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FLIGHT CONTROL SYNTHESIS FOR FLEXIBLE
AIRCRAFT USING EIGENSPACE ASSIGNMENT

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LIST OF SYMBOLS

Symbol	Meaning
$\mathbf{A}$	plant matrices
$\mathbf{B}$	control matrices
$\tilde{\mathbf{B}}$	controllability matrix
$\mathbf{C}$	output matrices (for states)
$\tilde{\mathbf{C}}$	observability matrix
$\mathbf{D}$	output matrices (for controls)
$\mathbf{E}$	disturbance matrix
$\mathbf{M}$	measurement matrix (for states)
$\mathbf{N}$	measurement matrix (for controls)
$\mathbf{F}$	filter gain matrix
$\mathbf{G}$	control gain matrices
$\mathbf{I}_{(\cdot)}$	identity matrix
J_i	objective functional
$\mathbf{P}_n$	projection matrix
$\mathbf{Q}$	state weighting matrix
$\mathbf{Q}_{d_i}$	weighting matrix for desired eigenvector
$\mathbf{R}$	control weighting matrix
$R_{(\cdot)}$	impulse residue for response (.)
$\mathbf{T}$	modal matrix (matrix of eigenvectors)
T	scaling matrix
U_o	forward flight speed (ft/sec)
$\mathbf{V}$	matrix of achievable eigenvectors
$\mathbf{V}_z$	intensity of white noise v_z
$\mathbf{W}_x$	intensity of white noise w_x
j	square root of negative one
l	number of measurements
l_k	k^{th} left eigenvector
m	number of controls

Symbol	Meaning
n	number of states
$n_{z(\cdot)}$	plunge acceleration (g's)
p	number of outputs
$\underline{q}$	modal state vector
u_f	perturbed forward velocity (ft/sec)
$\underline{u}_{(\cdot)}$	input vector
v_z	measurement disturbance vector
$\underline{w}_x$	state disturbance vector
$\underline{x}$	state vector
$\hat{\underline{x}}$	state estimate vector
$\tilde{\underline{x}}$	state error vector
$\underline{y}$	output vector
$z_{(\cdot)}$	zero of a transfer function
$\underline{z}$	measurement vector
$\Delta G_{(\cdot)}$	incremental change in loop gain
Λ	diagonal matrix of eigenvalues
$\Phi_{(\cdot)}$	phase of impulse residue $R_{(\cdot)}$
α	angle of attack, (rad)
γ	flight path angle, (rad)
$\delta_{(\cdot)}$	control surface deflection, (rad)
$\delta_{(\cdot)_c}$	commanded control surface deflection, (rad)
ζ	damping ratio
$\theta_{(\cdot)}$	attitude angle, (rad)
κ	measurement noise intensity weighting
$\lambda_{(\cdot)}$	eigenvalue
$\nu_{(\cdot)}$	eigenvector
$\xi_{(\cdot)}$	generalized deflection, (dimensionless)
ρ	LQ cost function control weighting
$\sigma_{(\cdot)}$	real part of eigenvalue λ
τ	actuator time constant
$\phi_{(\cdot)}$	mode shape, (ft)
$\phi'_{(\cdot)}$	mode slope, (ft/ft)
$\underline{\chi}$	partial state vector
ω	imaginary part (frequency) of eigenvalue λ
ω_n	natural frequency

Symbol	Meaning
	operations
$\lvert\cdot\rvert$	magnitude of $(\cdot)$
$\overline{(\cdot)}$	complex conjugate of $(\cdot)$
$(\dot{\cdot})$	time derivative of $(\cdot)$
	subscripts
I	imaginary
R	real
a	achievable value
cv	control vane
con	controller
d	desired value
e	elevator
p	pilot
r	rigid-body
ss	steady-state
t	total (rigid-body + elastic)
x	state
y	output
z	measurement
	superscripts
f	finite
T	transpose
-1	inverse
∞	infinite
*	complex conjugate transpose

SUMMARY

The use of eigenspace assignment techniques to synthesize flight control systems for flexible aircraft is explored. These eigenspace assignment techniques are used to achieve a specified desired eigenspace, chosen to yield desirable system impulse residue magnitudes for selected system responses. Two eigenspace assignment techniques are investigated. The first is a technique for directly determining constant measurement feedback gains that will yield a closed-loop system eigenspace "close" to a desired eigenspace. The second technique is a method for selecting quadratic weighting matrices in a linear quadratic control synthesis that will asymptotically yield the closed-loop achievable eigenspace. Finally, the possibility of using either of these techniques with state estimation is explored.

Application of the methods to synthesize integrated flight-control and structural-mode-control laws for a large flexible aircraft is demonstrated and results discussed. Eigenspace selection criteria based upon the design goals are discussed, and for the study case it would appear that a desirable eigenspace can be obtained. In addition, the importance of state-space selection is noted along with problems with reduced-order measurement feedback. Since the full-state control laws may be implemented with dynamic compensation (state estimation), the use of reduced-order measurement feedback is less desirable. This is especially true since no change in the transient response from the pilot's input results if state estimation is used appropriately. The potential is also noted for high actuator bandwidth requirements if the linear quadratic synthesis approach is utilized. Even with the actuator pole location selected, a problem with unmodeled modes is noted due to high bandwidth.

Some suggestions for future research include investigating how to choose an eigenspace that will achieve certain desired dynamics and stability robustness, determining how the choice of measurements effects synthesis results, and exploring how the phase relationships between desired eigenvector elements effects the synthesis results.

CHAPTER I
INTRODUCTION

Many aircraft control systems in use today are synthesized by treating the aircraft as if it were a rigid-body. This is appropriate when the aircraft are rigid enough so that the structural flexibility has little effect on the rigid-body dynamics. Because of the rapidly growing use of composites, the need for greater payload capability and increased range, future aircraft will become lighter and more flexible. Therefore, the aircraft's aeroelastic modes will begin to play a greater role in its dynamics.

These aeroelastic modes pose special problems with both the analysis of the flexible dynamics and in synthesizing control systems to achieve desired dynamics. Work done by D.K. Schmidt and M.R. Waszak [1,2,3] addressed the first problem, analyzing the dynamics of aircraft in which structural flexibility is significant. This report will build upon the work of Schmidt and Waszak and address the problem of how to synthesize control systems to achieve desired dynamics in flexible aircraft. In particular, these control systems will be synthesized using eigenspace assignment techniques [4,5,6,7,8]. This work will first explore two eigenspace assignment techniques then examine how these techniques can be used with state estimation.

This report is presented in six chapters. A brief summary of Schmidt and Waszak's work and how this work naturally leads to an eigenspace assignment control synthesis technique is presented in Chapter II. Chapter III presents Direct Eigenspace Assignment (DEA). DEA is a control synthesis technique for directly determining measurement feedback control gains that will yield a closed-loop system eigenspace as close as possible to a desired eigenspace [5,6,7,8]. In Chapter IV, Linear Quadratic Eigenspace Assignment (LQEA) is presented. This method is based upon the work of C.A. Harvey and G. Stein [9,10]. LQEA is a method for selecting quadratic weighting matrices that will asymptotically yield a closed-loop achievable eigenspace in a linear quadratic control synthesis. Chapter V explores the advantages of using either of these eigenspace assignment techniques with state estimation. In conclusion, Chapter VI presents a summary of the results and conclusions based on the results.

CHAPTER II
BACKGROUND

A control synthesis technique can be viewed as a mapping from a given set of poor dynamics into a set of desirable dynamics (See Figure 2.1). When considering a rigid aircraft, these desirable dynamics are well defined. They are stated in terms of handling qualities specifications, time response requirements, etc. These desirable dynamics can usually be expressed in a way that is compatible with a popular control synthesis technique, such as Root Locus, Frequency Analysis methods (Bode, Nyquist, etc), or Linear Quadratic optimization methods. Therefore, using one of these techniques a designer can usually synthesize a control system for a rigid aircraft that will achieve some desired dynamics in the closed-loop system.

In reality, aircraft are not rigid. The term "relatively rigid" only refers to aircraft in which the vibrational frequencies of the aircraft's aeroelastic modes are large compared to the frequencies of the rigid-body modes. As the frequencies of the aeroelastic modes becomes lower, these aeroelastic modes begin to significantly affect the aircraft's dynamics. These structural modes pose special problems, not only with describing the flexible dynamics, but also in synthesizing control designs to achieve desired dynamics.

One control synthesis approach that is too often used is just ignoring the flexible dynamics. But work done by M.G. Gilbert, D.K. Schmidt, and T.A. Weisshaar [11] has shown that ignoring the structural dynamics of these aircraft in flight control designs can lead to very poor if not disastrous results.

Modal Analysis

In work on the analysis of flexible aircraft dynamics, D.K. Schmidt and M.R. Waszak [1,2,3] present one solution to the problem of describing the effects of flexible dynamics. Schmidt and Waszak present a modal analysis technique that is especially useful for analyzing the dynamics of flexible vehicles. This technique involves calculating both the eigenvalues and impulse residues of the system. By comparing the magnitudes of the system's impulse

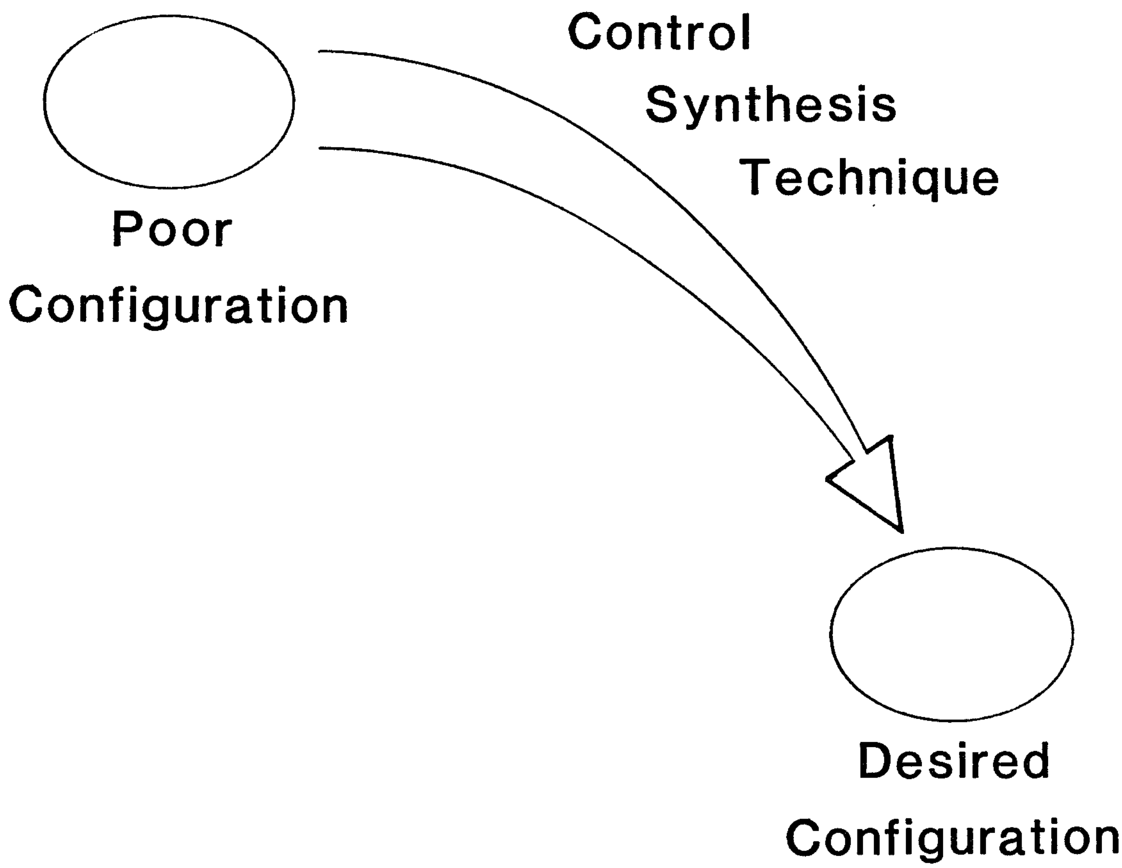

Figure 2.1
Control Synthesis Technique Mapping

residues for various outputs, this analysis shows how much the elastic modes contribute to the system's dynamics. The residues of a system are related to its dynamics in the following way.

Given the system

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \quad \text{(system dynamics)}$$

$$\underline{y} = \mathbf{C}\underline{x} \quad \text{(system responses)}$$

where $\underline{x} \in R^n$, $\underline{u} \in R^m$, and $\underline{y} \in R^p$. Assume all the eigenvalues of $\mathbf{A}$ are distinct. Transform this system to modal coordinates using the transformation

$$\underline{x} = \mathbf{T}\underline{q}$$

where $\mathbf{T}$ is the system modal matrix (a matrix of the eigenvectors of $\mathbf{A}$). One obtains

$$\dot{\underline{q}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\underline{q} + \mathbf{T}^{-1}\mathbf{B}\underline{u}$$

$$= \Lambda\, \underline{q} + \tilde{\mathbf{B}}\underline{u}$$

where Λ is a diagonal matrix of the eigenvalues of $\mathbf{A}$, and $\tilde{\mathbf{B}}$ is the system controllability matrix. Taking the Laplace transform of this equation (assuming $\underline{q}(0) = 0$) yields

$$s\underline{q}(s) = \Lambda\underline{q}(s) + \tilde{\mathbf{B}}\underline{u}(s)$$

$$\underline{q}(s) = [s\mathbf{I}_n - \Lambda]^{-1}\, \tilde{\mathbf{B}}\underline{u}(s)$$

The system input-output transfer function relationship is given by

$$\underline{y}(s) = \mathbf{C}\mathbf{T}\underline{q}(s) = \tilde{\mathbf{C}}[s\mathbf{I}_n - \Lambda]^{-1}\, \tilde{\mathbf{B}}\underline{u}(s)$$

Looking at the ith output, jth input

$$\frac{y_i(s)}{u_j(s)} = \tilde{c}_i\, [s\mathbf{I}_n - \Lambda]^{-1}\, \tilde{b}_j$$

$$= \sum_{k=1}^{n} \frac{\tilde{c}_{i,k}\tilde{b}_{k,j}}{s-\lambda_k} = \sum_{k=1}^{n} \frac{c_i \nu_k l_k b_j}{s-\lambda_k}$$

where

c_i = i^{th} row of $\mathbf{C}$

$\tilde{c}_i$ = i^{th} row of $\tilde{\mathbf{C}}$

b_j = j^{th} column of $\mathbf{B}$

$\tilde{b}_j$ = j^{th} column of $\tilde{\mathbf{B}}$

$\tilde{c}_{i,k}$ = (i,k) element of $\tilde{\mathbf{C}}$

$\tilde{b}_{k,j}$ = (k,j) element of $\tilde{\mathbf{B}}$

λ_k = k^{th} eigenvalue of $\mathbf{A}$

ν_k = k^{th} column of $\mathbf{T}$ (k^{th} right eigenvector of $\mathbf{A}$)

l_k = k^{th} row of $\mathbf{T}^{-1}$ (k^{th} left eigenvector of $\mathbf{A}$)

For an impulse in the j^{th} input, the i^{th} output is given by

$$u_j(t) = \text{impulse} \rightarrow u_j(s) = 1$$

$$y_i(s) = \sum_{k=1}^{n} \frac{\tilde{c}_{i,k}\tilde{b}_{k,j}}{s-\lambda_k} = \sum_{k=1}^{n} \frac{c_i \nu_k l_k b_j}{s-\lambda_k}$$

$$= \sum_{k=1}^{n} \frac{R_{i,j,k}}{s-\lambda_k}$$

The system output time response due to an impulsive input can be found by taking the inverse Laplace transform of this equation. This yields

$$y_i(t) = \sum_{k=1}^{n} R_{i,j,k}\, e^{\lambda_k t}$$

In this equation $R_{i,j,k}$ is the impulse residue for output i, associated with eigenvalue k, and due to an impulse in input j.

Example 2.1

This example demonstrates the role impulse residues play in a flexible aircraft's pitch rate dynamic response due to an impulsive elevator input. For this example, let $y_i(t) = \dot{\theta}(t)$ and $u_j(t) = \delta_e(t)$

$$\dot{\theta}(t) = \sum_{i=1}^{n} R_i \, e^{\lambda_i t}$$

$$= R_{ph} \, e^{\lambda_{ph} t} + \bar{R}_{ph} \, e^{\bar{\lambda}_{ph} t} + R_{sp} \, e^{\lambda_{sp} t} + \bar{R}_{sp} \, e^{\bar{\lambda}_{sp} t}$$

$$+ \sum_{i=1}^{k} \left(R_{\xi_i} \, e^{\lambda_{\xi_i} t} + \bar{R}_{\xi_i} \, e^{\bar{\lambda}_{\xi_i} t} \right)$$

$$= 2 \left| R_{ph} \right| \, e^{-\sigma_{ph} t} \cos \left(\omega_{ph} t + \Phi_{ph} \right) +$$

$$2 \left| R_{sp} \right| \, e^{-\sigma_{sp} t} \cos \left(\omega_{sp} t + \Phi_{sp} \right) +$$

$$\sum_{i=1}^{k} 2 \left| R_{\xi_i} \right| \, e^{-\sigma_{\xi_i} t} \cos \left(\omega_{\xi_i} t + \Phi_{\xi_i} \right)$$

where

$$\lambda_i = -\sigma_i + j\omega_i \, , \; \Phi_i = \arctan \left(\mathrm{Im}(R_i)/\mathrm{Re}(R_i) \right),$$

$$\left| R_i \right| = \left(\mathrm{Re}^2(R_i) + \mathrm{Im}^2(R_i) \right)^{1/2}$$

k = number of elastic modes

sp = short period mode

ph = phugoid mode

ξ_i = i^{th} elastic mode

As this example shows, a system's dynamics are dependent on both its eigenvalues and its residues. The eigenvalues determine the natural frequency and damping of each mode. The residues are an indicator of how much each mode of the system contributes to a given output.

Schmidt and Waszak's research involved analyzing a family of configurations of large flexible aircraft similar in geometry to the Rockwell B-1 (see Figure 2.2). The structural rigidity of these configurations was varied by parametrically varying the invacuo-structural vibration frequencies of the two lowest frequency aeroelastic modes of the vehicle. A summary of the analysis of two of the configurations will be presented to illustrate the method.

<u>Example 2.2</u>

Configuration One (Baseline)

This configuration is representative of an unmodified B-1. This configuration received good subjective ratings of the handling qualities † in a tracking task with pilot comments of "Good; no problem". The eigenvalues of this configuration are:

$$\lambda_{sp} = -1.49 \pm j\,2.37 = (\omega_n = 2.799\ \text{(rad/s)}\ ,\ \varsigma = 0.532)$$

$$\lambda_{ph} = -0.0015 \pm j\,0.0672 = (\omega_n = 0.067\ \text{(rad/s)}\ ,\ \varsigma = 0.022)$$

$$\lambda_{\xi_1} = -0.657 \pm j\,13.29 = (\omega_n = 13.31\ \text{(rad/s)}\ ,\ \varsigma = 0.049)$$

$$\lambda_{\xi_2} = -0.46 \pm j\,21.35 = (\omega_n = 21.35\ \text{(rad/s)}\ ,\ \varsigma = 0.022)$$

The impulse residue magnitudes (expressed as a percentage of the total response) of the rigid-body pitch and pitch rate outputs can be found in Figure 2.3.

† The Cooper Harper rating scale is a subjective rating (1 being best and 10 being worst) used to describe vehicle handling qualities in various tasks [12].

