NAG2-366 DAA/Ames

Approximating Linearizations for Nonlinear Systems

L.R. Hunt, R. Su, and G. Meyer

ABSTRACT

Given a nonlinear control system

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t))$$

on $\mathbf{R}^n$ and a point x_0 in $\mathbf{R}^n$, we want to approximate the system near x_0 by a linear system. Of course, one approach is to use the usual Taylor series linearization. However, the controllability properties of both the nonlinear and linear systems depend on certain Lie brackets of the vector field under consideration. This suggests that we should construct a linear approximation based on Lie bracket matching at x_0. In general, the linearizations based on the Taylor method and the Lie bracket approach are different. However, under certain mild assumptions, we show that there is a coordinate system for $\mathbf{R}^n$ near x_0 in which these two types of linearizations agree. We indicate the importance of this agreement by examining the time responses of the nonlinear system and its linear approximation and comparing the lower order kernels in Volterra expansions of each.

(NASA-TM-88772) APPROXIMATING LINEARIZATIONS FOR NONLINEAR SYSTEMS Semiannual Progress Report (NASA) 23 p CSCL 12A

N86-29578

Unclas G3/64 43229

TM-88772

dSNC 473659

Approximating Linearizations for Nonlinear Systems

L.R. Hunt*, R. Su** and G. Meyer+

I. Introduction

Suppose we have a nonlinear control system that we wish to simplify in some way. An approach that has received much attention in the literature is the exact linearization, whereby the nonlinear system is transformed to a linear system. Both theoretical problems [1], [2],[3],[4],[5],[6],[7] and practical applications [8],[9],[10],[11], [12],[13],[14],[15],[16],[17] concerning this method exist. Theoretically, a nonlinear control system in which the controls enter linearly, is "(locally) equivalent" to a controllable linear system if and only if (i) a certain set of vector fields is linearly independent and (ii) related sets of vector fields are involutive (see [3] and [5]. These two conditions will be explicitly stated later.

The ideas in this paper were generated by the desire to construct an approximate transformation (if an exact one cannot be found) for a nonlinear system that is transformable to a linear one. Meyer [12] used linear Taylor series expansions about certain points along a flight trajectory to discover approximate transformations. However, this type of linearization did not in general reflect the rich differential geometry inherent in the assumptions (i) and (ii) previously

*Research supported by NASA Ames Research Center under grants NAG2-189 and NAG2-366 and the Joint Services Electronics Program under ONR Contract N00014-76-C1136.

**Research supported by NASA Ames Research Center under grant NAG2-203 and the Joint Services Electronics Program under ONR Contract N00014-76-C1136.

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mentioned. Therefore, we introduced in [20] a linear approximation based on the important Lie brackets.

Given a nonlinear system which satisfies condition (i) (and not necessarily condition (ii)) and a point x_0 in state space, there are two types of linearizations of interest. One depends on Taylor series, the other on Lie brackets, and in most cases these do not agree. However, if x_0 is an equilibrium point of the drift term of the system or if we are operating about a known trajectory containing the point x_0, then the two linearizations are the same.

Recently, the authors have discovered for nonlinear systems satisfying the condition (i) that there exists a coordinate system (called the s coorinates) on state space in which the nonlinear system is in a particularly nice form [18], [19]. This state space coordinate system appears in the literature in [1] and [3], but the canonical system expansion of a general nonlinear system presented in [19] is new.

The main point of this paper is to show that the two kinds of linearizations coincide in the s coordinate system that we studied in [19]. We consider formal Volterra series and stress the importance of this agreement in computing Volterra kernels. From an input to output (input to state in our case) point of view, for any appropriate input the error between the time responses of the actual system and the approximating system (in the special coordinates) propagates like $0(|t|^3)$ in the single input case.

We show how to compute the coordinate changes to move from the original coordinate system to the one in which the linearizations agree, but it is not always possible to do this in practice. However, we are still able to find the linear part about x_0 of the nonlinear system in the special coordinates.

