NASA-CR-178,138

NASA Contractor Report 178138

ON THE SOLUTION OF INTEGRAL EQUATIONS WITH STRONGLY SINGULAR KERNELS

A. C. KAYA AND F. ERDOGAN

LEHIGH UNIVERSITY
BETHLEHEM, PENNSYLVANIA

GRANT NGR 39-007-011
JUNE 1986

NASA

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

ON THE SOLUTION OF INTEGRAL EQUATIONS WITH STRONGLY SINGULAR KERNELS

by

A.C. Kaya and F. Erdogan
Lehigh University, Bethlehem, PA

Abstract

In this paper some useful formulas are developed to evaluate integrals having a singularity of the form $(t-x)^{-m}$, $m \geq 1$. Interpreting the integrals with strong singularities in Hadamard sense, the results are used to obtain approximate solutions of singular integral equations. A mixed boundary value problem from the theory of elasticity is considered as an example. Particularly for integral equations where the kernel contains, in addition to the dominant term $(t-x)^{-m}$, terms which become unbounded at the end points, the present technique appears to be extremely effective to obtain rapidly converging numerical results.

1. Introduction

The mixed boundary value problems in physics and engineering may generally be expressed in terms of a "singular" integral equation of the form

$$\int_D k(\underline{t},\underline{x})f(\underline{t})d\underline{t} = g(\underline{x}) \ , \ \underline{x} \in D \tag{1}$$

where g is a known bounded function and the kernel k is usually singular. The nature of singularity of k is dependent on the choice of the density function f in formulating the problem. For example, in one dimensional integral equations arising from potential theory if f(t) is selected to be a "flux", then k has an ordinary Cauchy singularity $(t-x)^{-1}$. On the other hand if f is a potential, then k has a strong singularity of the form $(t-x)^{-2}$. Particularly in two dimensional integral equations, formulating the problem in terms of a potential rather than a flux type quantity has certain advantages. Because of this it is worthwhile to develop effective techniques for evaluating singular integrals with strong singularities. In actual physical problems the density function f is either bounded or may have integrable singularities on the boundary of D. Thus, in one dimensional integral equations the integral on the left hand side of (1) may be interpreted in Cauchy principal value sense for a Cauchy kernel, whereas it would be unbounded in the case of a strong singularity $(t-x)^{-2}$. Despite this in the latter case the physical problem can still be solved provided the integral is interpreted in Hadamard sense by retaining the finite part only.

The concept of finite part integrals was first introduced by Hadamard [1] in connection with divergent integrals of the form

$$\int_a^x \frac{f(t)}{(x-t)^{p+\frac{1}{2}}}\, dt \,, \tag{2}$$

where f is bounded and p is an integer. In spite of this relatively early beginning, the adoption of the concept in applications has been rather slow [2,3]. It is mainly due to Kutt's work [4]-[6] that in recent years the idea is finding relatively wide applications. To demonstrate Hadamard's basic idea we consider the following integral

$$S_0(x) = \int_x^b \frac{dt}{(t-x)^{\frac{1}{2}}} = 2(b-x)^{\frac{1}{2}} \,, \quad (x<b) \,, \tag{3}$$

from which, differentiating both sides separately, it follows that

$$\frac{d}{dx} S_0(x) = \frac{1}{2} \int_x^b \frac{dt}{(t-x)^{3/2}} - \frac{1}{(t-x)^{\frac{1}{2}}} \Big|_{t=x} = -\frac{1}{(b-x)^{\frac{1}{2}}} \,, \quad (x<b) \,. \tag{4}$$

In (4) it is seen that the derivative of S_0 (which is bounded) is the difference between a divergent integral and an unbounded integrated term. Noting that the integrated term is independent of b, we may now consider the derivative of S_0 as being the "finite part" of the divergent integral and define

$$⨍_x^b \frac{dt}{(t-x)^{3/2}} = \lim_{c\to x} \Big[\int_c^b \frac{dt}{(t-x)^{3/2}} - \frac{2}{(c-x)^{\frac{1}{2}}}\Big] = -\frac{2}{(b-x)^{\frac{1}{2}}} \,, \quad (x<c<b) \,. \tag{5}$$

Following are some other examples:

$$⨍_x^b \frac{dt}{(t-x)^{\alpha+1}} = \lim_{c\to x} \Big[\int_c^b \frac{dt}{(t-x)^{\alpha+1}} - \frac{1}{\alpha}\frac{1}{(c-x)^{\alpha}}\Big] = -\frac{1}{\alpha}(b-x)^{-\alpha} \,, \quad (\alpha>0), \tag{6}$$

$$\fint_x^b \frac{dt}{t-x} = \lim_{c\to x} \left[\int_c^b \frac{dt}{t-x} + \log(c-x)\right] = \log(b-x) \ , \tag{7}$$

$$\fint_a^b \frac{dt}{(t-x)^2} = -\frac{1}{b-x} - \frac{1}{x-a} \ , \ (a<x<b) \ , \tag{8}$$

$$\frac{d}{dx}\int_a^b f(t)\log|t-x|\,dt = -\fint_a^b \frac{f(t)}{t-x}\,dt \ , \ (a<x<b) \ , \tag{9}$$

$$\frac{d}{dx}\fint_a^b \frac{f(t)}{t-x}\,dt = \fint_a^b \frac{f(t)}{(t-x)^2}\,dt \ , \ (a<x<b) \ , \tag{10}$$

$$\frac{d}{dx}\fint_a^b \frac{f(t)}{(t-x)^{\alpha+1}}\,dt = \fint_a^b \frac{\partial}{\partial x}\left[\frac{1}{(t-x)^{\alpha+1}}\right]f(t)\,dt \ , \ (a<x<b, \ \alpha>0) \tag{11}$$

In this paper first some useful formulas for the evaluation of certain singular integrals are developed. The results are then used to obtain effective numerical solutions to integral equations having kernels with strong singularities and some examples are given.

