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OF ELASTICITY USING A COMPACT SCHEME

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MULTIGRID METHOD FOR THE EQUILIBRIUM EQUATIONS OF ELASTICITY USING A COMPACT SCHEME

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Abstract

A compact difference scheme derived in [3] for treating the equilibrium equations of elasticity is studied. The scheme turns out to be inconsistent and unstable. A multigrid method which takes into account these properties is described. The solution of the discrete equations, up to the level of discretization errors, is obtained by this method in just two multigrid cycles.

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1. INTRODUCTION

In this paper, we study a compact finite difference scheme for the equilibrium equations of elasticity derived in [3]. We focus here on the two-dimensional case only.

We begin in Section 2 by the derivation of the scheme for a general source term in the elasticity equations (i.e., a non-equilibrium case). This is needed later when a multigrid method is considered.

In the first step, equations for displacements and stresses in a cell are obtained. They consist of equations which approximate the equilibrium equations and those which represent some single-valuedness of displacements in cell centers. The second step is a process of elimination of stresses. This results in a set of equations involving displacements only. The equations divide the set of grid points into two dispoint sets; on each of them different equations are given.

Section 3 deals with the inconsistency of the resulting scheme. A Taylor expansion of the different terms for either of the sets of equations shows that both are inconsistent with the equations of elasticity. However, a closer look reveals that consistency-in-the-average exists. That is, the sum of the equations in a cell shows the desired consistency. This fact is used later when a multigrid method for that scheme is derived.

Section 4 deals with the instability of the scheme. It is shown by means of Fourier analysis that the interior equations admit highly oscillatory solutions for the homogeneous problem. This means that there exist non-smooth components that will not affect the residuals almost at all. This is a numerical instability since it does not have a differential analogue. Schemes with such a property have been studied in [2] and are referred to as quasi-elliptic schemes. In particular, a multigrid method for such schemes is described there.

Sections 5 and 6 describe the multigrid ideas and their implementation in the present context. A standard Gauss-Seidel relaxation was used for relaxing interior equations, while a modification is introduced when traction boundary conditions are relaxed. The coarsening used was done in a compatible way to the discretization. An averaging operator is applied to grid functions before interpolating them to finer levels. This is done in order to remove unstable components in the approximation. The resulting full multigrid algorithm (nested iteration) solves the discretized equation, up to the level of discretization errors in just two cycles. Thus a very efficient method of solution is obtained.

2. COMPACT SCHEME

In two dimensions, the stresses $\tau = (\tau_1, \tau_2)$ are given in terms of the displacements $u = (u_1, u_2)$ as

$$\tau_{11} = \zeta\partial_{x_1}u_1 + \eta\partial_{x_2}u_2 \tag{2.1a}$$

$$\tau_{22} = \eta\partial_{x_1}u_1 + \zeta\partial_{x_2}u_2 \tag{2.1b}$$

$$\tau_{21} = \tau_{12} = \frac{1}{2}\sigma(\partial_{x_2}u_1 + \partial_{x_1}u_2). \tag{2.1c}$$

The parameters ζ, η, σ are given in terms of Young's modulus E and Poisson's ratio ν by

$$\zeta = \frac{E}{(1-\nu^2)}, \qquad \eta = \frac{E\nu}{(1-\nu^2)}, \qquad \sigma = \frac{E}{(1+\nu)} .$$

The equations of elasticity in terms of the stresses are

$$\partial_{x_1}\tau_{11} + \partial_{x_2}\tau_{12} = \tilde{f}_1 \tag{2.2a}$$

$$\partial_{x_1}\tau_{21} + \partial_{x_2}\tau_{22} = \tilde{f}_2, \tag{2.2b}$$

where in equilibrium $\tilde{f}_1 = \tilde{f}_2 = 0$. We assume here a nonequilibrium case, i.e., $\tilde{f}_1$, $\tilde{f}_2$ are not zero. This is needed later when coarse grid equations are considered.