In Section 2 of this paper we present definitions, review the exact linearization results, and indicate the desired coordinate system for a nonlinear control system. Section 3 consists of an example, the main result, and the interpretation of this result in terms of Volterra kernels.

II. Preliminaries

For C^∞ vector fields $f(x)$ and $g(x)$ on $\mathbb{R}^n$ we denote the Lie bracket (this is the negative of the usual definition)

$$\frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f$$

by $[f,g]$, where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are Jacobian matrices. We also define

$$\begin{aligned} (ad^0 f,g) &= g \\ (ad^1 f,g) &= [f,g] \\ &\vdots \\ (ad^k f,g) &= [f,(ad^{k-1} f,g)]. \end{aligned}$$

Let $h(x)$ be a C^∞ real-valued function on $\mathbb{R}^n$ and

$$f(x) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

a C^∞ vector field. The Lie derivatives of h with respect to f, $L_f h(x)$, is defined to be

$$\langle dh, f\rangle = \frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 + \dots + \frac{\partial h}{\partial x_n} f_n.$$

We take

$$\begin{aligned} L_f^0 h(x) &= h(x) \\ L_f^1 h(x) &= L_f h(x) \\ &\vdots \\ L_f^k h(x) &= \langle dL_f^{k-1} h(x), f\rangle. \end{aligned}$$

Given a nonlinear control system on $\mathbb{R}^n$

$$(1) \qquad \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)),$$

where $f, g_1, \dots, g_m$ are C^∞ vector fields, and a point x_0 in $\mathbb{R}^n$, we are interested in finding linear (affine) approximations of the form

$$(2) \qquad \dot{x}(t) = f(x_0) - Ax_0 + Ax + \sum_{i=1}^{m} u_i(t) b_i$$

which are appropriate for use in control problems.

Let $\kappa_1, \kappa_2, \dots, \kappa_m$ be positive integers such that $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m \geq 1$ and $\kappa_1 + \kappa_2 + \dots + \kappa_m = n$. We take the sets

$$C = \{g_1, [f, g_1], \dots, (ad^{\kappa_1 - 1} f, g_1), g_2, [f, g_2], \dots, (ad^{\kappa_2 - 1} f, g_2), \dots, g_m, [f, g_m], \dots, (ad^{\kappa_m - 1} f, g_m)\}$$

$$C_j = \{g_1, [f,g_1], \ldots, (ad^{\kappa_j - 2} f, g_1), g_2, [f,g_2], \ldots,$$
$$(ad^{\kappa_j - 2} f, g_2], \ldots, g_m, [f,g_m], \ldots, (ad^{\kappa_j - 2} f, g_m)\}$$

for $j=1,2,\ldots,m$. Suppose near x_0 system (1) satisfies the two conditions:

(i) The set C spans n dimensional space and the span of C_j equals the span of $C_j \cap C$ for each $j=1,2,\ldots,m$.

(ii) Each set C_j, $j=1,2,\ldots,m$, is involutive; i.e. the Lie bracket of any two vector fields in C_j is a linear combination of the vector fields in C_j.

Then it is proved in [3] and [5] if $f(x_0)=0$ and [21] if $f(x_0)\neq 0$ that system (1) is locally equivalent to a controllable linear system with Kronecker indices $\kappa_1, \kappa_2, \ldots, \kappa_m$ (we can renumber the $g_1, g_2, \ldots, g_m$ to make $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_m$ if necessary). Hence there are new state and control variables in which system (1) is actually a linear system for x near x_0. This is called an <u>exact linearization</u> of (1).

If the above two conditions hold and the state and control transformations can be found (a method for constructing such transformations is described in [5]), then it is not necessary to approximate the nonlinear system (1) by a linear system because in the correct coordinates it is a linear system.