2. Evaluation of Finite Part Integrals

With an eye on applications to one dimensional mixed boundary value problems, in this section we will describe some simple techniques for evaluating the finite part integrals having $(t-x)^{-2}$ as the kernel. Let $F(t)$ be a bounded function with continuous first and second derivatives and the interval be normalized such that $-1<(x,t)<1$. The singular integral may then be expressed as

$$\fint_{-1}^{1} \frac{F(t)w(t)}{(t-x)^2}\,dt = \int_{-1}^{1} [F(t)-F(x)-(t-x)F'(x)]\,\frac{w(t)}{(t-x)^2}\,dt$$

$$+ F(x)\fint_{-1}^{1} \frac{w(t)dt}{(t-x)^2} + F'(x)\fint_{-1}^{1} \frac{w(t)dt}{t-x} \ , \ (-1<x<1) \ , \tag{12}$$

where w(t) is the fundamental function of the corresponding mixed boundary value problem and may be determined by using a suitable function theoretic method [7]. For simple physical problems w is given by

$$w(t) = 1,\ (1-t^2)^{\mp\frac{1}{2}}\ ,\ (1-t)^{\mp\frac{1}{2}}\ . \tag{13}$$

One may now note that the first integral on the right hand side of (12) is bounded (the integrand approaches $\frac{1}{2}$ F"(x)w(x) as $t \to x$) and the remaining integrals may be evaluated by using the following expressions:

$$\int_{-1}^{1} \frac{dt}{t-x} = \log\left|\frac{1-x}{1+x}\right|\ , \tag{14}$$

$$\int_{-1}^{1} \frac{dt}{(t-x)^2} = -\frac{1}{1-x} - \frac{1}{1+x}\ , \tag{15}$$

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}\,dt = -\pi x,\quad (-1<x<1)\ , \tag{16}$$

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-x)^2}\,dt = -\pi\ ,\ (-1<x<1)\ , \tag{17}$$

$$\int_{-1}^{1} \frac{dt}{(t-x)\sqrt{1-t^2}} = 0\ ,\ (-1<x<1)\ , \tag{18}$$

$$\int_{-1}^{1} \frac{dt}{(t-x)^2\sqrt{1-t^2}} = 0\ ,\ (-1<x<1)\ , \tag{19}$$

$$\int_{-1}^{1} \frac{\sqrt{1-t}}{t-x}\,dt = -2\sqrt{2}\,\left(1 - \frac{1}{2}\,\frac{\sqrt{1-x}}{2}\log B\right)\ ,\ (x<1)\ , \tag{20}$$

$$\int_{-1}^{1} \frac{\sqrt{1-t}}{(t-x)^2}\,dt = -\sqrt{2}\,\left(\frac{1}{1+x} + \frac{1}{4}\sqrt{\frac{2}{1-x}}\log B\right),\ (x<1)\ , \tag{21}$$

$$\int_{-1}^{1} \frac{dt}{(t-x)\sqrt{1-t}} = \frac{1}{\sqrt{1-x}} \log B \ , \ (x<1) \ , \tag{22}$$

$$\fint_{-1}^{1} \frac{dt}{(t-x)^2\sqrt{1-t}} = \frac{\sqrt{2}}{1-x} \left(- \frac{1}{1+x} + \frac{1}{4}\sqrt{\frac{2}{1-x}} \log B\right), \ (x<1) \ , \tag{23}$$

$$\fint_{-1}^{1} \frac{\sqrt{1-t}}{(t-x)^m} \, dt = - \frac{2\sqrt{2}\,(-1)^m}{(m-1)(1-x)(1+x)^{m-1}} + \frac{2m-5}{2(m-1)(1-x)} \fint_{-1}^{1} \frac{\sqrt{1-t}}{(t-x)^{m-1}} \, dt,$$

$$(x<1) \ , \tag{24}$$

$$\fint_{-1}^{1} \frac{dt}{(t-x)^m\sqrt{1-t}} = - \frac{\sqrt{2}\,(-1)^m}{(m-1)(1-x)(1+x)^{m-1}} + \frac{2m-3}{2(m-1)(1-x)} \fint_{-1}^{1} \frac{dt}{(t-x)^{m-1}\sqrt{1-t}} \ ,$$

$$(x<1) \ , \tag{25}$$

where m is an integer ($m \geq 2$) and

$$B = \left(1 + \sqrt{\frac{1-x}{2}}\right) \Big/ \left(1 - \sqrt{\frac{1-x}{2}}\right) \ . \tag{26}$$