A compact scheme for square cells has been derived in [3]. It is assumed that displacements and stresses are given on cell midfaces as shown in Figure 1. The scheme is given by

$$\mu_{x_1}\tau_{11} = \zeta\delta_{x_1}u_1 + \eta\delta_{x_2}u_2 \tag{2.3a}$$

$$\mu_{x_2}\tau_{22} = \eta\delta_{x_1}u_1 + \zeta\delta_{x_2}u_2 \tag{2.3b}$$

$$\mu_{x_1}\tau_{21} = \mu_{x_2}\tau_{12} = \frac{1}{2}\sigma(\delta_{x_2}u_1 + \delta_{x_1}u_2) \tag{2.3c}$$

$$\delta_{x_1}\tau_{11} + \delta_{x_2}\tau_{12} = f_1 \tag{2.3d}$$

$$\delta_{x_1}\tau_{21} + \delta_{x_2}\tau_{22} = f_2 \tag{2.3e}$$

$$\mu_{x_1}u_1 - kh^2\delta_{x_1}\tau_{11} = \mu_{x_2}u_1 - kh^2\delta_{x_2}\tau_{12} \tag{2.3f}$$

$$\mu_{x_1} u_2 - kh^2 \delta_{x_1} \tau_{21} = \mu_{x_2} u_2 - kh^2 \delta_{x_2} \tau_{22}, \tag{2.3g}$$

where f_1, f_2 are midcell values of $\tilde{f}_1$, $\tilde{f}_2$, respectively, and the operators μ_{x_i}, δ_{x_i} $(i = 1, 2)$ are defined by

$$\mu_{x_1} \phi(x) = (\phi(x_1 + h, x_2) + \phi(x_1 - h, x_2))/2$$

$$\delta_{x_1} \phi(x) = (\phi(x_1 + h_1 x_2) - \phi(x_1 - h_1 x_2))/2h$$

and similarly μ_{x_2}, δ_{x_2}. The parameter k is arbitrary.

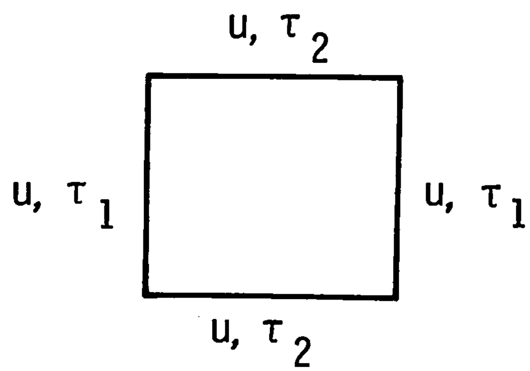

Figure 1

In actual computation the stresses are eliminated from the equations and one gets equations involving the displacement only. We will redo here the elimination of the stresses, for the case equation (2.3d) through (2.3e) has nonzero right-hand sides.

Using (2.3d), (2.3e) in (2.3f), and (2.3g) respectively, we get

$$h\delta_{x_1} \tau_{11} = \frac{1}{2kh} (\mu_{x_1} - \mu_{x_2}) u_1 + \frac{h}{2} f_1 \tag{2.4a}$$

$$h\delta_{x_2}\tau_{22} = \frac{1}{2kh}(\mu_{x_1} - \mu_{x_2})u_2 + \frac{h}{2} f_2. \tag{2.4b}$$

Also we have from (2.4) and (2.3d), (2.3e) the relations

$$h\delta_{x_1}\tau_{21} = \frac{1}{2kh}(\mu_{x_2} - \mu_{x_1})u_2 + \frac{h}{2} f_2 \tag{2.4c}$$

$$h\delta_{x_2}\tau_{12} = \frac{1}{2kh}(\mu_{x_2} - \mu_{x_1})u_1 + \frac{h}{2} f_1. \tag{2.4d}$$

