FINAL [illegible]CT REPORT

DAA/Ames

SEPARATED FLOWS NEAR THE NOSE OF A BODY OF REVOLUTION

Supported by NASA Grant NCC2-280

(NASA-CR-177107) SEPARATED FLOWS NEAR THE NOSE OF A BODY OF REVOLUTION Final Project Report, 1 Jan. 1984 - 28 Feb. 1986 (Clarkson Univ.) 89 p CSCL 20D

N86-30091

Unclas

G3/34 43217

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PART I - PROJECT IDENTIFICATION INFORMATION

1. Principal Investigator: Dr. S.P. Lin, Professor, Mechanical and Industrial Engineering

2. Institution: Clarkson University, Division of Research, Potsdam, NY 13676

3. NASA Program: NASA-Ames Cooperative Program, NCC2-280

4. Award Period: From 1/1/84 to 2/28/86

5. *Cumulative Award Amount: $80,063*

6. Project Title: Separated Flows Near the Nose of a Body of Revolution

7. Technical Program Officer: Dr. Gary Chapman, NASA-Ames

PART II - SUMMARY OF COMPLETED PROJECT

The solution of the Navier-Stokes equations for the problem of cross-flow separation about a deforming cylinder is achieved by a novel method of iteration. It is shown that the separation starts at the rear stagnation point and the point of primary separation moves upstream along the cylinder surface. The solution clearly indicates how the secondary separation may follow in time the primary separation. By invoking the analogy between the cross-flow structure around a three-dimensional body of revolution and the two-dimensional flow around a deforming cylinder, we postulate that the flow separation does not originate at the nose-tip. We also postulate how the open separation may originate near the nose-tip.

A general method of linear stability analysis for non-parallel external flows has been constructed. It consists of representing the eigenfunctions with complete orthogonal sets and forming characteristic equations with the Galerkin method. The method is applied to the Kovasznay flow which is an exact solution of the Navier-Stokes equation. The results show that when the critical parameter is exceeded, there are only a few isolated unstable eigen-frequencies. Hence the concept of neutral curve which is useful for parallel flows becomes irrelevant. Another exact solution, i.e. Taylor's vortex array is shown to be absolutely and monotonically stable with respect to infinitesimal disturbances of all frequencies. The flow is also globally, asymptotically and monotonically stable in the mean with respect to three-dimensional disturbances. This result forms the sound foundation of rigorous stability analysis for non-parallel flows, and provides an invaluable test ground for future studies of non-parallel flows in which the basic states do not possess exact solutions. The application of this method to the study of the formation of spiral vorticies near the nose of a rotating body of revolution is underway. The same method will be applied to the stability analysis of the reversed flow over a plate with suction. This flow is an exact solution of the Navier-Stokes equations, and was obtained by us.

PART III - TECHNICAL INFORMATION

1. Scientific Collaborators

The following individuals have collaborated with the principal investigator, Dr. S.P. Lin on this project:

David Mekala - Graduate Research Assistant, completed his M.S. thesis in June 1985

H.B. Chen - Graduate Research Assistant, completed his M.S. thesis in January 1986

K. El-Rais - Ph.D. Candidate

E. Ibrahim - Ph.D. Graduate Research Assistant

Prof. R. Hessel and Prof. D. Valentine of Clarkson University have been collaborating with the principal investigator on some numerical aspects of the project

M. Tobak and G. Chapman of NASA-Ames have provided valuable physical insight and scrutinized some technical detail.

2. Abstracts of Theses

A. "Migration of the Separation Point on a Deforming Cylinder"
M.S. Thesis by David Mekala

An iterative scheme of solving the Navier-Stokes equations for the two-dimensional cross-flow about a deforming cylinder is given. In the first approximation, the effect of vorticity convection is neglected. This effect is taken into account by successive iterations. Thus in the k-th iteration, the convected vorticity field appears as the source term in an equation of transient diffusion of vorticity. A novel integral transform is used to reduce this transient vorticity equation into a k-dimensional heat equation. The bounded solution of this equation with the appropriate boundary conditions is given. Based on analytically obtained flow field, the equation which gives the locations of zero shear stress at the surface of the deforming cylinder is found. The final solution for the separation angle is expressed as the sum of integrals involving, explicitly, the Reynolds number, and, implicitly the dimensionless cylinder deformation. Gauss-quadrature is used to evaluate these integrals for numerical evaluation. The specific forms of the cylinder motion used in the present numerical calculations include impulsively started uniform motion about a rigid cylinder, motion consisting of constant acceleration about a deforming cylinder and motion consisting of impulsively started uniform motion about deforming and contracting cylinders. The general solution is applied at the surface of the cylinder to determine the locations of primary separation points based on the first two iterations.

B. "Stability of a Non-Parallel Flow," M.S. Thesis by H.B. Chen

There have been many brilliant works on the stability of parallel flows during the past decades. Unlike the stability analysis of parallel flows, the stability analysis of non-parallel flows is not yet firmly established. Among other difficulties, the stability analysis of non-parallel flows suffers from some serious uncertainty associated with the approximate nature of the basic state of the non-parallel flows. Kovasznay obtained an exact solution to the Navier-Stokes equations for a non-parallel flow. He suggested that this flow may be used to describe the flow downstream of a two dimensional grid. We take this full non-parallel flow as our basic flow of stability analysis. The linear stability of the Kovasznay flow is analyzed. The solution to the obtained governing equation of the linear stability is expanded in two complete sets of orthogonal functions which satisfy the appropriate boundary conditions. The characteristic equation is then obtained by the Galerkin method. The stability of the flow is characterized by the Reynolds number R and another parameter B representing the length of the closed wake. It is shown that the Kovasznay flow is absolutely stable when B = 0 for any R. The numerical results show that there exists a critical B below which the flow is stable for a given R. Above the critical B, the flow becomes unstable only with respect to finite number of discrete frequencies of infinitesimal distrubances.

3. Publication Citations

a. S.P. Lin and M. Tobak, "Reversed Flow Above a Plate with Suction," AIAA Journal 24, 334, 1986.

b. S.P. Lin, D. Mekala, G. Chapman and M. Tobak, "Migration of the Separation Point on a Deforming Cylinder," submitted to AIAA Journal.

c. S.P. Lin and M. Tobak, "Spectral Stability of Taylor's Vortex Array," to appear in Physics of Fluids.

d. S.P. Lin and M. Tobak, "Stability of Taylor's Vortex Array," submitted to J. of Fluid Mech.

e. S.P. Lin, H.B. Chen and M. Tobak, "Stability of Non-Parallel Flows," submitted to Physical Review Letters.

The technical details of the above works are given in the Appendix.

4. Continuing Work

Building upon the results of the completed work listed above, we will complete the following unfinished works.

a. "Separated Flows about a rotating Body" (see Appendix for detailed description)

b. "Formation of Secondary Vorticies on a Circular Cylinder."

APPENDIX

TECHNICAL DESCRIPTION

* Instability of Non-Parallel Flows
* Spectral Theory of Taylor's Vortex Array
* Stability of Taylor's Vortex Array
* Reversed Flow Above a Plate with Suction
* Separated Flow about a Rotating Body
* Migration of the Separation Point on a Deforming Cylinder

Instability of Non-parallel Flows

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and

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This letter gives a general method of stability analysis of non-parallel flows which varies in the direction of flows. The solution to the governing partial differential equation is constructed with two complete orthogonal sets. The characteristic equation of disturbances is then obtained with the Galerkin method. The modified Kovasznay non-parallel flow which is an exact solution of the Navier-Stokes equation is used to demonstrate the method.[1] The results show that there are countably infinite numbers of stable eigenmodes but only a few isolated unstable modes. Unlike the parallel flows, a neutral curve does not exist.

Stability analysis for parallel flows is quite well developed, as can be seen from recent books on the subject[1-5]. A far less developed analysis is that for non-parallel flows which are encountered in almost every external flow. The importance of the non-parallel flow effect even on almost parallel flows are increasingly recognized[6-11]. In linear stability analysis of parallel flows, one obtains the neutral curve which divides the parameter space into stable and unstable regions. In such an analysis, the disturbance may be represented by the Fourier integral, and the governing equation of stability is reduced to an ordinary differential equation. This implies that the wave length of the Fourier component of disturbances may be varied continuously. Consequently, a continuous neutral curve may be obtained in the wave number vs. flow parameter space. In weakly non-parallel flows, the concept of a neutral curve is still applicable. However, in truely non-parallel flows the basic flow varies in the flow direction as well as in other directions, and the disturbance cannot be decomposed into independent Fourier components with arbitrarily close neighboring wave lengths. As a consequence, a continuous neutral curve does not exist in general. In fact, the disturbance must satisfy the partial differential equation which may admit only isolated eigen-modes. A general method of stability analysis for non-parallel flows has not yet appeared. Existing stability analysis for weakly non-parallel flows appear to be all applied to the basic states which can be described only approximately. While these studies are motivated by practical needs, they entail the theoretical disadvantage of having to tailor the method of analysis to fit the degree of approximation involved in the basic state. Thus, if we are to seek a general method of linear stability analysis for non-parallel flows, it may be more advantageous to try it on a non-parallel basic state which is an exact solution of the Navier-Stokes equation. This will free us from concern over the possible inadequency of the approximation involved in the basic state. Motivated by this thought, we apply our general method of linear stability analysis to the modified Kovasznay[12] wake flow which is an exact solution of the Navier-Stokes equation. Our method consists of constructing the disturbance with two complete orthogonal sets and solving the characteristic equation obtained by use of the Galerkin method. A similar method has been given by Orszag[13] for internal flows.

Consider two-dimensional flows of an incompressible Newtonian fluid. The X and Z Cartesian components of the velocity field are related respectively to the Stokes stream function Ψ by

$$U = \Psi_Z, \qquad V = -\Psi_X,$$

where subscripts X and Z denote partial differentiations. In terms of Ψ, the governing equation is

$$(\partial_t - \nu \underline{\nabla}^2)\underline{\nabla}^2\Psi = \Psi_X \underline{\nabla}^2\Psi_Z - \Psi_Z \underline{\nabla}^2\Psi_X \quad , \tag{1}$$

where t is time, ν the kinematic viscosity, $\underline{\nabla}^2$ is the Laplacian operator, the subscripts X and Z denote partial differentiations with respect to the space variables. By use of the following dimensionless variables

$$\psi = \Psi/(Ud) \quad , \quad (x, z) = (X, Z)/(d) \quad ,$$

$$\nabla = \underline{\nabla}(d) \quad , \quad \tau = t/(d/U) \quad ,$$

where U and d are respectively the characteristic velocity and length, (1) can be rewritten as

$$(\partial_\tau - \frac{1}{R}\nabla^2)\nabla^2\psi = \psi_x \nabla^2\psi_z - \psi_z \nabla^2\psi_x \quad , \tag{2}$$

where $R \equiv Ud/\nu$ is the Reynolds number.

Kovasznay found the following exact solution of (2)

$$\bar{\psi} = z - B \sin(2\pi z) \exp(Mx) \quad , \tag{3}$$

where

$$M = \frac{1}{2}(R-\sqrt{R^2+16\pi^2}) < 0 \quad .$$

He chose B to be $1/2\pi$ so that the stagnation points are located at $x=o$ and $z=\pm n$, n being any integers. Here, we chose B to be a free parameter, and consider only the region $x>o$ so that the velocity is bounded everywhere. The stream line pattern of this flow for $R=40$ and $B=10.0$ is given in Fig. 1. Note that the length of the closed wake, i.e. the distance between $x=0$ and the stagnation point is given by

$$L = \frac{1}{M} \ln \frac{1}{2\pi B} \quad .$$

Since $M<0$, for the existence of the closed wake we must have $2\pi B>1$. Thus, B is an independent parameter which represents the strength of suction and blow at $x=0$. When $B=0$, the flow is a uniform stream of speed U in the x-direction. When $0<B\leq 1/2\pi$, the flow is sucked around $z=n$ toward $x=0$ but blown away from $x=0$ around $z=2n-1$; however it is not large enough to form a closed wake. Kovasznay suggested that this solution can be used to describe the wake flow behind a grid of wires with uniform spacing d.

The stability of this modified Kovasznay flow is now analysed. Let the perturbed stream function ψ be written as

$$\psi = \bar{\psi} + \phi \quad , \tag{4}$$

where ϕ is the stream function perturbation. Substituting (4) into (2) and neglecting nonlinear terms, we have

$$(\partial_\tau - \frac{1}{R}\nabla^2)\nabla^2\phi = \bar{\psi}_x\nabla^2\phi_z - \bar{\psi}_z\nabla^2\phi_x + M[(\bar{\psi}_z - 1)\phi_x - \bar{\psi}_x\phi_z] \tag{5}$$

It follows from (3) that

$$\bar{\psi}_x = \bar{\psi}_{xo}B \sin(2\pi z) \text{ and } \bar{\psi}_z = 1 - \bar{\psi}_{zo}B \cos(2\pi z) \quad , \tag{6}$$

where

$$\bar{\psi}_{xo} = -MF(x), \quad \bar{\psi}_{zo} = 2\pi F(x) \quad , \quad F(x) = \exp(Mx) \quad .$$

The coefficients in (5) are functions of x and z. Thus, it cannot be reduced to an ordinary differential equation by Fourier decomposition as is customarily done in the parallel flow stability analysis. We seek the solution of (5) in the form

$$\phi = \sum_{n=1}^{N} Cn(\tau,x) \sin 2n\pi z + \sum_{n=1}^{N} Dn(\tau,x) \cos 2n\pi z$$

The solution is expected to approach the exact solution as $N \to \infty$. The chosen z-dependence in the above solution ensures that the stability behavior is the same in each wake. Physically, this implies that the disturbances are distributed randomly without preference of any particular wake. Substituting this assumed form of solution into (5), we have

$$(\partial_\tau - \frac{1}{R} E_n^2)E_n^2(Cn \sin 2n\pi z + Dn \cos 2n\pi z)$$

$$= 2n\pi\bar{\psi}_{xo}\ B \sin 2\pi z\,(E_n^2 - M)(Cn \cos 2n\pi z - Dn \sin 2n\pi z)$$

$$- [M\,\bar{\psi}_{zo}\ B \cos 2\pi z + (1 - \bar{\psi}_{zo} \cos 2\pi z)E_n^2](Cn \sin 2n\pi z$$

$$+ Dn \cos 2n\pi z)_x \quad , \qquad (7)$$

where

$$E_n^2 = \partial_{xx} - (2n\pi)^2 \quad .$$

The solution of (7) will be achieved by use of the Galerkin method. Multiplying (7) by each member of the orthogonal set sin 2mπz, m=1 to N, then integrating over one period in z and demanding the weighted residual over this period to be zero, we have

$$[(\partial_\tau - \frac{1}{R} E_m^2)E_m^2 + E_m^2 D]C_m = + B[\pi(m-1)\bar{\psi}_{xo}(E_{m-1}^2 - M) + \tfrac{1}{2}\bar{\psi}_{zo}(E_{m-1}^2 - M)D]C_{m-1} \quad ,$$

$$+ B[-\pi(m-1)\bar{\psi}_{xo}(E_{m-1}^2 - M) + \tfrac{1}{2}\bar{\psi}_{zo}(E_{m+1}^2 - M)D]C_{m+1} \qquad (8)$$

where $D = \partial_x$ and $N \geq m \geq 1$. Note $C_o = 0$.

Similarly, multiplying (10) by cos $2m\pi z$ and integrating over one period in z, we have

$$[(\partial_\tau - \frac{1}{R} E_m^{\;2}) E_m^{\;2} + E_m^{\;2} D] D_m = B[\tfrac{1}{2}\bar{\psi}_{zo} D - \pi(m+1)\bar{\psi}_{xo}](E^2_{m+1} - M) D_{m+1}$$

$$+ B[\tfrac{1}{2}\bar{\psi}_{zo} D - \pi(m-1)\bar{\psi}_{xo}](E^2_{m-1} - M) D_{m-1}] \qquad (9)$$

where $N \geq m \geq 0$, and $D_j = 0$ if $j < 0$.

Thus the odd mode and even mode disturbances are decoupled. The odd mode solution can be obtained from (8) which is a set of N linear fourth order ordinary differential equations in N unknowns C_m (m=1 to N). (9) is a set of (N+1) linear fourth order ordinary differential equations for the even mode solution D_m (m=0 to N). Note that the m-th Fourier mode is weakly coupled with its two immediate neighboring modes in (8) and (9).