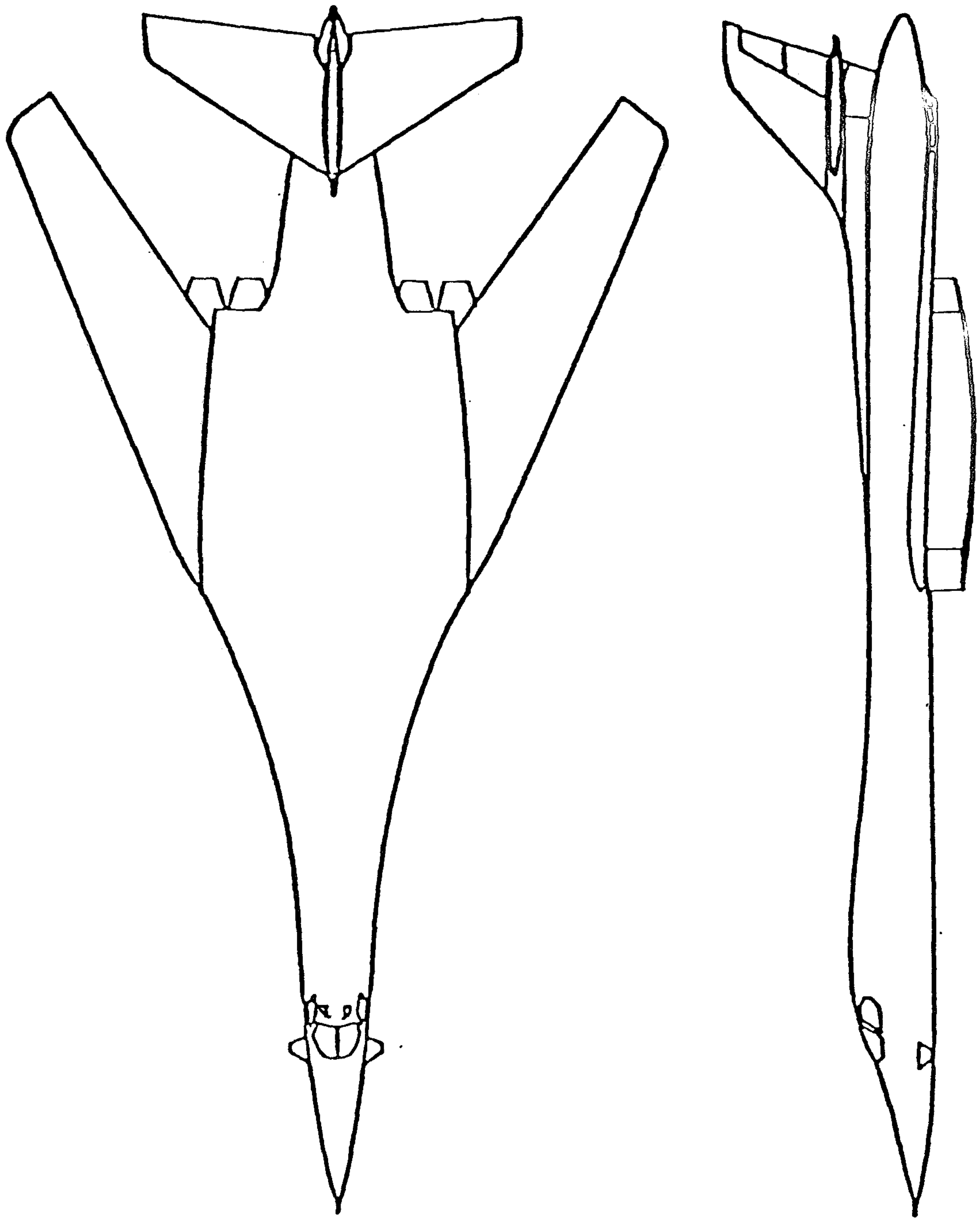

Figure 2.2
Rockwell B-1 Geometry

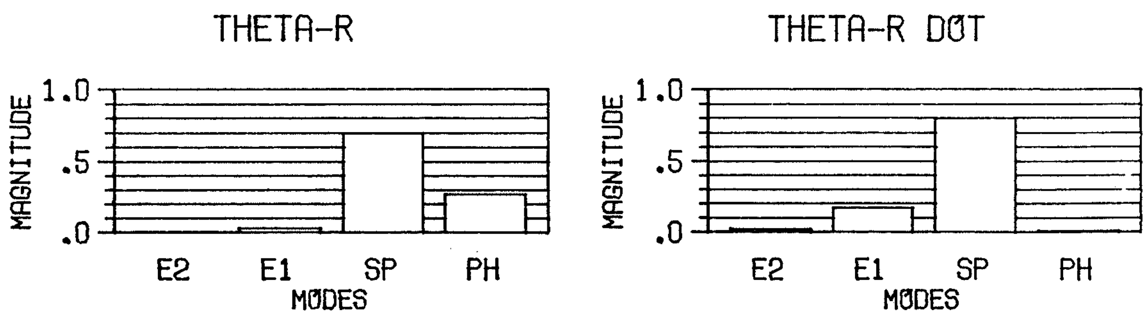

Figure 2.3
Impulse Residue Magnitudes - Configuration One

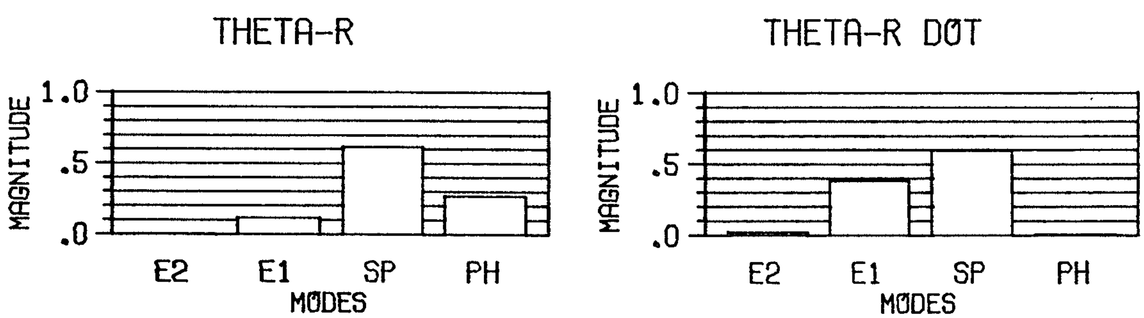

Figure 2.4
Impulse Residue Magnitudes - Configuration Two

Configuration Two

This configuration differs from Configuration One in that it has a lower first aeroelastic mode invacuo frequency. Therefore, this configuration is more flexible than Configuration One. The pilot ratings and tracking scores this configuration recieved (in the tracking task) were worse than those of Configuration One. The pilot comments were "little oscillation; slight control response lag". The eigenvalues of this configuration are:

$$\lambda_{sp} = -1.348 \pm j\ 2.193 = (\omega_n = 2.574\ \text{(rad/s)}\ ,\ \varsigma = 0.524)$$

$$\lambda_{ph} = -5.71 \times 10^{-5} \pm j\ 0.053 = (\omega_n = 0.053\ \text{(rad/s)}\ ,\ \varsigma = 0.001)$$

$$\lambda_{\xi_1} = -0.726 \pm j\ 8.757 = (\omega_n = 8.787\ \text{(rad/s)}\ ,\ \varsigma = 0.083)$$

$$\lambda_{\xi_2} = -0.456 \pm j\ 21.351 = (\omega_n = 21.35\ \text{(rad/s)}\ ,\ \varsigma = 0.021)$$

The impulse residue magnitudes of the rigid-body pitch and pitch rate outputs can be found in Figure 2.4. By comparing the impulse residue magnitudes of these two configurations one can see that Configuration Two, which received a lower pilot rating, has a greater contribution from the first elastic mode in these outputs. For a more detailed explanation of this method the reader is referred to Reference [3].

Schmidt and Waszak summarize their modal analysis research results by stating: "The results of modal analysis indicate when the magnitudes of the modal impulse residues of the aeroelastic modes become large compared to the residue magnitudes of the rigid-body modes for important outputs, the dynamics can change significantly and in such a way that the handling qualities of the vehicle may be degraded."†

† Reference [3], p.85.

Control Synthesis

This analysis leads to one possible approach to the flexible aircraft control synthesis problem. This approach would be to synthesize a control system that would control both the eigenvalue locations and impulse residue magnitudes associated with undesirable modes in certain outputs. For example, reducing the magnitude of impulse residues associated with aeroelastic modes in rigid-body outputs would make the flexible aircraft behave more like a rigid aircraft. This type of control strategy would reduce the undesirable effects of structural flexibility in the aircraft's dynamics.

This motivates interest in a control synthesis technique that will modify both the system's eigenvalues and its impulse residue magnitudes. But, as was shown, impulse residues are a function of the system's eigenvectors. This naturally leads to a control synthesis technique that involves achieving some desired eigenspace in the closed-loop system. This type of control synthesis technique will be referred to as an eigenspace assignment technique.

In the following chapters, two eigenspace assignment techniques will be presented. This report will investigate each of these with regard to level of design complexity, the utility afforded to the designer, and how well each of these techniques enhances the vehicle's dynamics. Each technique will be evaluated, demonstrated in examples, and compared to see how well it meets each of these control synthesis goals.

CHAPTER III
DIRECT EIGENSPACE ASSIGNMENT

In this chapter, a control synthesis technique for directly determining measurement feedback control gains that will yield an achievable eigenspace in the closed-loop will be developed. This technique will be referred to as the Direct Eigenspace Assignment (DEA) technique.

In papers on eigenspace assignment S. Srinathkumar [5,6]; A.N. Andry, E.Y. Shapiro, and J.C. Chung [7]; and T.B. Cunningham [8]; after B.C. Moore [4], have explored the limits of the achievable eigenspace for a system with measurement feedback. For a system that is observable and controllable and has n states, m controls, and l measurements; one can exactly place l eigenvalues and m elements of their associated eigenvectors in the closed-loop system †. If it is desired to place q elements ($m < q < n$) of the l eigenvectors associated with the l desired eigenvalues, the "best" that can be achieved is a least squares fit to the desired eigenvectors.

The following section will present the development of the DEA synthesis technique. Later sections will present a discussion of the achievable eigenspace and three examples that illustrate some of the strong and weak points of DEA.

DEA Technique Development

Given a system

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \qquad \text{(system dynamics)} \tag{3.1a}$$

$$\underline{y} = \mathbf{C}\underline{x} \qquad \text{(system responses)} \tag{3.1b}$$

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† This assumes $l > m$. For a general statement and proof of this property the reader is referred to Reference [5].

$$\underline{z} = \mathbf{M}\underline{x} \quad \text{(system measurements)} \tag{3.1c}$$

$$\underline{u}_c = \mathbf{G}\underline{z} \quad \text{(feedback control law)} \tag{3.1d}$$

$$\underline{u} = \underline{u}_c + \underline{u}_p \quad \text{(total control input)} \tag{3.1e}$$

$$\underline{u}_p = \text{pilot's inputs}$$

where $\underline{x} \in R^n$, $\underline{u} \in R^m$, $\underline{y} \in R^p$, and $\underline{z} \in R^l$. One can express the spectral decomposition of the open-loop system, **A**, by

$$\mathbf{A}\underline{\nu}_{o_i} = \lambda_i \underline{\nu}_{o_i} \qquad i=1,...,n \tag{3.2}$$

where

λ_{o_i} = i^{th} open–loop eigenvalue

$\underline{\nu}_{o_i}$ = i^{th} open–loop eigenvector

The system augmented with the control law is

$$\dot{\underline{x}} = (\mathbf{A} + \mathbf{BGM})\underline{x} + \mathbf{B}\underline{u}_p \tag{3.3}$$

and therefore the spectral decomposition of the augmented (closed-loop) system is given by

$$(\mathbf{A} + \mathbf{BGM})\underline{\nu}_{c_i} = \lambda_{c_i}\underline{\nu}_{c_i} \qquad i=1,...,n \tag{3.4}$$

Arranging this, one can obtain

$$\mathbf{BGM}\underline{\nu}_{c_i} = \lambda_{c_i}\underline{\nu}_{c_i} - \mathbf{A}\underline{\nu}_{c_i} \tag{3.5}$$

In Equation (3.5) let

$$\underline{w}_i \equiv \mathbf{GM}\underline{\nu}_{c_i} \tag{3.6}$$

and then solve for $\underline{\nu}_{c_i}$. One obtains

$$\mathbf{B}\underline{w}_i = \lambda_{c_i}\underline{\nu}_{c_i} - \mathbf{A}\underline{\nu}_{c_i}$$

$$= (\lambda_{c_i}\mathbf{I}_n - \mathbf{A})\underline{\nu}_{c_i}$$

and finally

$$\underline{\nu}_{c_i} = (\lambda_{c_i}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\underline{w}_i \tag{3.7}$$

= eigenvector achievable in the closed-loop system for a given λ_{c_i} and $\underline{w}_i$

$$\equiv \underline{\nu}_{a_i}$$

This equation describes the achievable i^{th} eigenvector of the closed-loop system as a function of the closed-loop eigenvalue, λ_{c_i}, and $\underline{w}_i$.

If one could calculate a value for $\underline{w}_i$ that would make the achievable eigenvector, $\underline{\nu}_{a_i}$, as close as possible to some desired eigenvector, $\underline{\nu}_{d_i}$, it could be used to determine a gain matrix that would yield this eigenspace. This is done by defining a cost function associated with the i^{th} mode of the system (assume for now that λ_{c_i}, $\underline{\nu}_{d_i}$, and $\underline{\nu}_{a_i}$ are real)

$$J_i = \frac{1}{2}\,(\underline{\nu}_{a_i} - \underline{\nu}_{d_i})^T \mathbf{Q}_{d_i}(\underline{\nu}_{a_i} - \underline{\nu}_{d_i}) \qquad i=1,\ldots,l \tag{3.8}$$

where

$\underline{\nu}_{a_i}$ = i^{th} achievable eigenvector associated with eigenvalue λ_i

$\underline{\nu}_{d_i}$ = i^{th} desired eigenvector

$\mathbf{Q}_{d_i}$ = i^{th} n-by-n symmetric positive semi-definite weighting matrix on eigenvector elements

This cost function represents the error between the achievable eigenvector and the desired eigenvector weighted by the matrix $\mathbf{Q}_d$.

By substituting the equation for the achievable closed-loop system eigenvector, Equation (3.7), into the cost function, Equation (3.8), one obtains

$$J_i = \frac{1}{2}\,(\underline{\nu}_{a_i} - \underline{\nu}_{d_i})^T \mathbf{Q}_{d_i}(\underline{\nu}_{a_i} - \underline{\nu}_{d_i})$$

$$= \frac{1}{2}\,(([\lambda_i \mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B}\underline{w}_i - \underline{\nu}_{d_i})^T \mathbf{Q}_{d_i}([\lambda_i \mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B}\underline{w}_i - \underline{\nu}_{d_i}))$$

$$= \frac{1}{2}\,(\underline{w}_i^T \mathbf{B}^T[\lambda_i \mathbf{I}_n - \mathbf{A}]^{-T}\mathbf{Q}_{d_i}[\lambda_i \mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B}\underline{w}_i$$

$$- \underline{\nu}_{d_i}^T \mathbf{Q}_{d_i}[\lambda_i \mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B}\underline{w}_i - \underline{w}_i^T \mathbf{B}^T[\lambda_i \mathbf{I}_n - \mathbf{A}]^{-T}\mathbf{Q}_{d_i}\underline{\nu}_{d_i}$$

$$+ \underline{\nu}_{d_i}^T \mathbf{Q}_{d_i}\underline{\nu}_{d_i}\,)$$

To find the $\underline{w}_i$ that will minimize J_i, take the gradient of J_i with respect to $\underline{w}_i$ and set it equal to zero.

$$\frac{\partial J_i}{\partial \underline{w}_i} = 0$$

$$= \underline{w}_i^T \mathbf{B}^T[\lambda_i \mathbf{I}_n - \mathbf{A}]^{-T}\mathbf{Q}_{d_i}[\lambda_i \mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B}$$

$$- \underline{\nu}_{d_i}^T \mathbf{Q}_{d_i}[\lambda_i \mathbf{I}_n - \mathbf{A}]^{-1}\mathbf{B} \tag{3.9}$$

Now solve Equation (3.9) for $\underline{w}_i^T$

$$\underline{w}_i^T = \underline{\nu}_{d_i}^T \mathbf{Q}_{d_i}\mathbf{L}_i[\mathbf{L}_i^T\, \mathbf{Q}_{d_i}\, \mathbf{L}_i]^{-1} \tag{3.10}$$

where

$\mathbf{L}_i = (\lambda_{d_i}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}$

λ_{d_i} = i^{th} desired eigenvalue

$\underline{\nu}_{d_i}$ = i^{th} desired eigenvector

$\mathbf{Q}_{d_i}$ = i^{th} n–by–n symmetric positive semi–definite

weighting matrix on eigenvector elements

Note that in this development λ_{d_i} cannot belong to the spectrum of $\mathbf{A}$.

By examing Equation (3.6), one can see that given values of $\underline{w}_i$ and $\underline{\nu}_{a_i}$ for $i=1, \ldots, l$ one can determine the feedback gain matrix, $\mathbf{G}$. This is done by using Equation (3.10) to calculate $\underline{w}_i$'s for $i=1,\ldots,l$. Concatenate the individual $\underline{w}_i$'s columnwise to form $\mathbf{W}$.

$$\mathbf{W} = [\ \underline{w}_1 \mid \underline{w}_2 \mid \ \cdots \ \mid \underline{w}_l\ ]$$

Use Equation (3.7) to calculate the closed-loop system achievable eigenvalues $\underline{\nu}_{a_i}$'s for $i=1, \ldots, l$. Concatenate the individual, $\underline{\nu}_{a_i}$'s, columnwise to form $\mathbf{V}$.

$$\mathbf{V} = [\ \underline{\nu}_{a_1} \mid \underline{\nu}_{a_2} \mid \ \cdots \ \mid \underline{\nu}_{a_l}\ ]$$

One can now solve for the gain matrix that will yield the l desired eigenvalues and l achievable eigenvectors in the closed-loop system by applying Equation (3.6).

$$[\ \underline{w}_1 \mid \ \cdots \ \mid \underline{w}_l\ ] = \mathbf{GM}[\ \underline{\nu}_{a_1} \mid \ \cdots \ \mid \underline{\nu}_{a_l}\ ]$$

$$\mathbf{W} = \mathbf{GMV}$$

Solving for $\mathbf{G}$, one obtains

$$\mathbf{G} = \mathbf{W}[\mathbf{MV}]^{-1} \tag{3.11}$$

The augmented system is given by

$$\dot{\underline{x}} = \mathbf{A}_{aug}\underline{x} + \mathbf{B}\underline{u}_p \tag{3.12}$$

where

$$\mathbf{A}_{aug} = \mathbf{A} + \mathbf{BGM} \tag{3.13}$$

The augmented system transfer functions are obtained from

$$s\underline{x}(s) = \mathbf{A}_{aug}\underline{x}(s) + \mathbf{B}\underline{u}_p(s)$$

$$(s\mathbf{I}_n - \mathbf{A}_{aug})\underline{x}(s) = \mathbf{B}\underline{u}_p(s)$$

$$\underline{x}(s) = (s\mathbf{I}_n - \mathbf{A}_{aug})^{-1}\mathbf{B}\underline{u}_p(s) \tag{3.14}$$

The system responses are given by

$$\underline{y}(s) = \mathbf{C}\underline{x}(s)$$

therefore,

$$\underline{y}(s) = \mathbf{C}\,(s\mathbf{I}_n - \mathbf{A}_{aug})^{-1}\mathbf{B}\,\underline{u}_p(s) \tag{3.15}$$

The solution method is similar for the case of control feed through in the measurements (i.e. $\underline{z} = \mathbf{M}\underline{x} + \mathbf{N}\underline{u}$) and for the case of complex eigenvalues and eigenvectors. The DEA control synthesis technique developments for both of these cases are given in Appendix A.1.