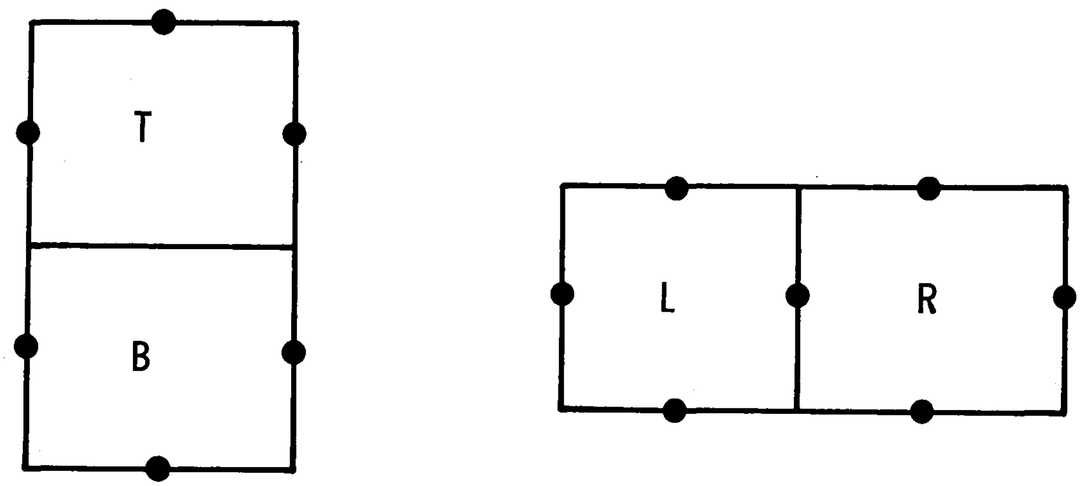

Figure 2

The idea of elimination is to set equal the values of τ that are computed from two neighboring cells. That is,

$$[\mu^R_{x_1} - \mu^L_{x_1} - (h\delta^R_{x_1} + h\delta^L_{x_1})]\ \tau_1 = 0 \tag{2.6a}$$

$$[\mu^T_{x_2} - \mu^B_{x_2} - (h\delta^T_{x_2} + h\zeta\delta^B_2)]\ \tau_2 = 0 \tag{2.6b}$$

where $\mu^R_{x_i}, \mu^L_{x_i}, \mu^T_{x_i}, \mu^B_{x_i}$ $(i = 1,2)$ denote averages on right, left, top, and bottom cells, respectively, and correspondingly $h\delta^R_{x_i}, h\delta^L_{x_i}, h\delta^T_{x_i}, h\delta^B_{x_i}$ denote differences (divided by 2) on right, left, top, and bottom cells respectively (see Figure 2).

Upon inserting (2.3a-c) and (2.4) into (2.6), we get the following equations:

$$\zeta(\delta^R_{x_1} - \delta^L_{x_1})u_1 + \eta(\delta^R_{x_2} - \delta^L_{x_2})u_2 - \left[\frac{1}{2kh}(\mu^R_{x_1} - \mu^R_{x_2}) + (\mu^L_{x_1} - \mu^L_{x_2})\right] u_1 - \frac{h}{2}(f^R_1 + f^L_1) = 0 \tag{2.7a}$$

$$\frac{1}{2}\sigma(\delta^R_{x_2} - \delta^L_{x_2})u_1 + \frac{1}{2}\sigma(\delta^R_{x_1} - \delta^L_{x_1})u_2 - \left[\frac{1}{2kh}(\mu^R_{x_2} - \mu^R_{x_1}) + (\mu^L_{x_2} - \mu^L_{x_1})\right] u_2 - \frac{h}{2}(f^R_2 + f^L_2) = 0 \tag{2.7b}$$

$$\frac{1}{2}\sigma(\delta^T_{x_2} - \delta^B_{x_2})u_1 + \frac{1}{2}\sigma(\delta^T_{x_1} - \delta^B_{x_1})u_2 - \left[\frac{1}{2kh}(\mu^T_{x_2} - \mu^T_{x_1}) + (\mu^B_{x_2} - \mu^B_{x_1})\right] u_1 - \frac{h}{2}(f^T_1 + f^B_1) = 0 \tag{2.7c}$$

$$\eta(\delta^T_{x_1} - \delta^B_{x_1})u_1 + \zeta(\delta^T_{x_2} - \delta^B_{x_2})u_2 - \frac{1}{2kh}\left[(\mu^T_{x_1} - \mu^T_{x_2}) + (\mu^B_{x_1} - \mu^B_{x_2})\right] u_2 - \frac{h}{2}(f^T_2 + f^B_2) = 0. \tag{2.7d}$$

Note that equations (2.7a) and (2.7b) are given on P-points, while (2.7c) and (2.7d) are given on Q-points (see Figure 3).