In solving integral equations it is often convenient to express the unknown function F(t) in terms of a polynomial with undetermined coefficients. In such problems the following expressions may be quite useful:

$$\int_{-1}^{1} \frac{P_n(t)}{t-x} \, dt = - 2Q_n(x) \ , \tag{27}$$

$$\fint_{-1}^{1} \frac{P_n(t)}{(t-x)^2} \, dt = - \frac{2(n+1)}{1-x^2} \left[xQ_n(x)-Q_{n+1}(x)\right] \ , \tag{28}$$

$$\int_{-1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{t-x} \, dt = -\pi \ T_{n+1}(x) \ , \ (n \geq 0) \ , \tag{29}$$

$$\int_{-1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{(t-x)^2}\, dt = -\pi(n+1)\; U_n(x) \;, \; (n\geq 0) \;, \tag{30}$$

$$\int_{-1}^{1} \frac{T_n(t)}{(t-x)\sqrt{1-t^2}}\, dt = \begin{cases} 0 \;, \; (n=0) \\ \pi U_{n-1}(x) \;, \; (n\geq 1) \;, \end{cases} \tag{31}$$

$$\int_{-1}^{1} \frac{T_n(t)}{(t-x)^2\sqrt{1-t^2}}\, dt = \begin{cases} 0 \;, \; (n=0,1) \\ \frac{\pi}{1-x^2}\left[-\frac{n-1}{2}\, U_n(x) + \frac{n+1}{2}\, U_{n-2}(x)\right], \; (n\geq 2), \end{cases} \tag{32}$$

where P_n, Q_n and T_n,U_n are the Legendre and Chebychev polynomials of first and second kind, respectively. Also

$$B_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{t^n\sqrt{1-t^2}}{t-x}\, dt = \sum_{k=0}^{n+1} b_k x^k \;, \; (n\geq 0) \tag{33}$$

$$C_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{t^n\sqrt{1-t^2}}{(t-x)^2}\, dt = \sum_{k=0}^{n} c_k x^k \;, \; (n\geq 0) \tag{34}$$

$$D_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{t^n\, dt}{(t-x)\sqrt{1-t^2}} = \begin{cases} 0 \;, \; (n=0) \\ \sum_{k=0}^{n-1} d_k x^k \;, \; (n\geq 1) \end{cases} \tag{35}$$

$$E_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{t^n\, dt}{(t-x)^2\sqrt{1-t^2}} = \begin{cases} 0 \;, \; (n=0,1) \;, \\ \sum_{k=0}^{n-2} e_k x^k \;, \; (n\geq 2) \end{cases} \tag{36}$$

$$R_n^{\lambda}(x) = \int_{-1}^{1} \frac{t^n\sqrt{1-t}}{(t-x)^{\lambda}}\, dt = \sum_{m=1}^{\lambda} \binom{n}{\lambda-m} x^{n-\lambda+m} \int_{-1}^{1} \frac{\sqrt{1-t}}{(t-x)^m}\, dt$$

$$+ \sum_{k=0}^{n-\lambda} A_k^{\lambda}\, x^{n-\lambda-k}, \; (x<1, \; n\geq 0) \;, \tag{37}$$

$$S_n^\lambda(x) = \int_{-1}^{1} \frac{t^n\,dt}{(t-x)^\lambda\sqrt{1-t}} = \sum_{m=1}^{\lambda} \binom{n}{\lambda-m} x^{n-\lambda+m} \int_{-1}^{1} \frac{dt}{(t-x)^m\sqrt{1-t}}$$

$$+ \sum_{k=0}^{n-\lambda} B_k^\lambda\, x^{n-\lambda-k}, \ (x<1,\ n\geq 0)\ , \tag{38}$$

where the coefficients b_k, c_k, d_k, e_k, A_k^λ, B_k^λ and the expressions for the polynomials B_n, C_n, D_n and E_n for $n=0,\ldots,5$ may be found in Appendix A. In (37) and (38) λ is a positive integer, $\binom{n}{\lambda-m}$ is the binomial coefficient and the integrals in the summations can be obtained from (20)-(26).

Even though there are also Gaussian type integration formulas developed by Kutt [5] for the evaluation of the singular integral

$$\int_{x}^{b} \frac{f(t)}{(t-x)^\lambda}\,dt\ ,\quad \lambda\geq 1\ , \tag{39}$$

they are not very convenient for solving integral equations by using the standard quadrature method which requires the use of fixed stations t_i, since in Kutt's formulas t_i vary as x is changed (see also [8]).

3. Solution of Integral Equations

Let us now assume that the mixed boundary value problem is reduced to the following one dimensional integral equation:

$$\int_a^b [k_s(t,x) + k(t,x)]f(t)dt = g(x),\ (a<x<b)\ , \tag{40}$$

where the kernel k is square integrable in [a,b] and g is a known bounded function. If the unknown function f is a "potential" type quantity, then the singular kernel k_s has a strong singularity (i.e., it contains terms of the order $(t-x)^{-n}$, $n>1$). The fundamental (or the weight) function w(t) of the problem may be determined from k_s and f may be expressed as

$$f(t) = F(t)w(t)\ ,\quad a<t<b\ , \tag{41}$$

where F is an unknown bounded function. In solving the integral equations with strong singularities the application of quadrature formulas do not seem to be very practical. In these problems the simplest and the most effective technique appears to be to approximate the unknown function F by a truncated series as

$$F(t) \cong \sum_{n=0}^{N} a_n \phi_n(t) \ , \tag{42}$$

and to determine the coefficients a_n by a weighted residual method. Here ϕ_n may be any convenient complete system of functions. Substituting from (41) and (42) into (40) we obtain

$$\sum_{n=0}^{N} a_n \, G_n(x) \cong g(x), \ (a<x<b) \ , \tag{43}$$

where

$$G_n(x) = \int_a^b k_s(t,x)\phi_n(t)w(t)dt + \int_a^b k(t,x)\phi_n(t)w(t)dt \ . \tag{44}$$