Suppose we assume that assumption (i) holds for system (1) but discard condition (ii). We present a coordinate system (called the s coordinates) on R^n near x_0 in which our nonlinear system takes a particularly nice form. In fact, if condition (ii) also holds we have a pure feedback system as in [19] (related results are in [3]), and it is trivial to move from this form to a linear system.

In a special case these pure feedback systems are called block triangular by Meyer and Cicolani [9].

Emphasizing that we are working under condition (i) only, we reorder the elements in the set C to reflect descending orders in the superscripts on the ad's and ascending orders in the subscripts of the g's. We call this reordering C' and the first element of C' is $(ad^{\kappa_1-1} f, g_1)$. If $\kappa_1=\kappa_2$, the second element of C' is $(ad^{\kappa_2-1} f, g_2)$, and if $\kappa_1 > \kappa_2$, it is $(ad^{\kappa_1-2} f, g_1)$. If $\kappa_1=\kappa_2=\kappa_3$, the third element is $(ad^{\kappa_3-1} f, g_3)$, if $\kappa_1=\kappa_2>\kappa_3$, it is $(ad^{\kappa_1-2} f, g_1)$, and if $\kappa_1>\kappa_2\geq\kappa_3$ it is $(ad^{\kappa_1-3} f, g_1)$ or $(ad^{\kappa_2-1} f, g_2)$ depending on whether $\kappa_1-1>\kappa_2$ or not. The process can be continued, and the last element in C' becomes g_m. This order is simply the opposite lexicographic order.

For $x_0=(x_{10},x_{20},\ldots,x_{n0})$ given and $s_0=(s_{10},s_{20},\ldots,s_{n0})$ we solve in order the following systems of ordinary differential equations with initial conditions;

(3)

$$\frac{dx(s_1)}{ds_1} = (ad^{\kappa_1-1} f, g_1),\ x(s_{10}) = x_0$$

$$\frac{dx(s_1,s_2)}{ds_2} = \text{2nd element of } C',\ x(s_1,s_{20}) = x(s_1)$$

$$\frac{dx(s_1,s_2,s_3)}{ds_3} = \text{3rd element of } C',\ x(s_1,s_2,s_{30}) = x(s_1,s_2)$$

$$\vdots$$

$$\frac{dx(s_1,s_2,\ldots,s_n)}{ds_n} = g_m,\ x(s_1,s_2,\ldots,s_{n-1},s_{n0}) = x(s_1,s_2,\ldots,s_{n-1}).$$

Since the set C' consists of linearly independent vector fields, by the inverse function theorem we can solve for $s_1, s_2, \ldots, s_n$ as functions of $x_1, x_2, \ldots, x_n$ near x_0. Moreover, we can take the point $s_0 = 0$, the origin in s-space.

We now view the s coordinates geometrically and introduce a sequence of manifolds $S_0, S_1, \ldots, S_n$ in the following manner. S_0 is the point 0 and S_1 is the one-dimensional integral manifold of $(ad^{\kappa_1 - 1} f, g_1)$ through the point 0. Similiarly, S_2 is the two-dimensional manifold constructed by taking the integral curves of the second element of C' through S_1. Likewise, S_3 is the three-dimensional manifold formed by merging the integral curves of the third element of C' through S_2. Continuing in this manner, the manifold S_n is a neighborhood of the point 0 in $\mathbb{R}^n$ on which the above process is guaranteed from the inverse function theorem. Hence in the s coordinates

$$g_m \text{ is the vector field } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

g_{m-1} (or the second to last element of C')

$$\text{is the vector field } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ when restricted to } S_{n-1},$$

g_{m-2} (or the third to last element of C') is the vector field $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$ when restricted to S_{n-2},

$$\vdots$$

the second element in C' is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ when restricted to S_2, and

$(ad^{\kappa_1 - 1} f, g_1)$ is $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$ on S_1.