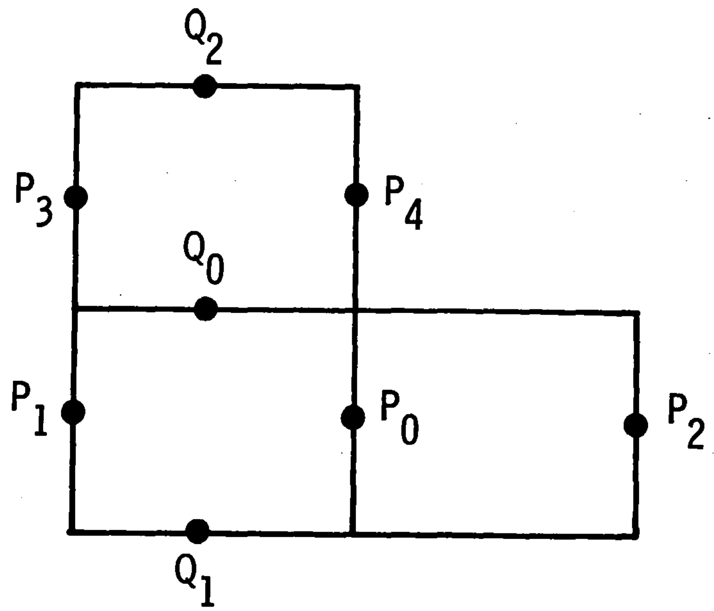

Figure 3

Since it is natural to expect that some of the boundary conditions will be given in terms of stresses, we have to express the stresses at the boundary in terms of the displacement. That is,

$$\tau^{T/B}_{22} = \eta\delta_{x_1}u_1 + \zeta\delta_{x_2}u_2 \pm \frac{1}{2kh}(\mu_{x_1} - \mu_{x_2})\, u_2 \pm \frac{h}{2} f_2 \tag{2.8a}$$

$$\tau_{12}^{T/B} = \frac{1}{2}\,\sigma(\delta_{x_2}u_1 + \delta_{x_1}u_2) \pm \frac{1}{2kh}(\mu_{x_2} - \mu_{x_1})\,u_1 \pm \frac{h}{2}\,f_1 \tag{2.8b}$$

$$\tau_{12}^{R/L} = \zeta\delta_{x_1}u_1 + \eta\delta_{x_2}u_2 \pm \frac{1}{2kh}(\mu_{x_1} - \mu_{x_2})\,u_1 \pm \frac{h}{2}\,f_1 \tag{2.8c}$$

$$\tau_{21}^{R/L} = \frac{1}{2}\,\sigma(\delta_{x_2}u_1 + \delta_{x_1}u_2) \pm \frac{1}{2kh}(\mu_{x_2} - \mu_{x_1})\,u_2 \pm \frac{h}{2}\,f_2. \tag{2.8d}$$

In the next sections we assume the parameter k to have the value $\frac{2}{\sigma}$.

3. INCONSISTENCY AND IMPLICATIONS

The equations of elasticity, in terms of displacements, are,

$$\zeta\partial_{x_1}^2 u_1 + \eta\partial_{x_2}^2 u_2 + \frac{1}{2}\,\sigma\partial_{x_2}^2 u_1 + \frac{1}{2}\,\sigma\partial_{x_1x_2}u_2 = \tilde{f}_1 \tag{3.1a}$$

$$\frac{1}{2}\,\sigma\partial_{x_1x_2}u_1 + \frac{1}{2}\,\sigma\partial_{x_1}^2 u_2 + \eta\partial_{x_1x_2}u_1 + \zeta\partial_{x_2}^2 u_2 = \tilde{f}_2. \tag{3.1b}$$

The compact scheme (2.7) is not <u>consistent</u> with this equation in the usual sense. In fact, if we expand the different terms in (2.7) in a Taylor expansion, we find the following: The equations at P-points are consistent with

$$\frac{\sigma}{4}\,\Delta u_1 + (\zeta - \frac{\sigma}{2})\partial_{x_1}^2 u_1 + \eta\partial_{x_1x_2}u_2 = \frac{1}{2}\,\tilde{f}_1 \tag{3.2a}$$

$$\frac{\sigma}{2}\,\partial_{x_1x_2}u_1 + \frac{\sigma}{4}\,\Delta u_2 = \frac{1}{2}\,\tilde{f}_2, \tag{3.2b}$$

and the Q-equations are consistent with

$$\frac{\sigma}{4}\,\Delta u_1 + \frac{1}{2}\,\sigma\partial_{x_1x_2}u_2 = \frac{1}{2}\,\tilde{f}_1 \tag{3.3a}$$

$$\eta\partial_{x_1x_2}u_1 + \frac{\sigma}{4}\,\Delta u_2 + (\zeta - \frac{\sigma}{2})\,\partial^2_{x_2}u_2 = \frac{1}{2}\,\tilde{f}_2. \tag{3.3b}$$

Neither (3.2) nor (3.3) are the equilibrium equation of elasticity. However, by summing (3.2) with (3.2) and using the definition of ζ, η, σ, we get back (3.1). That is, we have consistency in an averaged sense.