The coefficients a_n may then be determined from the following system of algebraic equations:

$$\sum_{n=0}^{N} a_n \int_a^b G_n(x)\psi_j(x)w_j(x)dx = \int_a^b g(x)\psi_j(x)w_j(x)dx, \ (j=0,1,\ldots,N) \ , \tag{45}$$

where ψ_j is a coordinate function in a complete system (e.g., a set of orthonormal polynomials) and w_j is the corresponding weight. The functions $\phi_n(t)$ and $\psi_j(x)$ are usually selected in such a way that their orthogonality properties may be utilized. In practice one may use trigonometric functions, Legendre polynomials, Chebychev polynomials, delta functions or any linearly independent set of polynomials such as t^n and x^j. Quite clearly the numerical work in (45) may be reduced considerably if we select

$$w_j(x) = 1, \ \psi_j(x) = \delta(x-x_j) \ , \ (j=0,1,\ldots,N) \ . \tag{46}$$

By doing so we can use a simple collocation method to reduce (43) to the following algebraic system:

$$\sum_{n=0}^{N} a_n G_n(x_j) = g(x_j) \ , \ (j=0,1,\ldots,N) \quad . \tag{47}$$

Although the collocation points x_j can be selected arbitrarily, in general they are chosen as the roots of Legendre or Chebychev polynomials. Even though there is no restriction on the choice of x_j, a symmetric distribution with respect to the origin with more points concentrated near the ends seems to help. One may also note that in case of a resulting ill-conditioned system one could select (M+1) coordinate functions ψ_j with M>N in (45) or (46) and determine (N+1) unknowns a_n from a set of (M+1) equations by using the method of least squares.

Needless to say, if the integral equation (40) contains only a dominant kernel $(t-x)^{-1}$ or $(t-x)^{-2}$, one may always obtain the closed form solution by expanding the functions g(x) and F(t) into appropriate series and by using the results given in the previous section and Appendix A.

4. Application: A Crack in an Infinite Strip

In fracture mechanics the problem of an infinite strip containing a crack perpendicular to its boundaries has been of wide interest since this geometry can be used as an approximation to a number of structural components and laboratory specimens. The related boundary value problem will be discussed below and the numerical treatment of the resulting singular integral equation will be given to demonstrate the solution technique that was outlined in the previous section.

As shown in Fig. 1, the crack lies perpendicular to the stress-free boundaries and is under prescribed surface tractions p(x). The problem requires solving the Navier's equations

$$\nabla^2 u_i + \frac{2}{\kappa-1} u_{k,k} = 0 \ , \ (i=1,2) \tag{48}$$

subject to

$$\sigma_{11}(0,y) = \sigma_{12}(0,y) = \sigma_{11}(h,y) = \sigma_{12}(h,y) = 0 \ , \ (-\infty<y<\infty) \ , \qquad (49)$$

$$\sigma_{12}(x,0) = 0 \ , \ (0<x<h) \ , \qquad (50)$$

$$\left.\begin{aligned} &\sigma_{22}(x,0) = p(x), \ (a<x<b) \\ &u_2(x,0) = 0 \ , \ (0<x<a, \ b<x<h) \end{aligned}\right\} \qquad (51\ a,b)$$

where u_1, u_2 are the x,y components of the displacement vector, σ_{ij} is the stress tensor referred to x,y coordinates and κ is an elastic constant ($\kappa=3-4\nu$ for plane strain and $\kappa=(3-\nu)/(1+\nu)$ for plane stress, ν being the Poisson's ratio. The stress and displacement components are related through

$$\sigma_{ij} = \mu(u_{i,j}+u_{j,i}) + \lambda u_{k,k}\delta_{ij} \ , \ (i,j,k=1,2) \qquad (52)$$

where μ and λ are Lame's constants. The solution of (48) satisfying the symmetry condition (50) may be expressed as [9]

$$u_i(x,y) = u_i^c + u_i^1 + u_i^2 \ , \ (i=1,2) \ , \qquad (53)$$

$$u_1^c(x,y) = \frac{1}{2\pi(1+\kappa)} \int_a^b V(t)[-(\kappa-1)\ \frac{x-t}{r^2} + \frac{4(x-t)y^2}{r^4}]dt \qquad (54)$$

$$u_2^c(x,y) = \frac{1}{2\pi(1+\kappa)} \int_a^b V(t)[(\kappa-1)\ \frac{y}{r^2} + \frac{4y^3}{r^4}]dt \qquad (55)$$

$$r^2 = (x-t)^2 + y^2 \ , \qquad (56)$$

$$u_1^1(x,y) = \frac{2}{\pi} \int_0^\infty [A_1 + (\frac{\kappa}{\alpha} + x)A_2]e^{-\alpha x} \cos\alpha y d\alpha \qquad (57)$$

$$u_1^1(x,y) = \frac{2}{\pi} \int_0^\infty (A_1+A_2 x)e^{-\alpha x} \sin\alpha y d\alpha \qquad (58)$$

$$u_1^2(x,y) = -\frac{2}{\pi}\int_0^\infty [B_1 + (\frac{\kappa}{\beta} + h-x)B_2]e^{-\beta(h-x)} \cos\beta y d\beta \qquad (59)$$

$$u_2^2(x,y) = \frac{2}{\pi}\int_0^\infty [B_1+B_2(h-x)]e^{-\beta(h-x)} \sin\beta y d\beta \qquad (60)$$