Achievable Eigenvectors

As was shown in the DEA development, Equation (3.7)

$$\underline{\nu}_{a_i} = (\lambda_{c_i}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\underline{w}_i \tag{3.7}$$

describes the achievable eigenspace of the system. In this equation, $(\lambda_{c_i}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}$ is an n-by-m matrix and $\underline{w}_i$ is an m-by-1 vector. By examing this equation, one can see that the number of control variables (m) determines the dimension of the subspace in which the achievable eigenvectors must reside.

If the desired eigenvector, $\underline{\nu}_{d_i}$, cannot be achieved, the closest one can come to it (the achievable eigenvector) is a projection of $\underline{\nu}_{d_i}$ onto the subspace spanned by the columns of $(\lambda_i \mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}$ (see Figure 3.1).

If one desires to exactly place a larger number of eigenvector elements, the rank of **B** must be increased (i.e. one must have more controls). If one is interested in having control over a larger number of modes (exactly placing eigenvalues and a least squares fit for the eigenvectors), the rank of **M** must be increased (i.e. more measurements).

System Scaling

In general, the state vector must be transformed and the units changed in the following way to aid in the physical interpretation of the eigenvector elements. Let $\underline{x}_s$ be the new state definition, where

$$\underline{x}_s = T\underline{x}$$

then

$$\dot{\underline{x}}_s = T\mathbf{A}T^{-1}\underline{x}_s + T\mathbf{B}\underline{u}$$

$$\underline{y} = \mathbf{C}T^{-1}\underline{x}_s$$

$$\underline{z} = \mathbf{M}T^{-1}\underline{x}_s$$

where

$$T = \text{diagonal matrix of scaling elements}$$

For the large flexible aircraft examples presented in the following section, T is chosen to be

$$T = \text{diag}(1.0,\ 1.0,\ 1/U_o,\ 1.0,\ \phi'_1,\ \phi'_2,\ \phi'_1,\ \phi'_2)$$

with

U_o = forward flight speed (ft/sec)

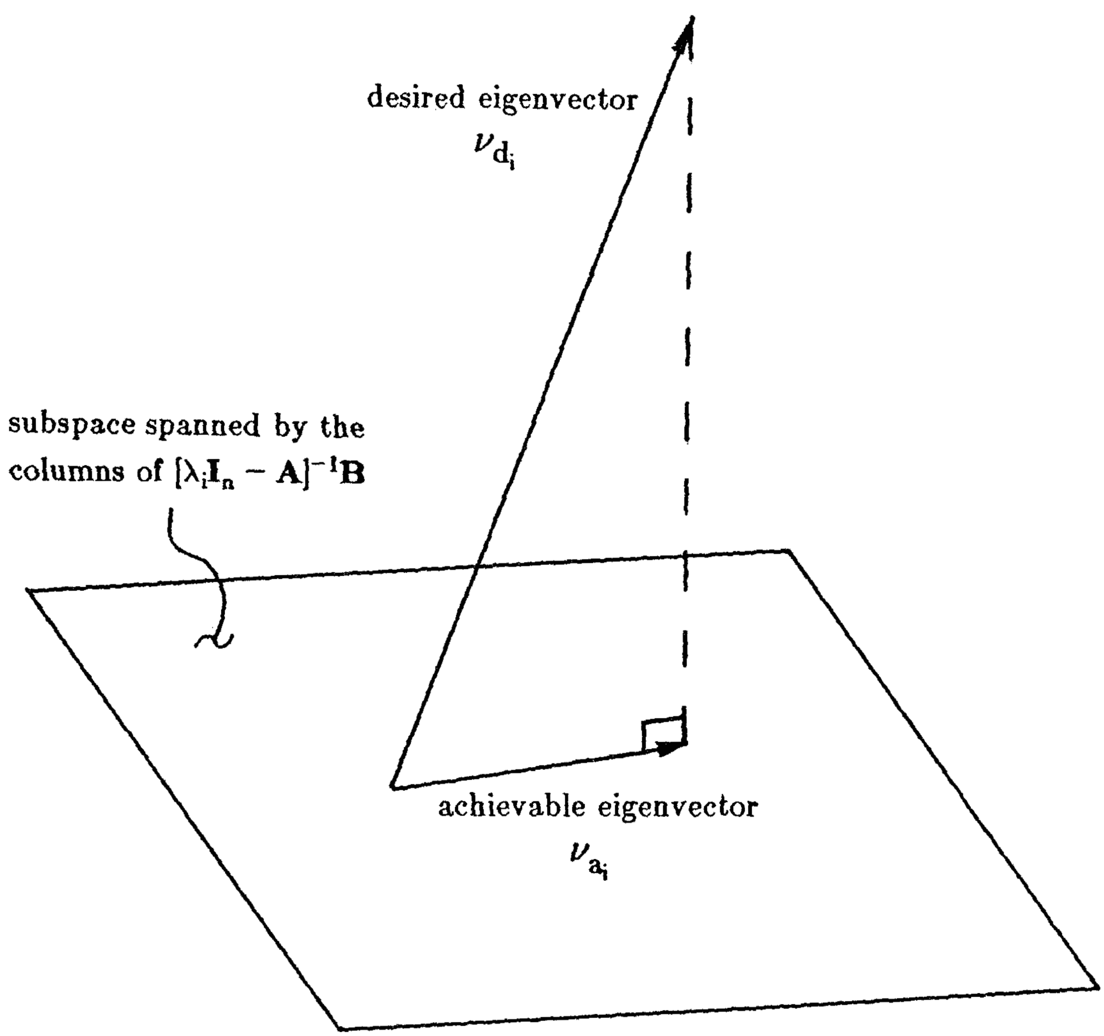


Figure 3.1
Achievable Subspace

ϕ'_i = mode slope of the i^{th} elastic mode

at the cockpit

The scaling matrix used in the examples is given in Appendix A.2. All of the example results are presented in the scaled units. The reader should note that a system's responses, eigenvalues, and impulse residues are independent of system transformations, but the eigenvectors are not. Therefore, the choice of the system scaling will effect the choice of the desired eigenvectors and the synthesis results.

DEA Examples

Three examples will be presented to demonstrate the method. All three will use the same model, the longitudinal dynamics of a large flexible aircraft (Schmidt and Waszak's Configuration Two). The model is based on a steady-state flight condition of forward cruise speed equaling 949 feet/second at sea level. It includes the four standard rigid-body degrees of freedom and the two lowest frequency symmetric aeroelastic modes of the vehicle. These two aeroelastic modes are the first fuselage bending mode and the second fuselage bending mode.

System

The unscaled model is as follows:

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \qquad \text{(system dynamics)}$$

$$\underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u} \qquad \text{(system responses)}$$

$$\underline{z} = \mathbf{M}\underline{x} + \mathbf{N}\underline{u} \qquad \text{(system measurements)}$$

$$\underline{u} = \underline{u}_c + \underline{u}_p \qquad \text{(total control input)}$$

$$\underline{u}_c = \mathbf{G}\underline{z} \qquad \text{(feedback control law)}$$

$$\underline{u}_p = \text{pilot's input}$$

States

$$\underline{x}^T = (\alpha \, , \dot{\theta}_r \, , u_f \, , \theta_r \, , \xi_1 \, , \xi_2 \, , \dot{\xi}_1 \, , \dot{\xi}_2)$$

where

α = angle of attack (radians)

$\dot{\theta}_r$ = rigid–body pitch rate (radians/sec)

u_f = forward velocity (ft/sec)

θ_r = rigid–body pitch angle (radians)

ξ_1 = mode one generalized deflection (dimensionless)

ξ_2 = mode two generalized deflection (dimensionless)

$\dot{\xi}_1$ = mode one generalized deflection rate (1/sec)

$\dot{\xi}_2$ = mode two generalized deflection rate (1/sec)

Controls

$$\underline{u}_c^T = (\delta_e \, , \delta_{cv})$$

$$\underline{u}_p^T = (\delta_e \, , 0.0)$$

where

δ_e = elevator deflection (radians)

δ_{cv} = forward control vane deflection (radians)

Outputs

$$\underline{y}^T = (u_f , \theta_r , \dot{\theta}_r , \gamma , \theta_t , \dot{\theta}_t , n_{z_p})$$

where

u_f = forward velocity (ft/sec)

θ_r = rigid–body pitch angle (radians)

$\dot{\theta}_r$ = rigid–body pitch rate (radians/sec)

γ = flight path angle (radians)

θ_t = total pitch angle (radians)

$\dot{\theta}_t$ = total pitch rate (radians/sec)

n_{z_p} = plunge acceleration at the cockpit (g' s)

The total pitch angle is the pitch attitude measured at the cockpit. It is the rigid-body pitch attitude plus contributions of the local structural deflections (Figure 3.2). The total pitch angle is defined by the equation

$$\theta_t = \theta_r - \sum_{i=1}^{k} \xi_i \phi'_i(l_x)$$

where

k = number of aeroelastic modes

l_x = distance between the c.g. and station x (ft)

ϕ'_i = mode slope of the i^{th} elastic mode (ft/ft)

ξ_i = generalized coordinate of the i^{th} elastic mode

The plunge acceleration at location x, n_{z_x}, is defined by

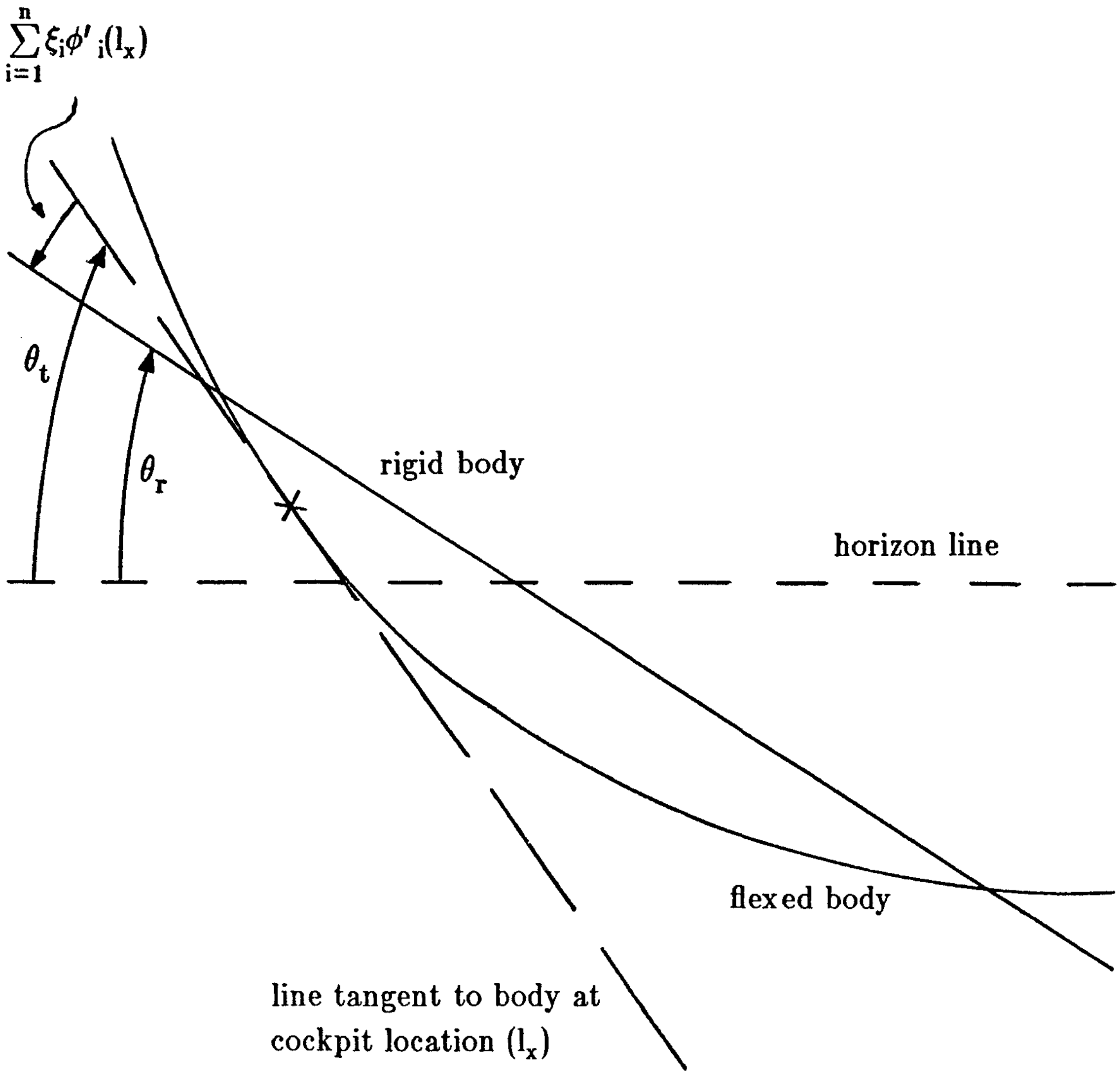


Figure 3.2
Rigid and Elastic Pitch Angles

$$n_{z_x} = \frac{1.0}{g} \left[U_o \dot{\gamma} + l_x \ddot{\theta}_r - \sum_{i=1}^{k} \ddot{\xi}_i \phi_i(l_x) \right]$$

where

g = gravitational acceleration (ft/sec^2)

U_o = cruise velocity (ft/sec)

l_x = distance between the c.g. and station x (ft)

ϕ_i = mode shape of the ith elastic mode (ft)

The flight path angle, γ, is given by

$$\gamma = \theta_r - \alpha$$

Measurements

The measurements considered for feedback are

$$\underline{z}^T = (\theta_t \, , \, \dot{\theta}_t \, , \, n_{z_p} \, , \, n_{z_t})$$

where

θ_t = total pitch angle measured at the cockpit (radians)

$\dot{\theta}_t$ = total pitch rate measured at the cockpit (radians/sec)

n_{z_p} = plunge acceleration at the cockpit (g' s)

n_{z_t} = plunge acceleration at an aft fuselage station (g' s)

For the examples, the aft fuselage station was chosen to be located at $l_x = -28.23$ ft. The open-loop system eigenvalues, the eigenvalues of **A**, are:

$$\lambda_{sp} = -1.348 \pm j\, 2.193 = (\omega_n = 2.574 \text{ (rad/s)} \, , \, \varsigma = 0.524)$$

$$\lambda_{ph} = -5.71 \times 10^{-5} \pm j\, 0.053 = (\omega_n = 0.053 \text{ (rad/s)}\ ,\ \varsigma = 0.001)$$

$$\lambda_{\xi_1} = -0.726 \pm j\, 8.757 = (\omega_n = 8.787 \text{ (rad/s)}\ ,\ \varsigma = 0.083)$$

$$\lambda_{\xi_2} = -0.456 \pm j\, 21.351 = (\omega_n = 21.35 \text{ (rad/s)}\ ,\ \varsigma = 0.021)$$

The open-loop system eigenvectors are given in Table 3.1. In this table, the eigenvectors have been scaled such that the first element is unity. The open-loop system **A**, **B**, **C**, **D**, **M**, and **N** matrices are given in Appendix A.2. The open-loop system impulse residue magnitudes for each output due to each mode are shown in Figure 3.3. The residue magnitudes for each response are normalized for plotting such that the sum of the residue magnitudes for each response is unity. By examining the relative impulse residue magnitudes one can note several things. The first elastic mode contributes significantly to the rigid-body pitch rate, $\dot{\theta}_r$. The first elastic mode also contributes to the rigid-body pitch angle, θ_r. The elastic modes contribute significantly to the total pitch rate, $\dot{\theta}_t$, total pitch angle, θ_t, and plunge acceleration at the cockpit, n_{z_p}, outputs.

The open-loop system time responses due to a unit step in $\underline{u}_p$, with initial conditions $\underline{x}(0) = \underline{0}$, are shown in Figure 3.4. The rigid-body pitch rate time response also shows the first elastic modes contribution to this output. The high frequency ripple is due to the first elastic mode.

Example 3.1 : Four Measurements

This control synthesis has two design objectives - to improve the aircraft's rigid-body dynamics and to reduce the undesirable effects of flexibility on the rigid-body dynamics. For this example $\dim(\underline{x})=n=8$, $\dim(\underline{u})=m=2$, and $\dim(\underline{z})=l=4$; therefore using DEA one can modify four eigenvalue/eigenvector pairs.

Specification of Synthesis Parameters

Desired Eigenvalues. For this control synthesis one may specify four real desired eigenvalues or two complex conjugate eigenvalue pairs. By examining

Table 3.1
Open-loop Eigenvectors

CONFIGURATION TWO : OPEN LOOP

EIGENVALUES

1:	-1.3479	+	J	2.1929
2:	-1.3479	-	J	2.1929
3:	-.0001	+	J	.0527
4:	-.0001	-	J	.0527

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

1		2	
(1.0000E+00,	0)	(1.0000E+00,	0)
(2.2949E+00,	-2.6416E+02)	(2.2949E+00,	2.6416E+02)
(2.1545E-02,	-3.1488E+02)	(2.1545E-02,	3.1488E+02)
(8.9153E-01,	-2.5741E+01)	(8.9153E-01,	2.5741E+01)
(2.8846E-01,	-3.3438E+02)	(2.8846E-01,	3.3438E+02)
(5.1928E-02,	-1.7132E+02)	(5.1928E-02,	1.7132E+02)
(7.4252E-01,	-2.1280E+02)	(7.4252E-01,	2.1280E+02)
(1.3366E-01,	-4.9746E+01)	(1.3366E-01,	4.9746E+01)

3		4	
(1.0000E+00,	0)	(1.0000E+00,	0)
(1.7976E-01,	-1.9265E+02)	(1.7976E-01,	1.9265E+02)
(2.1292E+00,	-1.7933E+02)	(2.1292E+00,	1.7933E+02)
(3.4097E+00,	-2.8271E+02)	(3.4097E+00,	2.8271E+02)
(2.5276E-01,	3.6763E-01)	(2.5276E-01,	-3.6763E-01)
(5.2150E-02,	-1.7985E+02)	(5.2150E-02,	1.7985E+02)
(1.3326E-02,	-2.6957E+02)	(1.3326E-02,	2.6957E+02)
(2.7494E-03,	-8.9783E+01)	(2.7494E-03,	8.9783E+01)

State vector (unscaled): $\mathbf{x}^T = (\ \alpha\ ,\ \dot{\theta}_r\ ,\ u_f\ ,\ \theta_r\ ,\ \xi_1\ ,\ \xi_2\ ,\ \dot{\xi}_1\ ,\ \dot{\xi}_2\ )$

Table 3.1, concluded

```
EIGENVALUES
 5:      -.7264 + J    8.7584
 6:      -.7264 - J    8.7584
 7:      -.4564 + J   21.3506
 8:      -.4564 - J   21.3506


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                            5                               6
(  1.0000E+00,             0)      (  1.0000E+00,             0)
(  7.7404E+00,  6.8135E+01)      (  7.7404E+00, -6.8135E+01)
(  6.2283E-03,  7.1265E+01)      (  6.2283E-03, -7.1265E+01)
(  8.8075E-01, -2.6606E+01)      (  8.8075E-01,  2.6606E+01)
(  8.8783E+00, -2.3329E+02)      (  8.8783E+00,  2.3329E+02)
(  2.4390E-01, -2.1916E+02)      (  2.4390E-01,  2.1916E+02)
(  7.8027E+01, -1.3855E+02)      (  7.8027E+01,  1.3855E+02)
(  2.1435E+00, -1.2442E+02)      (  2.1435E+00,  1.2442E+02)

                            7                               8
(  1.0000E+00,             0)      (  1.0000E+00,             0)
(  2.0127E+01,  6.2180E+01)      (  2.0127E+01, -6.2180E+01)
(  2.6446E-03,  7.2886E+01)      (  2.6446E-03, -7.2886E+01)
(  9.4247E-01, -2.9045E+01)      (  9.4247E-01,  2.9045E+01)
(  3.3014E+00,  2.3713E+02)      (  3.3014E+00, -2.3713E+02)
(  6.0949E+01,  2.8848E+02)      (  6.0949E+01, -2.8848E+02)
(  7.0502E+01, -3.1647E+01)      (  7.0502E+01,  3.1647E+01)
(  1.3016E+03,  1.9709E+01)      (  1.3016E+03, -1.9709E+01)
```