The above notation assumes that we have written the nonlinear system (1) in the s coordinates as

$$\dot{s} = f(s) + \sum_{i=1}^{m} u_i g_i(s), \tag{4}$$

where the new f is $(\frac{\partial x}{\partial s})^{-1} f(x(s))$ and each new g_i is $(\frac{\partial x}{\partial s})^{-1} g_i(x(s)$. We remark that in the s coordinates,

$$g_m = \frac{\partial}{\partial s_n}$$

$$g_{m-1} = \frac{\partial}{\partial s_{n-1}} \text{ on } S_{n-1}$$

$$\vdots$$

$$(ad^{\kappa_1 - 1} f, g_1) = \frac{\partial}{\partial s_1} \text{ on } S_1.$$

These facts will prove to be useful in our later work.

We shall return to the nonlinear system (1) and introduce two kinds of approximate linearizations for nonlinear systems.

III. Approximate Linearizations

Given system (1) and point x_0, we suppose that condition (i) is satisfied. We can do the usual Taylor series approach to find the linear approximation (about x_0 and zero controls)

$$\dot{x}(t) = f(x_0) - Ax_0 + Ax + \sum_{i=1}^{m} u_i(t)b_i, \tag{5}$$

where $A = \frac{\partial f}{\partial x}(x_0)$ and $b_i = g_i(x_0)$, $i=1,2,\ldots,m$. This is called the <u>tangent model</u> in [12]. Using Lie bracket matching at x_0 we arrive at the <u>modified tangent model</u> [20]

$$\dot{x}(t) = f(x_0) - Ax_0 + Ax + \sum_{i=1}^{m} u_i(t)b_i, \tag{6}$$

where $A, b_1, \ldots, b_m$ are defined by

$$\begin{aligned} A^k b_1 &= (\mathrm{ad}^k f, g_1)(x_0), \; k = 0,1,\ldots,\kappa_1 \\ A^k b_2 &= (\mathrm{ad}^k f, g_2)(x_0), \; k = 0,1,\ldots,\kappa_2 \\ &\vdots \\ A^k b_m &= (\mathrm{ad}^k f, g_m)(x_0), \; k = 0,1,\ldots,\kappa_m. \end{aligned} \tag{7}$$

In general the A matrices defined by (5) and (7) are different.

The advantages and disadvantages of each of these two types of linearizations will be stressed in terms of the formal Volterra series introduced later. We now show that for classical problems in control theory, these linearizations agree.

Suppose x_0 is an equilibrium point of the system $\dot{x}=f(x)$ in (1) (i.e. $f(x_0)=0$) and assume $x_0=0$. The system (5) given by Taylor series has the property that $A^k b_i=(ad^k f,g_i)(0)$, $k=0,1,2,\ldots,\kappa_i$, and $i=1,2,\ldots,m$. Thus the tangent model and modified tangent agree in this case.

Suppose φ is a trajectory of system (1) corresponding to all $u_i=0$; in other words $\dot{\varphi}=f(\varphi(t))$. We let

$$\dot{z} = A(t)z + \sum_{i=1}^{m} u_i(t)b_i(t),$$

where

$$z = x-\varphi(t)$$

$$A = \frac{\partial f}{\partial x}(\varphi(t))$$

$$b_i = g_i(\varphi(t)).$$

Setting $\varphi(t_0)=x_0$ and $\Gamma=A(t)-\frac{d}{dt}$, a time varying Lie derivative, we find (as Hermes did in [22]) that

$$b_i(t_0) = g_i(x_0),\ i = 1,2,\ldots,m,$$

$$\Gamma b_i(t_0) = A(t_0)b_i(t_0) - \frac{\partial b_i(t_0)}{\partial t}$$

$$= \frac{\partial f}{\partial x}(x_0)g_i(x_0) - \frac{\partial g_i}{\partial x}(x_0)f(x_0)$$

$$= [f,g_i](x_0),\ i = 1,2,\ldots,m,$$

$$\Gamma^2 b_i(t_0) = (ad^2 f,g_i)(x_0),\ i = 1,2,\ldots,m,$$

$$\vdots \quad .$$

Hence linearizing about the trajectory $\varphi(t)$ and evaluating at $x_0=\varphi(t_0)$

show that the Taylor series approach and Lie bracket matching method yield the same result.