where V is the auxiliary function defined by

$$V(x) = u_2(x,+0) - u_2(x,-0) , (a<x<b) . \qquad (61)$$

In this solution u_i^c, u_i^1 and u_i^2 are respectively associated with an infinite plane with a crack and the half planes x>0 and x<h. Using the homogeneous boundary conditions (49), the unknown functions $A_1(\alpha)$, $A_2(\alpha)$, $B_1(\beta)$ and $B_2(\beta)$ can be expressed in terms of V(x) and the mixed boundary conditions (51) may be shown to reduce to the following integral equation [9]:

$$\oint_a^b \frac{V(t)}{(t-x)^2} dt + \int_a^b V(t)k(t,x)dt = -\pi (\frac{1+\kappa}{2\mu})p(x) , a<x<b , \qquad (62)$$

where the kernel k(t,x) is given by

$$k(t,x) = k_1(t,x) + k_1(h-t,h-x) + k_2(t,x) + k_2(h-t,h-x) , \qquad (63)$$

$$k_1(t,x) = -\frac{1}{(t+x)^2} + \frac{12x}{(t+x)^3} - \frac{12x^2}{(t+x)^4} , \qquad (64)$$

$$k_2(t,x) = \int_0^\infty [f_1(t,x,\alpha)e^{-\alpha(t+x)} + f_2(t,x,\alpha)e^{-\alpha(2h+x-t)}]d\alpha , \qquad (65)$$

$$f_1(t,x,\alpha) = \frac{\alpha}{D} e^{-2\alpha h}\{8\alpha^4h^2tx-12\alpha^3h^2(t+x)+2\alpha^2[9h^2+h(t+x)+tx]$$

$$- 3\alpha[2h+t+x]+5+e^{-2\alpha h}[-2\alpha^2tx+3\alpha(t+x)-5]\} ,$$

$$f_2(t,x, \ ) = \frac{\alpha}{D} \{-4\alpha^3[hx(h-t)]+6\alpha^2[h^2+h(x-t)]$$

$$+ \alpha[-10h+t-x]+3+e^{-2\alpha h}[\alpha(x-t)-3]\} ,$$

$$D = 1-(4\alpha^2h^2+2)e^{-2\alpha h}+e^{-4\alpha h} . \qquad (66a-x)$$

Note that for $h\to\infty$ k_2 vanishes and the integral equation for the half plane is recovered.

Normalizing the interval (a,b) by defining

$$t = (\frac{b-a}{2})r + (\frac{b+a}{2}) , \quad x = (\frac{b-a}{2})s + (\frac{b+a}{2}) , \qquad (67)$$

$$V(t) = (\frac{b-a}{2})f(r) \qquad (68)$$

the integral equation (62) becomes

$$\int_{-1}^{1} \frac{f(r)}{(r-s)^2} dr + \int_{-1}^{1} f(r)K(r,s)dr = g(s), \ -1<s<1 , \qquad (69)$$

where

$$K(r,s) = (\frac{b-a}{2})^2 \ k(t,x) \ , \ g(s) = \pi(\frac{1+\kappa}{2\mu})p(x) . \qquad (70)$$

The cases $a>0$ and $a=0$ represent the internal and the edge crack, respectively, and these two problems will be treated separately. In each case the solution will be assumed to be of the form

$$f(r) = F(r)w(r) \qquad (71)$$

where the fundamental solution $w(r)$ can be determined from the dominant behavior of the singular kernels in the integral equation and is found to be

$$w(r) = \sqrt{1-r^2} \quad \text{, internal crack ,} \tag{72}$$

$$w(r) = \sqrt{1-r} \quad \text{, edge crack.} \tag{73}$$

Internal Crack: $a>0$

Following the procedure described in the previous section, $F(r)$ is now approximated in terms of a truncated series of Chebychev polynomials,

$$F(r) = \sum_{n=0}^{N} a_n U_n(r) \ , \tag{74}$$

By substituting from (71), (72) and (74) into (69) and by using (30) we obtain

$$\sum_{n=0}^{N} a_n[-\pi(n+1)U_n(s)+h_n(s)] = g(s) \ , \ -1<s<1 \ , \tag{75}$$

where

$$h_n(s) = \int_{-1}^{1} U_n(r)K(r,s)\sqrt{1-r^2}\, dr \ . \tag{76}$$

The unknown coefficients a_n are then determined from equation (75) by selecting a convenient set of collocation points such as

$$T_{n+1}(s_j) = 0 \ , \ s_j = \cos(\frac{2j+1}{N+1}\frac{\pi}{2}) \ , \ (j=0,1,...,N) \ , \tag{77}$$

Once the solution is obtained, the stress intensity factors which are the main parameters of interest in fracture problems, can be calculated from

$$\begin{aligned} k_1(a) &= \lim_{x\to a} \sqrt{2(a-x)}\, \sigma_{22}(x,0) \ , \ (x<a) \\ &= (\frac{2\mu}{\kappa+1}) \lim_{t\to a} \frac{V(t)}{\sqrt{2(t-x)}} \ , \ (t>a) \\ &= (\frac{2\mu}{\kappa+1}) \quad \frac{b-a}{2} F(-1) \ , \end{aligned} \tag{78}$$

$$k_1(b) = \lim_{x \to b} \sqrt{2(x-b)}\, \sigma_{22}(x,0) \quad , \; x>b$$

$$= \left(\frac{2\mu}{\kappa+1}\right) \lim_{t \to b} \frac{V(t)}{\sqrt{2(b-t)}} \quad , \; t<b$$

$$= \left(\frac{2\mu}{\kappa+1}\right) \quad \frac{b-a}{2} F(1) \quad . \tag{79}$$

Equations (78) and (79) are obtained from (62) by observing that the left-hand side in (62) gives the stress component $\sigma_{22}(x,0)$ outside as well as inside the cut (a,b).