The instability of the basic flow with respect to the even mode disturbance will be investigated first. We assume that the ultimate stability or instability does not depend on the initial transience but depends on the property of the discrete modes

$$D_m = \sum_{k=0}^{K} \exp(\omega\tau)\psi_k^s(x)\, B_m^k \quad , \qquad (10)$$

where ω is the complex eigen-frequency, $B_m^k \psi_k^s(x)$ is the complex disturbance amplitude which varies with x. We assume that the disturbance does not occur at x=0 but vanishes at infinity, and thus chose the representation

$$\psi_k^s = [x^s \exp(-x)]^{\frac{1}{2}} L_k^s(x)/[k!\Gamma(s+k+1)]^{\frac{1}{2}} \quad ,$$

where Γ is the Gama function, and $L_k^s(x)$ is the modified Lagurre polynomial[14] of order k. s is chosen to be 4 in this analysis. Note that ψ_k^s form a complete orthonormal set of functions which satisfy the vanishing boundary conditions at x=0 and $x \to \infty$. The eigen-frequency and the amplitude coefficient B_m^k will be determined by use of the Galerkin method. Substituting (10) into (8), we have

$$L_{m1}(B^k_m\psi^s_k) = BR[L_{m2}(B^k_{m-1}\psi^s_k)+L_{m3}(B^k_{m+1}\psi^s_k)] \quad , \tag{11}$$

where the summation sign over k is omitted, and

$$L_{m1} = (\omega R-E^2_m)E^2_m + RE^2_m D \quad ,$$

$$L_{m2} = \pi(m-1)\bar{\psi}_{xo}(E^2_{m-1}-M) + \tfrac{1}{2}\bar{\psi}_{zo}(E^2_{m-1}-M)D \quad ,$$

$$L_{m3} = \tfrac{1}{2}\bar{\psi}_{zo}(E^2_{m+1}-M)D - \pi(m+1)\bar{\psi}_{xo}(E^2_{m+1}-M)$$

Multiplying (11) with ψ^s_ℓ, (ℓ=0 to K), and integrating over $0\leq x<\infty$, we have

$$Q_{mk\ell}B^k_{m-1} + R_{mk\ell}B^k_m + S_{mk\ell}B^k_{m+1} - R\omega\bar{R}_{mk\ell}B^k_m = 0 \tag{12}$$

where

$$Q_{mk\ell} = BR\int_0^\infty \psi^s_\ell L_{m2}(\psi^s_k)dx \quad ,$$

$$R_{mk\ell} = \int_0^\infty \psi^s_\ell L_{m1}(\psi^s_k)dx \quad , \quad \bar{R}_{mk\ell} = \int_0^\infty \psi^s_\ell E^2_m(\psi^s_k)dx \quad , \tag{13}$$

$$S_{mk\ell} = BR\int_0^\infty \psi^s_\ell L_{m3}(\psi^s_k)dx \quad .$$

It can be seen from (12) that the Fourier modes (m) are weakly coupled but the Lagurre modes (k) are strongly coupled. The system (12) can be written in a matrix form as

$$\{[G] + R\,\omega_i[H]\}(V) = 0 \quad , \tag{14}$$

where

$$[G] = \begin{bmatrix} R_{ok\ell} & S_{ok\ell} & & & \\ & \cdots & & & \\ & Q_{mk\ell} & R_{mk\ell} & S_{mk\ell} & \\ & & & \cdots & \\ & & & Q_{Nk\ell} & R_{Nk\ell} \end{bmatrix} , \quad [H] = \begin{bmatrix} \bar{R}_{ok\ell} & & \\ & \ddots & \\ & & \bar{R}_{mk\ell} & \\ & & & \ddots \\ & & & & \bar{R}_{Hk\ell} \end{bmatrix} ,$$

and (V) is the eigen-vector defined by

$$(V)^T = (B^k_o, B^k_1 \ldots B^k_{m-1}, B^k_m, B^k_{m+1} \ldots B^k_{N-1}, B^k_N) \quad ,$$

in which the superscript T denotes transpose, and in each m-th mode k ranges from 0 to K. (14) is a system of (K+1)xN equations in (K+1)x(N) unknown eigen-vector components. Note $B^k_{-1} = 0$, and B^k_{N+1} are truncated to form a determinate system (14). The non-trivial solution of (14) exists only if the determinant of its coefficient matrix vanishes, i.e.

$$\left| [G] + \omega_i R[H] \right| = 0 \quad . \tag{15}$$

The solution of (15) was achieved by use of the IMSL routine EIGZF. The required integrals given in (13) were evaluated with the adaptive Gauss quadrature[15]. All computations have been carried out with double precision on the Gould PN9780. Table I gives the complex eigen-frequencies of unstable modes for which the real parts are positive. The rest of eigen-frequencies, which are not listed, have negative real parts. At R=1000, no unstable modes with finite frequencies exist for values of B smaller than 6. Hence for R=1000, the critical value of B below which the flow remains stable with respect to the even mode of disturbances is 6. However, referring to Table I, we note that when the flow becomes unstable it does so only at a few isolated frequencies. This is quite different from the situation in parallel flows. When the flow parameter exceeds the critical value in a parallel flow, the flow becomes unstable with respect to disturbances of infinitely many frequencies within a given band-width. Comparing the eigenvalues in Table 1 obtained with two different sets of K and N, we see that K=N=10 is not yet large enough to yield accurate eigenvalues. However, the convergence of the method is expected. The main point in this letter is to point the irrelevance of the notion of neutral curve in non-parallel flows, and that there are only a few eigen frequencies at the onset of instability.

The stability analysis for odd mode can be carried out along the same line starting with (8). It turns out that the flow is more stable with respect to the odd mode disturbances. Exact eigenvalues for both modes will be obtained with more extensive computation and reported elsewhere in the near future.

This work was supported in part by a NASA Grant CC-280.

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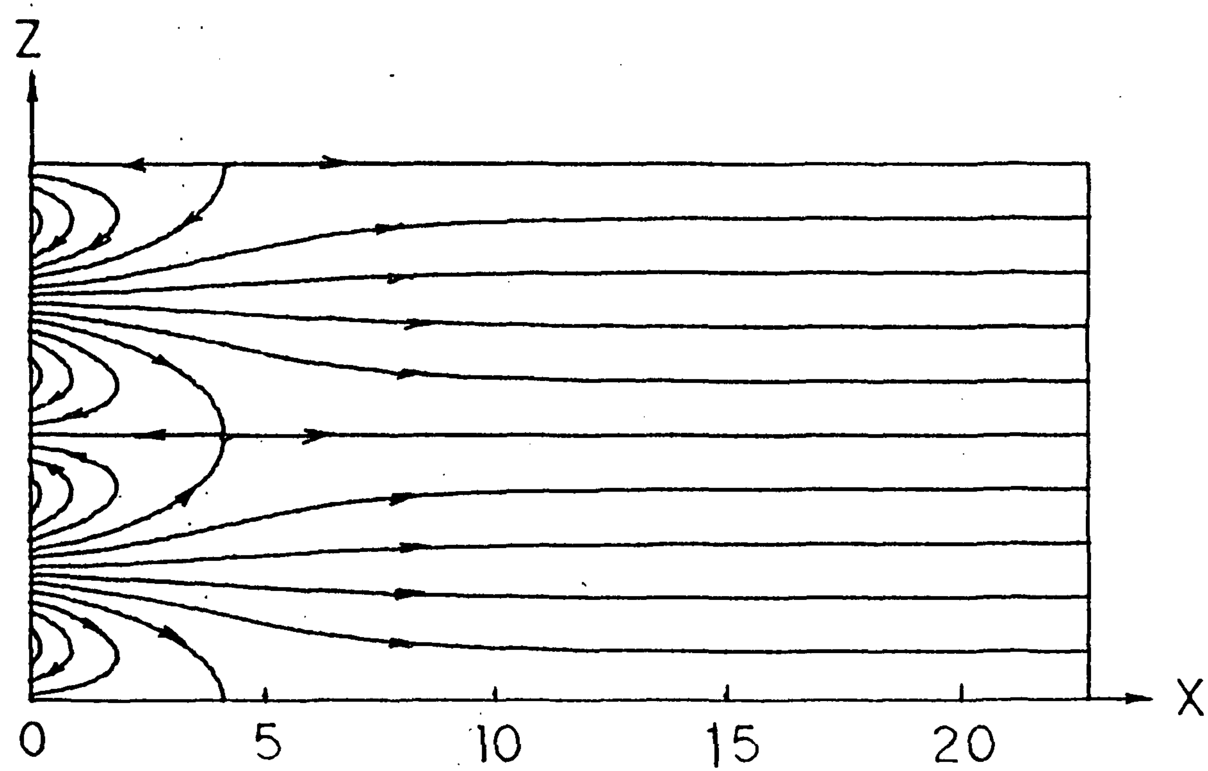
TABLE I. Unstable modes, R=1000.

B	N,K	ω_r	ω_i
6	9	-	-
	10	0.16	10.4
8	9	0.10	10.0
	10	0.69	13.8
10	9	0.60	12.3
	10	1.18	17.3

Figure Caption

Figure 1. Streamline pattern of Kovasznay flow, R=40, B=10.


Z
X
0
5
10
15
20

Spectral stability of Taylor's vortex array

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Abstract

It is proved that the two-dimensional Taylor vortex array, which is an exact solution of the Navier-Stokes equation, is absolutely and monotonically stable with respect to infinitesimal disturbances of all discrete frequencies.

PACS (47,51,44)

Consider the two-dimensional flows of an incompressible Newtonian fluid. The governing Navier–Stokes equations written in terms of the Stokes stream function is

η eta ∇² Lapl
Ψ psi(capt)
ν nu

$$\eta_t + \Psi_Z \eta_X - \Psi_X \eta_Z = \nu \nabla^2 \eta \qquad (1)$$

where t is time, (X, Z) are the Cartesian coordinates, ν is the kinematic viscosity, ∇^2 is the Laplacian operator, the subscripts stand for partial differentiation, and η is the sole component of the vorticity perpendicular to the X–Z plane of flow, i.e.

$$\eta = \nabla^2 \Psi \ . \qquad (2)$$

In terms of the following dimensionless variables

τ Tau ψ psi (l/c) ζ zeta

$$\tau = t/(d^2/\nu) \ , \ (X,Z) = (x,z)/d \ , \ \psi = \Psi/\nu \ , \ \zeta = \eta/(\nu/d^2) \ ,$$

where d is the characteristic length, (1) can be written as

$$\zeta_\tau + \psi_z \zeta_x - \psi_x \zeta_z = \nabla^2 \zeta \ . \qquad (3)$$

$$\zeta = \nabla^2 \psi \ .$$

Taylor[1] obtained an exact solution of (3) given by

π pi

$$\bar{\psi} = A \cos \pi x \cos \pi z \exp(-2\pi^2 \tau) \ . \qquad (4)$$

where A is a constant representing the strength of the flow. This stream function represents a time dependent two-dimensional array of counter-rotating vorticies enclosed in squares of a dimensional side-length d. Note that $x = (K+1)/2$ and $z = (K+1)/2$ are stream lines, where K is any integers. Note also that the x- and z-components of the velocity are respectively given by

$$\bar{u} = \bar{\psi}_z = -A\pi\cos\pi x\ \sin\pi z\ \exp(-2\pi^2\tau)$$

and

$$\bar{v} = -\bar{\psi}_x = A\pi\sin\pi x\ \cos\pi z\ \exp(-2\pi^2\tau)\ .$$

Hence the above mentioned two sets of stream lines intersect at saddle points, and the centers of vorticies are located at the stagnation points $(x,z) = (M,N)$ where M and N are any integers.

The stability analysis of this flow does not seem to have yet appeared. Most of the exact solutions of the Navier-Stokes equations represent parallel flows. The linear stability of parallel flows is governed by the well-known Orr - Sommerfeld type ordinary differential equation.[2,3] The linear stability of the non-parallel flow described by (4) is governed by the partial differential equation

$$\zeta'_\tau + \bar{\psi}_z\zeta'_x - \bar{\psi}_x\zeta'_z + \psi'_z\bar{\zeta}_x - \psi'_x\bar{\zeta}_z = \nabla^2\zeta'\ , \qquad (5)$$

where $\bar{\zeta} = \nabla^2\bar{\psi}$, and ψ' and ζ' are respectively the perturbation stream function and vorticity. (5) is obtained by substituting

$$\psi = \bar{\psi} + \psi' \text{ and } \zeta = \bar{\zeta} + \zeta'$$

into (3) and retaining only the linear terms. Consider disturbances with the characteristic frequency ω, i.e.

ω omega

$$\psi' = \phi e^{\omega\tau} \; ; \; \zeta' = \xi e^{\omega\tau} \; , \; \xi = \nabla^2\phi . \qquad (6)$$

φ ph. ξ xi

It is also assumed that the perturbation vanishes at infinity. Substitution of (6) into (5) yields

$$\omega\xi + (\bar{\psi}_z\xi)_x - (\bar{\psi}_x\xi)_z + (\phi_z\bar{\xi})_x - (\phi_x\bar{\zeta})_z - \nabla^2\xi = 0 \; , \qquad (7)$$

Multiplying (7) with ξ^*, where the upper star denotes the complex conjugate, and then taking the sum of the resulting equation with its complex conjugate, we have

$$2\omega_r\xi\xi^* + \{\, \xi^*(\bar{\psi}_z\xi)_x - \xi^*(\bar{\psi}_x\xi)_z + \xi^*(\phi_z\bar{\zeta})_x - \xi^*(\phi_x\bar{\zeta})_z - \xi^*\nabla^2\xi \,\} \qquad (8)$$

$$+ \{c.c\} = 0 \; ,$$

where ω_r is the real part of ω. Integration of (8) gives

$$- \langle 2\omega_r\xi\xi^* \rangle = \langle -\bar{\psi}_z(\xi\xi^*_x + \xi_x\xi^*) + \bar{\psi}_x(\xi\xi^*_z + \xi_z\xi^*)$$

$$- \bar{\zeta}(\xi^*_x\phi_z + \xi_x\phi^*_z) + \bar{\zeta}(\xi^*_z\phi_x + \xi_z\phi^*_x) - \xi^*\nabla.\nabla\xi - \xi\nabla.\nabla\xi^* \rangle \qquad (9)$$

where $\langle . \rangle$ signifies integration over the infinite domain. In arriving at (9), integration by parts was applied to every term in (8) except the first one, and then the condition that the perturbation vanishes at infinity was invoked. Integrating by parts once again, and invoking the divergence theorem we have from (9)

$$- \langle 2\omega_r |\xi^2| \rangle = \langle 2\nabla\xi^* \cdot \nabla\xi \rangle \quad , \tag{10}$$

where ξ is the absolute value of ξ. Hence

$$\omega_r = - \frac{\langle |\nabla\xi|^2 \rangle}{\langle |\xi|^2 \rangle} \tag{11}$$

This completes the proof that Taylor's vortex array is absolutely and monotonically stable with respect to infinitesimal disturbances of all frequencies as long as the viscosity is positive. The last condition follows from the definition of τ and (6).

The convective diffusion of perturbed vorticity happens to cancel out in Taylor's vortex array to yield (11). The perturbed flow is dominantly dissipative. However, this is not obvious a priori. It should be pointed out that the stability we have proved is the large time asymptotic stability of the discrete spectrum. The extension of the notion of asymptotic stability of steady basic flow to the unsteady mean flow demonstrated in this note does not involve any physical or mathematical ambiguity.

ACKNOWLEDGMENTS

This work was supported in part by a NASA Grant NCC2-280.

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Stability of Taylor's Vortex Array

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Abstract

It is proved that the two-dimensional Taylor vortex array, which is an exact solution of the Navier-Stokes equation, is globally, asymtotically and monotonically stable in the mean with respect to three-dimensional disturbances.

Consider the two-dimensional flows of an incompressible Newtonian fluid. The governing Navier-Stokes equations written in terms of the Stokes stream function is

$$\eta_t + \Psi_Z \eta_X - \Psi_X \eta_Z = \nu \nabla^2 \eta \qquad (1)$$

where t is time, (X, Z) are the Cartesian coordinates, ν is the kinematic viscosity, ∇^2 is the Laplacian operator, the subscripts stand for partial differentiation, and η is the sole component of the vorticity perpendicular to the X-Z plane of flow, i.e.

$$\eta = \nabla^2 \Psi \quad . \qquad (2)$$

In terms of the following dimensionless variables

$$\tau = t/(d^2/\nu), \ (x,z) = (X,Z)/d, \ \ \psi = \Psi/\nu \ , \ \ \zeta = \eta/(\nu/d^2) \ ,$$

where d is the characteristic length, (1) can be written as

$$\zeta_\tau + \psi_z \zeta_x - \psi_x \zeta_z = \nabla^2 \zeta \ . \qquad (3)$$

$$\zeta = \nabla^2 \psi \quad .$$

Taylor (1923) obtained an exact solution of (3) given by

$$\bar{\psi} = A \cos \pi x \cos \pi z \exp(-2\pi^2 \tau) \ . \qquad (4)$$

where A is a constant representing the strength of the flow. Without loss of generality, A is taken to be positive. This stream function represents a time dependent two-dimensional array of counter-rotating vorticies enclosed in squares of a dimensional side-length d. Note that $x = (K+1)/2$ and $z = (K+1)/2$ are stream lines, where K is an integer .

Note also that the x- and z-components of the velocity are respectively given by

$$\bar{u} = \bar{\psi}_z = -A\pi\cos\pi x \ \sin\pi z \ \exp(-2\pi^2\tau)$$

and

$$\bar{v} = -\bar{\psi}_x = A\pi\sin\pi x \ \cos\pi z \ \exp(-2\pi^2\tau) \ .$$

Hence the above mentioned two sets of stream lines intersect at saddle points, and the centers of vorticies are located at the stagnation points $(x,z) = (M,N)$ where M and N are any integers.