We now present an example in R^3 to show that these approximating linearizations can be different.

Example 3.1. On R^3 we take the single input system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2+x_1x_3 \\ x_3 \\ x_1 \end{bmatrix} + u\begin{bmatrix} x_3 \\ 0 \\ 1 \end{bmatrix} = f(x)+ug(x)$$

and $x_0=(x_{10},x_{20},x_{30})$. The tangent model in this case is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_{20}+x_{10}x_{30} \\ x_{30} \\ x_{10} \end{bmatrix} + \begin{bmatrix} x_{30} & 1 & x_{10} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1-x_{10} \\ x_2-x_{20} \\ x_3-x_{30} \end{bmatrix} + u\begin{bmatrix} x_{30} \\ 0 \\ 1 \end{bmatrix}$$

$$= f(x_0) + A(x-x_0) + ub.$$

Computing we find that

$$Ab = \begin{bmatrix} x_{30}^2+x_{10} \\ 1 \\ x_{30} \end{bmatrix}, [f,g](x_0) = \begin{bmatrix} x_{30}^2 \\ 1 \\ x_{30} \end{bmatrix},$$

and if $x_{10}\neq 0$, the tangent model and modified tangent model cannot coincide.

The work of Hermes [22] on controlling a system along a trajectory indicates for this example that the important Lie brackets are $g(x_0), [f,g](x_0), (ad^2 f,g)(x_0)$. These are linearly independent at any point x_0. Thus if we are operating near a point x_0 with $x_{10} \neq 0$, the modified tangent model seems appropriate. On the other hand, if the control $u \equiv 0$, the standard linear Taylor series approach seems reasonable. In our later discussion of Volterra series, these observations will be explained. We remark that in our s coordinate system, we do not have to choose between these types of linearizations. The authors wish to thank a reviewer for shortening the proof of the following result.

Theorem 3.1. Given the nonlinear control system (1) satisfying condition (i) near the point $x_0 \in \mathbb{R}^n$, there exist a local coordinate system on $\mathbb{R}^n$ at x_0 for which the tangent model and modified tangent model agree.

Proof. The s coordinates are the obvious candidates so we assume our nonlinear system is given by equations (4)

$$\dot{s} = f(s) + \sum_{i=1}^{m} u_i g_i(s).$$

First we construct the tangent model at the point 0 where $x(0)=x_0$. We write

$$f(s) = f(s_1, s_2, \ldots, s_n)$$

and expand in a Taylor series to find

$$f(s) = f(0) + \sum_{i=1}^{n} \frac{\partial f}{\partial s_i}(0)\, s_i + 0(s^2).$$

Since g_m is equal to $\frac{\partial}{\partial s_n}$, g_{m-1} is equal to $\frac{\partial}{\partial s_{n-1}}$ on $S_{n-1}, \ldots,$ $(\mathrm{ad}^{\kappa_1 - 1} f, g)$ is equal to $\frac{\partial}{\partial s_1}$ on S_1, we have

$$\frac{\partial f}{\partial s_n}(0) = [f,g_m](0)$$

$$\frac{\partial f}{\partial s_{n-1}}(0) = [f,g_{m-1}](0)$$

$$\vdots$$

$$\frac{\partial f}{\partial s_1}(0) = [f,(ad^{\kappa_1 -1} f,g_1)](0) = (ad^{\kappa_1} f,g_1)(0).$$

Expanding

$$\dot{s} = f(s) + \sum_{i=1}^{m} u_i g_i(s)$$

in a Taylor series with zero controls yields the tangent model

$$\dot{s} = f(0) + As + \sum_{i=1}^{m} u_i b_i,$$

with $b_i = g_i(0), i = 1,2,\ldots,m$ and $A = \frac{\partial f}{\partial s}(0)$. Then equations (7) are easily verified for A and b_i. □