Table 1 shows the stress intensity factors for an internal crack in a half-plane under uniform loading, $p(x) = -P_o$ as an example.

Table 1. Normalized stress intensity factors for an internal crack in a half-plane. (N+1) terms are used in approximating the unknown function.

$(\frac{b+a}{b-a})$	$\frac{k_1(a)}{P_o \frac{b-a}{2}}$	$\frac{k_1(b)}{P_o \frac{b-a}{2}}$	N+1
1.01	3.6387	1.3298	15
1.05	2.1547	1.2536	10
1.1	1.7585	1.2108	10
1.2	1.4637	1.1626	6
1.3	1.3316	1.1331	6
1.4	1.2544	1.1123	6
1.5	1.2035	1.0967	4
2.0	1.0913	1.0539	4
3.0	1.0345	1.0246	4
4.0	1.0182	1.0141	4
5.0	1.0112	1.0092	4
10.0	1.0026	1.0024	4
20.0	1.0006	1.0006	4

Edge Crack: a=0

The solution of the integral equation (62) for a=0 needs more care. This is due to the fact that the kernel k(t,x) becomes singular as t and x approach 0 simultaneously (similarly in (69) K(r,s) becomes unbounded as r and s approach -1.)

For a weight function $\sqrt{1-r}$ certain relations involving singular integrals of power series have been presented in Section 2. Therefore, if we express the unknown function F(r) as

$$F(r) = \sum_{n=0}^{N} a_n r^n \quad , \tag{80}$$

the singular integrals may be evaluated from (37) by letting $\lambda=2$. The integral equation (69) now becomes

$$\sum_{n=0}^{N} a_n G_n(s) = g(s) \quad , \; -1<s<1 \; , \tag{81}$$

where

$$G_n(s) = \mathop{\not\!\!\int}_{-1}^{1} \frac{r^n\sqrt{1-r}}{(r-s)^2}\, dr + \int_{-1}^{1} r^n\sqrt{1-r}\, K(r,s)dr \; , \tag{82}$$

or using the notation of (37),

$$G_n(s) = R_n^{\,2}(s) + \int_{-1}^{1} r^n\sqrt{1-r}\, K(r,s)dr \quad . \tag{83}$$

The integral in (83) can be evaluated numerically, however, as $s\to-1$, the value of the integral becomes unbounded. It may be observed that for s=-1 $R_n^{\,2}(s)$ is also unbounded resulting in a bounded value for $G_n(-1)$. To determine the coefficients a_n the collocation points may be selected as in (77).

For $h\to\infty$ the kernel K(r,s) is simply

$$K(r,s) = -\frac{1}{(r+s+2)^2} + \frac{12(s+1)}{(r+s+2)^3} - \frac{12(s+1)^2}{(r+s+2)^4} \quad , \tag{84}$$

from which it follows that

$$G_n(s) = R_n^2(s)-R_n^2(-s-2)+12(s+1)R_n^3(-s-2)-12(s+1)^2R_n^4(-s-2) \quad (85)$$

In the limiting case we find

$$G_n(-1+0) = -\sqrt{2}\,(4n+1)(-1)^n \quad (86)$$

The stress intensity factor is given by

$$\begin{aligned} k_1(b) &= \lim_{x\to b} \sqrt{2(x-b)}\,\sigma_{22}(x,0)\ , \ x>b \\ &= \left(\frac{2\mu}{\kappa+1}\right) \lim_{t\to b} \frac{V(t)}{\sqrt{2(b-t)}}\ , \ t<b \\ &= \left(\frac{2\mu}{\kappa+1}\right) \frac{\sqrt{b}}{2} F(1) \ . \end{aligned} \quad (87)$$

As a first example we again consider a semi-infinite plane with an edge crack. In this case the kernel of the integral equation is given in closed form (see (69) and (84)) and the numerical analysis can be carried out quite accurately. For a uniform crack surface pressure $p(x) = -P_0$ and for various values of N the calculated stress intensity factor k(b) and the relative crack opening displacement V(0) are given in Table 2. The table also shows the correct value of k(b) which was calculated from the infinite integral given in [10] (see Appendix B). It is seen that the convergence of the method is extremely good.

The second example is concerned with a long strip of finite width h which contains an edge crack of length b and is subjected to a uniform tension P_0 ($p(x)=-P_0$) (table 3) or pure bending M ($p(x) = -\frac{6M}{h^2}(1-\frac{2x}{h})$) (table 4) away from the crack region. In the numerical analysis the number of collocation points was increased until the accuracy of the last significant digits given in tables 3 and 4 were verified. In no case more than 20 points were needed.

Aside from providing accurate answers to some very practical questions, the results given in Tables 3 and 4 are important in that they follow

Table 2. Normalized stress intensity factor and crack opening displacement for an edge crack in a half-plane.

N+1	$\frac{k_1(b)}{P_o\sqrt{b}}$	$(\frac{2\mu}{1+\kappa})\ \frac{V(0)}{P_o\ b}$
1	1.062652	1.502816
2	1.126950	1.423476
3	1.124283	1.457747
4	1.121818	1.455918
5	1.121442	1.454520
6	1.121451	1.454224
7	1.121483	1.454211
8	1.121504	1.454241
9	1.121514	1.454264
10	1.121518	1.454278
15	1.121522	1.454298
20	1.121522	1.454298
	1.121522*	

*The correct value of stress intensity factor (calculated from the infinite integral given by Koiter [10].