The nonlinear stability analysis of this flow does not seem to have yet appeared. Most of the exact solutions of the Navier-Stokes equations represent parallel flows. The linear stability of parallel flows is giverned by the well-known Orr - Somerfeld type ordinary differential equation. (Joseph, 1976; Drazin and Reid, 1983) The linear stability of the non-parallel flow described by (4) is giverned by the partial differential equation

$$\zeta'_\tau + \bar{\psi}_z\zeta'_x - \bar{\psi}_x\zeta'_z + \psi'_z\bar{\zeta}_x - \psi'_x\bar{\zeta}_z = \nabla^2\zeta' \ , \qquad (5)$$

where $\bar{\zeta} = \nabla^2\bar{\psi}$, and ψ' and ζ' are respectively the perturbation stream function and vorticity. (5) is obtained by substituting

$$\psi = \bar{\psi} + \psi' \quad \text{and} \quad \bar{\zeta} \quad \bar{\zeta} + \zeta'$$

into (3) and retaining only the linear terms. Based on this equation, it was shown recently by Lin and Tobak (1986) that Taylor's vortex array is absolutely and monotonically stable with respect to two-dimensional infinitesimal disturbances which vanish at infinity for all frequencies.

It turns out that Taylor's vortex array is also globally stable with respect to three- as well as two-dimensional disturbances of any finite amplitude. This will be proved presently. The Navier-Stokes equations can be written as

$$\partial_t(\underline{U}+\underline{u}) + (\underline{U}+\underline{u})\cdot\underline{\nabla}(\underline{U}+\underline{u}) = -\frac{1}{\rho}\underline{\nabla}(P+p)+\nu\nabla^2(\underline{U}+\underline{u}) \tag{6}$$

$$\underline{\nabla}\cdot(\underline{U}+\underline{u}) = 0 \ , \tag{7}$$

where $\underline{U}$ is the basic flow field the stability of which is of interest, $\underline{u}$ is the velocity perturbation, ρ is the density, p is the pressure field in the basic flow and p is the pressure perturbation. The basic flow field satisfies

$$\partial_t\underline{U} + \underline{U}\cdot\underline{\nabla U} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\underline{U} \tag{8}$$

$$\underline{\nabla}\cdot\underline{U} = 0 \tag{9}$$

Substrating (8) from (6), forming the inner product of the resulting equation with $\underline{u}$ and using the fact that the velocity field is divergence free, we have

$$[\partial_t + (\underline{U}+\underline{u})\cdot\underline{\nabla}]e(t) = -\underline{u}(\underline{u}\cdot\underline{\nabla})\underline{U}-\underline{\nabla}\cdot(\underline{u}p) + \{\underline{\nabla}\cdot(\underline{u}\cdot\underline{\nabla})\underline{u}]-\underline{\nabla u}:\underline{\nabla u}\} \ , \tag{10}$$

where $e(t) = \underline{u}\cdot\underline{u}/2$ is the kinetic energy of the disturbance per unit mass of fluid. Integrating (10) over the entire flow domain, using the divergence theorem and the boundary condition

$$\underline{u}|_s = 0,$$

where the subscript s denotes that $\underline{u}$ is to be evaluated at the boundary s of the flow domain, we have

$$\frac{d\langle E\rangle}{dt} = -\nu\langle\underline{\nabla u}:\underline{\nabla u}\rangle - \langle\underline{u}\cdot(\underline{u}\cdot\underline{\nabla})\underline{U}\rangle \ , \tag{12}$$

where $d/dt = \partial_t + (\underline{U}+\underline{u})\cdot\underline{\nabla}$. For two dimensional basic flows, such as the Taylor vortex array, we have

$$\langle \underline{u}\cdot(\underline{u}\cdot\underline{\nabla})\underline{U}\rangle = \langle u^2\partial_X U + v^2\partial_Z V + uv(\partial_X V + \partial_Z U)\rangle \ , \tag{13}$$

where U and V are respectively the X- and Z-components of the basic flow field $\underline{U}$, and u and v are the corresponding components of the disturbances. It should be pointed out that $\underline{u}$ was not assumed to be two-dimensional in (13). It can be seen from the Taylor solution (4) that the shear rate of deformation in the basic flow vanishes, i.e.,

$$\partial_X V + \partial_Z U = 0 \ , \tag{14}$$

and

$$\partial_X U = -\partial_Z V = \nu(\pi/d^2)A\cos(\pi X/d).\cos(\pi Z/d)\exp(-2\pi^2\nu t/d^2) \ .$$

Hence the minimum of $\partial_X U$ and $\partial_Z V$ is $-(\nu\pi/d^2)A\exp(-2\pi^2\nu t/d^2)$. It follows from (13) and (14) that

$$\langle \underline{u}\cdot(\underline{u}\cdot\underline{\nabla})\underline{U}\rangle \geq -\langle(\nu\pi/d^2)(u^2+v^2)A\exp(-2\pi^2\nu t/d^2)\rangle \geq -(\nu\pi/d^2) \ .$$
$$\langle(u^2+v^2+w^2)\rangle A\exp(-2\pi^2\nu t/d^2) \ . \tag{15}$$

On the other hand, it can be shown by use of Schwarz's inequality that (see e.g. Joseph 1976)

$$\langle \underline{\nabla u}:\underline{\nabla u}\rangle \geq \frac{2}{\ell^2}\langle(\underline{u}\cdot\underline{u})\rangle \ , \tag{16}$$

where $\ell = Kd$, K being any arbitrarily large integer, is the distance between any arbitrarily far apart parallel boundaries of the Taylor vortex array where condition (11) is invoked. It follows from (12), (15) and (16) that

$$\frac{d}{dt} \langle E\rangle \leq -2\langle E\rangle \left[\frac{2}{\ell^2} + \frac{\pi}{d^2} A \exp(-2\pi^2 \nu t/d^2)\right]. \tag{17}$$

Integration of this inequality yields

$$\langle E(t)\rangle \leq \langle E(0)\rangle \exp[I(t)] , \tag{18}$$

$$I(t) = - \frac{4\nu}{K^2 d^2} t - \frac{A}{\pi} [1 - \exp(-2\pi^2 \nu t/d^2)] ,$$

where $\langle E(0)\rangle$ is the energy integral of any kinematically admissible initial disturbances. Since the inequalities (15) and (16) used to arrive at (18) hold for both two- and three-dimensional disturbances, the decay law described by (18) holds for both types of disturbances.

In summary, we state the following results. Taylor's two-dimensional vortex array, which is an exact solution of the Navier-Stokes equations, is *asymptotically stable in the mean in a sense that*

$$\lim_{t\to\infty} \langle E(t)\rangle/\langle E(0)\rangle \to 0 \tag{19}$$

with respect to two- as well as three-dimensional disturbances satisfying the kinematic conditions

$$\underline{\nabla}\cdot\underline{u} = 0 , \quad u|_S = 0 , \quad \nu > 0 \quad \text{and} \quad \langle E(0)\rangle < \infty . \tag{20}$$

Moreover, the right side of (17) is negative for all t. Hence the disturbance energy integral decays monotonically. The stability is global in a sense that the initial disturbance is allowed to exist everywhere in the infinite flow domain without limitation except that given by (20). This can be seen easily from the fact that (19) remains true even when $K\to\infty$. It can be shown easily from (4) and (18) that as $K\to\infty$, the initial decay rate of the disturbance is greater than that of the basic flow if $A>\pi$. If $A<\pi$, the converse is true. *It follows from (18) that the decay rate decreases* as K and d increase, but never vanishes for finite d.

Finally, we note that this work, as far as the authors are aware, is the first nonlinear stability analysis of a non-parallel flow which is an exact solution of the Navier-Stokes equation. For this reason this work will probably serve as a valuable testing ground for any method of stability analysis of non-parallel flows for which an exact solution is not available. Unlike the parallel flow theory, the general method of stability analysis of non-parallel flow is yet to be firmly established.

Acknowledgments

This work was supported in part by a NASA grant NCC2-280.

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Reversed Flow above a Plate with Suction

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Introduction

Most of the known exact analytical solutions of the Navier-Stokes equations have been obtained for parallel laminar flows for which the Navier-Stokes equations can be linearized. G.I. Taylor[1] observed that the nonlinear convective terms in the two-dimensional Navier-Stokes equations vanish when the vorticity is a function of the Stokes stream function alone. For the special case of vorticity being proportional to the stream function, he obtained an exact solution which represented a double infinite array of vorticies decaying exponentially with time. Kovasznay[2] extended G.I. Taylor's method, and considered the flow in which the local vorticity is proportional to the stream function perturbed by a uniform stream. He was able to linearize the Navier-Stokes equation and obtained an exact solution which may be used to represent the flow down stream of a two-dimensional grid.

In this note we make a minor extension of Kovasznay's solution, and present an exact solution of the Navier-Stokes equations, which may represent the reversed flow above a flat plate with suction. The present exact solution grew out of our search for a simple exact solution which may serve as an exact basic flow for a model stability analysis of non-parallel flows involving a flow reversal. Unlike the stability analysis of parallel flows, the stability analysis of non-parallel flows is not yet firmly established. Among other difficulties, the stability analysis of nonparallel flows suffers from some serious uncertainty associated with the approximate nature of non-parallel basic flows.

Exact Solution

Consider two-dimensional flows of an incompressible Newtonian fluid. The X and Z Cartesian components of the velocity field are related respectively to the Stokes stream function Ψ by

$$U = \Psi_Z , \qquad V = - \Psi_X ,$$

where subscripts X and Z denote partial differentiations. In terms of Ψ, the governing equation is

$$(\partial_t - \nu \underline{\nabla}^2)\underline{\nabla}^2\Psi = \Psi_X \underline{\nabla}^2 \Psi_Z - \Psi_Z \underline{\nabla}^2 \Psi_X , \tag{1}$$

where t is time, ν the kinematic viscosity, $\underline{\nabla}^2$ is the Laplacian operator, the subscripts X and Z denote partial differentiations with respect to the space variables. By use of the following dimensionless variables

$$\psi = \Psi/\nu , \qquad (x,z) = (X,Z)/(\nu/U) ,$$

$$\nabla = \underline{\nabla}(\nu/U) , \qquad \tau = t/(\nu/U^2) ,$$

where U is the characteristic velocity, Eq. (1) can be rewritten as

$$(\partial_\tau - \nabla^2)\nabla^2\psi = \psi_X \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x . \tag{2}$$

This nonlinear equation can be linearized for flows with the following particular vorticity distribution

$$\nabla^2\psi = k(\psi - Rz) , \tag{3}$$

where k is a constant to be determined, and R is a flow parameter to be explained. Substitution of Eq. (3) into Eq. (2) gives the following linear equation

$$(\partial_\tau - \nabla^2)\psi = -R\psi_x \ . \tag{4}$$

A special steady solution, ψ, of Eq. (4) is given by

$$\psi = Rz + B\exp(mx-nz) \ , \tag{5}$$

where B and n are constants, and

$$m = \tfrac{1}{2}(R \pm \sqrt{R^2-4n^2}) \ . \tag{6}$$

The coefficient of vorticity distribution, k, appearing in Eq. (3), can now be determined by substituting Eqs. (5) and (6) into Eq. (3). It is found that

$$k = mR.$$

When $n=2\pi i$, and R is identified as the Reynolds number, the solution given by Eqs. (5) and (6) reduces to the Kovasznay flow. Kovasznay suggested that the solution corresponding to the negative root of m in Eq. (6), i.e. $m = 0.5\ (R-\sqrt{R^2+16\pi^2})$, can be used to describe the wake flow in the region $x>0$ behind a grid at $x=0$. The solution corresponding to the positive root of m, i.e. $m = 0.5(R+\sqrt{R^2+16\pi^2})$, can be used to describe the two-dimensional uniform flow, in $x<0$, perturbed by a grid of blunt bodies with suction or blowing whose magnitude is proportional to B near the front stagnation points.

When n is real, the new exact solution given by Eqs. (5) and (6) represents flows over a plate with suction or blowing of fluid at the plate. For example, when $n>0$ and $R^2-4n^2>0$ the exact solution describes the flow over a porous plate $z=0$, $x<0$ with velocity components given by

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$$u = \psi_z = R - nB \exp(mx-nz) \quad , \tag{7}$$

$$v = -\psi_x = -mB \exp(mx-nz) \quad . \tag{8}$$

It is seen from Eqs. (7) and (8) that R can be put to 1 without loss of generality, since it amounts to rescaling the velocity by a factor of R. It is seen from Eq. (8) that B>0 corresponds to suction and B<0 corresponds to blowing at the bottom plate z=0. Fig. 1 gives the streamlines of the flow for R=1, n=0.05 and B=80. Note the flow reversal due to suction near the trailing edge. It should be pointed out that the flow reversal may not take place when B is sufficiently small.

When n>0 but $R^2-4n^2 = -q^2<0$, the real part of Eq. (5) gives

$$\psi = Rz + B \exp(Rx/2-nz) \cos qx.$$

Hence

$$u = R-nB \exp(Rx/2-nz) \cos qx \ ,$$

$$v = -B[(R/2)\cos qx - q \sin qx] \exp(Rx/2-nz) \quad .$$

This solution represents the wavy flow over a plate z=0, x<0 with a periodic variation of suction and blowing along the plate. The possible flow represented by Eqs. (5) and (6), with n neal, are delineated in Fig. 2. Note that R is put to 1 without loss of generality.

Discussion

A new exact solution of the Navier-Stokes equation is given. The solution can be used to describe the flow with or without flow reversal above a plate with suction and/or blowing. Aside from its own intrinsic interest, it provides us with a simple but exact non-parallel basic flow which could be used for a model analysis of stability of non-parallel flow. The stability analysis of non-parallel flows usually suffers from the uncertainty associated with modeling the non-parallel flows with approximate solutions.

Acknowledgments

This work was supported in part by NASA Research Grant NCC2-280. Thanks are due to Mr. Chen Haibo for his help in producing Fig. 1.