The fact that we have only consistency in-the-average is important when coarsening has taken place in a multigrid process. Residuals that are transferred to coarser levels should better be related to equations which are consistent with the differential equation.

This can be achieved for equations (2.7) if we sum the proper equations to ensure consistency with (3.1). That is, by summing (2.7a) with (2.7d) and (2.7b) with (2.7c), consistency with (3.1a) and (3.1b), respectively, is obtained.

If boundary conditions are given in terms of stresses, residual transfer should be done carefully. By looking at (2.8) we see that the discretization is such that the boundary condition has contribution from the right hand side of the interior equation. Residual transfer to coarser levels should maintain this.

4. INSTABILITY

Another important property of the scheme (2.7) is its instability. That is, there are highly oscillatory displacements which satisfy the interior equations with zero right hand side. This means that large changes in these components, or components close to them, will not affect the residual almost at all. Hence, a small change in the equations can introduce large changes in the solution in these unstable components. This is a numerical instability, since a correspondingly large change in the differential solution cannot occur.

We show below how to find the unstable components by means of Fourier analysis. We consider the homogeneous equation and write it in the form

$$\mathcal{L}\begin{pmatrix} U_P \\ U_Q \\ V_P \\ V_Q \end{pmatrix} = 0$$

where U_P (U_Q) denotes the u_1 values on $P(Q)$ points and similarly V_P (V_Q) denotes u_2 values on $P(Q)$ points. Consider displacements of the form

$$\begin{pmatrix} U_P \\ U_Q \\ V_P \\ V_Q \end{pmatrix} = \begin{pmatrix} A_P \\ A_Q \\ B_P \\ B_Q \end{pmatrix} \exp(i\underline{\theta}\cdot\underline{x}/H)$$

where $\theta = (\theta_1,\theta_2)$, $H = 2h$ (the size of a cell), and $|\theta| < \pi$, where $|\theta| = \max(|\theta_1|, |\theta_2|)$. For this choice of displacements we have

$$\mathcal{L}\begin{pmatrix} U_P \\ U_Q \\ V_P \\ V_Q \end{pmatrix} = \hat{\mathcal{L}}(\theta)\begin{pmatrix} A_P \\ A_Q \\ B_P \\ B_Q \end{pmatrix} \exp(i\underline{\theta}\cdot\underline{x}/H)$$

where (θ) is a 4×4 matrix of functions depending on $\underline{\theta}$. By looking at (θ) we can examine the stability of our scheme. If there exists a $\underline{\theta} \neq 0$ such that det $(\underline{\theta}) = 0$, it means that there are high frequency components which solve the homogeneous equation. For our scheme

$$\det \hat{\mathcal{L}}(\theta) = 0$$

implies $\underline{\theta} = (0,0)$ or $\underline{\theta} = (\pi,\pi)$. The unstable Fourier component is therefore $\theta = (\pi, \pi)$. Its amplitude is obtained by solving the equation

$$\hat{\mathcal{L}}((\pi, \pi))\begin{pmatrix} A_P \\ A_Q \\ B_P \\ B_Q \end{pmatrix} = 0$$

for a nontrivial solution. This gives us

$$A_P = 0, \quad A_Q = 1, \quad B_P = -1, \quad B_Q = 0.$$

This corresponds to the following displacements (in a cell).

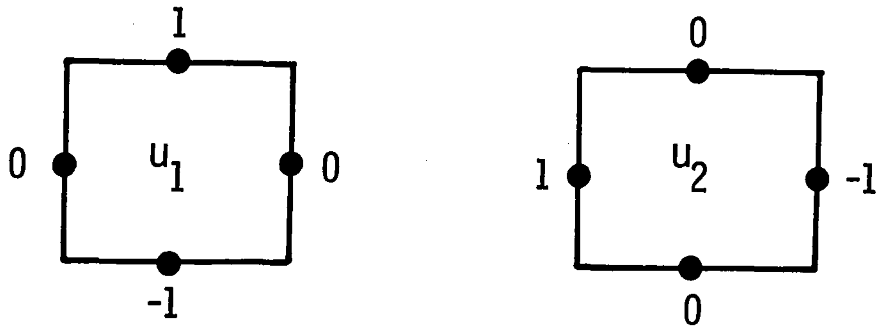

Figure 4

The computation above shows that these are the <u>only</u> unstable components. Hence, this scheme satisfies the definition of quasi-ellipticity given [2].