Table 3. Normalized stress intensity factor and crack opening displacement for an edge crack in a strip under uniform tension.

b/h	$\frac{k_1(b)}{p_o\sqrt{b}}$	$(\frac{2}{1+})\frac{V(0)}{p_o\ h}$
→ 0	1.12152226	
0.00001	1.121522	$0.14543\text{x}10^{-4}$
0.001	1.121531	$0.14543\text{x}10^{-2}$
0.1	1.1892	0.15490
0.2	1.3673	0.36543
0.3	1.6599	0.70358
0.4	2.1114	1.3048
0.5	2.8246	2.4702
0.6	4.0332	4.9746
0.7	6.3569	11.246
0.8	11.955	31.840
0.85	18.628	63.288
0.9	34.633	158.94
0.95	99.14	708.8

Table 4. Normalized stress intensity factor and crack opening displacement for an edge crack in a strip under pure bending. $\sigma_n = 6M/h^2$

b/h	$\frac{k_1(b)}{\sigma_n \sqrt{b}}$	$(\frac{2\mu}{1+\kappa}) \frac{V(0)}{\sigma_n h}$
$\to 0$	1.12152226	
0.00001	1.1215	0.14543×10^{-4}
0.001	1.1202	0.14535×10^{-2}
0.1	1.0472	0.14529
0.2	1.0553	0.31822
0.3	1.1241	0.56141
0.4	1.2606	0.94130
0.5	1.4972	1.5924
0.6	1.9140	2.8387
0.7	2.7252	5.6432
0.8	4.6764	13.989
0.85	6.9817	25.990
0.9	12.462	60.965
0.95	34.31	253.7

routinely from the technique presented in this paper, for very deep cracks (b>0.8h) are not available in literature, and are extremely difficult to obtain by using other methods. For example, the solution of the corresponding singular integral equation having a Cauchy type dominant kernel by using a Gaussian integration formula requires much greater computational effort than the technique presented here for the same accuracy and for b>0.8h has an extremely slow convergence.

Acknowledgements: The research reported in this paper was supported by NSF under the Grant MEA-8209083 and by NASA-Langley under the Grant NGR 39-007-011 and was completed when the second author was an Alexander von Humboldt Senior U.S. Scientist Awardee in Freiburg (i.Br.), Germany.

References

1. J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press (1923).

2. L. Schwartz, Theorie et Distributions, Vols. I and II, Hermann, Paris (1957-1959).

3. K.W. Mangler, Improper Integrals in Theoretical Aerodynamics, Royal Aircraft Establishment, Farnborough, Report No. 2424 (1951).

4. H.R. Kutt, "The Numerical Evaluation of Principal Value Integrals by Finite-Part Integration", Numer. Math. 242, pp. 205-210 (1975).

5. H.R. Kutt, Quadrature Formulae for Finite-Part Integrals, Special Report WISK 178, Pretoria, National Research Institute for Mathematical Sciences (1975).

6. H.R. Kutt, On the Numerical Evaluation of Finite-Part Integrals Involving an Algebraic Singularity, Special Report, WISK 179, Pretoria, National Research Institute for Mathematical Sciences (1975).

7. N.I. Muskhelishvili, Singular Integral Equations, Noordhoff, Groningen, The Netherlands (1953).

8. N.I. Ioakimidis, "On the Numerical Evaluation of Derivatives of Cauchy Principal Value Integrals", Computing, 27, pp. 81-88 (1981).

9. A.C. Kaya, Applications of Integral Equations with Strong Singularities in Fracture Mechanics, Ph.D. Thesis, Lehigh University (1984).

10. W.T. Koiter, Discussion of "Rectangular Tensile Sheet with Symmetrical Edge Cracks", by O.L. Bowie, J. Appl. Mech., 32, Trans. ASME, 87, Series E, p. 237 (1965).

APPENDIX A

The coefficients given in the equations (33)-(38):

$$b_k = \begin{cases} 0 \text{ , for } n-k = \text{even,} \\ \dfrac{1}{2\sqrt{\pi}} \dfrac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n-k+3}{2})} \text{ , for } n-k = \text{odd ,} \end{cases} \quad (n \geq 0,\ n \geq k) \ ,$$

$$c_k = \begin{cases} 0 \text{ , for } n-k = \text{odd} \\ \dfrac{(k+1)}{2\sqrt{\pi}} \dfrac{\Gamma(\frac{n-k-1}{2})}{\Gamma(\frac{n-k+2}{2})} \text{ , for } n-k = \text{even} \end{cases} \quad (n \geq 0,\ n \geq k) \ ,$$

$$d_k = \begin{cases} 0 \text{ , for } n-k = \text{even} \\ \dfrac{1}{\sqrt{\pi}} \dfrac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n-k+1}{2})} \text{ , for } n-k = \text{odd} \end{cases} \quad (n \geq 1,\ n \geq k)$$

$$e_k = \begin{cases} 0 \text{ , for } n-k = \text{odd} \\ \dfrac{(k+1)}{\sqrt{\pi}} \dfrac{\Gamma(\frac{n-k-1}{2})}{\Gamma(\frac{n-k}{2})} \text{ , for } n-k = \text{even} \end{cases} \quad (n \geq 2,\ n \geq k)$$

$$A_k^\lambda = 4\sqrt{2} \binom{n-k-1}{\lambda-1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(2)^j}{2j+3} \ , \quad (n \geq 0)$$

$$B_k^\lambda = 2\sqrt{2} \binom{n-k-1}{\lambda-1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(2)^j}{2j+1} \ , \quad (n \geq 0)$$

where $\binom{n}{\lambda}$ is the binomial coefficient and for the gamma function we have $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$.