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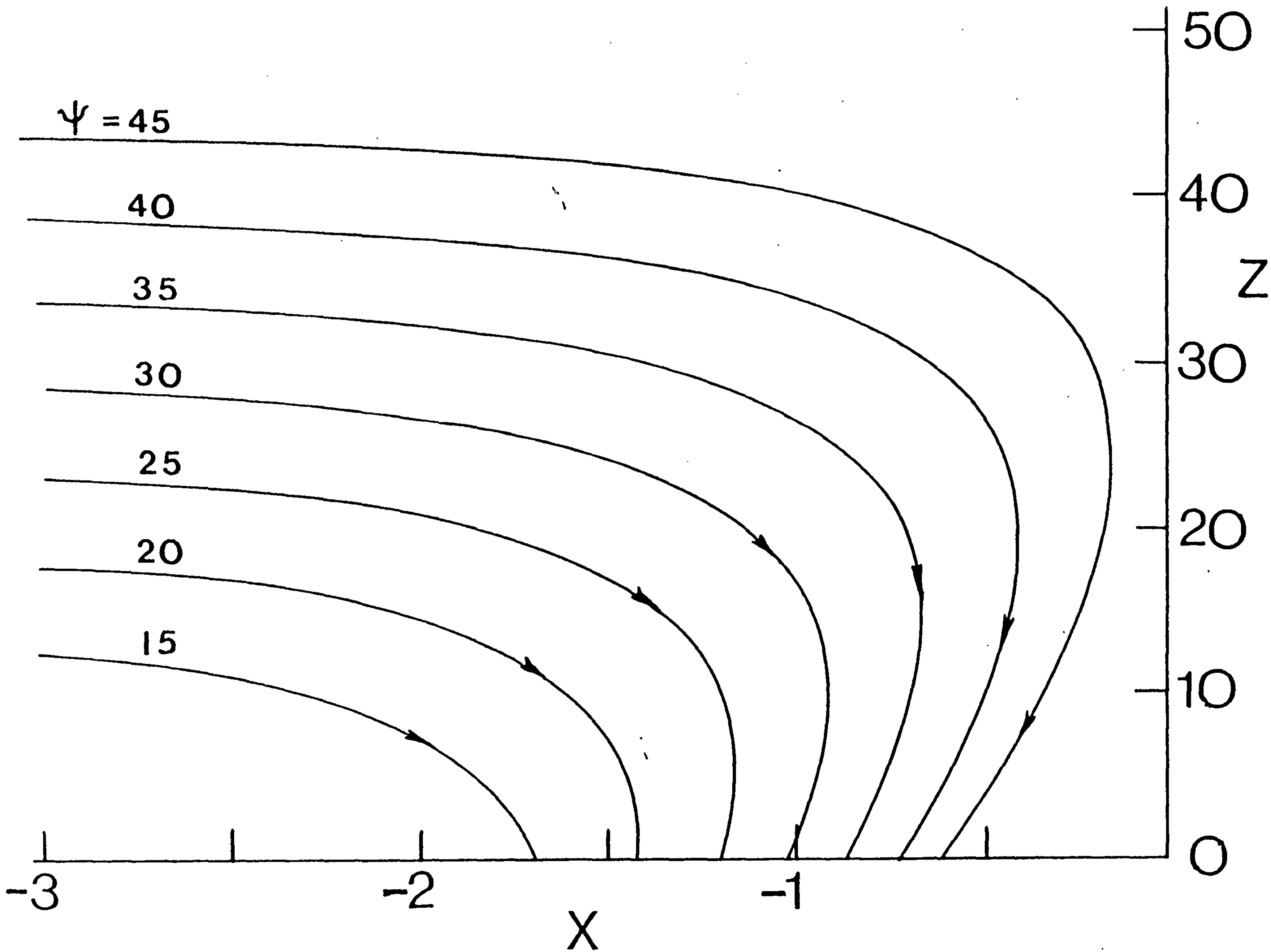
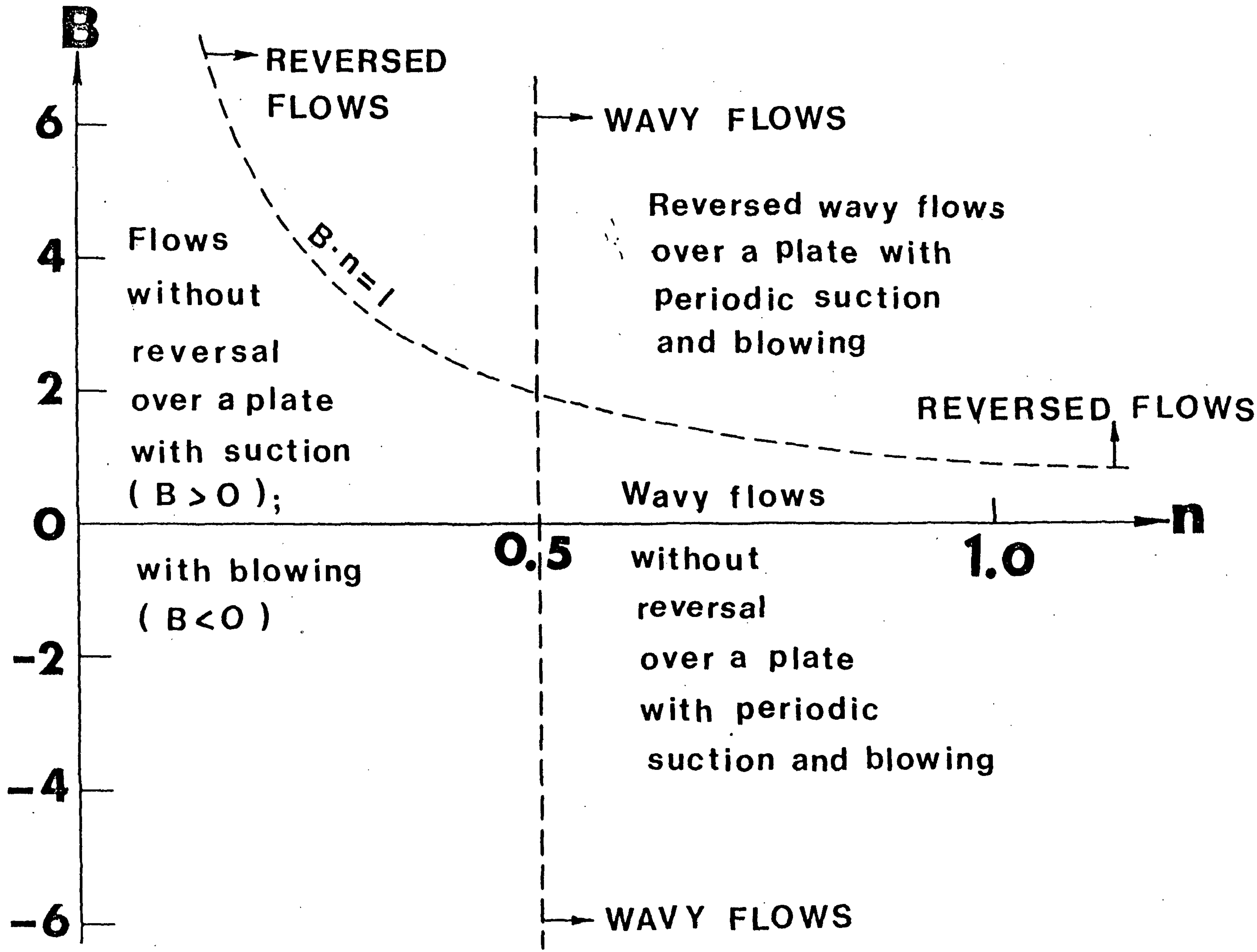
FIG. 1. Streamline pattern for $R=1$, $B=80$, $C=0.05$.

FIG. 2. Exact solutions of the Navier-Stokes equations for flows over a plate $z=0$, $x<0$.



Ψ = 45
40
35
30
25
20
15
50
40
30
20
10
0
Z
-3
-2
-1
X



B
REVERSED FLOWS
WAVY FLOWS
Reversed wavy flows over a plate with periodic suction and blowing
Flows without reversal over a plate with suction (B > O); with blowing (B<O)
B·n=1
REVERSED FLOWS
Wavy flows without reversal over a plate with periodic suction and blowing
WAVY FLOWS
6
4
2
0
-2
-4
-6
0.5
1.0
n

SEPARATED FLOW ABOUT A ROTATING BODY

- VARIOUS METHODS OF SOLUTION

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1. Introduction

The laminar boundary-layer oscillation on a rotating disk was investigated by Smith (1947). The spiral vortices on a rotating disk were discovered by Gregory, Stuart & Walker (1955). They theorized that the sprial vorticies were the products of flow instability. Their study has been expanded and improved by many authors Gregory & Walker, 1960; Brown, 1961; Tobak, 1973; Kobayashi, 1980). The agreement between theories and experiments remain inconclusive. Similar spiral vortex motions are observed on other surfaces where the centrifugal force plays an important role; spiral vorticies on a swept wing are an example (see Chapman & Tobak, 1984). The boundary-layer and side force characteristics of a spinning axisymmetric body were investigated experimentally by Kegelman, Nelson, & Muller (1980) and Muller et al. (1981). They observed, in addition to the Tollmien Schlichting waves, regular vortices roping tightly around the rotating cone-cylinder. Similar vortex motion has been observed by Kobayashi (1980), Kobayashi and Izumi (1983), Kobayashi et al. (1983) on a rotating cone. They also analyzed the flow with the aid of linear stability theories. Definitive comparisons between theories and experiments cannot yet be made.

The lack of definitive comparisons between experiments and theories mentioned in the previous paragraph is not surprising. This is because in all of the theoretical studies, flow instability was assumed to be a local phenomena. The basic flows were "frozen" locally in the analysis which disregards the possible up-stream and down-stream influence. The concern over this type of analysis for non-parallel flows has already been raised by Morkovin (1979). Note that all of the basic flows mentioned in the previous paragraph are non-parallel. The stability theories for parallel flows are far more well developed than the theories for non-parallel flows (Joseph, 1976). The parallel basic flows possess translational invariance in the direction of flow. No such invariance exists for non-parallel basic flows. While it is reasonable to assume that the physical characteristics of the disturbance retains the same invariance in the case of parallel flows, no such assumption can be made for the case of non-parallel flows. The mathematical consequence is that the governing equation of stability is an ordinary differential equation for the former case but it is a partial differential equation for the latter one. The analysis of the latter case is made even more difficult by the fact that while the parallel basic flow is independent of the parameter, the critical value of which is to be determined by stability analysis, the non-parallel basic flow depends crucially on the stability parameter itself. Stability analysis for non-parallel flows is still in an exploration stage (Goldstein, 1981; Goldstein et al., 1983; Hall, 1983).

The purpose of this work is to investigate the precise mechanism of the flow separation from a rotating body of revolution. The spiral vortices around a rotating body are viewed as the consequence of the instability of the steady non-parallel laminar basic flow about a body of revolution.

2. Basic Flows

Two different basic flows are considered. The first is the flow around an axisymmetric body rotating at a constant angular velocity Ω in an otherwise quiescent fluid of infinite extent as shown in figure 1. The second is the flow around the same body placed in a coaxial cylinder of infinite length. The axial stream far from the nose-tip may be imposed externally for this latter flow. For sufficiently small Reynolds numbers based on Ωa, a being the maximum radius of the body, the flow is laminar and axisymmetric. For these basic flows the Navier-Stokes equations are reduced to

$$\frac{DU}{Dt} - \frac{V^2}{r} = -\partial_r P + \frac{1}{R}(\nabla^2 U - \frac{U}{r^2}) , \quad (1)$$

$$\frac{DV}{Dt} + \frac{UV}{r} = \frac{1}{R}(\nabla^2 V - \frac{V}{r^2}) , \quad (2)$$

$$\frac{DW}{Dt} = - \partial_z P + \frac{1}{R}\nabla^2 W , \quad (3)$$

$$\partial_r(rU) + \partial_z(rW) = 0 , \quad (4)$$

where t is the time nondimensionalized by Ω^{-1}, (U,V,W) are the radial, tangential and axial velocity components nondimensionalized by Ωa in the cylindrical coordinates (r,θ,z); all lengths are normalized with a, R is the Reynolds number $\Omega a^2/\nu$, ν being the fluid kinematic viscosity, and

$$\frac{D}{Dt} = \partial_t + U\partial_r + \frac{V}{r}\partial_\theta + W\partial_z , \quad (5)$$

$$\nabla^2 = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \partial_{zz} . \quad (6)$$

The pressure terms in (1) and (3) can be eliminated by cross-differentiation which yields, upon utilizing (4), the vorticity equation

$$\frac{D\xi}{D\tau} - \frac{U}{r}\xi = \frac{2V}{r}\partial_z V + \frac{1}{R}(V^2 - \frac{1}{r^2})\xi \quad , \tag{7}$$

where ξ is the vorticity given by

$$\xi = \partial_z U - \partial_r W \ .$$

The velocity components can be obtained from (2), (4) and (7). Then, if desired, the pressure field is obtainable from (1) and (3).

A Galerkin method similar to that used by Munson & Joseph (1971), Munson & Menguturk (1975), Yakushin (1969, 1972, 1973) and Khlebutin (1968) will be used to construct the solution. The velocity components will be represented by a complete set of orthogonal functions which satisfy the no-slip boundary conditions on the rigid surface. For the convenience of applying the boundary condition, we introduce the new variable $y = r-h(z,\tau)$. Thus r and ∂_r in (1) to (6) must be replaced by $y+h$ and ∂y respectively, and the differentiation with respect to z must be transformed according to

$$(\partial_z)_{r,\tau} = D - h'd$$

$$(\partial_{zz})_{r,\tau} = (D-h'd)(D-h'd) = D^2-2h'dD + h'^2 d^2 - h''d, \text{ etc.}$$

where primes denote differentiation with respect to z, and

$$D = (\partial_z)_{y,\tau} \quad \text{and} \quad d = (\partial_y)_{z,\tau}$$

2.1. Unbounded domain

Consider the steady axisymmetric flow around a body of revolution in an unbounded fluid as shown in figure 1. The velocity components are represented by a complete orthogonal set

$$U = Un(z)\ Yn(y)$$

$$V - F = Vn(z)\ Yn(y) \quad , \qquad F = h^2/(y+h) \tag{8}$$

$$W = Wn(z)\ Yn(y) \ ,$$

where the double indecies imply summation over n, from n=1 to N, and

$$Y_n(y) = \frac{(y^s e^{-s})^{\frac{1}{2}} L_n^{\ s}(y)}{[n!\Gamma(s+n+1)]^{\frac{1}{2}}} \ , \quad s>-1 \ . \tag{9}$$

$L_n^{\ s}(y)$ in (8) is the generalized Laguerre polynomial (Aizenshtadt, 1966) possessing the following orhtogonality relation

$$\int_0^\infty y^s\ e^{-s}\ L_m^{\ s} L_n^{\ s} dy = \begin{cases} 0 \ , & n \neq m \\ n!\Gamma(s+n+1), & n = m; \end{cases}$$

and $\Gamma(s+n+1)$ denotes the Gama function.

$$\Gamma(s+n-1) = (s+n)!.$$

Note that the velocity components given in (7) satisfies the boundary condition at the body surface,

$$U = W = 0 \quad \text{and} \quad V = h \quad \text{at} \quad y = 0 \ ,$$

and the condition that the velocity vanishes at infinity.

Substituting (8) into (2), (4) and (7), we have

$$\{[(D-h'd)^2+d^2+d/r-r^{-2}]-R[U_n Y_n d+W_n Yn(D-h'd)+U_n Y_n/r]\}(V_n Y_n+F) = 0 \ ,$$

$$dU_n Y_n+U_n Y_n/r+(D-h'd)W_n Y_n = 0 \ ,$$

$$\{[(D-h'd)^2+d^2+d/r-r^{-2}]-R[U_n Y_n d+w_n Y_n(D-h'd)-U_n Y_n/r]\}[(D-h'd)U_n Y_n-dW_n Y_n]$$

$$+ [2R(V_n Y_n+F)/r](D-h'd)(V_n Y_n+F) = 0 \ .$$

Multiplying ach of the previous three equations with Y_m, integrating over $0 \leq y < \infty$, and using the orthogonality property of Y_m we have

$$V_n''\delta_{mn} = 2h'V_n' \int Y_m Y_n' - V_n \int Y_m \bar{d}^2 Y_n$$

$$+ R \{V_n' W_e \int Y_e Y_m Y_n + V_n [(U_e - h'W_e) \int Y_e Y_n' Y_m + U_e \int (Y_e Y_m Y_n / r)\}$$

$$+ R \int Y_n Y_m [U_n dF + W_n (DF - h'dF) + FU_n / r]$$

$$- \int (\bar{d}^2 + D^2 - 2h'dD) FY_m \tag{10}$$

$$W_n'\delta_{mn} = h'W_n \int Y_m Y_n' - U_n [\int Y_m Y_n' + \int (Y_m Y_n / r)] \tag{11}$$

$$U_n'''\delta_{mn} = U_n'' \cdot 3h' \int Y_m Y_n' - U_n' [2h'^2 \int Y_m Y_n'' - 2h'' \int Y_m Y_n' + \int Y_m \bar{d}^2 Yn]$$

$$- U_n [-h''' \int Y_m Y_n dy - h' \int Y_m \bar{d}^2 Y_n + 2h'h'' \int Y_m Y_n''] + W_n'' \int Y_m Y_n'$$

$$- W_n' \cdot 2h' \int Y_m Y_n'' + W_n \int Y_m \bar{d}^2 Y_n'$$

$$+ R\{U_n'' \cdot W_e \; Y_m Y_e Y_n$$

$$+ U_n' [U_e \int Y_m Y_e Y_n' - 2W_e h' \int Y_m Y_e Y_n' - U_e \int Y_m Y_e Y_n / r$$

$$- h'W_e \int Y_m Y_e Y_n']$$

$$+ U_n[-U_e h'\int Y_m Y_e Y_n'' - h'U_e\int r^{-1} Y_m Y_e Y_n' + h'^2 W_e\int Y_m Y_e Y_n''$$

$$- W_e h''\int Y_m Y_e Y_n']$$

$$+ W_n'[-W_e\int Y_m Y_e Y_n']$$

$$- W_n[U_e\int Y_m Y_e Y_n'' - U_e\int r^{-1} Y_m Y_e Y_n' - h'W_e\int Y_m Y_e Y_n'']$$

$$- 2V_n'[V_e\int Y_m Y_e + \int r^{-1} F Y_m Y_n]$$

$$+ 2V_n[h'V_e\int Y_m Y_e Y_n' + h'\int r^{-1} F Y_m Y_n'$$

$$- \int Y_m Y_n DF + \int Y_m Y_n h' dF]$$

$$- \int r^{-1} Y_m F(DF - h'dF) = 0 \quad , \qquad (12)$$

where primes denote differentiation with respect to the arguments of the functions, the integration is understood to be between $y=0$ and $y\to\infty$, each of e m and n range from 0 to N, and

$$\bar{d}^2 = d^2 + \frac{d}{r} + h'^2 d^2 - h''d - r^{-2} \quad .$$

(10), (11) and (12) are a system of 3N equations in 3N unknowns Un, $\bar{\text{Yn}}$ and Wn (n=1 to N). The independent variable is z. This system is third order in U_n, second order in V_n and first order in W_n. Note the second order term of W_n in (12) can be reduced to the first order terms by use of (11). This system can be reduced to a system of the first order differential equations given by

$$U_m' = U_m^{(1)} ,$$

$$[U_m^{(1)}]' = U_m^{(2)} ,$$

$$[U_m^{(2)}]' = S_1(U_n, U_n^{(1)}, U_n^{(2)}, V_n, V_n^{(1)}, W_n, z), \tag{13}$$

$$V_m' = V_m^{(1)} ,$$

$$[V_m^{(1)}]' = S_2(U_n, V_n, V_n^{(1)}, W_n, z) ,$$

$$[W_m]' = S_3(U_n, W_n, z) .$$

where S_1, S_2 and S_3 are given respectively by the right sides of (12), (10) and (11). The solution of this system with appropriate upstream conditions has been attempted by the use of the Runge-Kutta method, for various values of R. Seveor numerical instability has been encountered, possibly because of the extreme spatial resolution required. Note that (13) is a dynamical system in the variable z. The fixed points of the system in the phase space are given by the solution of (13) with all elements on the left side put to zero. Note that this system is non-autonomous, since z appears explicitely through the body surface $r = h(z)$. Thus the character of the fixed points is expected to depend on z. To overcome the numerical instability encountered with the Runge-Kutta method, we expand the amplitude functions in (8) in terms of another complete orthonormal set $P_k(z)$ and apply the Galerkin method in the z-direction. Hence

$$[U_n(z), V_n(z), W_n(z)] = [U_{nk}, V_{nk}, W_{nk}] \, P_k(z) \tag{14}$$

where the coefficients of $P_k(z)$ are constants, and the repeated subscript k denotes summation over k. Substituting (14) into (10) to (12), multiplying the resulting equations with $P_\ell(z)$, and integrating from $z=-\infty$ to $z=+\infty$, we have

$$\delta_{mn} \int V_{nk} P_k'' P_\ell = \int S_1(m,n,k) P_\ell$$

$$\delta_{mn} \int W_{mk} P_k' P_\ell = \int S_2(m,n,k) P_\ell \qquad (15)$$

$$\delta_{mn} \int U_{nk} P_k''' P_\ell = \int S_3(m,n,k) P_\ell$$

where S_1, S_2 and S_3 are the respectively the right side of (10), (11) and (12) with the velocity amplitude functions properly expanded in terms of P_k. P_k can be any set of orthonormal functions which vanish at infinity and render the right side of (14) integrable over $-\infty<z<\infty$. (15) is a system of nonlinear algebra equations in the amplitude functions U_{kn}, V_{kn} and W_{kn}. The solution of this system can be achieved in principle by the Newton-Raphson method. For the convergence of this method, as usual, a sufficiently close initial guess of the solution is required. For steady nonlinear flows, such a close initial guess is difficult. To avoid this difficulty, we may treat the amplitude functions as time dependent and solve the initial value problem. The solution then may be started from any kinematically compatible initial condition without the need of a close initial guess. For the initial value problem, we need only to add to the left sides of the first and the last of (15) respectively the following terms,

$$-\delta_{mn} \delta_{k\ell} \partial_\tau V_{kn} ,$$

and

$$[\delta_{k\ell} \int h' Y_n' Y_m - \delta_{mn} \int P_k' P_\ell] \partial_\tau U_{nk} ,$$

The resulting equations form a system of ordinary differential equations in $[U_{nk}(\tau), V_{nk}(\tau), W_{nk}(\tau)]$. The solution of this system can be achieved by use of the finite difference method successfully employed by Aidun (1985).