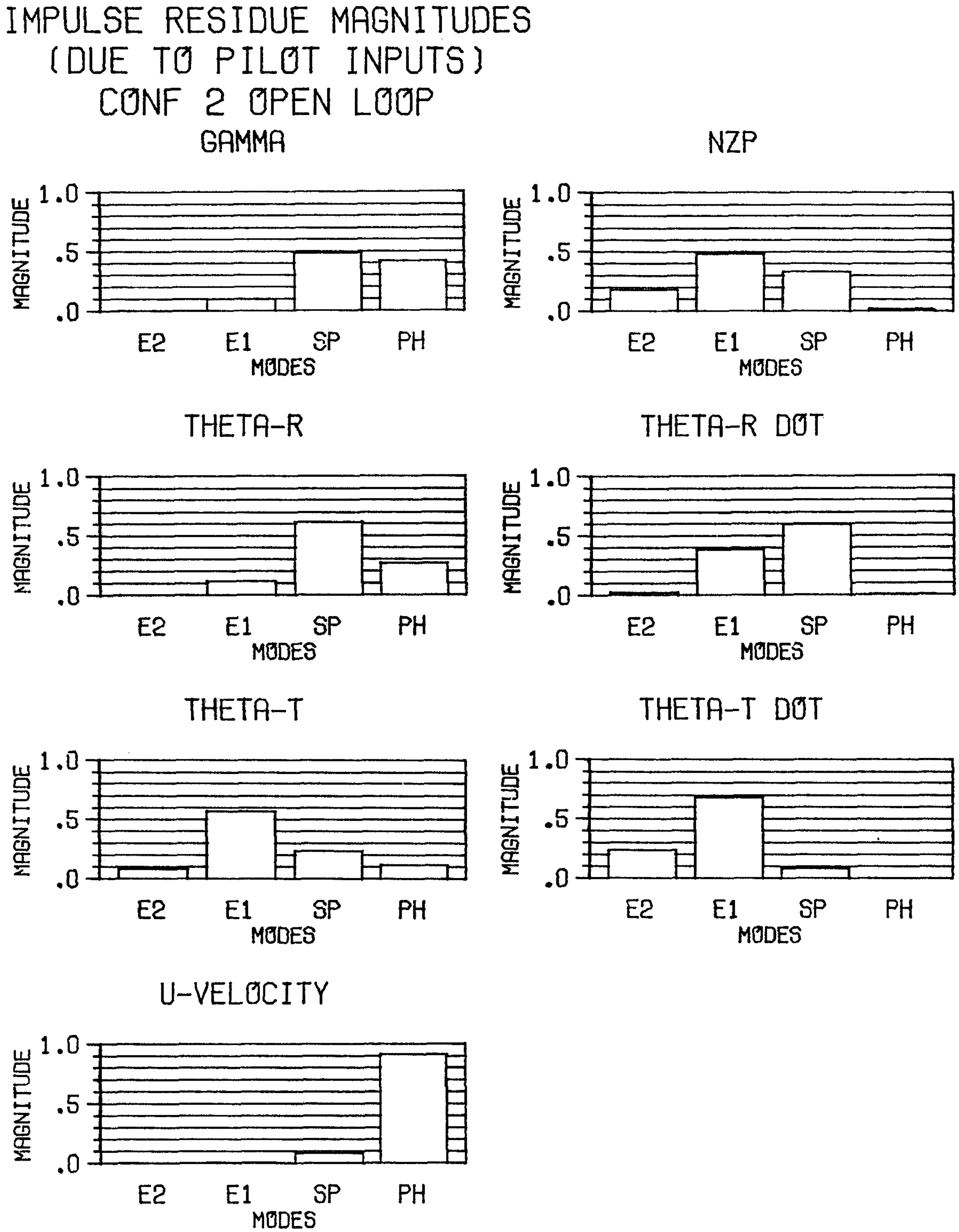


Figure 3.3
Impulse Residue Magnitudes - Open-Loop

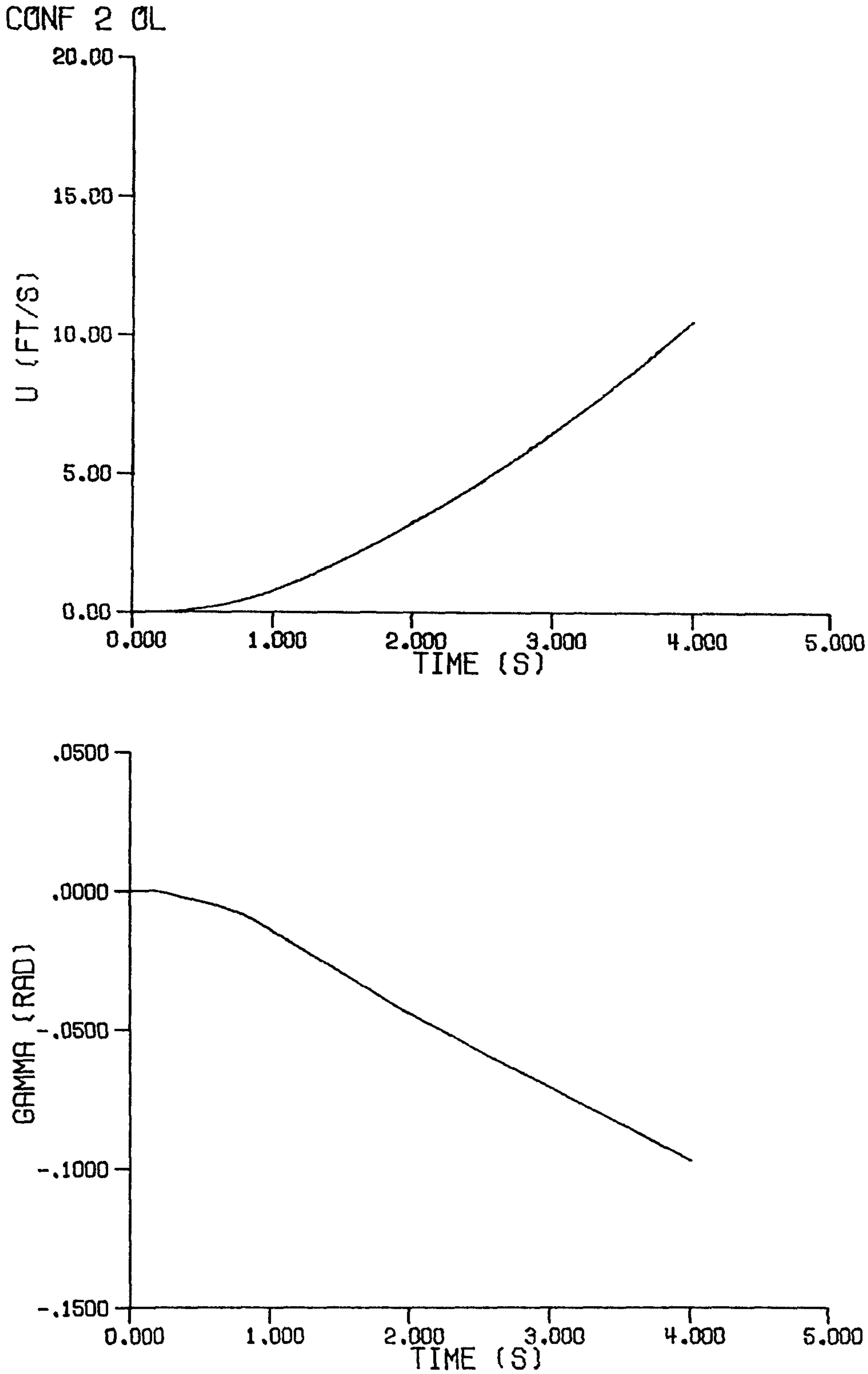
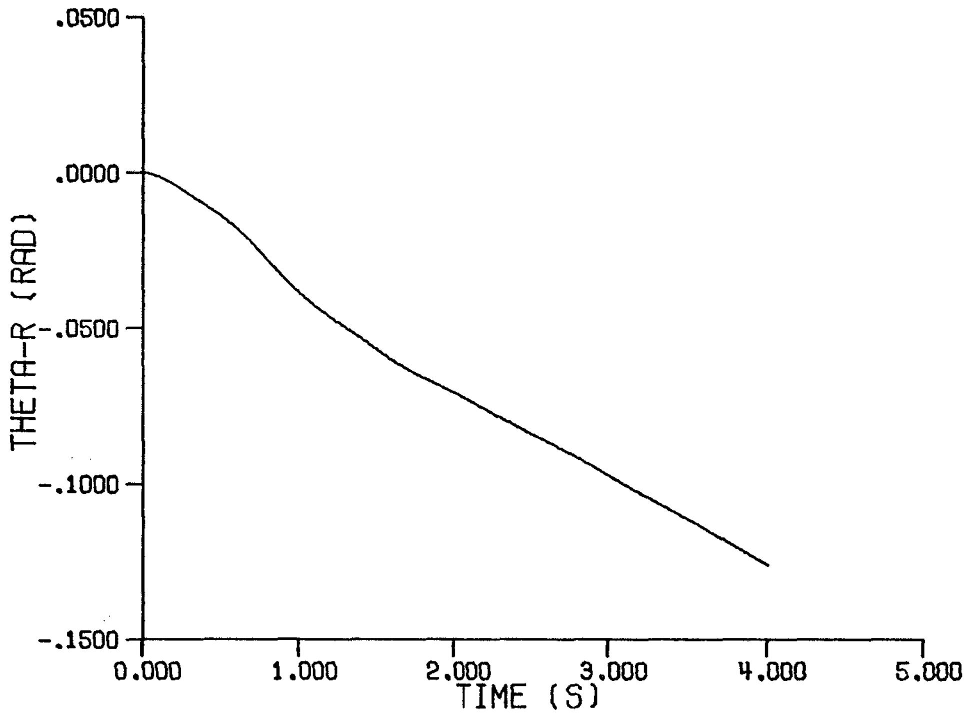


Figure 3.4
Output Time Responses - Open-loop



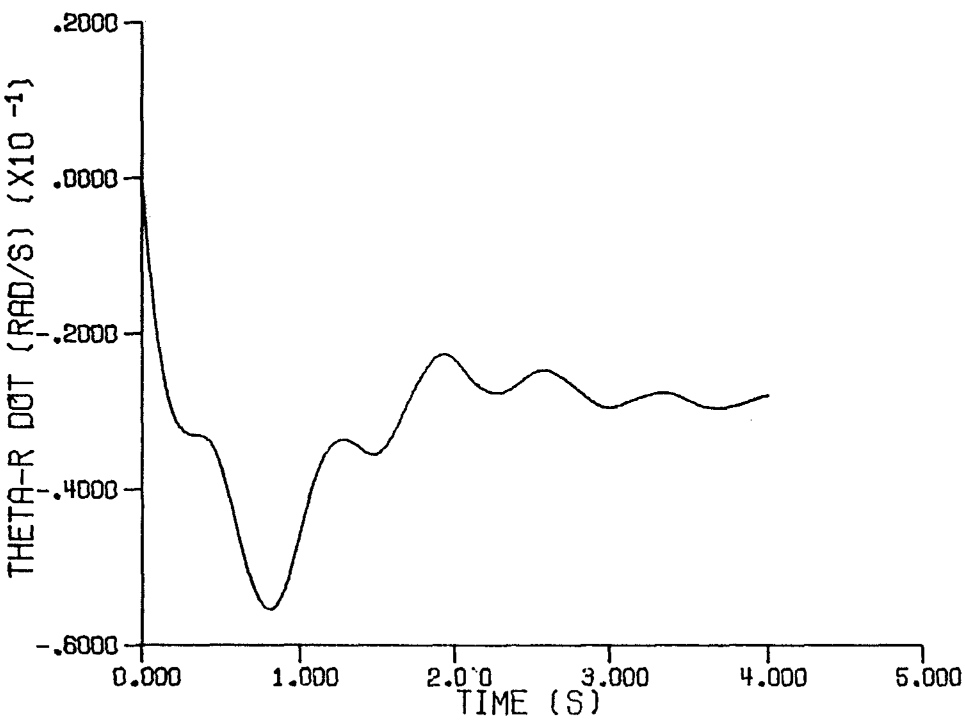


Figure 3.4, continued

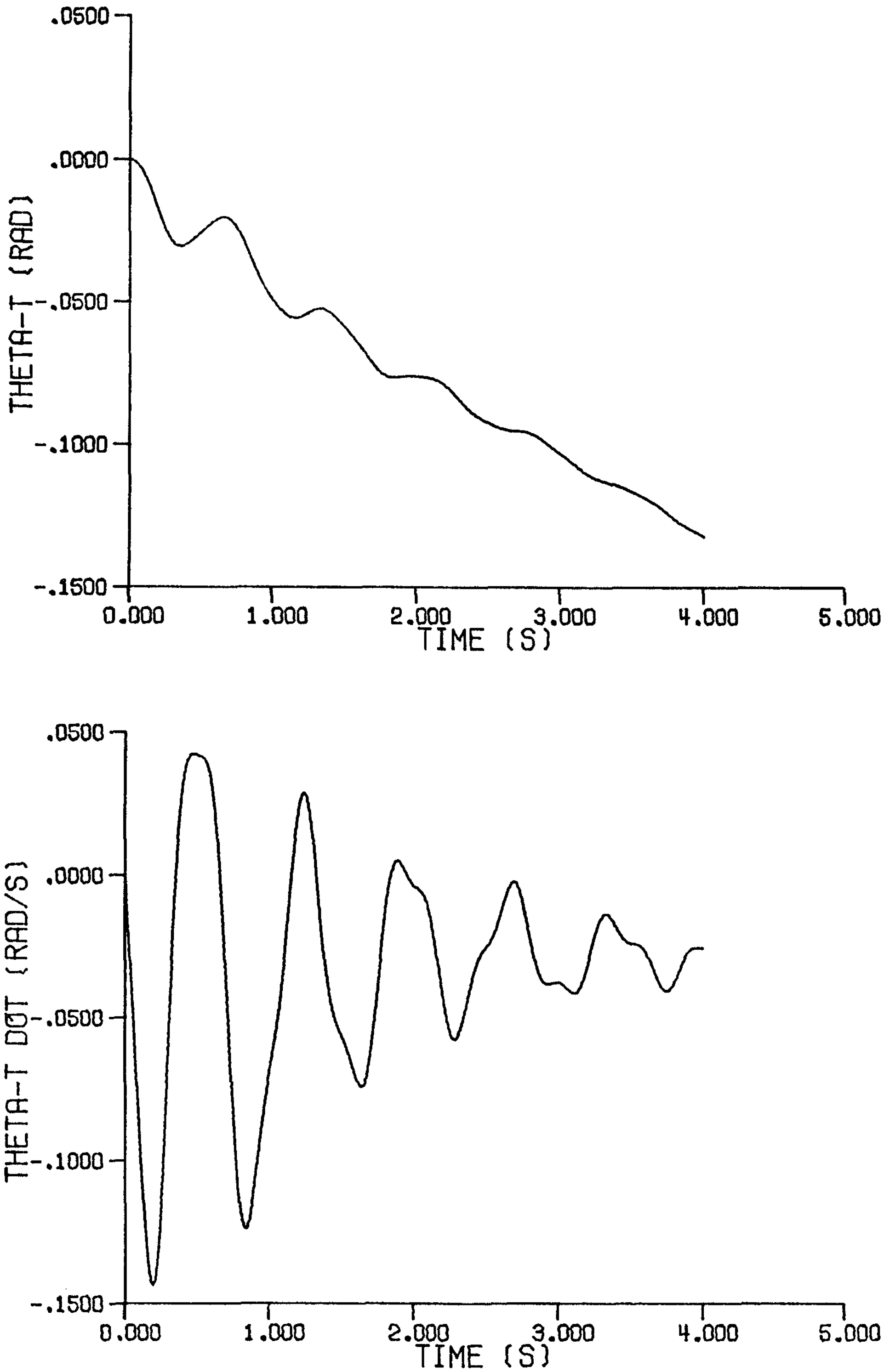


Figure 3.4, continued

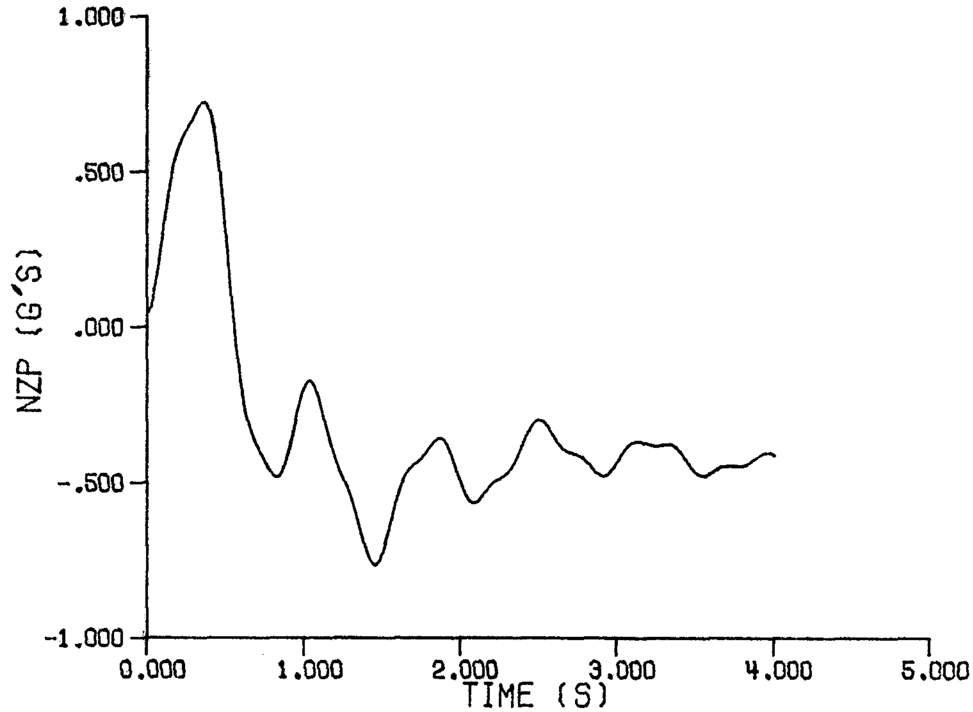


Figure 3.4, concluded

the open-loop impulse residue magnitudes, one can see that the rigid-body dynamics are dominated by the short period mode, and the first elastic mode contributes much more than the second elastic mode to the overall system dynamics. Therefore, for this synthesis, choose the short period and first elastic eigenvalues as the ones to be specified.

The desired short period complex conjugate eigenvalue pair will be placed at the same location as in the "more rigid" baseline configuration. In an attempt to reduce the contribution of the first elastic mode to the overall system dynamics, the first elastic complex conjugate eigenvalue pair will be placed at a location such that the mode's natural frequency is the same as the open-loop natural frequency and the mode has increased damping.

Therefore choose:

$$\lambda_{d_{1,2}} = \lambda_{d_{sp}} = -\,1.49 \pm j\,2.37 = (\omega_n = 2.799\ (\text{rad/s})\ ,\ \varsigma = 0.532)$$

$$\lambda_{d_{3,4}} = \lambda_{d_{\xi_1}} = -\,1.758 \pm j\,8.611 = (\omega_n = 8.789\ (\text{rad/s})\ ,\ \varsigma = 0.200)$$

Desired Eigenvectors. Specify desired eigenvectors for the short period and first elastic modes. Choosing proper desired eigenvectors is very important. For this example, they must be chosen such that the closed-loop system will retain an "aircraft like" eigenspace.