Since det $(\theta) = 0$ for some $\underline{\theta} \neq 0$, in an infinite space, or under periodic boundary conditions, there exists a highly oscillatory function $\nu^h(x) = A \exp(i\underline{\theta} \cdot \underline{x}/H)$ which satisfies the homogeneous equation $L^h \nu^h \equiv 0$. Hence, the solution, unlike the corresponding differential solution, is not unique (up to an additive constant); it contains an undetermined highly oscillatory component. Similarly in any bounded domain with any boundary conditions, functions $W^h(x)$ close to $\nu^h(x)$ (e.g., $W^h = \phi_1 \nu^h + \phi_2$, ϕ_j being smooth) exist which satisfy the boundary condition and for which $L^h W^h$ is everywhere small. Such W^h, therefore, forms an unstable mode: a small change in the equation can introduce a large change proportional to W^h.

This is a kind of numerical instability, since a correspondingly large change in the differential solution cannot occur. The numerical instability need not hurt much: If the differential system is $LU = f$ and the discrete system is $L^h U^h = f^h$, all one has to do is to define $F^h = I^h F$, say, through an averaging operator which liquidates the unstable modes. Another way to remove the instability is by averaging the solution, that is, by replacing the computed solution U^h by S^hU^h, where S^h is an averaging operator which removes all the unstable components but retains the accuracy of smooth components.

When derivatives are calculated, much greater loss of accuracy can occur than in the solution itself. The averaging process discussed above reduces this inaccuracy in derivatives.

In the problem we treat here, the stresses, as computed from the displacements, are the important physical quantities. Since they involve derivative of displacements, we might expect to see degradation of accuracy as a result of the instability. However, the unstable components for the scheme discussed here are such that they produce zero stresses. That is, no loss of accuracy in the stresses occurs as a result of the instability.

The mentioned instability has strong implications on the multigrid method to be used. Usual multigrid solvers yield poor asymptotic rates when applied to quasi-elliptic schemes. The reason is simple: slow to converge are the unstable modes. They cannot converge by coarse-grid correction, since they are high-frequency modes, essentially invisible on coarser levels. Neither can they significantly converge by any type of local relaxation since these unstable modes show a very small residual function (compared with residuals shown by other modes with comparable amplitude) and the correction introduced by relaxation is proportional to the size of the residuals (see [1]). The smoothing factor for such schemes is 1, and it is achieved at the unstable modes.

The poor asymptotic convergence is not important. The modes which are slow to converge are exactly those unstable ones for which algebraic convergence is not really desired, their amplitudes in the algebraic solution being unrelated to their amplitude in the differential equation. The only concern is that these amplitudes will remain suitably small.

5. PRACTICAL IMPLEMENTATION

In this section we describe the elements of a multigrid procedure defined in Section 6.

Relaxation

As a relaxation we have used Gauss-Seidel for the interior equation, where the P-points are relaxed first followed by the Q-points.

Relaxing the traction boundary conditions is done slightly differently (see [1], Section 5.3). Instead of performing Gauss-Seidel for the traction boundary condition $BU = g$, we perform Gauss-Seidel on the equation $\frac{\partial^2}{\partial s^2} BU = \frac{\partial^2}{\partial s^2} g$, where $\frac{\partial}{\partial s}$ is derivative tangential to the boundary. Practically, it means that instead of satisfying a given equation at a boundary point, we only change it such that its error is the average of the errors at neighboring boundary points. This relaxation preserves the smoothness of the interior approximation, unlike the straightforward Gauss-Seidel.

Coarsening

The coarsening method we have used here is referred in [1] as compatible coarsening. That is, the coarsening procedure is analogous to the fine-grid discretization.