2.2 Bounded domain

For the flow around a body of revolution, bounded internally by a cylinder of radius b as shown in figure 2, we represent each velocity component by an orthogonal set, each member of which satisfies the no-slip condition at solid walls. Thus

$$
\begin{aligned}
U &= U_n(z)Y_n(y,h) \\
V-F &= V_n(z)Y_n(y,h) \\
W-G &= W_n(z)Y_n(y,h)
\end{aligned}
\tag{16}
$$

where

$$Y_n = \sin n\pi \left(\frac{y}{b-h}\right)$$

$$F = A(y+h) + B/(y+h), \quad A = -h^2/(b^2-h^2), \quad B = b^2h^2/(b^2-h^2);$$

$$G = (1-H)(1-y/b)\Sigma n\pi W_n(-\infty)$$

$$H = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

Note that G is so chosen that the axial velocity is allowed to exist at $z=\pm\infty$, and $dW/dy = 0$ in $z \leq 0$. F is so chosen that the circumferential velocity at $z=\infty$ approaches that between two concentric cylinders with the inner cylinder rotating and the outer one fixed. The forms of Y_n and F allow the no-slip condition to be satisfied at the solid surfaces.

Substituting (16) into (2), (4) and (7) and integrating between y=0 and y=b-h with the weighting function Y_m

$$V_n''\delta_{mn}(b-h)/2 = V_n'[2h' \int dY_nY_m - 2\int DY_nY_m]$$

$$- V_n[\int \bar{d}^2 Y_nY_m + \int D^2 Y_nY_m - 2h'\int dDY_nY_m] - \int (D^2 - 2h'dD + \bar{d}^2)FY_m$$

$$+ R\{V_n'[\int W_eY_eY_nY_m + G\delta_{mn}] + V_n[U_e\int Y_e dY_nY_m + W_e\int Y_e DY_nY_m$$

$$- h' W_e \int Y_e dY_n Y_m - h' \int G dY_n Y_m + \int G D Y_n Y_m$$

$$+ U_e \int r^{-1} Y_e Y_n Y_m] + [W_n \int Y_n Y_m (D - h'd) F$$

$$+ \int G Y_m (D - h'd) F + U_n \int r^{-1} F Y_n Y_m + U_n \int Y_n Y_m dF \} \qquad (17)$$

$$W_n' \delta_{mn} (b-h)/2 = W_n [h' \int dY_n Y_m + \int D Y_n Y_m] - U_n (\int dY_n Y_m + \int r^{-1} Y_n Y_m)$$

$$- \int (D - h') G Y_m \qquad (18)$$

$$U_n''' \delta_{mn} (-bh)/2 = 3U_n'' [h' \int dY_n Y_m + \int D Y_n Y_m] + 3U_n' [-\int (\bar{d}^2) Y_n Y_m$$

$$- \int D^2 Y_n Y_m + 2h' \int d D Y_n Y_m] + U_n [-\int \bar{d}_1^{\,3} Y_n Y_m$$

$$- \int D^3 Y_n Y_m + 3h' \int d D^2 Y_n Y_m - 3 \int (\bar{d}^2) D Y_n Y_m$$

$$+ W_n'' \int d Y_n Y_m + W' [-2h' \int d^2 Y_n Y_m + 2 \int d D Y_n Y_m]$$

$$+ W_n [\int \bar{d}_2^{\,2} d Y_n Y_m + \int D^2 d Y_n Y_m - 2h' \int d^2 D Y_n Y_m]$$

$$- \int (D - 2h' d D + \bar{d}^2) d G Y_m$$

$$+ R\{-U_n'' [W_e \int Y_e Y_n Y_m + \int G Y_n Y_m]$$

$$+ U_n' [2h' W_e \int Y_e d Y_n Y_m + 2h' \int G d Y_n Y_m + U_e \int Y_e d Y_n Y_m$$

$$- U_e \int r^{-1} Y_e Y_n Y_m - 2 W_e \int Y_e D Y_n Y_m - 2 \int G D Y_n Y_m]$$

$$+ U_n [U_e \int Y_e d D Y_n Y_m - U_e \int r^{-1} Y_e D Y_n Y_m - h' U_e \int Y_e d^2 Y_n Y_m$$

$$+ h' U_e \int r^{-1} Y_e d Y_n Y_m + \int Y_n d^2 G Y_m - \int r^{-1} Y_n d G Y_m$$

$$+ 2h' W_e \int Y_e D d Y_n Y_m + 2h' \int G D d Y_n Y_m$$

$$- W_e \int Y_e D^2 Y_n Y_m - \int G D^2 Y_n Y_m$$

$$+ W_n' [W_e \int Y_e d Y_n Y_m + \int G d Y_n Y_m]$$

$$+ W_n[U_e\int r^{-1}Y_e dY_n Y_m - \int Y_e d^2 Y_n Y_m + W_e \int Y_e DdY_n Y_m$$

$$+ \int GDdY_n Y_m - W_e \int Y_e h' d^2 Y_n \bar{Y}_m - \int Gh' d^2 Y_n Y_m + W_n \int Y_n (D-h'd) dGY_m]$$

$$- 2V_n'[V_e \int r^{-1} Y_e Y_n Y_m + \int r^{-1} FY_n Y_m] - 2V_n [V_e \int r^{-1} Y_e DY_n Y_m + \int r^{-1} FDY_n Y_m$$

$$- h' V_e \int r^{-1} Y_e dY_n Y_m - h' \int r^{-1} FdY_n Y_m$$

$$+ \int r^{-1} Y_n (D-h'd) FY_m - 2\int r^{-1} F(d-h'd) FY_m\} \quad (19)$$

where

$$\bar{d}_1^{\,3} = -(h'+h'^3)d^3 + (3h'h'' - r^{-1}h')d^2 - h'''d .$$

Again the differential system can be casted into a system of 6N nonlinear ordinary differential equations of the first order of the same form as (13) where S_1, S_2 and S_3 are now respectively given by the right sides of (19), (17) and (18).

The extremely high spatial resolution required for the Runge-Kutta method to converge was also encountered in the bounded domain. The alternate methods of minimizing the residual over the entire flow domain and treating the flow establishment as an initial value problem described for the unbounded domain can be equally applied to the bounded case.

3. Stability Analysis

The solution to the differential system given in the previous subsection depends on the flow parameter R. The assumed axisymmetric flow may not exist due to instability at Reynolds numbers exceeding a critical value R_c. The R_c at the onset of instability may be, in principle, determined from extensive repetitive solutions of the differential system for different increasingly large R values until the axisymmetric solution becomes impossible. New bifurcated non-axisymmetric solutions must be found. However, such computation is costly and not physically very revealing. For this reason, we will investigate the vortex structure in the separated flow as the consequence of the instability of the basic flow given by (8) and (16).

According to the known observations (Muller et al., 1981, Kobayashi et al., 1983), the perturbed flow is not axisymmetric. Hence the governing equations of the non-axisymmetric flow perturbations are

$$\frac{Du}{Dt} + u\partial_y U + w\partial_z U - \frac{2Vv}{y+h} = -\partial_y p + \frac{1}{R}\left[\nabla^2 u - \frac{u}{(y+h)^2} - \frac{2}{(y+h)^2}\partial_\theta v\right],$$

$$\frac{Dv}{Dt} + u\partial_y V + w\partial_z V + \frac{Uv+Vu}{y+h} = -\frac{1}{y+h}\partial_\theta p + \frac{1}{R}\left[\nabla^2 v - \frac{v}{(y+h)^2} + \frac{2}{(y+h)^2}\partial_\theta u\right],$$

$$\frac{Dw}{Dt} + u\partial_y W + w\partial_z W = -\partial_z p + \frac{1}{R}\nabla^2 w \tag{20}$$

$$\partial r(ru) + \partial_z(rw) + \partial_\theta v = 0$$

where (u,v,w) are perturbations of (U,V,W). The continuous septrum of the above system is assumed to be damped, and the discrete spectrum which portrays the spiral vorticies is periodic in θ. Hence, we represent the solution by the complete set

$$u = u_n(\tau,z)Y_n\exp[i\beta\theta],$$

$$v = v_n(\tau,z)Y_n\exp[i\beta\theta], \tag{21}$$

$$w = w_n(\tau,z)Y_n\exp[i\beta\theta],$$

where β is the real wave number in the tangential direction. Note that (21) satisfies the no-slip condition at the solid surface and/or the vanishing of disturbances at $y=\infty$. Substitution of (21) into (20) and elimination of the pressure terms by cross-differentiation then yield a system of 3N partial differential equations with (U_n, V_n, W_n) as dependent variables and τ and z as independent variables. Solution of this system with an appropriate initial condition and an upstream condition but without specifying the downstream condition yields the information sought about the instability.

An alternate approach is to solve (21) as an eigenvalue problem. To do this, we expand the amplitude functions in a complete orthonormal set $P_k(z)$,

$$u_n = u_{nk} P_k(z) \exp(\omega\tau) \quad ,$$

$$v_n = v_{nk} P_k(z) \exp(\omega\tau) \quad ,$$

$$w_n = w_{nk} P_k(z) \exp(\omega\tau) \quad , \tag{22}$$

where ω is the complex eigen-frequency. Substituting (21) and (22) into (20), multiplying the resulting equations by P_ℓ and integrating from $z=-\infty$ to $+\infty$, we obtain a system of homogeneous linear equations in $[u_{nk}, v_{nk}, w_{nk}]$. The existence condition of the solution then determined the eigen-frequency which determines the stability criteria. The details of this approach are demonstrated in the accompanying report "Stability of the Kovasznay Flow."

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Migration of the Separation Point on a Deforming Cylinder***

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Abstract

An iterative scheme of solving the Navier-Stokes equations for unsteady two-dimensional flow about a deforming and translating cylinder is given. In the k-th iteration, the convected vorticity appears as the source term in an equation of transient diffusion of vorticity. A novel integral transform is used to reduce this transient vorticity equation to a k-dimensional heat equation. The bounded solution of this equation is obtained with a general method of superposition for problems involving a moving boundary. An equation describing the migration of the separation point on a deforming cylinder in unsteady cross-flows is derived from the analytically obtained velocity field. The radial contraction of the cylinder surface is shown to postpone the separation time and to reduce the separation angle. The results are applied to infer that the separation on a body of revolution translating steadily with an angle of attack in a fluid does not originate at the nose-tip of the body.

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***Being revised for publication in AIAA Journal

I. INTRODUCTION

THE unsteady flow separation about a radially deforming cylinder starting an arbitrary translation from rest is analyzed. This problem is relevant to several fundamental issues of considerable importance. These issues include the control of flow separation by moving an aerodynamic surface[1] the possible drag reduction by surface deformation[2] and the issue on the appropriate definition of an unsteady flow separation point[3]. However, this study was originally motivated by the need to understand the origin of flow separation in a steady three-dimensional flow about a body of revolution translating with an angle of attack in an otherwise quiescent Newtonian fluid. As one moves downstream along the axis of revolution and views the flow pattern on successive planes perpendicular to the axis, one sees the evolution of a sectionally two-dimensional vortical flow structure about a circle whose radius varies with time. Thus there is a certain analogy between the steady three-dimensional flow about a slender body of revolution and the unsteady two dimensional flow about a circular cylinder whose radius changes with time in a cross-flow. This topological analogy was demonstrated by Tobak and Peak[4], and Chapman and Tobak[5].

Creeping flow around a deforming sphere was studied by Lin and Gautesen[2] on the basis of the unsteady Stokes equation. The linear unsteady Stokes equation is known to be also a valid first order approximate equation even at finite Reynolds numbers for small amplitude high frequency fluctuating flows[6-8] as well as for initial transient flows[9]. The nonlinear convective acceleratic terms were later treated as source terms in perturbation solutions to the Navier-Stokes equation by Tseng and Lin[10,11] in their study of transient heat transfer from a heated wire and for constructing a theory of a heat sensing velocimeter. While these perturbation solutions are useful for describing the initial transience of flows in which the convective acceleration is weaker than the local acceleration, they are obviously inadequate for the description of separated flows in which the convective and local accelerations may become equally important at a later stage of the flow development.

An iterative scheme of solving the Navier-Stokes equations for unsteady two-dimensional flow about a deforming and translating cylinder is given in the next section. The method is novel to the best knowledge of the writers[12]. In the first approximation, the effect of vorticity convection is neglected. This effect is taken into account by successive iterations. Thus, in the k-th iteration, the convected vorticity appears as the source term in an equation of transient diffusion of vorticity. A novel integral transform is used to reduce this transient vorticity equation to a k-dimensional heat equation. The bounded solution of this equation with the appropriate boundary conditions is given in section 3. A general method of superposition for problems involving a moving boundary is given in the Appendix. An equation which describes the migration of the separation point on a deforming cylinder is obtained in section 4. The numerical results are then presented and discussed. The present iterative solution is expected to be valid during the initial transience before a large amount of vorticity is convected to infinity. Then the well known difficulty associated with the Whithead paradox[13]

surfaces. The numerical results show that the flow separation establishes itself long before the vorticity has time to populate itself at infinity.

II. Formulation

Consider the flow of an incompressible Newtonian fluid about a circular cylinder. The fluid is initially quiescent. The translational velocity of the cylinder is $-\underline{i}U(t)$, where t is time and $\underline{i}$ is the unit vector in the negative x-direction as shown in Fig. 1. The instantaneous radius of the deforming cylinder is given by $a_1(t)$.

The governing equations with respect to a reference frame attached to the cylinder are

$$\begin{aligned} &\underline{\nabla}\cdot\underline{V} = 0, \\ &\partial_t\underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\frac{1}{\rho}\underline{\nabla}P + \nu\nabla^2\underline{V} + \underline{i}\,\dot{U}(t) - \underline{j}g\ , \end{aligned} \tag{1}$$

where $\underline{V}$ is the velocity, P is the pressure, ρ is density, ν is the kinematic viscosity, and $\underline{j}g$ is the gravitational force per unit mass of fluid. Let the characteristic time and distance be respectively ω^{-1} and δ. The corresponding characteristic velocity is $\omega\delta$. Nondimensionalizing the distance, time, velocity and pressure respectively by a_o, a_o^2/ν, $\omega\delta$ and $\rho(\omega\delta)^2$, Eq. (1) can be written as

$$\partial_\tau\underline{v} + \epsilon R(\underline{v}\cdot\underline{\nabla})\underline{v} = -\epsilon R\underline{\nabla}p + \nabla^2\underline{v}\ , \tag{2a}$$

$$\underline{\nabla}\cdot\underline{v} = 0\ , \tag{2b}$$

where

$$\underline{v} = V/\omega\delta, \qquad \tau = t/[a_o^2/\nu], \qquad \epsilon = \delta/a_o,$$

$$p = (P + gy - \dot{U}(t)x)/\rho(\omega\delta)^2, \qquad R = \omega a_o^2/\nu\ ,$$

and a_o is the average cylinder radius i.e., $a_o = (a_{min} + a_{max})/2$. Equation (2b) is the necessary and sufficient condition for the existence of the stream function ψ such that the radial and tangential velocity components of two-dimensional velocity can be written respectively as

$$v_r = -\frac{1}{r}\partial_\theta\psi , \qquad v_\theta = \partial_r\psi .$$

In terms of the stream functions, Eqs. (2a) and (2b) can be written as

$$(\partial_\tau - \nabla^2)\nabla^2\psi = -\epsilon \frac{R}{r}\frac{\partial(\psi,\nabla^2\psi)}{\partial(r,\theta)} = -\epsilon \frac{R}{r}J , \tag{3}$$

where

$$\nabla^2 = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} ,$$

$$J = \partial_r\psi\partial_\theta\nabla^2\psi - \partial_\theta\psi\partial_r\nabla^2\psi .$$

The initial condition is

$$\psi(r,\theta,\tau) = 0 , \qquad \tau \leq 0 .$$

The boundary conditions are

$$-(\frac{1}{r}\partial_\theta\psi)_{r=a(\tau)} = \frac{da}{d\tau} = \dot{a}, \quad (\partial_r\psi)_{r=a(\tau)} = 0 , \tag{4}$$

and

$$-(\frac{1}{r}\partial_\theta\psi)_{r\to\infty} = u(\tau)\cos\theta , \quad (\partial_r\psi)_{r\to\infty} = -u(\tau)\sin\theta , \tag{5}$$

where $u(\tau) = U(t)/[U(t)]_{max}$.