In order to make the flexible aircraft behave more like a rigid aircraft, the desired eigenvectors are chosen in such a way as to eliminate the elastic mode's contribution to the rigid-body pitch angle and rigid-body pitch rate outputs. Therefore, choose the elements of the desired eigenvectors so as to reduce the impulse residue magnitudes associated with the flexible modes in the rigid-body degrees of freedom, and to reduce the impulse residue magnitudes associated with the rigid-body modes in the flexible degrees of freedom. One way this can be accomplished is by choosing the elements of the short period eigenvector that are associated with the (scaled) states ξ_1, ξ_2, $\dot{\xi}_1$, and $\dot{\xi}_2$ to have small magnitudes. The desired short period eigenvector element associated with $\dot{\theta}_r$ is chosen to be unit magnitude as a reference value.

The elements of the desired first elastic eigenvector that are associated with the (scaled) rigid-body states u_f, θ_r, γ, and $\dot{\theta}_r$ are chosen to have small magnitudes. The element associated with the state ξ_1 is chosen to be unit magnitude as a reference value. The phase relationships of these desired eigenvectors are arbitrarily chosen to be similar to those of the "more rigid" baseline configuration's short period and first elastic eigenvectors, respectively. Values chosen for the desired eigenvectors are given in Table 3.2. In this table, the * denotes elements that are not weighted in the cost function. Therefore, the desired value for these elements is taken as arbitrary. The weighting on each of the placed elements is chosen to be unity.

Calculation of $\underline{w}_i$*'s.* Values for $\underline{w}_i$ $i=1,...,l$ were calculated using Equation (3.10) to be:

$$\underline{w}_{1,2}^T = \underline{w}_{sp}^T = (-0.0550 - j\ 0.0288\ , -1.606 - j\ 0.7783\ )$$

$$\underline{w}_{3,4}^T = \underline{w}_{\xi_1}^T = (0.0301 + j\ 0.0174\ , 0.0221 - j\ 1.099\ )$$

Achievable Eigenvectors. The achievable short period and first elastic eigenvectors were calculated using Equation (3.7). These eigenvectors are given in Table 3.3. Note that in this table the eigenvectors have been scaled such that the first element is unity.

Closed-Loop System Analysis

The gain matrix yielding the achievable eigenspace in the closed-loop system and the closed-loop system matrices for this control synthesis are given in Appendix A.3. The augmented system eigenvalues, the eigenvalues of $\mathbf{A}_{aug}$ are:

$$\lambda_{sp} = -\ 1.49\ \pm\ j\ 2.37\ = (\omega_n = 2.799\ \text{(rad/s)}\ ,\ \varsigma = 0.532)$$

$$\lambda_{ph1} = +\ 0.0117$$

$$\lambda_{ph2} = +\ 0.3273$$

$$\lambda_{\xi_1} = -\ 1.758\ \pm\ j\ 8.611\ = (\omega_n = 8.789\ \text{(rad/s)}\ ,\ \varsigma = 0.200)$$

Table 3.2
Desired Eigenvectors - Example 3.1

```
DESIRED EIGENVECTORS

EIGENVALUES
 1:     -1.4900 + J     2.3700
 2:     -1.4900 - J     2.3700
 3:     -1.7577 + J     8.6109
 4:     -1.7577 - J     8.6109


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                                1                                     2
(      *     ,       *      )       (      *     ,       *      )
(  1.0000E+00,  1.0000E+02)       (  1.0000E+00, -1.0000E+02)
(      *     ,       *      )       (      *     ,       *      )
(      *     ,       *      )       (      *     ,       *      )
(  5.0000E-03,  3.0000E+01)       (  5.0000E-03, -3.0000E+01)
(  5.0000E-04, -1.7000E+02)       (  5.0000E-04,  1.7000E+02)
(  1.0000E-02,  1.5000E+02)       (  1.0000E-02, -1.5000E+02)
(  5.0000E-03, -5.0000E+01)       (  5.0000E-03,  5.0000E+01)

                                3                                     4
(  1.0000E-03,           0)       (  1.0000E-03,           0)
(  1.0000E-02,  6.0000E+01)       (  1.0000E-02, -6.0000E+01)
(  1.0000E-04,  7.0000E+01)       (  1.0000E-04, -7.0000E+01)
(  1.0000E-03, -3.0000E+01)       (  1.0000E-03,  3.0000E+01)
(      *     ,       *      )       (      *     ,       *      )
(      *     ,       *      )       (      *     ,       *      )
(  1.0000E+00,  1.4000E+02)       (  1.0000E+00, -1.4000E+02)
(      *     ,       *      )       (      *     ,       *      )
```

State vector (unscaled): $\mathbf{x}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2)$

Table 3.3
Achievable Eigenvectors - Example 3.1

```
ACHIEVABLE EIGENVECTORS

EIGENVALUES
 1:     -1.4900 + J    2.3700
 2:     -1.4900 - J    2.3700
 3:     -1.7577 + J    8.6109
 4:     -1.7577 - J    8.6109


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                         1                                2
(  1.0000E+00,           0)     (  1.0000E+00,            0)
(  2.5129E+00,  9.8270E+01)     (  2.5129E+00, -9.8270E+01)
(  1.9947E-02,  4.5456E+01)     (  1.9947E-02, -4.5456E+01)
(  8.9765E-01, -2.3888E+01)     (  8.9765E-01,  2.3888E+01)
(  9.0783E-03,  2.7450E+01)     (  9.0783E-03, -2.7450E+01)
(  5.2106E-02, -1.7034E+02)     (  5.2106E-02,  1.7034E+02)
(  2.5414E-02,  1.4961E+02)     (  2.5414E-02, -1.4961E+02)
(  1.4587E-01, -4.8185E+01)     (  1.4587E-01,  4.8185E+01)

                         3                                4
(  1.0000E+00,           0)     (  1.0000E+00,            0)
(  1.9832E+00,  9.2307E+01)     (  1.9832E+00, -9.2307E+01)
(  3.8622E-03,  7.6548E+01)     (  3.8622E-03, -7.6548E+01)
(  2.2566E-01, -9.2295E+00)     (  2.2566E-01,  9.2295E+00)
(  2.3543E+01,  7.0110E+01)     (  2.3543E+01, -7.0110E+01)
(  1.0039E-01,  3.6417E+01)     (  1.0039E-01, -3.6417E+01)
(  2.0691E+02,  1.7165E+02)     (  2.0691E+02, -1.7165E+02)
(  8.8227E-01,  1.3795E+02)     (  8.8227E-01, -1.3795E+02)
```

State vector (unscaled): $\mathbf{x}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2)$

$$\lambda_{\xi_2} = -0.393 \pm j\,20.85 = (\omega_n = 20.85\ (\mathrm{rad/s})\ ,\ \varsigma = 0.019)$$

The eigenvectors of $\mathbf{A}_{aug}$ are given in Table 3.4. The augmented system residue magnitudes, for $\underline{u}_p$ = impulse, for each output are given in Figure 3.5. The augmented system output time responses due to a unit step input in $\underline{u}_p$, with zero initial conditions, are shown in Figure 3.6.

Comments on Example 3.1

This example illustrates some of the properties of the DEA synthesis technique. As can be seen by examining the desired and achievable eigenvector elements, all the desired elements were not exactly obtained (as the theory states). But overall, the achievable eigenspace is close to the desired eigenspace. Choosing the desired eigenvectors to uncouple the rigid and flexible modes has achieved the desired synthesis result. This is reflected in the closed-loop system impulse residue magnitudes. The control law has reduced the contributions of the first elastic mode in the rigid-body pitch, rigid-body pitch rate and plunge acceleration outputs. The impulse residue magnitudes show that the aircraft's flexible modes are not significantly affecting the θ_r and $\dot{\theta}_r$ outputs. Therefore, in this respect, this control law is making the augmented aircraft behave more like a rigid aircraft.

One disadvantage that is readily apparent from this example is that this technique does not guarantee a stable closed-loop system. Experience has shown that the choice of the desired eigenspace can significantly affect the movement of the $(n - l)$ unplaced poles. In this example, the desired eigenspace has caused the unplaced phugoid mode to split into two unstable real eigenvalues (0.0117 and 0.3273). As was shown in the DEA development, with this technique only l of the closed-loop system eigenvalues are placed to desired locations. The designer has no direct control over the remaining $(n - l)$ eigenvalues and no way of predicting a priori their directions of movement.

At this point the designer has two options: 1) increase the number of measurements and thereby place a larger number of eigenvalues or 2) use physical insight into the problem and conventional control techniques to achieve stability by making individual loop gain adjustments and hopefully not disturb the achieved eigenspace too much. Option one is the simplest from a theoretical viewpoint, but difficult and costly from a practical viewpoint. This method

Table 3.4
Closed-loop Eigenvectors - Example 3.1

```
DEA 4 MEASUREMENTS

EIGENVALUES
 1:      -1.4900 + J     2.3700
 2:      -1.4900 - J     2.3700
 3:        .3273 + J          0
 4:        .0117 + J          0


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                               1                                      2
(  1.0000E+00,                0)      (  1.0000E+00,                 0)
(  2.5129E+00,  -2.6173E+02)      (  2.5129E+00,   2.6173E+02)
(  1.9947E-02,  -3.1454E+02)      (  1.9947E-02,   3.1454E+02)
(  8.9765E-01,  -2.3888E+01)      (  8.9765E-01,   2.3888E+01)
(  9.0783E-03,  -3.3255E+02)      (  9.0783E-03,   3.3255E+02)
(  5.2106E-02,  -1.7034E+02)      (  5.2106E-02,   1.7034E+02)
(  2.5414E-02,  -2.1039E+02)      (  2.5414E-02,   2.1039E+02)
(  1.4587E-01,  -4.8185E+01)      (  1.4587E-01,   4.8185E+01)

                               3                                      4
(  1.0000E+00,                0)      (  1.0000E+00,                 0)
(  1.1652E+00,                0)      (  1.3844E-02,                 0)
(  4.1763E-01,   1.8000E+02)      (  1.8182E+00,  -1.8000E+02)
(  3.5600E+00,                0)      (  1.1878E+00,                 0)
(  6.7394E-01,                0)      (  4.1801E-01,                 0)
(  4.4342E-02,   1.8000E+02)      (  5.1817E-02,  -1.8000E+02)
(  2.2059E-01,                0)      (  4.8720E-03,                 0)
(  1.4514E-02,   1.8000E+02)      (  6.0393E-04,  -1.8000E+02)
```

State vector (unscaled): $\mathbf{x}^T = (\ \alpha\ ,\ \dot{\theta}_r\ ,\ u_f\ ,\ \theta_r\ ,\ \xi_1\ ,\ \xi_2\ ,\ \dot{\xi}_1\ ,\ \dot{\xi}_2\ )$

Table 3.4, concluded

```
EIGENVALUES
 5:    -1.7577 + J    8.6109
 6:    -1.7577 - J    8.6109
 7:     -.3931 + J   20.8499
 8:     -.3931 - J   20.8499


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                               5                                    6
(  1.0000E+00,                0)      (  1.0000E+00,                0)
(  1.9832E+00, -2.6769E+02)      (  1.9832E+00,  2.6769E+02)
(  3.8622E-03, -2.8345E+02)      (  3.8622E-03,  2.8345E+02)
(  2.2566E-01, -9.2295E+00)      (  2.2566E-01,  9.2295E+00)
(  2.3543E+01, -2.8989E+02)      (  2.3543E+01,  2.8989E+02)
(  1.0039E-01, -3.2358E+02)      (  1.0039E-01,  3.2358E+02)
(  2.0691E+02, -1.8835E+02)      (  2.0691E+02,  1.8835E+02)
(  8.8227E-01, -2.2205E+02)      (  8.8227E-01,  2.2205E+02)

                               7                                    8
(  1.0000E+00,                0)      (  1.0000E+00,                0)
(  1.0405E+02, -2.0100E+02)      (  1.0405E+02,  2.0100E+02)
(  8.6732E-03, -2.1085E+02)      (  8.6732E-03,  2.1085E+02)
(  4.9898E+00,  6.7918E+01)      (  4.9898E+00, -6.7918E+01)
(  9.2935E+01, -1.4567E+01)      (  9.2935E+01,  1.4567E+01)
(  2.4828E+02, -5.2830E+01)      (  2.4828E+02,  5.2830E+01)
(  1.9380E+03,  7.6513E+01)      (  1.9380E+03, -7.6513E+01)
(  5.1776E+03,  3.8250E+01)      (  5.1776E+03, -3.8250E+01)
```

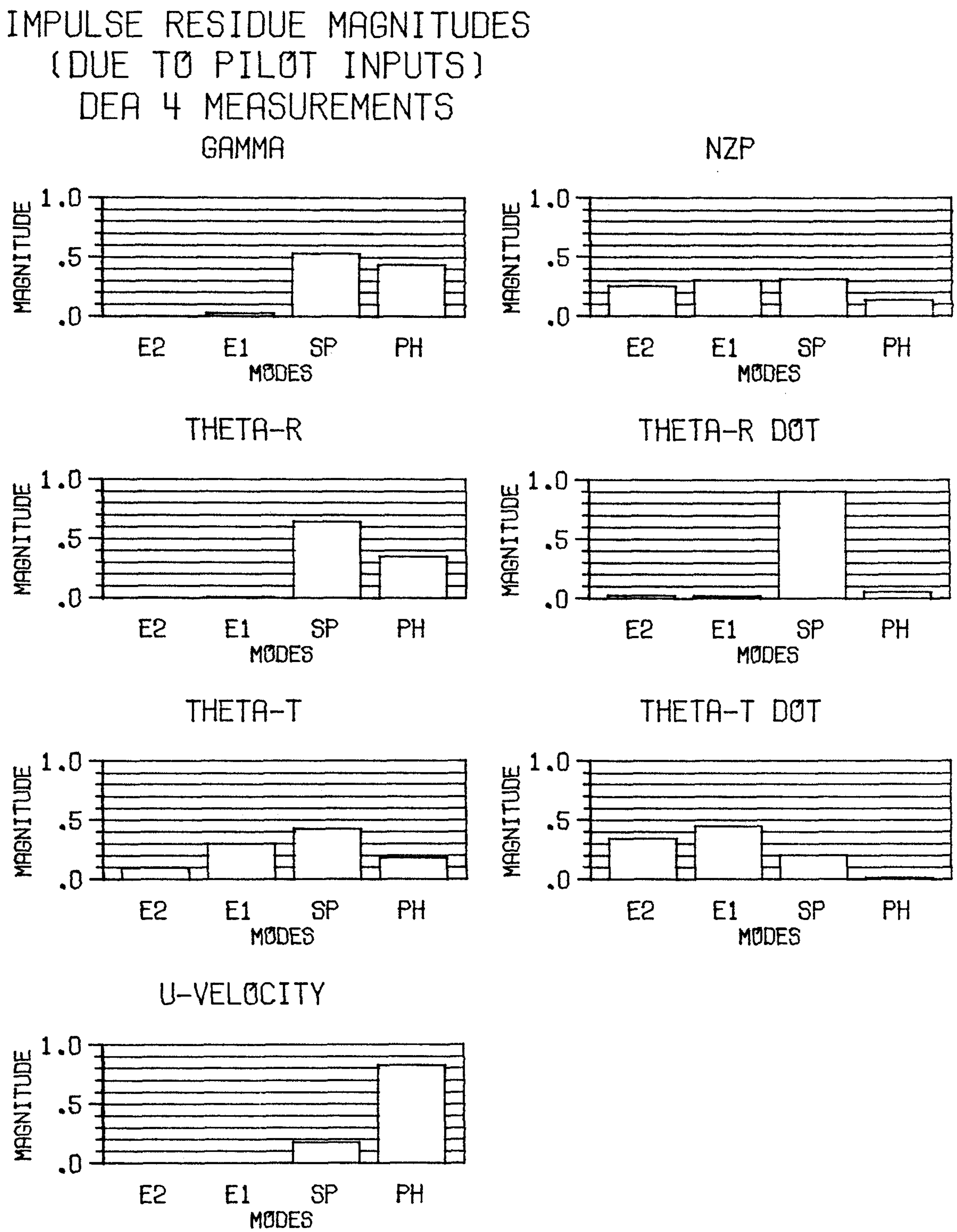


Figure 3.5
Impulse Residue Magnitudes - Example 3.1

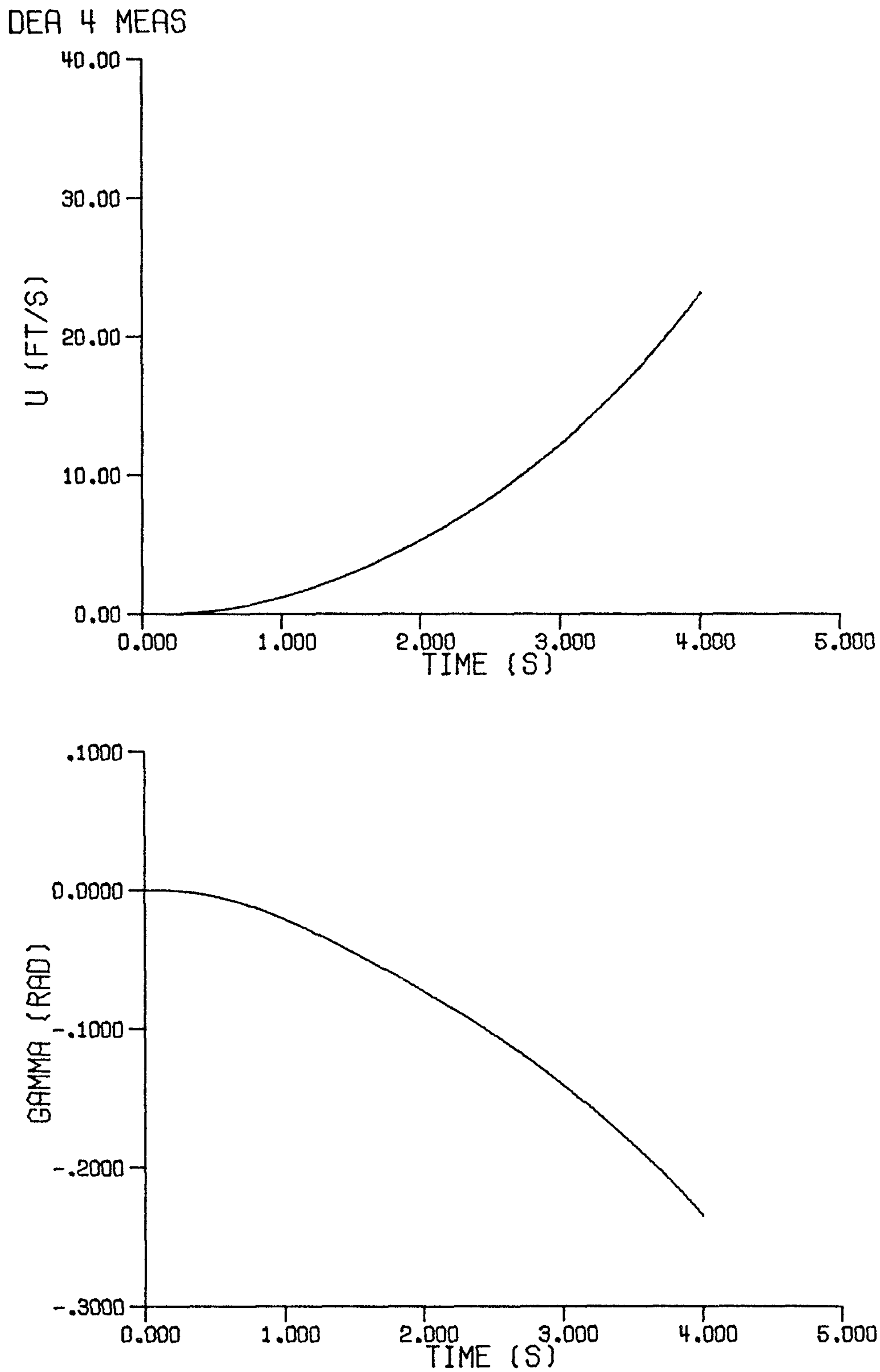
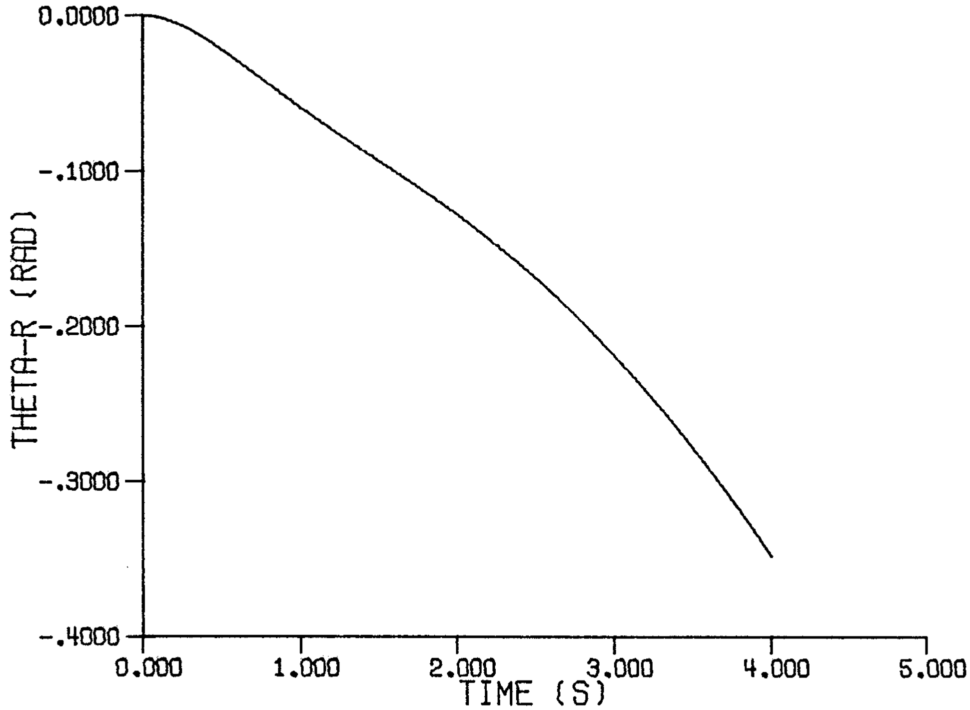