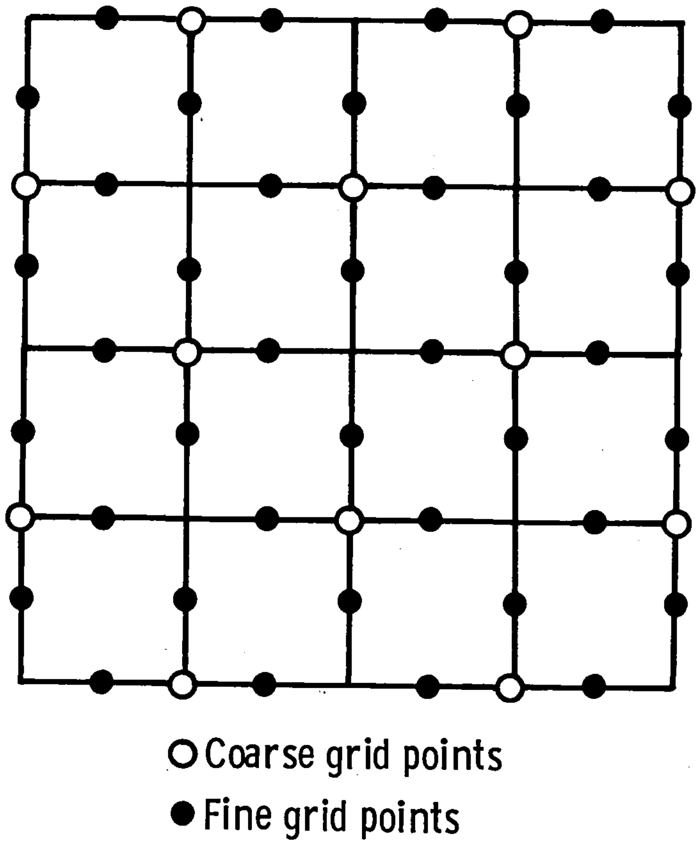

Figure 5

Each 2 x 2 fine-grid cell forms a coarse grid cell [see figure 5]. Coarse grid equations are defined using formulas (2.7) where f_1, f_2 are replaced by residuals of finer levels, which are consistent with differential residuals (as mentioned in section 4).

Interpolation

In interpolating corrections from a coarse to a fine grid, we used the following procedure:

(a) Define values at midcells and at vertices on coarse level by linear interpolation.

(b) Linearaly interpolate corrections to the fine grid, using midcells and vertex values defined in (a), and original coarse-grid function.

Averaging

An averaging operator was applied to corrections before interpolating them. This was done in order not to introduce high frequency errors which are unstable. The averaging was such that it killed the unstable mode but retained the accuracy of the scheme. It is given by the following formula:

$$(S^h u_i)(x) = \frac{1}{8}\left[4u_i(x,y) + u_i(x + h,\ y + h) + \right.$$

$$\left. u_i(x + h_1,\ y - h) + u_i(x - h,\ y + h) + u_i(x - h,\ y - h)\right],$$

$$(i = 1,2).$$

6. FMG SOLUTION TO TRUNCATION LEVEL

Since the multigrid cycling is inefficient in reducing unstable mode errors, the multigrid solver should take care not to start with an initial solution which contains large amplitudes of such errors. The overall initial error in unstable modes should be smaller than the overall truncation error. This is easily obtained by taking a first approximation from a coarser grid, employing interpolation of suitable order. The usual "Full multigrid" (FMG, also called "nested iteration") algorithm can therefore be used, with slight modifications described in the following. For a flowchart, and a detailed discussion of FMG algorithms and the order of the first interpolation, see Secs. 1.6 and 7 in [1]. We describe here the Correction Scheme (CS) version of the algorithm since our problem is linear, and issues of local refinement are not discussed here.

6.1 Multigrid cycle.

Suppose a sequence of grids is given with meshsizes h_k (k = 1, 2, 3,...) where $h_{k+1} = h_k/2$. On the h_k grid the discrete equations have the form

$$L^k U^k = F^k \tag{6.1}$$

where L^k approximates L^{k+1}. Given u_0^k, an approximate solution to (6.1), the multigrid cycle MG for producing an improved approximation u_1^k

$$u_1^k \leftarrow MG(k,\ u_0^k,\ F^k) \tag{6.2}$$

is recursively defined as follows:

If $k = 1$ solve (6.1) by any direct or iterative method, yielding the final result u_1^k. Otherwise do (A) through (D).