A discussion of formal Volterra series, in which questions concerning convergence are ignored, is appropriate. We take the nonlinear system (1) and add as output the identity function on $\mathbb{R}^n$, i.e.

$$y = h(x) = (h_1(x),h_2(x),\ldots,h_n(x))$$

(8)
$$h_1(x) = x_1$$
$$h_2(x) = x_2$$
$$\vdots$$
$$h_n(x) = x_n.$$

If we are concerned with convergence, then we must take $f,g_1,\ldots,g_m$ to be real-analytic. However, since we are interested in low order

Volterra kernels, we stay with the C^∞ assumption and consider expansions of low order plus remainder as in [23].

We assume for the rest of our discussion on Volterra series that the initial value problem $\dot{x}=f(x), x(0)=x_0$ has a solution on $[0,T]$, for some $T>0$, and the real-valued inputs considered are in $L^1([0,T])$. This allows us to discuss finite Volterra expansions with remainder (analogously to that of Taylor series) as in [23]. The time t will be restricted to the set $[0,T]$.

From [24] we take the Volterra expansions for the system (1) with output $y=h(x)$

(9)
$$\begin{aligned} y(t) &= w_0(t) + \sum_{i=1}^{m} \int_0^t w_i(t,\tau_1)u_i(\tau_1)d\tau \\ &+ \sum_{i_1,i_2=1}^{m} \int_0^t \int_0^{\tau_1} w_{i_1 i_2}(t,\tau_1,\tau_2)u_{i_1}(\tau_1)u_{i_2}(\tau_2)d\tau_1 d\tau_2 + O(t^3) \end{aligned}$$

where

$$\text{(using } \langle dh,f\rangle = \begin{bmatrix} \langle dh_1,f\rangle \\ \langle dh_2,f\rangle \\ \vdots \\ \langle dh_n,f\rangle \end{bmatrix} = L_f^1 h, \text{ etc.)}$$

$$w_0(t) = \sum_{k=0}^{\infty} L_f^k h|_{x_0} \frac{t^k}{k!} = e^{tf}h|_{x_0}$$

(10)
$$w_i(t,\tau_1) = \sum_{k_2,k_1=0}^{\infty} L_f^{k_2} L_{g_i} L_f^{k_1} h|_{x_0} \frac{(t-\tau_1)^{k_1}\tau_1^{k_2}}{k_1!k_2!}, \quad i=1,2,\ldots,m,$$

$$w_{i_1 i_2}(t,\tau_1,\tau_2) = \sum_{k_3,k_2,k_1=0}^{\infty} L_f^{k_3} L_{g_{i_2}} L_f^{k_2} L_{g_{i_1}} L_f^{k_1} h|_{x_0} \frac{(t-\tau_1)^{k_1}(\tau_1-\tau_2)^{k_2}\tau_2^{k_3}}{k_1!k_2!k_3!},$$

$$i_1, i_2 = 1,2,\ldots,m.$$

Here the notation from [24] has been extended to multi-input systems, and the infinite series are to be taken formally and will be truncated in our discussion.

Now

$$\begin{aligned} L_f^0 h(x_0) &= h(x_0) = x_0, \\ L_f^1 h(x_0) &= \langle dh, f\rangle(x_0) = f(x_0) \\ L_f^2 h(x_0) &= \langle df, f\rangle(x_0) = \frac{\partial f}{\partial x}(x_0) f(x_0). \end{aligned} \tag{11}$$