The iterative solution of Eqs. (3), (4) and (5) will be obtained for the case of symmetric flow. The generalization of the method to the case of asymmetric flows will be mentioned later. The m-th iterated solution will be written as

$$\psi^{(m)} = \sum_{k=1}^{m} (\varepsilon R)^{k-1} \dot{\psi}_k^{(m)} \sin k\theta - a\dot{a}\theta \qquad (6)$$

Substituting Eq. (6) into the Jacobian in Eq. (3), we have

$$J^{(m-1)} = \partial(\psi^{(m-1)}, \nabla^2\psi^{(m-1)})/\partial(r,\theta)$$

$$= \sum_{q=1}^{m-1} \sum_{\ell=1}^{m-1} (\varepsilon R)^{(q+\ell-2)} \left\{ A_{q\ell}^{(m-1)} \sin q\theta \cos \ell\theta - B_{q\ell}^{(m-1)} \cos q\theta \sin \ell\theta \right\} + \sum_{\ell=1}^{m-1} (\varepsilon R)^{\ell-1} C_\ell^{(m-1)} \sin \ell\theta, \qquad (7)$$

where

$$A_{q\ell}^{(m-1)} = (\partial_r \psi_q^{(m-1)})(\ell D_\ell^2 \psi_\ell^{(m-1)}) , \quad C_\ell^{(m-1)} = a\dot{a}\partial_r D_\ell^2 \psi_\ell^{(m-1)} ,$$

$$B_{q\ell}^{(m-1)} = (q\psi_q^{(m-1)})(\partial_r D_\ell^2 \psi_\ell^{(m-1)}), \quad m \geq 2$$

$$D_\ell^2 = \partial_{rr} + r^{-1} \partial_r - \ell^2 r^{-2}, \quad J^{(0)} = 0 .$$

By use of the relation $\sin q\theta \cos \ell\theta = [\sin (q+\ell)\theta + \sin (q-\ell)\theta]/2$, we can rewrite Eq. (7) as

$$J^{(m-1)} = \sum_{\ell=1}^{m-1} (\varepsilon R)^{\ell-1} C_\ell^{(m-1)} \sin \ell\theta$$

$$+ \tfrac{1}{2} \sum_{q=1}^{m-1} \sum_{\ell=1}^{m-1} (\varepsilon R)^{(q+\ell-2)} \left\{ (A_{q\ell}^{(m-1)} - B_{q\ell}^{(m-1)}) \sin(q+\ell)\theta + (A_{q\ell}^{(m-1)} + B_{q\ell}^{(m-1)}) \sin(q-\ell)\theta \right\} \qquad (8)$$

Equation (8) can be rearranged into a form of truncated Fourier series

$$J^{(m-1)} = \sum_{k=1}^{m-1} (\varepsilon R)^{k-1} C_k^{(m-1)} \sin k\theta$$

$$+ \tfrac{1}{2} \sum_{k=2}^{2m-2} \sum_{\ell=1}^{k-1} (\varepsilon R)^{k-2} (A_{(k-\ell)\ell}^{(m-1)} - B_{(k-\ell)\ell}^{(m-1)}) \sin k\theta \bullet h_{k(\ell+m)} \cdot h_{\ell(m)}$$

$$+ \tfrac{1}{2} \sum_{k=1}^{m-2} \sum_{\ell=1}^{m-1-k} (\varepsilon R)^{2\ell+k-2} (A_{(k+\ell)\ell}^{(m-1)} + B_{(k+\ell)\ell}^{(m-1)}) \sin k\theta$$

$$+ \tfrac{1}{2} \sum_{k=1}^{m-2} \sum_{q=1}^{m-1-k} (\varepsilon R)^{2q+k-2} (A_{q(q+k)}^{(m-1)} + B_{q(q+k)}^{(m-1)}) \sin (-k\theta) \ . \qquad (9)$$

Equation (9) can be further reduced to

$$J^{(m-1)} = \sum_{k=1}^{m} (\varepsilon R)^{k-2} \Big\{ (\varepsilon R) C_k^{(m-1)} h_{k(m)} + \tfrac{1}{2} \sum_{\ell=1}^{k-1} \Big[(1-\delta_{k1}) (A_{(k-\ell)\ell}^{(m-1)} - B_{(k-\ell)\ell}^{(m-1)})$$

$$+ \sum_{\ell=1}^{m-1-k} (\varepsilon R)^{2\ell} (A_{(\ell+k)\ell}^{(m-1)} + B_{(\ell+k)\ell}^{(m-1)}) h_{k(m-1)}$$

$$- \sum_{\ell=1}^{m-1-k} (\varepsilon R)^{2\ell} (A_{\ell(\ell+k)}^{(m-1)} + B_{\ell(\ell+k)}^{(m-1)}) h_{k(m-1)} \Big] \Big\} \sin k\theta$$

$$+ \tfrac{1}{2} \sum_{k=m+1}^{2m-2} \sum_{\ell=1+k-m}^{m-1} (\varepsilon R)^{k-2} (A_{(k-\ell)\ell}^{(m-1)} - B_{(k-\ell)\ell}^{(m-1)}) h_{(k-\ell)(m)} \sin k\theta \ , \qquad (10)$$

where

$$h_{k(m)} = \begin{cases} 1 & \text{if } k<m \\ 0 & \text{if } k \geq m. \end{cases}$$

We assume that the coefficients of $\sin k\theta$ with $k>m$ are negligibly small in the (m-1)-th iteration, and write $J^{(m-1)}$ as

$$J^{(m-1)} = \sum_{k=1}^{m} (\varepsilon R)^{k-2} J_k^{(m-1)} \sin k\theta \ , \qquad (11)$$

where $J_k^{(m-1)}$ is given by the sum of terms in the curly bracket in Eq. (10). Thus, the last double summation term in Eq. (10) will be neglected.

Substituting Eq. (6) and Eq. (11) respectively into the left and right sides of Eq. (3), and equating the coefficients of the same harmonics on on both sides we have

$$(\partial_\tau - D_k^2)D_k^2\psi_k^{(m)} = -r^{-1}J_k^{(m-1)} \quad . \qquad (k\geq 1) \tag{12}$$

By use of the following transform

$$\psi_k^{(m)} = u(\tau)(a^2r^{-1}-r)\delta_{k1} + r^{-k}\int_a^r \chi_k^{(m)}(s,\tau)s^{2k-1}ds \ , \tag{13}$$

Eq. (12) can be reduced to the 2k-dimensional heat equation

$$(\partial_\tau - \nabla_k^2)\chi_k^{(m)} = G_k^{(m-1)} \ , \tag{14}$$

where

$$G_k^{(m-1)} = -\int_{a(\tau)}^r s^{-k} J_k^{(m-1)}(s,\tau)ds \ ,$$

$$\nabla_k^2 = r^{1-2k}\ \partial_r(r^{2k-1}\ \partial_r) \ .$$

Note that Eq. (6) with Eq. (13) satisfies the first condition of Eq. (4). In order to satisfy the second condition of Eq. (4), we must have

$$\chi_k^{(m)}[a(\tau),\tau] = 2u(\tau)a^{1-k}(\tau)\delta_{k1} \quad . \tag{15}$$

Substituting the bounded solution of Eq. (14) with Eq. (15) into the boundary condition Eq. (5), we have

$$-\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)_{r\to\infty} = u(\tau)\cos\theta + f(\tau)\cos k\theta \quad ,$$

$$\left(\frac{\partial\psi}{\partial r}\right)_{r\to\infty} = -u(\tau)\sin\theta + g(\tau)\sin k\theta \ ,$$

where $f(\tau)$ and $g(\tau)$ are the "penalty functions" given by

$$f(\tau) = \lim_{r\to\infty}\left(r^{-k-1}\int_a^r \chi_k^{(m)}(s,\tau)s^{2k-1}ds\right) \ ,$$

$$g(\tau) = \lim_{r\to\infty}\left(-kr^{-k-1}\int_a^r \chi_k^{(m)}(s,\tau)s^{2k-1}ds + \chi_k^{(m)}(r,\tau)r^{k-1}\right) \ .$$

Thus the bounded solution of Eq. (14) with Eq. (15) will not satisfy the bounda condition (5) unless $f(\tau)\to 0$ and $g(\tau)\to 0$. For the special case of harmonic oscillation with m=k=1, it was shown by Lin[6] that $f(\tau)\to 0$ and $g(\tau)\to 0$. In general, it is expected, on a physical ground, that there exists a small time τ_c below which $f(\tau)$ and $g(\tau)$ remain much smaller than $u(\tau)$. In any event this expectation can be verified rigorously aposteriori.

3. Solution

III. Solution

The solution of Eq. (14) with its boundary condition Eq. (15) can be achieved as follows. First, the solution of Eq. (14) which satisfies the quasi steady boundary condition corresponding to Eq. (15) will be obtained. This solution can be obtained from

$$(\partial_\tau - \nabla_k^2)\, X_k^{(m)}(r,\ell,\tau) = G_k^{(m-1)}(r,\ell,\tau) \ , \tag{16}$$

$$X_k^{(m)}[a(\ell),\ell,\tau] = 2u(\ell)a^{1-k}(\ell)\delta_{k1}H(\tau) \ , \tag{17}$$

where $H(\tau)$ is the Heaviside unit step function, i.e.

$$H(\tau) = \begin{cases} 1, & \tau > 0 \ , \\ 0, & \tau \le 0 \ . \end{cases}$$

It is shown in the Appendix that the solution of Eq. (14) with the unsteady boundary condition is then given by

$$X_k^{(m)}(r,\tau) = \int_0^\tau \partial_\tau X_k^{(m)}[r,\ell,\tau-\ell]\,d\ell \ . \tag{18}$$

Equation (14) can be solved by use of the Laplace transform. The Laplace transform of Eq. (14) is

$$[v^2 - d_r{}^2 - (2k-1)r^{-1}d_r]\bar{X}_k^{(m)} = \bar{G}_k^{(m-1)} \ , \tag{19}$$

where v^2 is the Laplace transform variable. By use of the new variables

$$\bar{X}_k^{(m)} = S_k^{(m)}/r^{k-1} \quad \text{and} \quad \zeta = vr \ , \tag{20}$$

Eq. (19) can be reduced to the non-homogeneous modified Bessel equation of the (k-1) - th order.

$$\{d_\zeta^2 + \zeta^{-1}d_\zeta - [1+(k-1)^2\zeta^{-2}]\}S_k^{(m)} = -\bar{G}_k^{(m-1)} r^{k-1} v^{-2} \tag{21}$$

The solution of this equation is given by

$$\begin{aligned} S_k^{(m)} &= [b_k^{(m)}(v) + B_k^{(m)}(vr)]I_{k-1}(vr) \\ &+ [c_k^{(m)}(v) + C_k^{(m)}(vr)]K_{k-1}(vr) \end{aligned} \tag{22}$$

where

$$B_k^{(m)} = -\int_\infty^r s^k\ K_{k-1}(vs)\bar{G}_k^{(m-1)}(s,v)ds,$$

$$C_k^{(m)} = \int_o^r s^k\ I_{k-1}(vs)\bar{G}_k^{(m-1)}(s,v)ds.$$

For the bounded solution, we chose $b_k^{(m)}(v) = 0$. Taking the inverse Laplace transform of Eq. (20), and applying the convolution theorem, we have from Eq. (20) and Eq. (22)

$$\begin{aligned} \chi_k^{(m)} &= r^{1-k}\{\int_o^\tau q_k^{(m)}(\ell,\lambda)\ H_k(r,\tau-\lambda)d\lambda \\ &+ \int_o^\tau \int_o^\infty V_k(r,s,\tau-\lambda)\ G_k^{(m-1)}(s,\lambda)s^k ds d\lambda\} \end{aligned} \tag{23}$$

where $q_k^{(m)}$ is the inverse Laplace transform of $v^2K_{k-1}[a(\ell)v]c_k^{(m)}(v)$, i.e.

$$q_k^{(m)} = \mathcal{L}^{-1}\ v^2\ K_{k-1}[a(\ell)v]c_k^{(m)}\ (v), \text{ and}$$

$$H_k(r,\tau) = \mathcal{L}^{-1}\ \frac{K_{k-1}(vr)}{K_{k-1}[a(\ell)v]\ v^2} = \bar{H}_k(r,\tau)H(\tau)$$

$$\bar{H}_k(r,\tau) = 1 + \frac{2}{\pi}\int_o^\infty \exp[-u^2\tau]\ \frac{J_{k-1}(ur)Y_{k-1}[a(\ell)u]-Y_{k-1}(ur)J_{k-1}[a(\ell)u]}{J_{k-1}^2[a(\ell)u] + Y_{k-1}^2[a(\ell)u]}\ \frac{du}{u}$$

and V_k is the inverse Laplace transform of $I_{k-1}(vr)K_{k-1}(vs)$ and $I_{k-1}(vs)$ $K_{k-1}(vr)$, i.e.,

$$V_k = \frac{1}{2\tau} I_{k-1} (rs/2\tau)\exp[-(r^2+s^2)/4\tau]H(\tau) .$$

In order to satisfy the boundary condition (17), $q_k^{(m)}$ in Eq. (23) must satisfy the following integral equation

$$\int_o^\tau q_k^{(m)}[a(\ell),\lambda]\, H_k[a(\ell),\tau-\lambda]d\lambda = 2u(\ell)\delta_{k1}$$

$$-\int_o^\tau\int_{a(\ell)}^\infty V_k[a(\ell),s,\tau-\lambda]\, G_k^{(m-1)}(s,\ell,\lambda)s^k dsd\lambda \quad . \qquad (24)$$

Note the lower integration limit in the last integral in Eq. (24) is changed from 0 to $a(\ell)$, since $G_k^{(m-1)}$ vanishes for $s\leq a$.

Recall that

$$G_k^{(m-1)}(r,\ell,\tau) = -\int_{a(\ell)}^r s^{-k} J_k^{(m-1)}(s,\tau)ds.$$

The solution of Eq. (24) for $q_k^{(m)}$ can be easily obtained by differentiating both sides of Eq. (24) with τ

$$\int_{-\infty}^{\infty} \partial_\tau q_k^{(m)}[a(\ell),\lambda]\bar{H}_k[a(\ell),\tau-\lambda][H(\lambda)-H(\lambda-\tau)]d\lambda = 2u(\ell)\delta(\tau)\delta_{k1}$$

$$- \partial_\tau \int_{-\infty}^{\infty} [H(\lambda)-H(\lambda-\tau)]\int_{a(\ell)}^{\infty} V_k[a(\ell),s,\tau-\lambda]G_k^{(m-1)}(s,\ell,\lambda)s^k dsd\lambda \; .$$

where $\delta(\tau)$ is the Dirac delta function. Performing the differentiation and using the relations $\bar{H}_k[a(\ell),0] = 1$, $V[a(\ell),s,0] = 0$, we reduce this equation to

$$q_k^{(m)}[a(\ell),\tau] = 2u(\ell)\delta_{k1}\delta(\tau)$$

$$- \int_0^\tau \int_{a(\ell)}^\infty \partial_\tau V_k[a(\ell),s,\tau-\bar{\lambda}]G_k^{(m)}(s,\ell,\bar{\lambda})s^k dsd\bar{\lambda} \quad . \qquad (25)$$

It follows from Eq. (23) that

$$r^{k-1}\chi_k^{(m)}(r,\ell,\tau) = 2u(\ell)\delta_{k1}H_k(r,\tau)$$

$$-\int_0^\tau \int_0^\lambda \int_{a(\ell)}^\infty \partial_\lambda V_k[a(\ell),s,\lambda-\bar{\lambda}]G_k^{(m-1)}(s,\ell,\bar{\lambda})H_k(r,\tau-\lambda)s^k ds d\bar{\lambda} d\lambda$$

$$+\int_0^\tau \int_{a(\ell)}^\infty V_k(r,s,\tau-\lambda)G_k^{(m-1)}(s,\ell,\lambda)s^k ds d\lambda \quad . \tag{26}$$

Substituting this into Eq. (18), we have after some simple manipulation

$$r^{k-1}\chi_k^{(m)}(r,\tau) = \int_0^\tau 2u(\ell)\partial_\tau H_k(r,\ell,\tau-\ell)d\ell\,\delta_{k1}$$

$$-\int_0^\tau \int_0^{\tau-\ell} \int_0^\lambda \int_{a(\ell)}^\infty \partial_\lambda V_k[a(\ell),s,\lambda-\bar{\lambda}]G_k^{(m-1)}(s,\ell,\bar{\lambda})\partial_\tau H_k(r,\ell,\tau-\ell-\lambda)ds d\bar{\lambda} d\lambda d$$

$$+\int_0^\tau \int_0^{\tau-\ell} \int_{a(\ell)}^\infty \partial_\tau V_k(r,s,\tau-\ell-\lambda)G_k^{(m-1)}(s,\ell,\lambda)s^k ds d\lambda d\ell \quad . \tag{27}$$

With $\chi_k^{(m)}(r,\tau)$ given by Eq. (27), the m-th iterative solution given by Eqs. (6) and (13) is now complete. The convergence of the iterative solution (6) may be proved by showing that there exists a time τ_c below which $(\varepsilon k)^{k-1}$ $\psi_k^{(m)} \leq M$, M being a bounded constant, for given $U(\tau)$ and $a(\tau)$.