Some examples of the polynomials B_n, C_n, D_n and E_n are given below [see (33)-(36)]:

$B_0 = -x \; ,$

$B_1 = -x^2 + \frac{1}{2} \; ,$

$B_2 = -x^3 + \frac{x}{2} \; ,$

$B_3 = -x^4 + \frac{x^2}{2} + \frac{1}{8} \; ,$

$B_4 = -x^5 + \frac{x^3}{2} + \frac{x}{8} \; ,$

$B_5 = -x^6 + \frac{x^4}{2} + \frac{x^2}{8} + \frac{1}{16} \; ,$

$C_0 = -1 \; ,$

$C_1 = -2x \; ,$

$C_2 = -3x^2 + \frac{1}{2} \; ,$

$C_3 = -4x^3 + x \; ,$

$C_4 = -5x^4 + \frac{3}{2}x^2 + \frac{1}{8} \; ,$

$C_5 = -6x^5 + 2x^3 + \frac{x}{4} \; ,$

$D_0 = 0 \; ,$

$D_1 = 1 \; ,$

$D_2 = x \; ,$

$D_3 = x^2 + \frac{1}{2} \; ,$

$D_4 = x^3 + \frac{x}{2} \; ,$

$D_5 = x^4 + \frac{x^2}{2} + \frac{3}{8} \; ,$

$E_0 = 0 \; ,$

$E_1 = 0 \; ,$

$E_2 = 1 \; ,$

$E_3 = 2x \; ,$

$E_4 = 3x^2 + \frac{1}{2} \; ,$

$E_5 = 4x^3 + x \; .$

APPENDIX B

Stress Intensity Factor for the Edge Crack Calculated from Koiter's Results

The edge crack problem in a semi-infinite plane has been considered in the literature many times and mostly for comparison reasons. The stress intensity factor 1.1215 has become a standard when comparing numerical techniques for the solution of singular integral equations. For uniform pressure p_0 applied on the crack surface, a closed form expression for the stress intensity factor in terms of an infinite integral is given by Koiter [10]:

$$\frac{k_1}{p_0\sqrt{b}} = \frac{\sqrt{2(B+1)}}{\sqrt{\pi}\,A} ,$$

where A is calculated from

$$\log A = -\frac{1}{\pi}\int_0^\infty \frac{1}{1+\alpha^2}\log\left(\frac{\alpha \sinh \pi\alpha}{\sqrt{B^2+\alpha^2}\,[\cosh\pi\alpha-2\alpha^2-1]}\right) d\alpha$$

and B is an arbitrary constant greater than 1.

The result is independent of the choice for B and numerical calculations show that

$$\frac{k_1}{p_0\sqrt{b}} \cong 1.12152226 ,$$

where there may be an error only in the last digit.

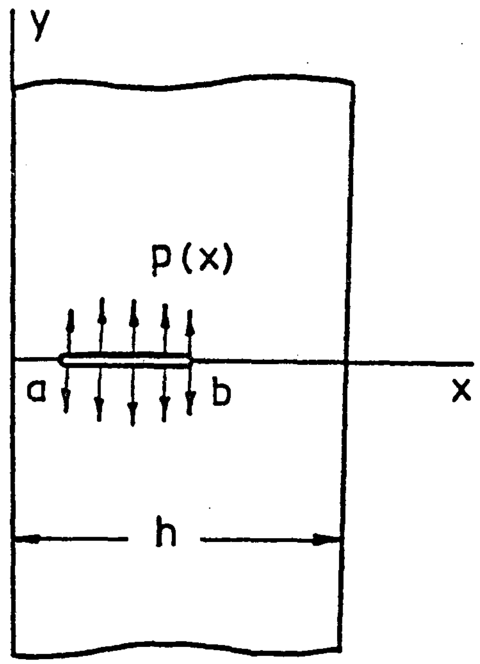

Fig. 1 A Crack in an Infinite Strip

Standard Bibliographic Page

1. Report No. NASA CR- 178138	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle On The Solution Of Integral Equations With Strongly Singular Kernels		5. Report Date June 1986
		6. Performing Organization Code
7. Author(s) A. C. Kaya and F. Erdogan		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address Lehigh University Bethlehem, Pennsylvania 18015		11. Contract or Grant No. NGR 39-007-011
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code 506-43-11-04

15. Supplementary Notes

Langley Technical Monitor: Dr. C. A. Bigelow

16. Abstract

In this paper some useful formulas are developed to evaluate integrals having a singularity of the form $(t-x)^{-m}, m\geq 1$. Interpreting the integrals with strong singularities in Hadamard sense, the results are used to obtain approximate solutions of singular integral equations. A mixed boundary value problem from the theory of elasticity is considered as an example. Particularly for integral equations where the kernel contains, in addition to the dominant term $(t-x)^{-m}$, terms which become unbounded at the end points, the present technique appears to be extremely effective to obtain rapidly converging numerical results.

17. Key Words (Suggested by Authors(s)) Formulaes Hadamard Mixed boundary value	18. Distribution Statement Unclassified-Unlimited Subject Category 67

19. Security Classif.(of this report) Unclassified	20. Security Classif.(of this page) Unclassified	21. No. of Pages 25	22. Price A02

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA Langley Form 63 (June 1985)

End of Document