Recall that the important consideration in this paper is the approximation of the nonlinear system (1) by a linear system of the form (2). Suppose we consider Volterra expansions in the form (9) for systems (1) and (2), both having the identity as output. If (2) is formed by the usual Taylor approach (i.e. we have the tangent model), then for the first three terms in $w_0(t)$ for the system (2) we obtain

$$\begin{aligned} L^0_{f(x_0)+A(x-x_0)} h(x_0) &= x_0, \\ L^1_{f(x_0)+A(x-x_0)} h(x_0) &= f(x_0) \\ L^2_{f(x_0)+A(x-x_0)} h(x_0) &= Af(x_0) = \frac{\partial f}{\partial x}(x_0) f(x_0). \end{aligned} \tag{12}$$

Hence we have agreement in the kernels $w_0(t)$ for the nonlinear system and the tangent model through order t^2 terms. This is a significant characteristic of the linear Taylor series expansion for our nonlinear system. If A in (2) is not $\frac{\partial f}{\partial x}(x_0)$, as can occur in the modified tangent model, this agreement through order t^2 is not assured.

Letting $e^{tf}h|_{x_0}$ denote the flow of the system $\dot{x}=f(x(t))$ starting at x_0, we can rewrite the kernels $w_i(t,\tau_1)$ as (see [24])

$$w_i(t,\tau_1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(ad^k \tau_1 f, g_i) e^{tf} h \,|_{x_0}, \quad i = 1,2,\ldots,m. \tag{13}$$

We emphasize the appearance of the Lie brackets $(ad^k f, g_i)$ in these kernels. The modified tangent model appears to be more natural than the tangent model because of this Lie bracket matching at x_0 through order $k=\kappa_i$ in w_i.

It should be obvious that by working in the s coordinate system, where the two types of linearizations agree, we have nice approximation from the input to state map point of view. Assume that system (1) is in the s coordinates (i.e. let x=s) and suppose y and y^L are the Volterra expansions for systems (1) and (2) respectively. Then, since the Taylor approach and Lie bracket matching method agree in (2),

$$y-y_L = O(t^3) + \sum_{i=1}^{m} \int_0^t \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}((ad^k \tau_1 f, g_i) e^{tf} h|_{x_0} - \tau_1{}^k A^k b_i e^{t(f(x_0)+A(x-x_0))}$$
$$h|_{x_0}) u_i(\tau_i) d\tau_1 + \sum_{i_1,i_2=1}^{m} \int_0^{\tau_1} w_{i_1 i_2}(t_1 \tau_1, \tau_2) u_{i_1}(\tau_1) u_{i_2}(\tau_2) d\tau_1 d\tau_2 + O(|t|^3) \tag{14}$$

Here the $w_{i_1 i_2}(t_1\tau_1,\tau_2)$ are for system (1), and the corresponding kernels for the linear system (2) are of course zero. We are interested in those terms that contribute to degree t^2 or less, the remaining terms being moved to $O(|t|^3)$. Hence we consider $\sum_{k=0}^{1}$ in (14), and in fact examine only $k_1=0$ and $k_2=0$, $k_1=1$ and $k_2=0$, $k_1=0$ and $k_2=1$ for

w_i in (10) and $k_1=k_2=k_3=0$ in $w_{i_1 i_2}$.

Computing we find

$$y-y_L=\sum_{i=1}^{m}\int_0^t (g_i(x_0)+\frac{\partial f}{\partial x}(x_0)g_i(x_0)(t-\tau_1)+\frac{\partial g_i}{\partial x}(x_0)f(x_0)\tau_i-b_i-Ab_i(t-\tau_1))u(\tau_1)$$

$$+\sum_{i_1,i_2=1}^{m}\int_0^t\int_0^{\tau_1}\frac{\partial g_{i_1}}{\partial x}(x_0)g_{i_2}(x_0)u_{i_1}(\tau_1)u_{i_2}(\tau_2)d\tau_1 d\tau_2 + O(|t|^3)$$