The extension of the present method to non-symmetric flow is immediate. In place of Eq. (6), the solution now can be written as

$$\psi^{(m)} = -a\dot{a}\theta + \sum_{k=1}^{m} (\varepsilon R)^{k-1}\psi_k^{(m)} \sin k\theta$$

$$+ \sum_{k=1}^{m} (\varepsilon R)^{k-1} \phi_k^{(m)} \cos k\theta \quad .$$

The Jacobian in Eqs. (7) to (11) will have to be modified to include cos kθ-terms. A set of integral transforms similar to that given by Eq. (13) will then reduce the partial differential equations for $\psi_k^{(m)}$ and $\phi_k^{(m)}$ into a set of 2k-dimensional heat equations. The solution of $\phi_k^{(m)}$ and $\psi_k^{(m)}$ by the Laplace transform method will follow the line described for the solution of $\psi_k^{(m)}$ given above.

IV. Separation Point

An appropriate definition of separation points in unsteady two-dimensional flows is of great current interest. For the discussions on the controversy and the relevant references, we refer the readers to the work of Ho[3]. Here, we are concerned with the separation points on a radially deforming cylindrical surface. With respect to this surface the velocity components (v_r', v_θ') are given by

$$v_r' = v_r - \dot{a} , \quad v_\theta' = v_\theta .$$

If we define a separation point at such a surface to be the point where the flow relative to the moving surface just starts to reverse its direction, then we must have

$$\partial_r v_\theta' = 0 \quad \text{at} \quad r = a(\tau) .$$

Since $v_\theta' = v_\theta$, this condition can be written as

$$(\partial_{rr}\psi)_{r=a} = 0.$$

It follows from Eq. (6) that at the separation points we must have

$$\sum_{k=1}^{m} \varepsilon^{k-1}(\partial_{rr} R^{k-1}\psi_k^{(m)})_{r=a} \sin k\theta(\tau) = 0 , \tag{29}$$

where, with the aid of Eq. (13),

$$(\partial_{rr}\psi_k^{(m)})_{r=a} = \frac{2u(\tau)}{a(\tau)} \delta_{k1} - a^{k-2}\chi_k^{(m)}(a,\tau) + a^{k-1}(\partial_r \chi_k^{(m)})_{r=a} . \tag{30}$$

It is easily verified that at $r=a(\tau)$, the last two integrals in Eq. (27) vanish, and Eq. (27) gives

$$a^{k-1}\chi_k^{(m)}[a(\tau),\tau] = 2u(\tau)\delta_{ik}$$

as is required by the boundary condition Eq. (15). Thus the first two terms in Eq. (30) cancel each other. The last term in Eq. (30) can be obtained from

Eq. (27) by a simple differentiation. It can be shown that the shear stress also vanishes at the separation points for this particular problem.

For the ease of comparisons with some known results, we define a new Reynolds number based on the uniform velocity $U = \omega\delta$ and the radius a_o, i.e.,

$$R_e = \frac{(\omega\delta)a_o}{\nu} = (\frac{\delta}{a_o})(\frac{\omega a_o^2}{\nu}) = \varepsilon R.$$

Combination of Eqs. (29) and (30) then gives

$$\sum_{k=1}^{m} (R_e a)^{k-1} (\partial_r \chi_k^{(m)})_{r=a} \sin k\theta = 0 \quad . \tag{31}$$

Several comments concerning the general feature of two-dimensional flow separation can now be made. In the limiting case of creeping flows $R_e \to 0$, and Eq. (31) is reduced to

$$(\partial_r \chi_1^{(m)})_{r=a} \sin\theta = 0 \ .$$

Thus the only points on the cylinder where shear stress vanish are at the forward and rear "stagnation" points respectively at $\theta = \pi$ and $\theta = 0$. For a moderately large R_e such that the second term in Eq. (31) is comparable to the first one, only the second iteration may be adequate to approximate the separated flow. Eq. (31) then gives

$$\sin\theta\, (\partial_r \chi_1^{(2)} + 2R_e \partial_r \chi_2^{(2)} \cos\theta)_{r=a} = 0 \tag{32}$$

Thus, in addition to the "stagnation" points there will be two symmetrically positioned primary separation points given by

$$\cos\theta = -\left(\frac{\partial_r \chi_1^{(2)}}{2R_e \chi_2^{(2)}}\right)_{r=a} \quad . \tag{33}$$

Eq. (33) has a solution only after a finite separation time is reached when the quotient in Eq. (33) becomes less than one in absolute value. At higher

Reynolds numbers, higher harmonic terms in θ must be retained. This may allow secondary and further separation points to be obtained as the roots of the transcendental equation of order higher than two. It is obvious that separation may occur in mid-stream. This can be seen easily from Eq. (29) when r is evaluated at a point inside the flow rather than at $r=a$.

It should be pointed out that if vortex shedding takes place, the flow is no longer symmetric, and separated flow must be based on Eq. (28). Asymmetric flow separation is not included in this study.

V. Results

To obtain numerical results, we used short-time expansions of Bessel functions appearing in the integrands of the analytical results given in the previous sections. For the evaluation of the multiple integrals, we used the M-point Gauss quadrature formula[14]. For each integration interval, the value of M was increased until the values of the given integral corresponding to the two successive values of M differ from each other by less than 100th of a percent. All numerical computation was carried out with double precision on an IBM 4341.

To compare our iterative solution with some known experimental and theoretical results, we obtained numerical results for the case of a rigid cylinder first. The time dependent angle of separation was determined from Eq. (33) for the case of an impulsively started uniform motion of a rigid cylinder. The results are given in Fig. 2 together with other known results. The finite separation times for various values of Re are all slightly larger than those given by Collins & Dennis[15] and Thoman & Szewezyk[16]. The difference may be due to the different initial conditions used. The initial flow obtained by Collins & Dennis was based on a boundary-layer approximation. Thoman & Szewczyk and others who used pure numerical solutions assumed either that the flow was initially irrotational everywhere or at some finite distance from the cylinder. The initial condition used in this study is that the fluid is completely quiescent everywhere. The separation angle for a given R_e increases rapidly after the onset of separation and approaches an almost steady value within a small fraction of viscous diffusion time. Note that the dimensionless time t_1 of the quoted theoretical studies are related to ours by $\tau=t_1/R_e$. The authors are not aware of any measurements of separation angles for the initial time smaller than $\tau=0.1$. We extrapolate the measurements of Coutanceau and Bouard[17] for Re=20 in a tank whose diameter is 8.3 times that of the test cylinder. Our predicted angle is considerably larger than their measured value. This is quite to be expected, since the wall of the test tank was shown by them to have the effect of reducing the separation angles. There is also considerable uncertainty involved in the extrapolation. It should be pointed out that the non-dimensional time t^* of Coutanceau & Bouard is related to ours by $\tau=2t^*/R_e$.

Figure 3 demonstrates the effect of the radial deformation of the cylinder, which impulsively starts a constant velocity translation, on the separation angle. It is seen that, for a given R_e, the radial expansion has an effect of reducing the onset time of separation and increasing the separation angle at the same instant. The larger the rate of expansion the larger is the effect. The earlier separation and larger separation angle are all associated with larger deceleration of fluid induced by radial expansion. Figure 4 shows that radial contraction has an opposite effect. Figure 5 shows the same effect demonstrated in Figs. 3 and 4 except that the cylinder translation is now not an impulsively started uniform motion but is given by a constant acceleration followed by a constant velocity after some finite time τ_c. This latter cylinder motion is obviously more easily attainable in experiments.

VI. Discussion

An equation which describes the evolution of the separation angle on a deforming cylinder in an unsteady cross-flow was derived from an iterative solution of the Navier-Stokes equation for finite Reynolds numbers. The validity of the iterative solution hinges on the condition that the penalty functions remain negligible. This can be verified aposteriori. These penalty functions were found to be negligible in a larger period of time for the case of constant acceleration than for the case of impulsively started motion. Thus the time periods during which the iterative solution is valid can be ascertained aposteriori at each iteration. The advantage of the present method over other known methods is that only one computer program needs to be written for any $u(\tau)$ and $a(\tau)$. The solutions of the evolutin equation of separation point on a circular cylinder reveal that a radial contraction retards separation due to the enhanced acceleration along the wall. The converse is true for the case of radial expansion. In the limit of $R_e \to 0$, it was shown that no separation at the wall other than that at the rear "stagnation" point can take place. All these results suggest that the flow separation on the body of revolution translating at an angle of attack in a fluid does nto originate at the nose-tip of the body, but originates somewhere downstream from the tip of the body. The results reported in this work were based on the second interative solution. Hence only the primary flow separation points were found . For the follow-up work, we intend to investigate the onset and evolution of the symmetric secondary flow separation[21] at the deforming wall as well as in the mid-stream and the growth of the closed-wake, by use of the third iterative solution. The iterative solution of the form given by Eq. (28) for asymmetric flows will also be developed to study the breakup of a closed-wake and the ensuing vortex shedding.

Acknowledgments

This work was supported in part by a NASA-Ames grant NCC2-280.

Appendix

A General Method of Superposition for Problems Involving a Moving Boundary

We wish to solve for the unknown function F from the following partial differential equation

$$(\partial_\tau - L)F(\underline{r},\tau) = G(\underline{r},\tau) \quad , \tag{A-1}$$

with the boundary condition

$$F[\underline{a}(\tau),\tau] = \begin{cases} 2u(\tau) \quad , & \tau > 0 \\ 0 \quad , & \tau \leq 0 \end{cases} \tag{A-2}$$

and the initial condition

$$F(\underline{r},\tau) = 0 \; , \qquad \tau \leq 0 \quad , \tag{A-3}$$

where τ is time, $\underline{r}$ is a position vector, $G(\underline{r},\tau)$ is a given function, L is any linear differential operator in space variables, and $\underline{r}=\underline{a}(\tau)$ specifies the instantaneous position of the moving boundary at any time τ.

It is presently shown that the solution of this system is given by

$$F(\underline{r},\tau) = \int_0^\tau \partial_\tau F(\underline{r},\ell,\tau-\ell)d\ell \quad , \tag{A-4}$$

where $F(\underline{r},\ell,\tau)$ is the solution of (A-1) subject to a fixed boundary condition

$$F[\underline{a}(\ell),\ell,\tau] = 2u(\ell)H(\tau) \; , \quad H(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau \leq 0 \end{cases} \; , \tag{A-5}$$

and the initial condition

$$F[\underline{r},\ell,\tau] = 0 \quad , \qquad \tau \leq 0 \tag{A-6}$$

where ℓ is a constant parameter. Since the differential system (A-1), (A-5) and (A-6) is invariant with respect to time translation,

$$F(\underline{r},\ell,\tau-\ell) \quad \text{and} \quad F(\underline{r},\ell,\tau-\ell-d\ell)$$

are also solution of (A-1). Each satisfies (A-5) and (A-6) with τ in both of them replaced respectively by $\tau-\ell$ and $\tau-\ell-d\ell$. It follows that dF given by

$$\begin{aligned} dF &= F(\underline{r},\ell,\tau-\ell) - F(\underline{r},\ell,\tau-\ell-d\ell) \\ &= \partial_\tau F(\underline{r},\ell,\tau-\ell)d\ell \end{aligned} \tag{A-7}$$

satisfies (A-1), since it is merely a linear superposition of two solutions of (A-1). Moreover (A-7) satisfies the boundary condition

$$dF[\underline{a}(\ell),\ell,\tau-\ell] = 2\dot{u}(\ell)[H(\tau-\ell)-H(\tau-\ell-d\ell)] \tag{A-8}$$

and the initial condition

$$dF = 0 \quad , \qquad \tau \leq \ell \quad . \tag{A-9}$$

The integral of dF from $\ell=0$ to $\ell=\tau$, i.e.

$$F(\underline{r},\tau) = \int_0^\tau \partial_\tau F(\underline{r},\ell,\tau-\ell)d\ell \tag{A-10}$$

being merely a linear superposition of solutions of (A-1), each satisfying (A-8) instantaneously from $\ell=0$ to $\ell=\tau$ for any $\tau>0$, is itself a solution of (A-1). Moreover it satisfies the boundary condition (A-2), since the integration of (A-8) gives

$$\int_0^\tau dF[\underline{a}(\ell),\ell,\tau-\ell]d\ell = F[\underline{a}(\ell),\tau] = \int_0^\tau 2u(\ell)[H(\tau-\ell)-H(\tau-\ell-d\ell)]d\ell = 2u(\tau)$$

It also follows from (A-9), (A-10) and the fact that $\ell \geq 0$, the initial condition (A-3) is satisfied by (A-10).

Note that (A-2) is the Dirichlet boundary condition. It is obvious that the same method applies to the Neumann or mixed boundary-value problems when the boundary conditions (A-2) and (A-5) are replaced by byoundary conditions specifying the spatial derivative of F or the linear combination of F and its spatial derivative.

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Figure Captions

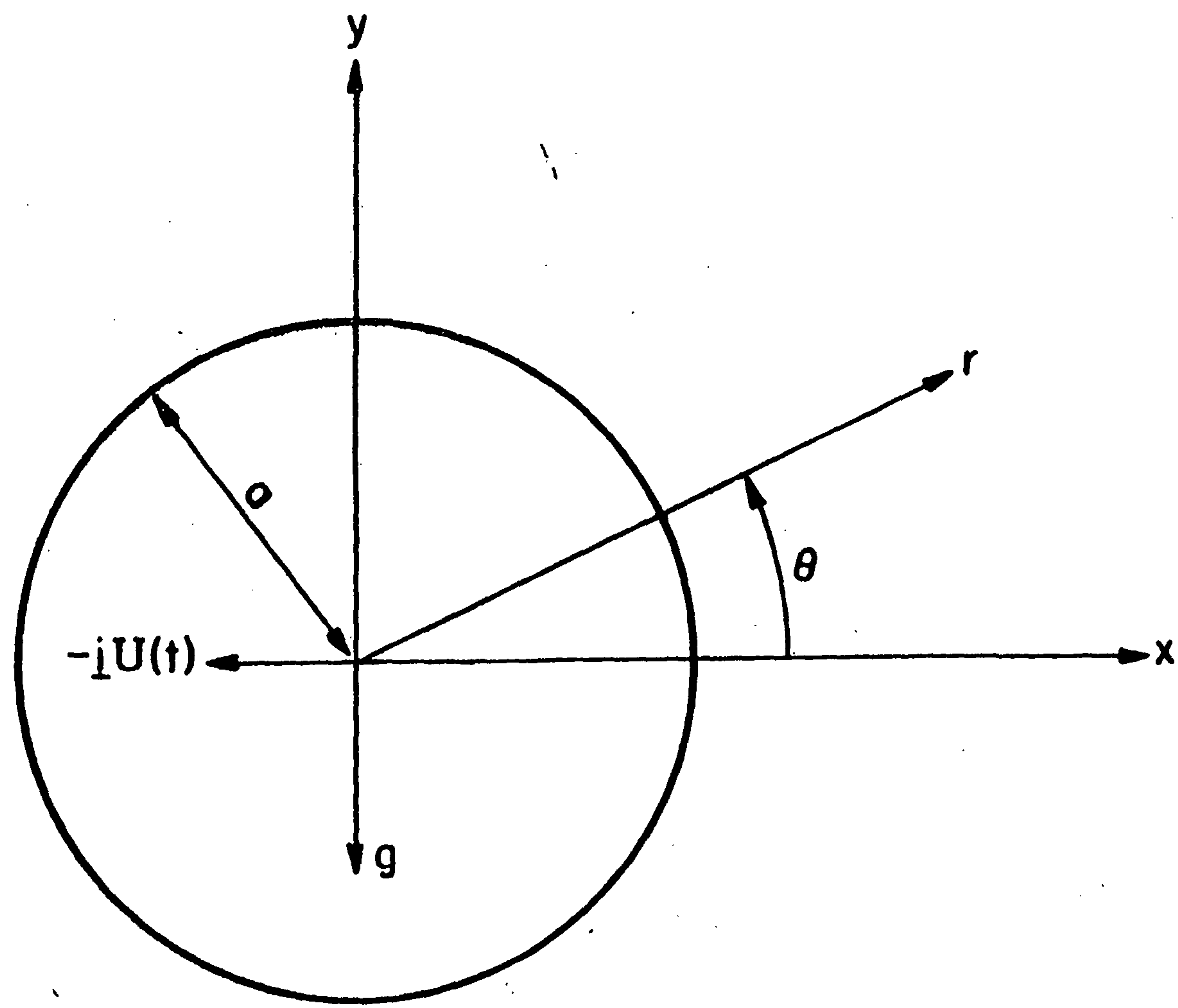
Figure 1. Definition sketch.