Figure 3.6
Output Time Responses - Example 3.1



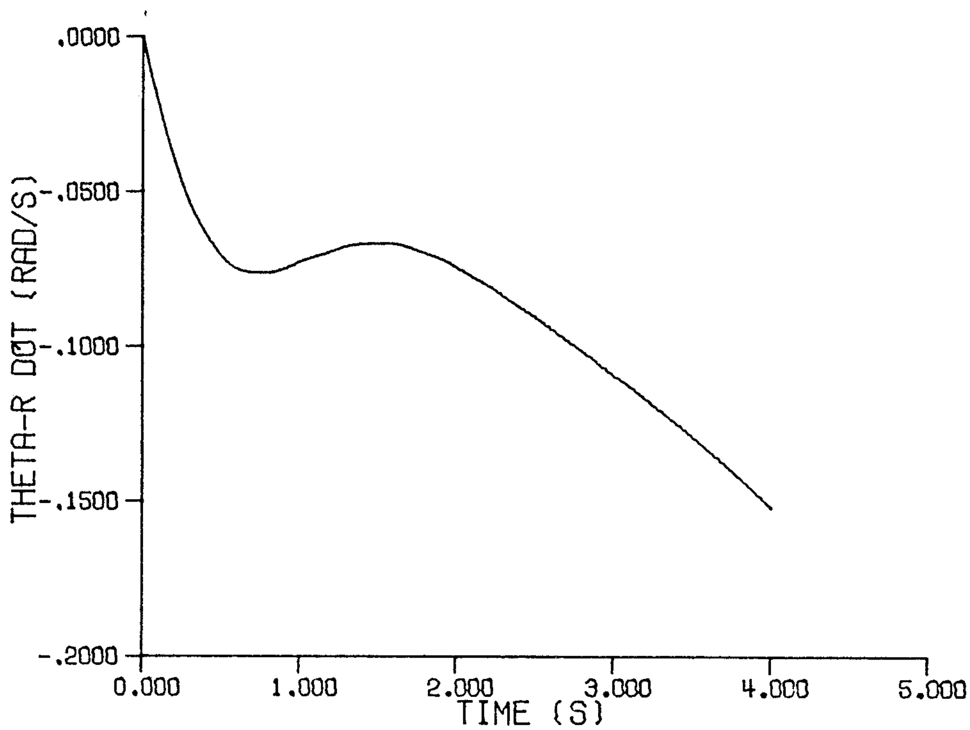
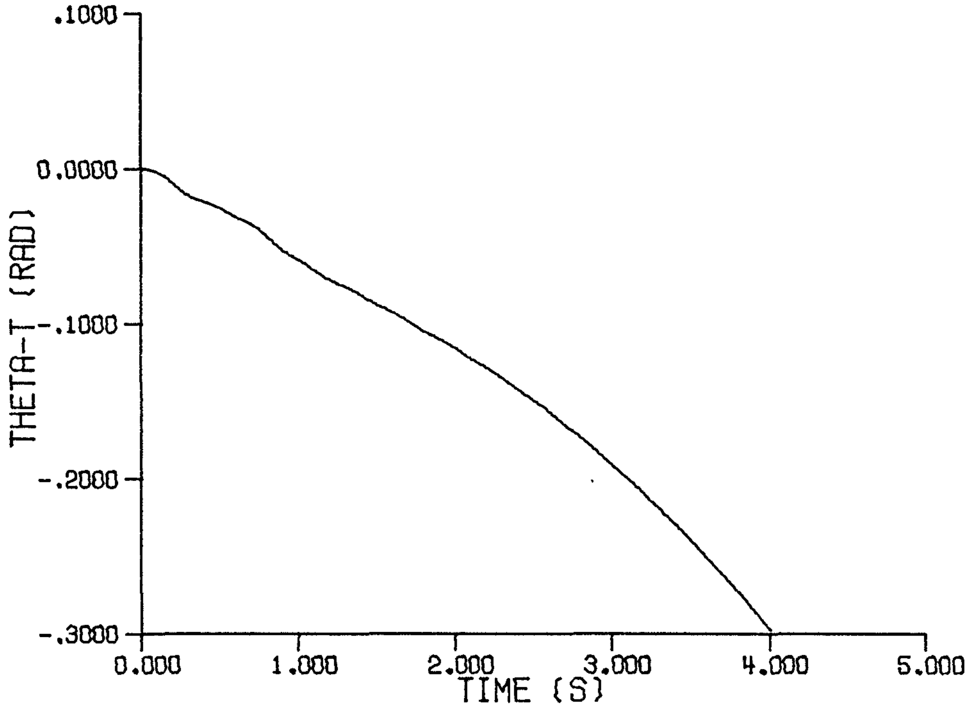


Figure 3.6, continued



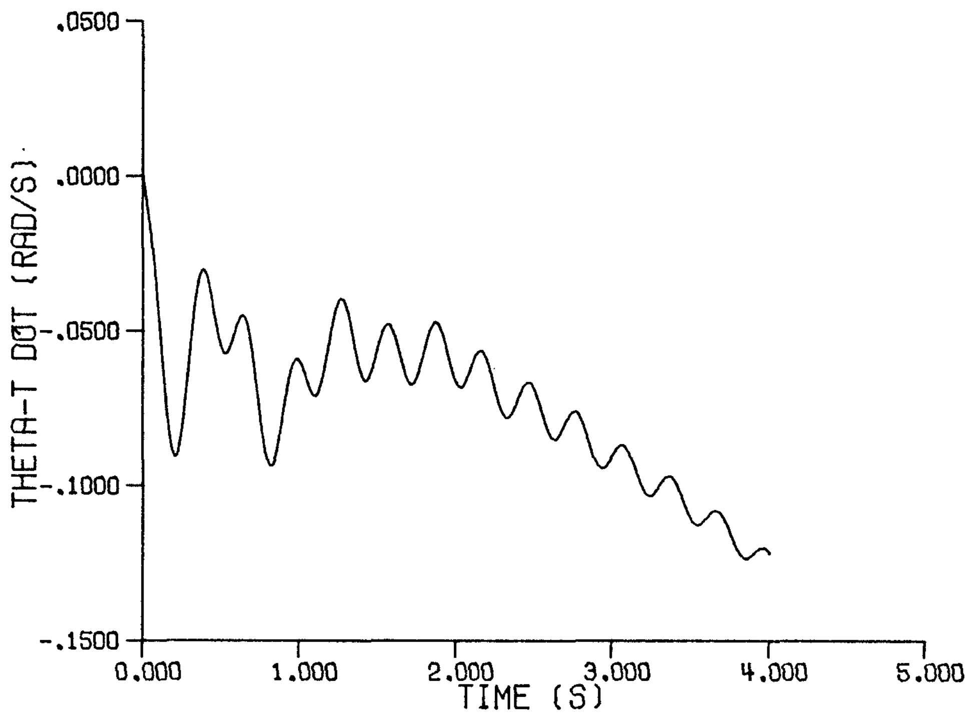


Figure 3.6, continued

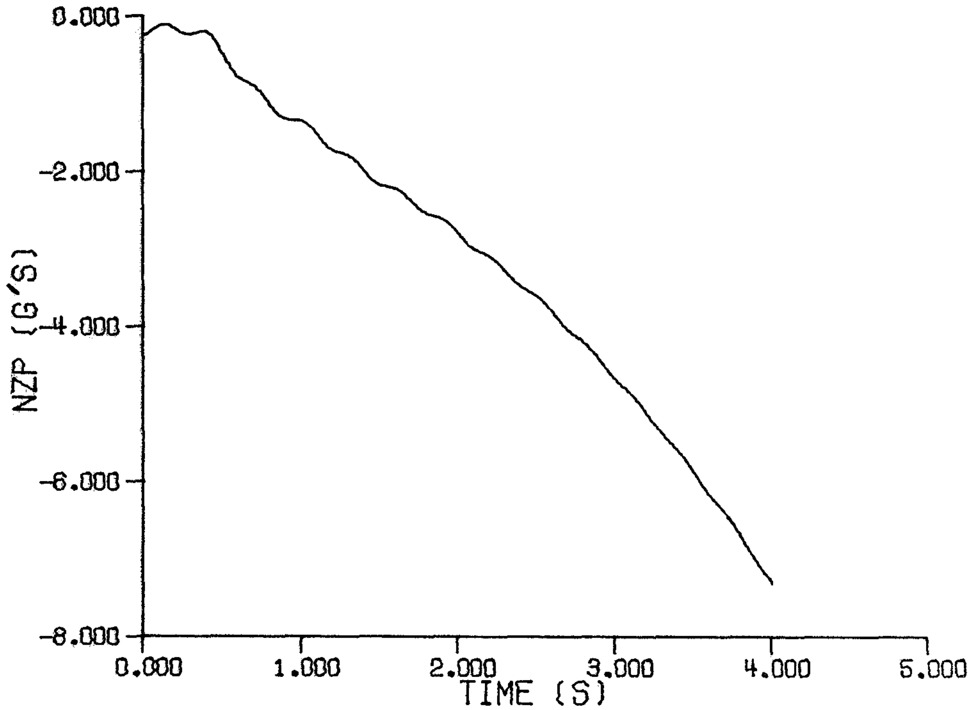


Figure 3.6, concluded

is demonstrated in Example 3.2. Option two can be useful to the designer, but it requires enough physical insight into the problem to know which loop gains to adjust and by how much. Such a situation is not undesirable but, many times this requires an iterative control synthesis strategy. However, this option does yield a control law that does not require additional sensors. This individual loop gain adjustment technique is illustrated in Example 3.3.

Example 3.2 : Six Measurements

This example will illustrate the DEA control synthesis technique when the model is eighth order, with two controls, but now has six measurements available for feedback. The design objectives for this example are the same as those in Example 3.1. Now the designer can place six eigenvalues. This synthesis will take advantage of this extra freedom by placing the phugoid poles (which were unstable in Example 3.1) to more desirable locations. The measurement vector for this example will consist of the four measurements used in Example 3.1, plus two velocity measurements. The measurement vector is

$$\underline{z}^{T} = (\theta_t , \dot{\theta}_t , n_{z_p} , n_{z_t} , v_{z_p} , v_{z_t})$$

where

θ_t = total pitch angle (radians)

$\dot{\theta}_t$ = total pitch rate (radians/sec)

n_{z_p} = plunge acceleration at the cockpit (g' s)

n_{z_t} = plunge acceleration at an aft fuselage station (g' s)

v_{z_p} = velocity at the cockpit = $\int n_{z_p}$ (n_{z_p} in g' s)

v_{z_t} = velocity at an aft fuselage station = $\int n_{z_t}$ (n_{z_t} in g' s)

Specification of Synthesis Parameters

Desired Eigenvalues. The desired short period and first elastic eigenvalue locations are chosen to be the same as in Example 3.1. The phugoid poles will be specified to be at the same locations as in the baseline configuration. Therefore choose:

$$\lambda_{d_{1,2}} = \lambda_{d_{sp}} = -\ 1.49\ \pm\ j\ 2.37\ = (\omega_n = 2.799\ \text{(rad/s)}\ ,\ \varsigma = 0.532)$$

$$\lambda_{d_{3,4}} = \lambda_{d_{ph}} = -\ 0.0015\ \pm\ j\ 0.0672\ = (\omega_n = 0.067\ \text{(rad/s)}\ ,\ \varsigma = 0.022)$$

$$\lambda_{d_{5,6}} = \lambda_{d_{\xi_1}} = -\ 1.758\ \pm\ j\ 8.611\ = (\omega_n = 8.789\ \text{(rad/s)}\ ,\ \varsigma = 0.200)$$

Desired Eigenvectors. The desired short period and first elastic eigenvectors are specified to have the same values as in Example 3.1. The desired phugoid eigenvector is chosen using the same strategy as in Example 3.1. The elements of this eigenvector that are associated with the (scaled) elastic states are chosen to have small magnitudes. The element associated with the the (scaled) state u_f is chosen to be unit magnitude as a reference value. The phase relationships of this desired eigenvector are arbitrarily chosen to be similar to that of the "more rigid" baseline configuration's phugoid eigenvector. The desired eigenvectors for this example are given in Table 3.5.

Calculation of $\underline{w}_i$*'s.* Because the desired short period and first elastic eigenvalues and eigenvectors are chosen to be the same as in Example 3.1, the values for $\underline{w}_{1,2}^T = \underline{w}_{sp}^T$ and $\underline{w}_{3,4}^T = \underline{w}_{\xi_1}^T$ are also the same. Values for $\underline{w}_{5,6}$, the $\underline{w}_i$ associated with the desired phugoid eigenvector, were calculated to be:

$$\underline{w}_{5,6}^T = \underline{w}_{ph}^T = (0.057\ +\ j\,8.56 \times 10^{-4}\ ,\ -1.553\ +\ j\,6.95 \times 10^{-3})$$

Achievable Eigenvectors. The achievable eigenvectors $\underline{\nu}_{a_{1,2}} = \underline{\nu}_{a_{sp}}$ and $\underline{\nu}_{a_{3,4}} = \underline{\nu}_{a_{\xi_1}}$ are the same as those obtained in Example 3.1. The achievable eigenvectors for this example are given in Table 3.6. Again, the eigenvectors have been scaled such that their first element is unity.

Closed-Loop System Analysis

The gain matrix to obtain the achievable eigenspace in the closed-loop system and the augmented system matrices are given in Appendix A.3. The system closed-loop eigenvalues, the eigenvalues of $\mathbf{A}_{aug}$ are:

Table 3.5
Desired Eigenvectors - Example 3.2

```
DESIRED EIGENVECTORS

EIGENVALUES
 1:     -1.4900 + J     2.3700
 2:     -1.4900 - J     2.3700
 3:      -.0015 + J      .0672
 4:      -.0015 - J      .0672
 5:     -1.7577 + J     8.6109
 6:     -1.7577 - J     8.6109


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                               1                                      2
(     *      ,       *       )      (      *      ,       *       )
(  1.0000E+00,  1.0000E+02)      (  1.0000E+00, -1.0000E+02)
(     *      ,       *       )      (      *      ,       *       )
(     *      ,       *       )      (      *      ,       *       )
(  5.0000E-03,  3.0000E+01)      (  5.0000E-03, -3.0000E+01)
(  5.0000E-04, -1.7000E+02)      (  5.0000E-04,  1.7000E+02)
(  1.0000E-02,  1.5000E+02)      (  1.0000E-02, -1.5000E+02)
(  5.0000E-03, -5.0000E+01)      (  5.0000E-03,  5.0000E+01)

                               3                                      4
(     *      ,       *       )      (      *      ,       *       )
(     *      ,       *       )      (      *      ,       *       )
(  1.0000E+00,  1.8000E+02)      (  1.0000E+00, -1.8000E+02)
(     *      ,       *       )      (      *      ,       *       )
(  1.0000E-02,  1.0000E+00)      (  1.0000E-02, -1.0000E+00)
(  5.0000E-03,  1.8000E+02)      (  5.0000E-03, -1.8000E+02)
(  5.0000E-04,  9.1000E+01)      (  5.0000E-04, -9.1000E+01)
(  1.0000E-04, -9.0000E+01)      (  1.0000E-04,  9.0000E+01)

                               5                                      6
(  1.0000E-03,            0)      (  1.0000E-03,            0)
(  1.0000E-02,  6.0000E+01)      (  1.0000E-02, -6.0000E+01)
(  1.0000E-04,  7.0000E+01)      (  1.0000E-04, -7.0000E+01)
(  1.0000E-03, -3.0000E+01)      (  1.0000E-03,  3.0000E+01)
(     *      ,       *       )      (      *      ,       *       )
(     *      ,       *       )      (      *      ,       *       )
(  1.0000E+00,  1.4000E+02)      (  1.0000E+00, -1.4000E+02)
(     *      ,       *       )      (      *      ,       *       )
```

State vector (unscaled): $\mathbf{x}^T = (\ \alpha\ ,\ \dot{\theta}_r\ ,\ u_f\ ,\ \theta_r\ ,\ \xi_1\ ,\ \xi_2\ ,\ \dot{\xi}_1\ ,\ \dot{\xi}_2\ )$

Table 3.6
Achievable Eigenvectors - Example 3.2

```
ACHIEVABLE EIGENVECTORS

EIGENVALUES
 1:     -1.4900 + J     2.3700
 2:     -1.4900 - J     2.3700
 3:      -.0015 + J      .0672
 4:      -.0015 - J      .0672
 5:     -1.7577 + J     8.6109
 6:     -1.7577 - J     8.6109


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                            1                               2
(  1.0000E+00,             0)      (  1.0000E+00,             0)
(  2.5129E+00,  9.8270E+01)      (  2.5129E+00, -9.8270E+01)
(  1.9947E-02,  4.5456E+01)      (  1.9947E-02, -4.5456E+01)
(  8.9765E-01, -2.3888E+01)      (  8.9765E-01,  2.3888E+01)
(  9.0783E-03,  2.7450E+01)      (  9.0783E-03, -2.7450E+01)
(  5.2106E-02, -1.7034E+02)      (  5.2106E-02,  1.7034E+02)
(  2.5414E-02,  1.4961E+02)      (  2.5414E-02, -1.4961E+02)
(  1.4587E-01, -4.8185E+01)      (  1.4587E-01,  4.8185E+01)

                            3                               4
(  1.0000E+00,             0)      (  1.0000E+00,             0)
(  3.2477E-01,  1.7162E+02)      (  3.2477E-01, -1.7162E+02)
(  2.3926E+00, -1.7913E+02)      (  2.3926E+00,  1.7913E+02)
(  4.8317E+00,  8.0354E+01)      (  4.8317E+00, -8.0354E+01)
(  2.3432E-02,  2.0432E+00)      (  2.3432E-02, -2.0432E+00)
(  5.1519E-02,  1.8023E+02)      (  5.1519E-02, -1.8023E+02)
(  1.5750E-03,  9.3305E+01)      (  1.5750E-03, -9.3305E+01)
(  3.4629E-03, -8.8511E+01)      (  3.4629E-03,  8.8511E+01)

                            5                               6
(  1.0000E+00,             0)      (  1.0000E+00,             0)
(  1.9832E+00,  9.2307E+01)      (  1.9832E+00, -9.2307E+01)
(  3.8622E-03,  7.6548E+01)      (  3.8622E-03, -7.6548E+01)
(  2.2566E-01, -9.2295E+00)      (  2.2566E-01,  9.2295E+00)
(  2.3543E+01,  7.0110E+01)      (  2.3543E+01, -7.0110E+01)
(  1.0039E-01,  3.6417E+01)      (  1.0039E-01, -3.6417E+01)
(  2.0691E+02,  1.7165E+02)      (  2.0691E+02, -1.7165E+02)
(  8.8227E-01,  1.3795E+02)      (  8.8227E-01, -1.3795E+02)
```

State vector (unscaled): $\mathbf{x}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2)$

$$\lambda_{sp} = -1.49 \pm j\ 2.37 = (\omega_n = 2.799\ (\text{rad/s})\ ,\ \varsigma = 0.532)$$

$$\lambda_{ph} = -0.0015 \pm j\ 0.0672 = (\omega_n = 0.067\ (\text{rad/s})\ ,\ \varsigma = 0.022)$$

$$\lambda_{\xi_1} = -1.758 \pm j\ 8.611 = (\omega_n = 8.789\ (\text{rad/s})\ ,\ \varsigma = 0.200)$$

$$\lambda_{\xi_2} = -0.3531 \pm j\ 20.97 = (\omega_n = 20.97\ (\text{rad/s})\ ,\ \varsigma = 0.017)$$

The eigenvectors of $\mathbf{A}_{aug}$ are given in Table 3.7. The augmented system impulse residue magnitudes (expressed as a percentage of the total respons for each output are shown in Figure 3.7, and the output time responses du a unit step in $\underline{u}_p$, with zero initial conditions, are given in Figure 3.8.