(A) Perform ν_1 relaxation sweeps on (6.1), resulting in a new approximation $\bar{u}^k$.

(B) Starting with $u_0^{k-1} = 0$, make γ successive cycles

$$u_j^{k-1} \leftarrow MG(k-1,\ u_{j-1}^{k-1},\ I_k^{k-1}(F^k - L^k\bar{u}^k)), \qquad (j = 1, \ldots, \gamma)$$

where I_k^{k-1} is a transfer ("reduction") of residuals from grid h_k to grid h_{k-1}.

(C) Calculate $\tilde{u}^k = \bar{u}^k + I_{k-1}^k\ S^{k-1}\ u_\gamma^{k-1}$, where I_{k-1}^k is a suitable interpolation ("prolongation") from grid h_{k-1} to grid h_k, and S^{k-1} is a suitable averaging operator.

(D) Perform ν_2 relaxation sweeps on (6.1), starting with $\tilde{u}^k$ and yielding the final result u_1^k.

The cycle with $\gamma = 1$ is called V cycle or $V(\nu_1, \nu_2)$, and the one with $\gamma = 2$ is called W cycle or $W(\nu_1, \nu_2)$.

6.2 Full Multigrid (FMG).

The N-FMG is an algorithm for calculating an approximate solution

$$u_N^k = \text{FMG}(k, F^k, N) \tag{6.4}$$

to equation (6.1), defined recursively as the following two successive steps.

(a) Calculating a first approximation u_0^k:
if $k = 1$, put $u_o^k = 0$. Otherwise, put

$$u_0^k = \Pi_{k-1}^k \text{FMG}(k-1, I_k^{k-1} F^k, N), \quad (6.5)$$

where Π_{k-1}^k is an interpolation operator from grid h_{k-1} to grid h_k, and I_k^{k-1} is a transfer from grid k to grid $k - 1$. In our experiments, the interpolation operator $\Pi^k{}_{k-1}$ was the same as the one for interpolating corrections (defined in Section 5).

(b) Improve the first approximation by N successive MG cycles

$$u_j^k \leftarrow \text{MG}(k, u_{j-1}^k, F^k), \qquad (j = 1,\dots, N)$$

as defined in Sec. 6.1.

7. RESULTS AND DISCUSSION

A domain $\{(x, y): 0 < x < 1, 0 < y < 1\}$ is considered. Boundary conditions are given in terms of displacements on the two boundaries $x = 0$, and $x = 1$. Stresses are given on the boundaries $y = 0$, $y = 1$. Source terms are used such that the exact solution is given by

$$u_1 = u_2 = \sin(.2(x + 2y - 2)).$$

The finest level uses a mesh size of 1/32, and 4 levels were used in the multigrid process. In the notation of Section 6 the following parameters were used: $\gamma = 2$, $\nu_1 = 3$, $\nu_2 = 3$, $N = 4$. The table below shows the dynamic L_2-norm of the residuals on the currently finest level, as well as the error in the approximation at that stage. It is seen that after two cycles the problem is solved to the level of discretization errors. We show results for $N = 4$ not because it is really needed, but in order to show that indeed, $N = 2$ is enough to obtain approximate solution whose errors are below the level of discretization errors. Observe that the asymptotic convergence rate, as predicted, is not very fast. It is related to the unstable components. This should not bother us since we do not want the unstable component to converge. All we need from these components is to have errors below the level of discretization errors.

The results clearly show that a very efficient method for dealing with compact scheme is obtained in spite of the fact that such schemes have instability and inconsistency properties.

Level #	Cycle #	$\|Residuals\|_2$	$\|u^h - v\|$
2	5	.505(-4)	.406(-3)
3	1	.482(-2)	.429(-4)
	2	.109(-2)	.274(-4)
	3	.279(-3)	.252(-4)
	4	.861(-4)	.247(-4)
4	1	.132(-2)	.676(-5)
	2	.520(-3)	.558(-5)
	3	.288(-3)	.558(-5)
	4	.162(-3)	.558(-5)

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Final Report

16. Abstract

A compact difference scheme derived in [3] for treating the equilibrium equations of elasticity is studied. The scheme turns out to be inconsistent and unstable. A multigrid method which takes into account these properties is described. The solution of the discrete equations, up to the level of discretization errors, is obtained by this method in just two multigrid cycles.

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