Figure 2. Migration of separation point for the case of impulsively started uniform velocity.

Figure 3. Effect of radial expansion on separation angle for the case of impulsively started uniform velocity. Rigid cylinder, ——. Deforming cylinder: — — —, $a(\tau)=1+0.01\tau$; —·—, $a(\tau)=1+0.02\tau$.

Figure 4. Effect of radial contraction on separation angle. Rigid cylinder, ——. Deforming cylinder: — —, $a(\tau)=1-0.1\tau$; — - —, $a(\tau)=1-0.02\tau$.

Figure 5. Effects of acceleration of a deforming cylinder on separation angle; $a(\tau)=1+0.01\tau$. ----, $\tau_c=0.1$; — - —, $\tau_c=0.01$.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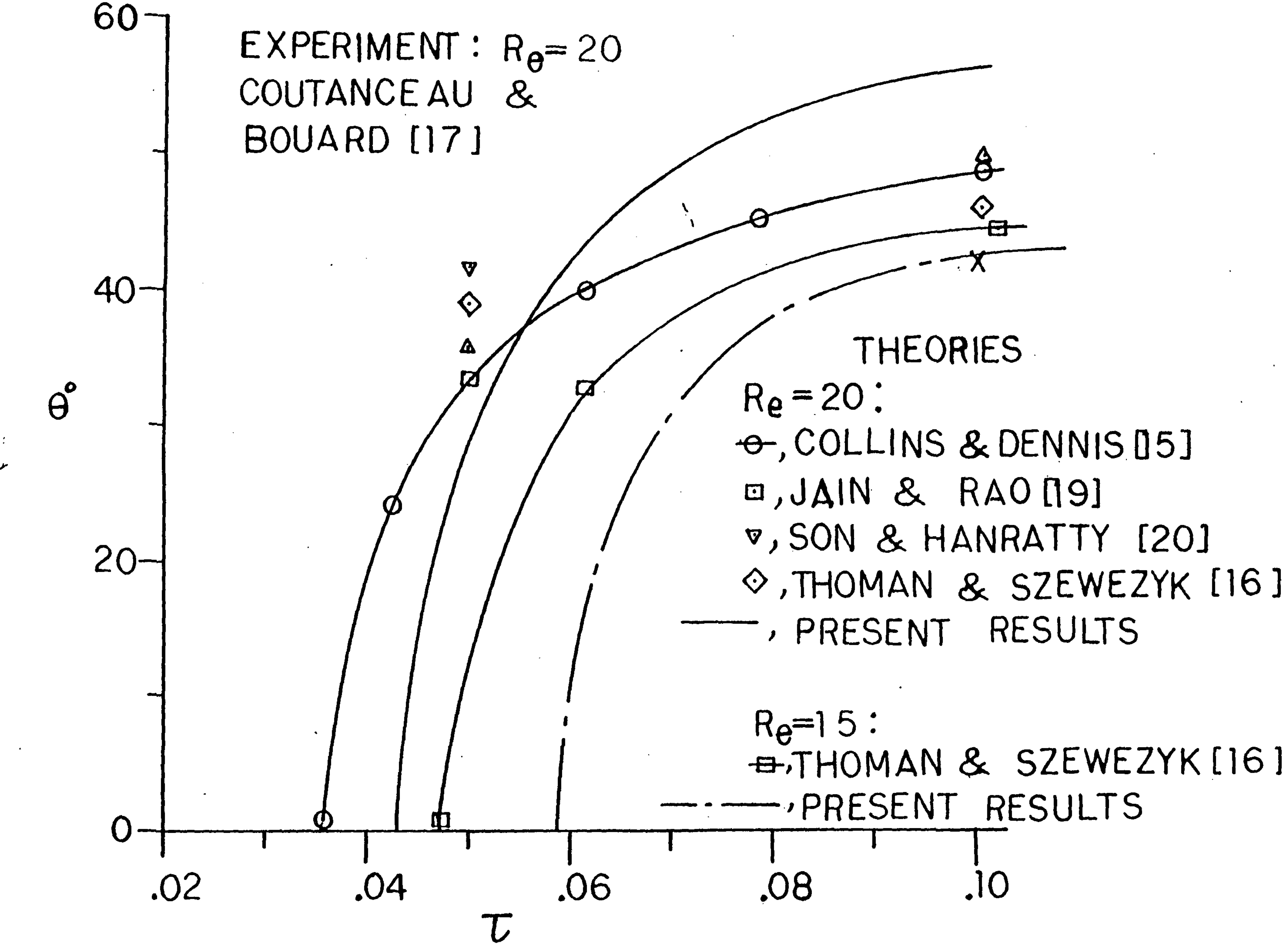
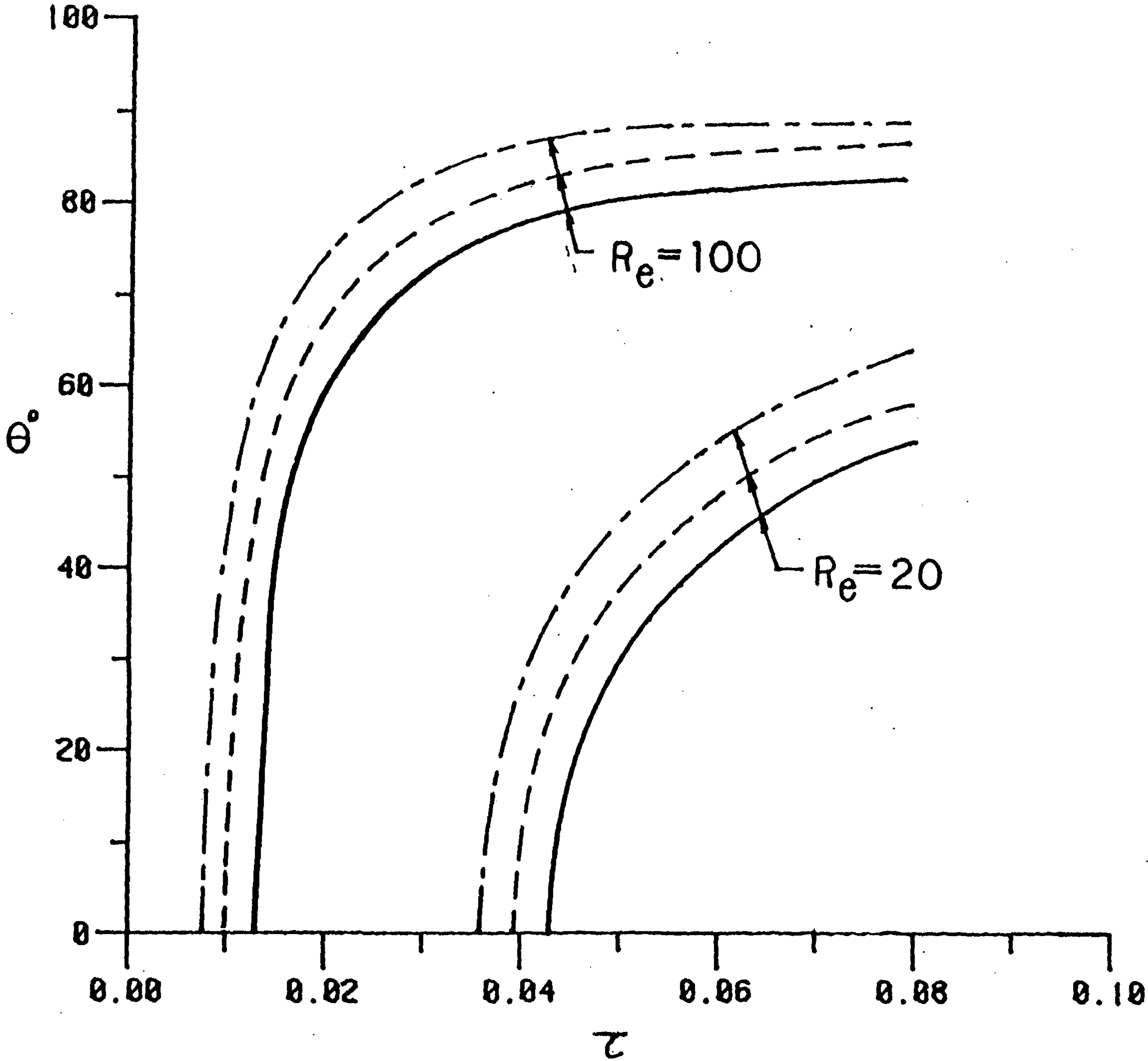


FIG. 2. S.P. LIN, D. MEKALA, G.T. GAPMAN & M. TOBAK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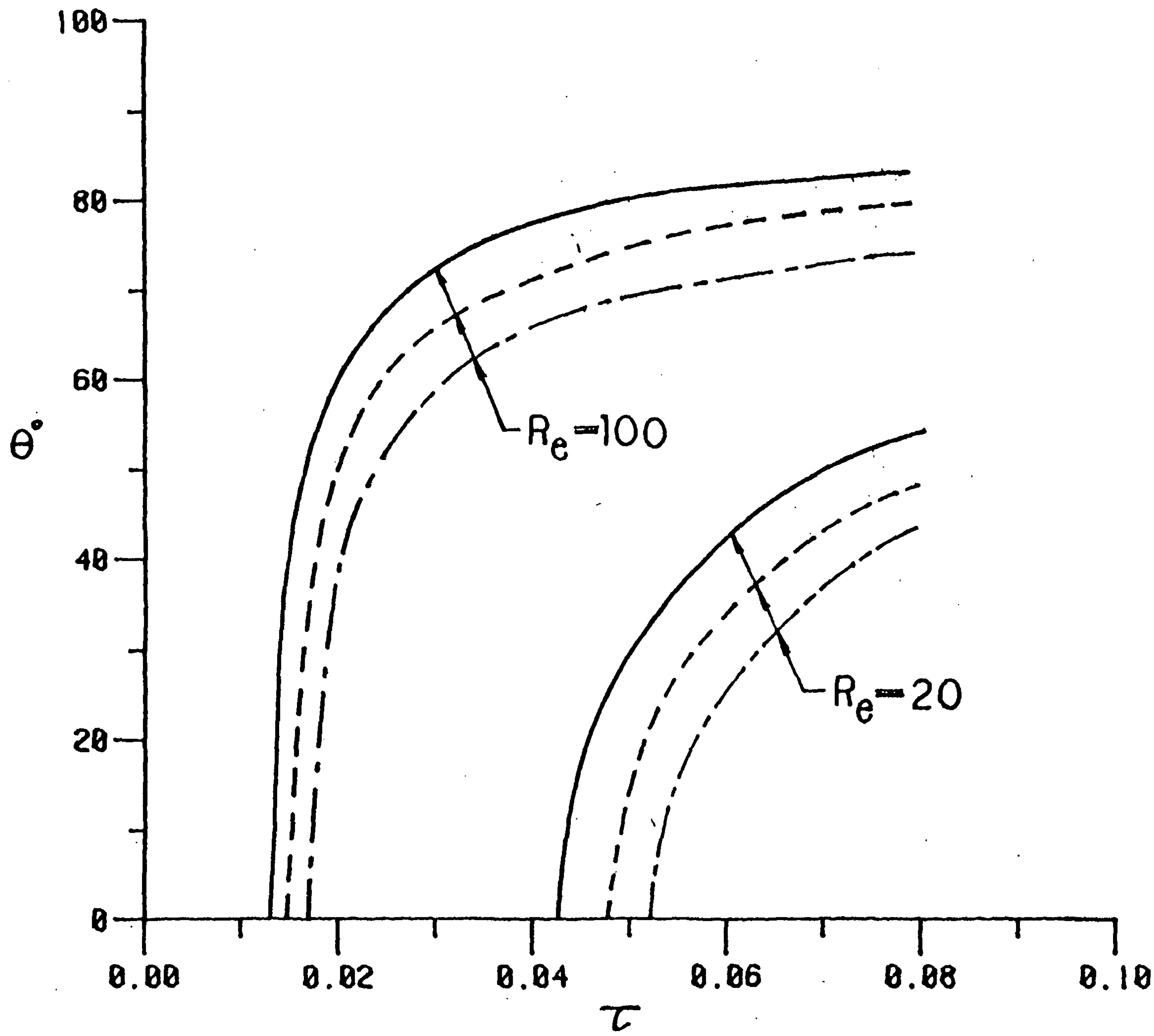
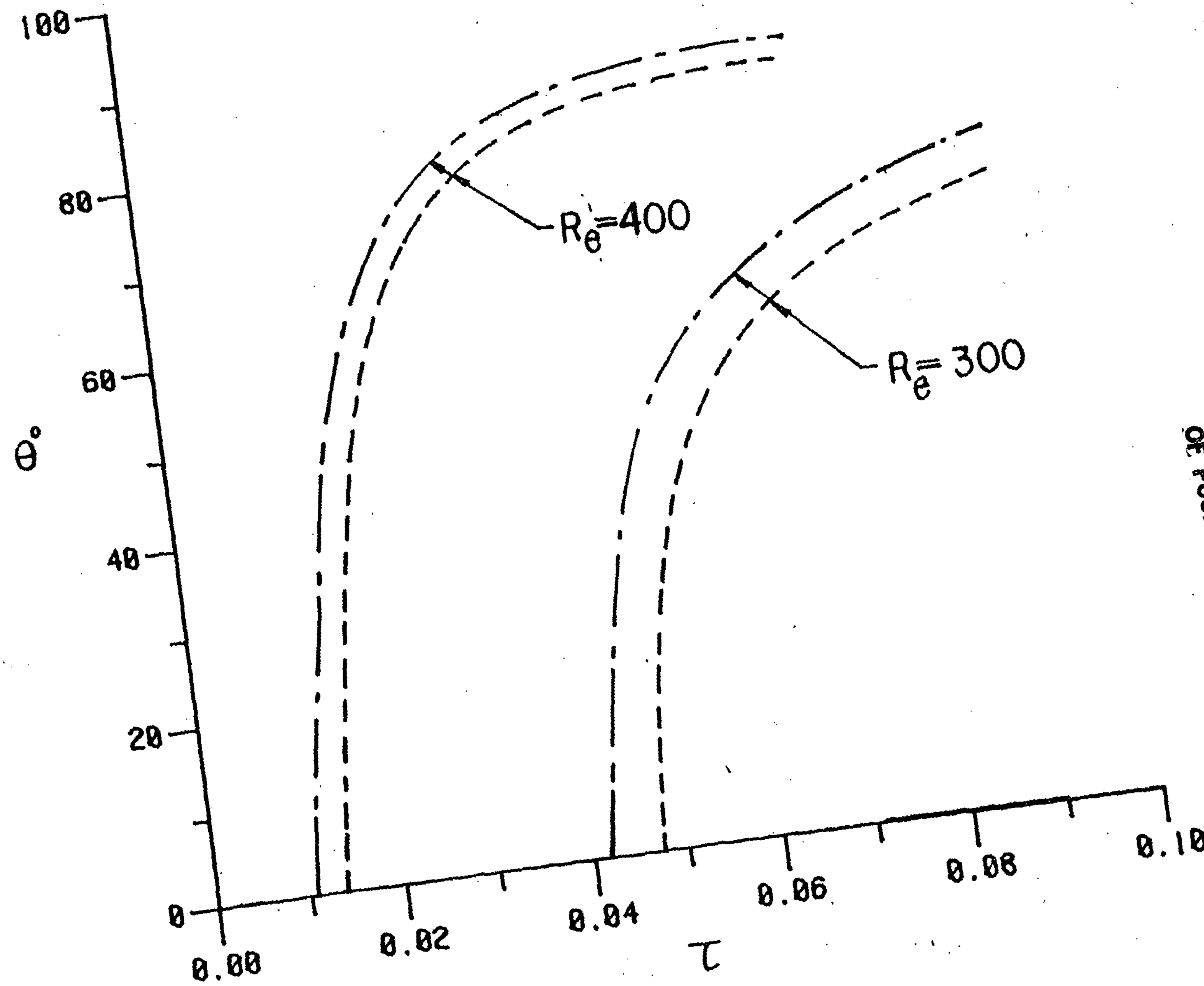


FIG. 4 S.P. LIN; D. MEKALA, G.T. CHAPMAN & M. TOBAK



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