(15)

$$=\sum_{i=1}^{m}\int_0^t ((-[f,g_i](x_0) + Ab_i)\tau_1 + (\frac{\partial f}{\partial x}(x_0)g_i(x_0)- Ab_i)(t))u(\tau_1)d\tau_1$$

$$+\sum_{i_1,i_2=1}^{m}\int_0^t\int_0^{\tau_1}\frac{\partial g_{i_1}}{\partial x}(x_0)g_{i_2}(x_0)u_{i_1}(\tau_1)u_{i_2}(\tau_2)d\tau_1 d\tau_2 + O(|t|^3).$$

Using the fact that the tangent model and modified tangent model agree we obtain

(16) $$y-y_L = \sum_{i_1,i_2=1}^{m}\int_0^t\int_0^{\tau_1}\frac{\partial g_{i_1}}{\partial x}(x_0)g_{i_2}(x_0)u_{i_1}(\tau_1)u_{i_2}(\tau_2)d\tau_1 d\tau_2+O(|t|^3$$

Suppose that we have a single input system (i.e. m=1). In the s coordinates, g_1 is the constant vector field

$$\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}.$$

and $y-y_L=O(|t|^3)$. For a two input system,

$$g_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ on } S_n \text{ and } g_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ on } S_{n-1}.$$

Hence

$$y-y_L=\int_0^t \int_0^{\tau_1} \frac{\partial g_1}{\partial x}(x_0)(g_1(x_0)u_1(\tau_1)u_1(\tau_2)+g_2(x_0)u_1(\tau_1)u_2(\tau_2))d\tau_1 d\tau_2+O(|t$$

However, if g_1 and g_2 are both constant vector fields in the s coordinate (e.g. this can be done if $[g_1,g_2]=0$), then $y-y_L=O(|t|^3)$.

Thus a pattern emerges which can be extended to a system having any number of inputs. The importance of the s coordinate system (and of the agreement of the tangent model and modified tangent model in these coordinates) in time response studies has been proved.

Starting with system (1) satisfying condition (i) in any x coordinate system, how do we find the tangent model, and thus the modified tangent model, in the s coordinate system? It certainly is not always possible to solve in closed form the systems of ordinary differential equations (3).

Given $\dot{x} = f(x) + \sum_{i=1}^{m} u_i g(x)$ and a point $x_0=x(0)$ we have

$$\dot{s} = (\frac{\partial x}{\partial s})^{-1} f(x(s)) + \sum_{i=1}^{m} u_i (\frac{\partial x}{\partial s})^{-1} g(x(s)).$$

From (3) we know $\frac{\partial x}{\partial s}$ where the entries are functions of $x_1,x_2,\ldots,x_n$. Since $\frac{\partial x}{\partial s}$ is invertible we obtain $(\frac{\partial x}{\partial s})^{-1}$ with entries as functions of $x_1,x_2,\ldots,x_n$. The tangent model at 0 is

$$\dot{s} = \left(\frac{\partial x}{\partial s}\right)^{-1}(x_0)f(x_0)$$

$$+ \frac{\partial\left(\left(\frac{\partial x}{\partial s}\right)^{-1} f(x)\right)(x_0)}{\partial x}\left(\frac{\partial x}{\partial s}\right)(x_0)\, s$$

$$+ \sum_{i=1}^{m} u_i\left(\left(\frac{\partial x}{\partial s}\right)^{-1}(x_0)g(x_0)\right).$$

The paper [20] is written from the point of view that the modified tangent model is more natural than the tangent model for constructing approximate transformations to linear systems for exactly linearizable nonlinear systems. However, since these two models agree in the s coordinates, no choice need be made. We simply find the tangent model in the s coordinates and apply the approximate transformation theory of [12]. In designing a trajectory autopilot for VSTOL aircraft, the method of [12] has been successfully tested in flight simulation.

In this article we have considered two types of linearizations of a nonlinear system about a point x_0. We have found a coordinate system in which these agree and have shown the value of this in examining input to state time response through Volterra expansions. Some of the results of this paper are presented in preliminary form in [25].

Recent results by Krener [26] on approximate linearization by state feedback and coordinate changes are quite interesting.

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