Comments on Example 3.2

For this example, using DEA with the freedom to place six eigenvalues has yielded a stable closed-loop system. The short period, phugoid, and first elastic eigenvalues have been placed to desired locations. The second elastic eigenvalues are at an acceptable location. They have remained close to their open-loop position.

The achieved eigenspace is close to the desired eigenspace. The impulse residue magnitudes show that the control synthesis has achieved the desired result; reducing the residue magnitudes of the first and second elastic modes in the rigid-body pitch angle and pitch rate outputs. In this respect, the control law is making the flexible aircraft behave more like a rigid aircraft.

These improved results are obtained at the "cost" of more sensors. This synthesis requires six measurements and is therefore more complex and costly than the four measurement synthesis of Example 3.1.

The next example will demonstrate a technique that will stabilize the system without increasing the number of measurements. This technique shall be referred to as Individual Loop Gain Adjustment (ILGA).

Table 3.7
Closed-loop Eigenvectors - Example 3.2

DEA 6 MEASUREMENTS

EIGENVALUES

1:	-1.4900	+ J	2.3700
2:	-1.4900	- J	2.3700
3:	-.0015	+ J	.0672
4:	-.0015	- J	.0672

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

1		2	
(1.0000E+00,	0)	(1.0000E+00,	0)
(2.5129E+00,	9.8270E+01)	(2.5129E+00,	-9.8270E+01)
(1.9947E-02,	4.5456E+01)	(1.9947E-02,	-4.5456E+01)
(8.9765E-01,	-2.3888E+01)	(8.9765E-01,	2.3888E+01)
(9.0783E-03,	2.7450E+01)	(9.0783E-03,	-2.7450E+01)
(5.2106E-02,	1.8966E+02)	(5.2106E-02,	-1.8966E+02)
(2.5414E-02,	1.4961E+02)	(2.5414E-02,	-1.4961E+02)
(1.4587E-01,	-4.8185E+01)	(1.4587E-01,	4.8185E+01)

3		4	
(1.0000E+00,	0)	(1.0000E+00,	0)
(3.2477E-01,	1.7162E+02)	(3.2477E-01,	-1.7162E+02)
(2.3926E+00,	1.8087E+02)	(2.3926E+00,	-1.8087E+02)
(4.8317E+00,	8.0354E+01)	(4.8317E+00,	-8.0354E+01)
(2.3432E-02,	2.0432E+00)	(2.3432E-02,	-2.0432E+00)
(5.1519E-02,	1.8023E+02)	(5.1519E-02,	-1.8023E+02)
(1.5750E-03,	9.3305E+01)	(1.5750E-03,	-9.3305E+01)
(3.4629E-03,	-8.8511E+01)	(3.4629E-03,	8.8511E+01)

State vector (unscaled): $\mathbf{x}^T = (\ \alpha\ ,\ \dot{\theta}_r\ ,\ \mathrm{u_f}\ ,\ \theta_r\ ,\ \xi_1\ ,\ \xi_2\ ,\ \dot{\xi}_1\ ,\ \dot{\xi}_2\ )$

Table 3.7, concluded

```
EIGENVALUES
 5:     -1.7577 + J    8.6109
 6:     -1.7577 - J    8.6109
 7:       -.3531 + J  20.9739
 8:       -.3531 - J  20.9739


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                              5                               6
(  1.0000E+00,               0)    (  1.0000E+00,               0)
(  1.9832E+00,  9.2307E+01)    (  1.9832E+00, -9.2307E+01)
(  3.8622E-03,  7.6548E+01)    (  3.8622E-03, -7.6548E+01)
(  2.2566E-01, -9.2295E+00)    (  2.2566E-01,  9.2295E+00)
(  2.3543E+01,  7.0110E+01)    (  2.3543E+01, -7.0110E+01)
(  1.0039E-01,  3.6417E+01)    (  1.0039E-01, -3.6417E+01)
(  2.0691E+02,  1.7165E+02)    (  2.0691E+02, -1.7165E+02)
(  8.8227E-01,  1.3795E+02)    (  8.8227E-01, -1.3795E+02)

                              7                               8
(  1.0000E+00,               0)    (  1.0000E+00,               0)
(  1.8194E+01,  1.0645E+02)    (  1.8194E+01, -1.0645E+02)
(  2.6346E-03,  9.7280E+01)    (  2.6346E-03, -9.7280E+01)
(  8.6732E-01,  1.5490E+01)    (  8.6732E-01, -1.5490E+01)
(  5.1347E+00,  1.7445E+01)    (  5.1347E+00, -1.7445E+01)
(  2.5863E+01,  2.1957E+02)    (  2.5863E+01, -2.1957E+02)
(  1.0771E+02,  1.0841E+02)    (  1.0771E+02, -1.0841E+02)
(  5.4252E+02, -4.9466E+01)    (  5.4252E+02,  4.9466E+01)
```

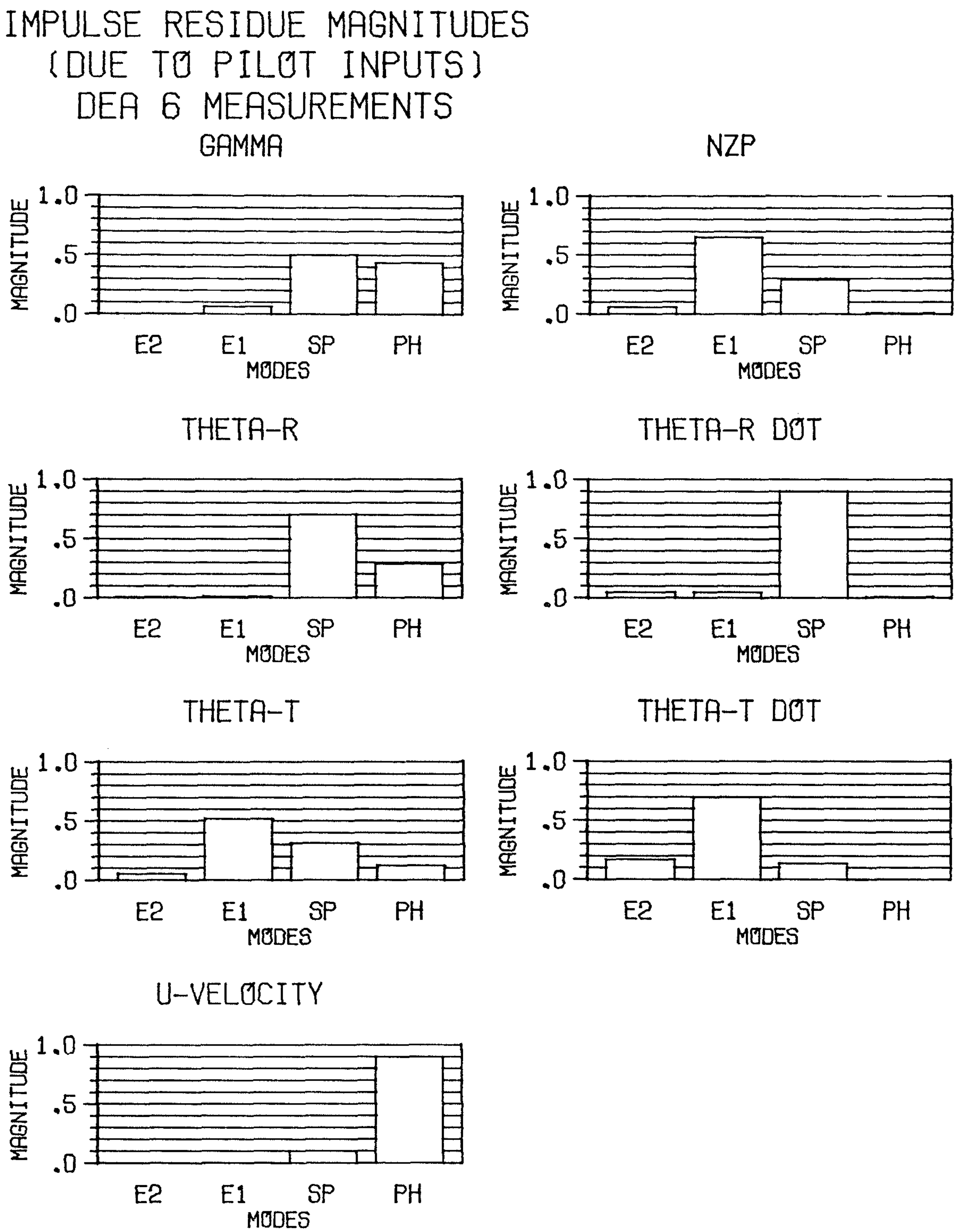


Figure 3.7
Impulse Residue Magnitudes - Example 3.2

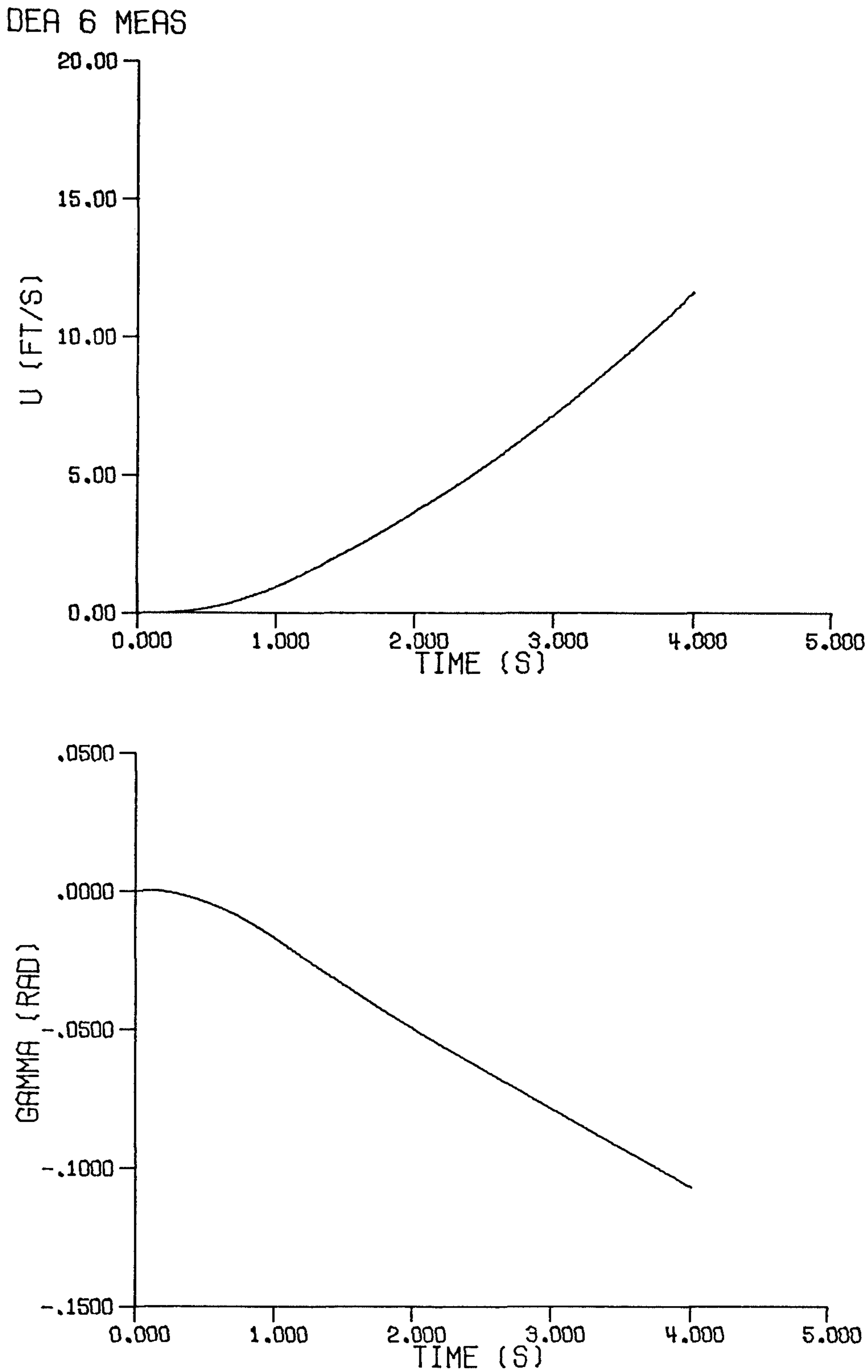
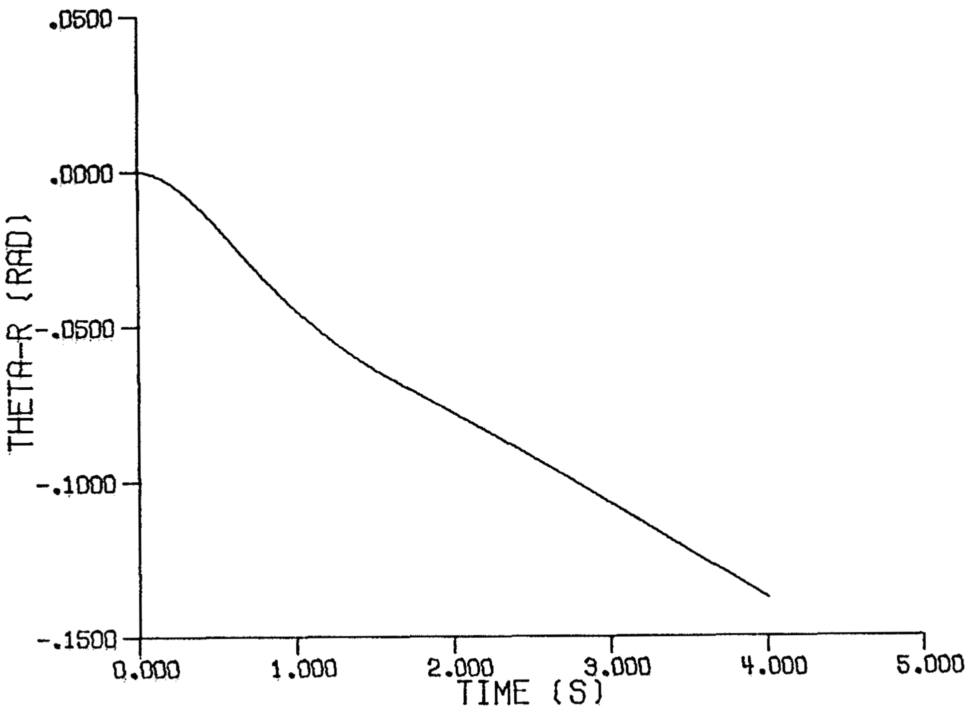


Figure 3.8
Output Time Responses - Example 3.2



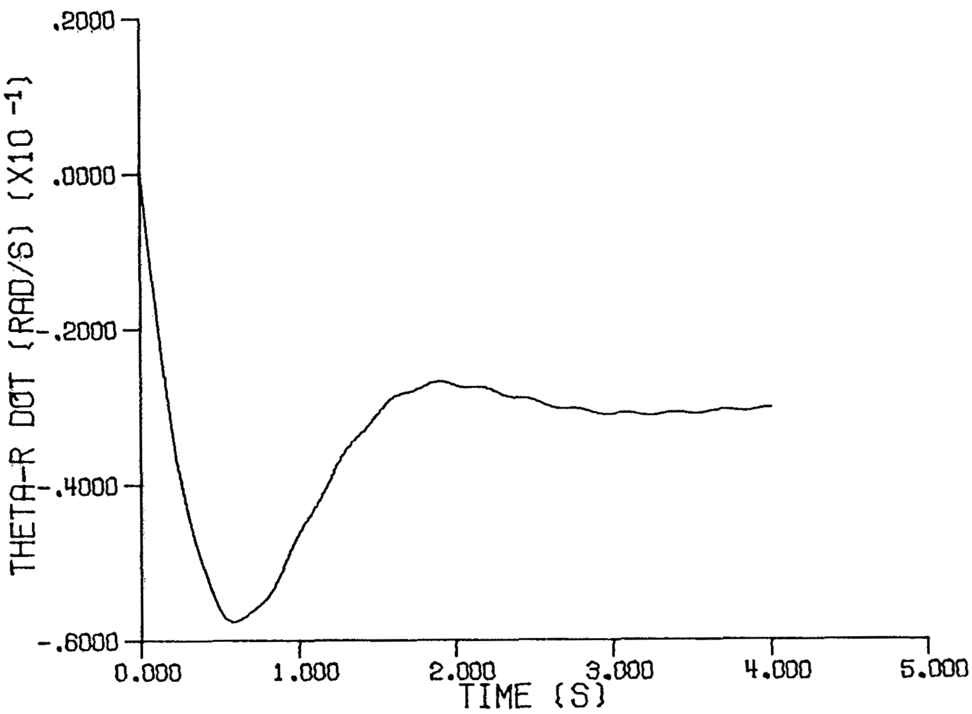
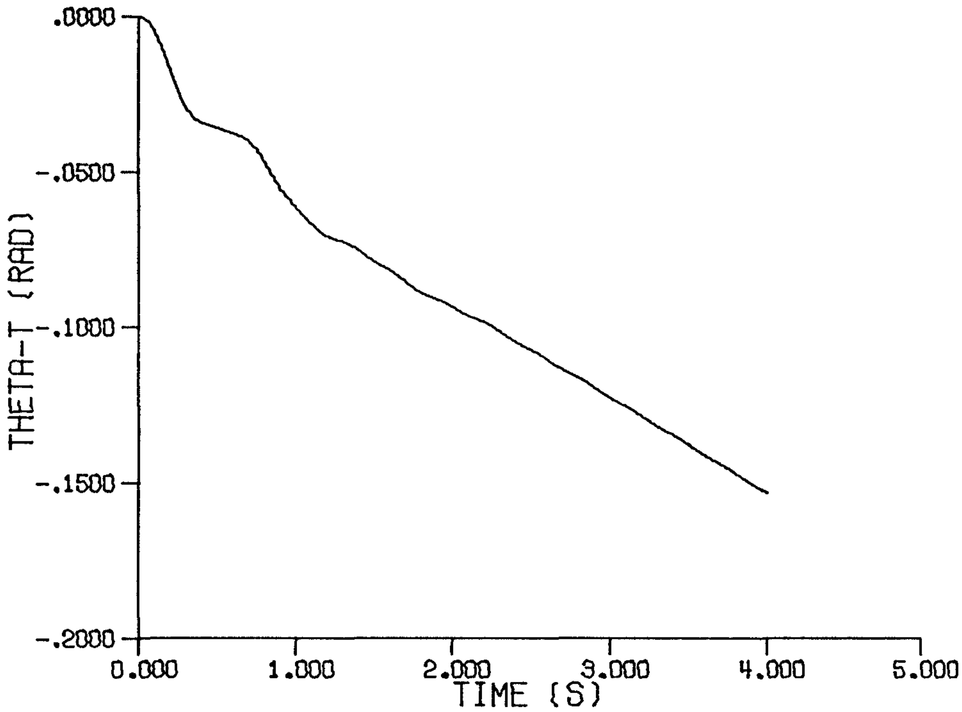


