint axis system.
The Denavit-Hartenberg parameters define the relative location of successive joint axis systems in a robot arm. A recent justifiable criticism is that one of these parameters becomes extremely large when two successive joints have near-parallel rotational axes. Geometrically, this parameter then locates a joint axis system at an excessive distance from the robot arm and, computationally, leads to an ill-conditioned transformation matrix. In this paper, a simple modification (which results from constraining a transverse vector between successive joint rotational axes to be normal to one of the rotational axes, instead of both) overcomes this criticism and favorably locates the joint axis system.

An example is given for near-parallel rotational axes of the elbow and shoulder joints in a robot arm. The regular and modified parameters are extracted by an algebraic method with simulated measurement data. Unlike the modified parameters, extracted values of the regular parameters are very sensitive to measurement accuracy.
<table>
<thead>
<tr>
<th>NAME</th>
<th>MS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chen</td>
<td>230</td>
</tr>
</tbody>
</table>