Figure 3.8, continued



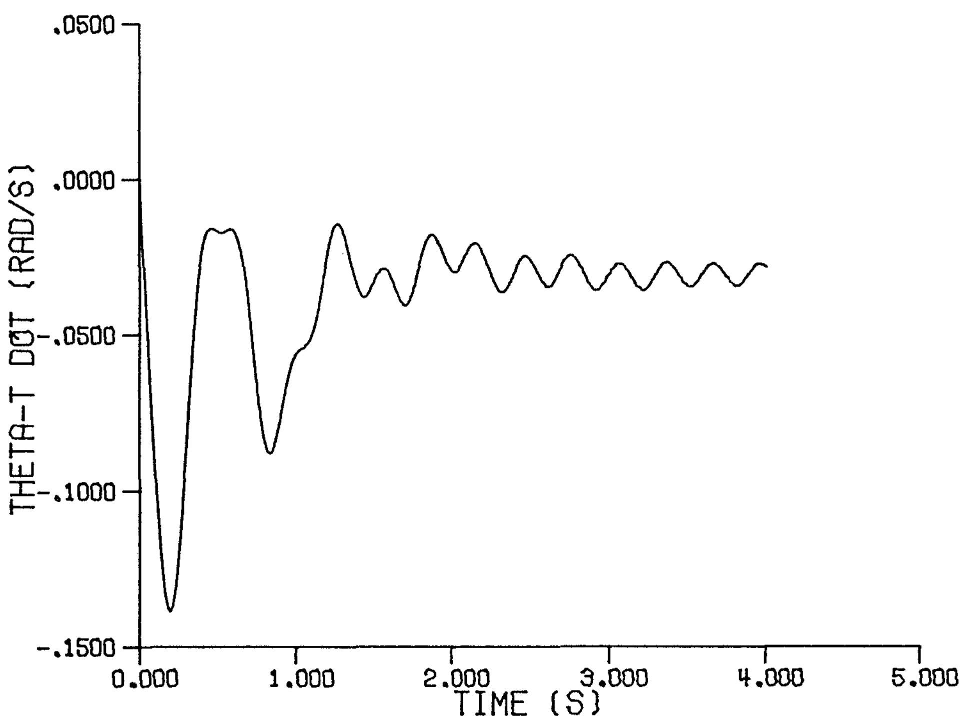


Figure 3.8, continued

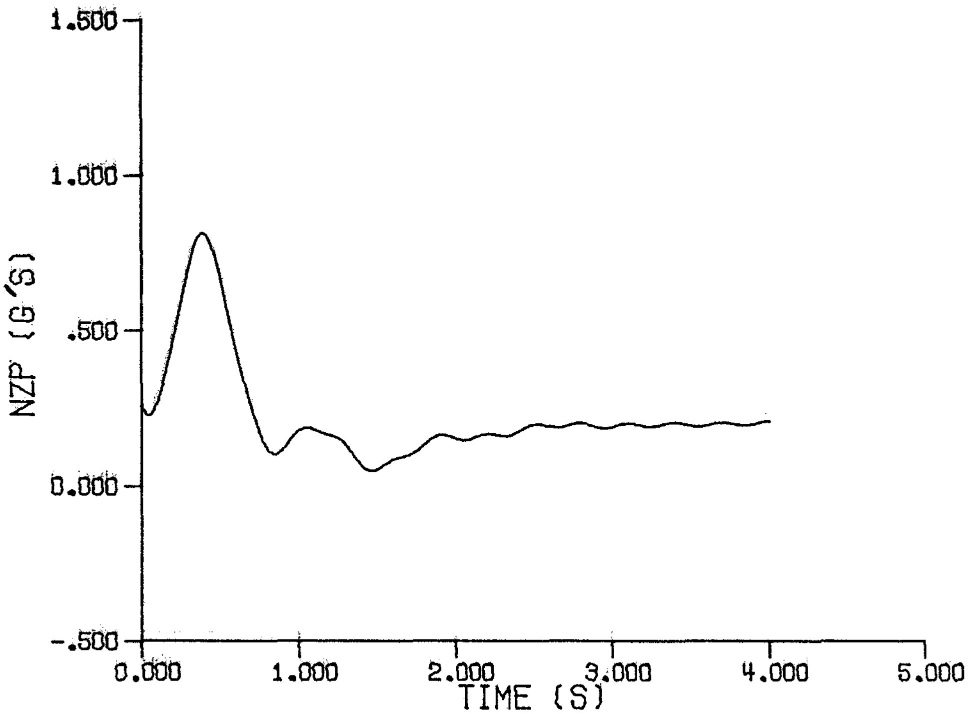


Figure 3.8, concluded

Example 3.3 : Individual Loop Gain Adjustment

This example will illustrate the use of an individual loop gain adjustment technique to stabilize the unstable system that resulted in Example 3.1. This method relys upon the designer's physical insight into the problem. The designer must choose which loop gains to vary to stabilize the system. If only a small change in loop gain is needed to stabilize the system, hopefully the resulting eigenspace will not be disturbed too much from the desired eigenspace.

This method involves choosing which loop gain to vary based upon the transfer function characteristics for this measurement to input and then using the Root Locus method to determine the change in the loop gain required to obtain acceptable eigenvalue locations. The new closed-loop eigenspace must then be checked to see how close it is to the original achievable eigenspace.

This example will start with the results from Example 3.1. The closed-loop system eigenvalues obtained were:

$$\lambda_{sp} = -\ 1.49\ \pm\ j\ 2.37\ = (\omega_n = 2.799\ (\mathrm{rad/s})\ ,\ \varsigma = 0.532)$$

$$\lambda_{ph1} = +\ 0.0117$$

$$\lambda_{ph2} = +\ 0.3273$$

$$\lambda_{\xi_1} = -\ 1.758\ \pm\ j\ 8.611\ = (\omega_n = 8.789\ (\mathrm{rad/s})\ ,\ \varsigma = 0.200)$$

$$\lambda_{\xi_2} = -\ 0.393\ \pm\ j\ 20.85\ = (\omega_n = 20.85\ (\mathrm{rad/s})\ ,\ \varsigma = 0.019)$$

As can be seen, it is the phugoid mode that is unstable. This mode "shape" is predominantly pitch attitude, θ and forward velocity, u_f, and therefore the pole locations should be affected more by the elevator than the control vane. Therefore, as a first attempt, the effect of varying the δ_e-to-θ_t loop gain will be explored. The first step of this process is to calculate the θ_t-to-δ_e transfer function. The zero's of the θ_t-to-δ_e transfer function were calculated using the method of Sandburg and So [14] to be:

$$z_1 = -0.00917 \qquad z_2 = -0.745$$

$$z_{3,4} = -0.856 \pm j\ 5.365$$

$$z_{5,6} = -0.864 \pm j\ 29.8$$

Therefore, the θ_t-to-δ_e transfer function is

$$\frac{\theta_t}{\delta_e} = \frac{k\,(s - z_1)(s - z_2)(s - z_{3,4})(s - z_{5,6})}{(s - \lambda_{ph1})(s - \lambda_{ph2})(s - \lambda_{sp})(s - \lambda_{\xi_1})(s - \lambda_{\xi_2})}$$

Using the Root Locus method, the θ_t-to-δ_e feedback gain was varied in an attempt to stabilize the system (see Figure 3.9 and 3.10). At a gain of, $\Delta G_{\theta_t,\delta_e}$ = 0.28 (dimensionless) the system eigenvalues are:

$$\lambda_{sp} = -\ 1.243 \pm j\ 2.907 = (\omega_n = 3.16\ (\text{rad/s})\ ,\ \varsigma = 0.393)$$

$$\lambda_{ph} = -\ 0.0253 \pm j\ 0.068 = (\omega_n = 0.073\ (\text{rad/s})\ ,\ \varsigma = 0.345)$$

$$\lambda_{\xi_1} = -\ 1.844 \pm j\ 9.100 = (\omega_n = 9.28\ (\text{rad/s})\ ,\ \varsigma = 0.199)$$

$$\lambda_{\xi_2} = -\ 0.3589 \pm j\ 20.69 = (\omega_n = 20.70\ (\text{rad/s})\ ,\ \varsigma = 0.017)$$

The closed-loop system gain to achieve this eigenvalue placement is

$$\mathbf{G'} = \mathbf{G} + \left[\begin{array}{c|c|c|c} \Delta G_{\theta_t,\delta_e} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

where **G** is the gain matrix obtained in Example 3.1

This modified gain matrix and the augmented system matrices are given in Appendix A.3. The eigenvectors of the augmented system are given in Table 3.8. The augmented system impulse residue magnitudes (normalized for plotting) for each output are shown in Figure 3.11, and the output time responses due to a unit step in $\underline{u}_p$, with initial conditions $\underline{x}(0) = \underline{0}$, are shown in Figure 3.12.

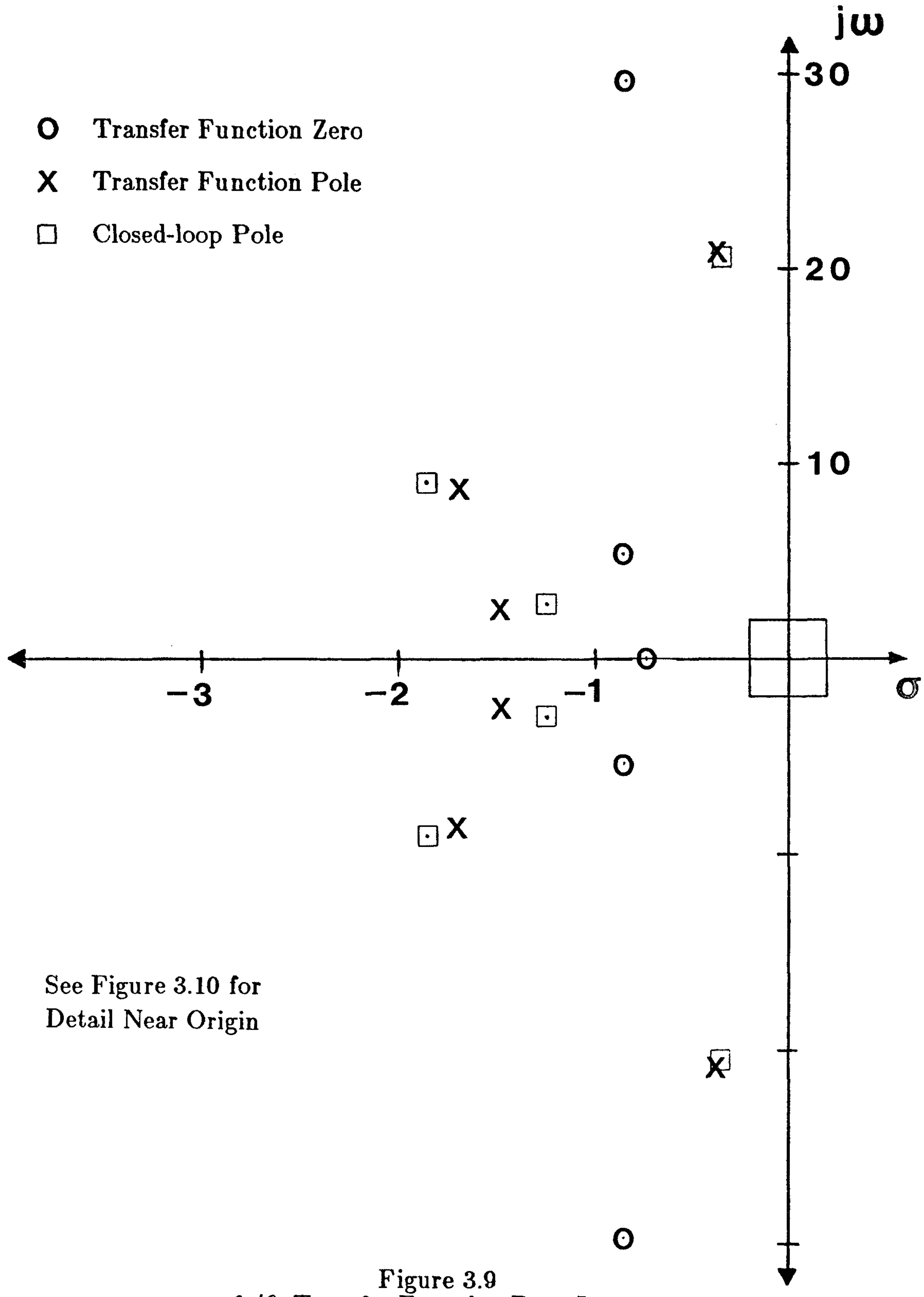


Figure 3.9
θ_t/δ_e Transfer Function Root Locus

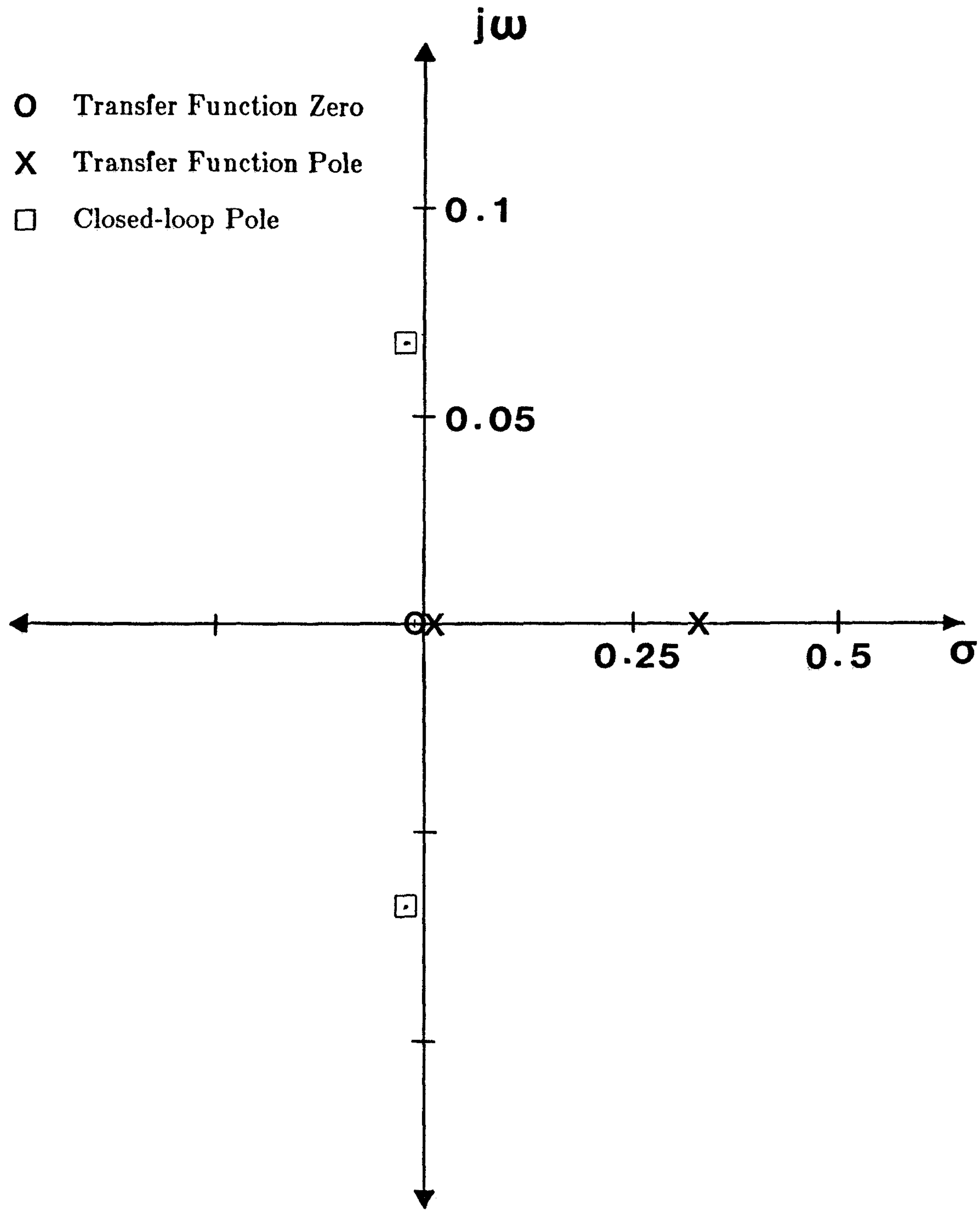


Figure 3.10
Root Locus, Detail near Origin

Table 3.8
Closed-loop Eigenvectors - Example 3.3

DEA ILGA

EIGENVALUES

1:	-1.2432	+ J	2.9073
2:	-1.2432	- J	2.9073
3:	-.0253	+ J	.0688
4:	-.0253	- J	.0688

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

1		2	
(1.0000E+00,	0)	(1.0000E+00,	0)
(3.0365E+00,	9.1774E+01)	(3.0365E+00,	-9.1774E+01)
(1.8373E-02,	5.5431E+01)	(1.8373E-02,	-5.5431E+01)
(9.6030E-01,	-2.1378E+01)	(9.6030E-01,	2.1378E+01)
(1.3229E-01,	-6.2820E+00)	(1.3229E-01,	6.2820E+00)
(6.1202E-02,	1.8662E+02)	(6.1202E-02,	-1.8662E+02)
(4.1830E-01,	1.0687E+02)	(4.1830E-01,	-1.0687E+02)
(1.9352E-01,	-6.0227E+01)	(1.9352E-01,	6.0227E+01)

3		4	
(1.0000E+00,	0)	(1.0000E+00,	0)
(3.6050E-01,	-1.4212E+02)	(3.6050E-01,	1.4212E+02)
(2.3365E+00,	1.8849E+02)	(2.3365E+00,	-1.8849E+02)
(4.9170E+00,	1.0771E+02)	(4.9170E+00,	-1.0771E+02)
(1.3442E+00,	9.7165E+01)	(1.3442E+00,	-9.7165E+01)
(5.1354E-02,	-1.6234E+02)	(5.1354E-02,	1.6234E+02)
(9.8555E-02,	-1.5266E+02)	(9.8555E-02,	1.5266E+02)
(3.7651E-03,	-5.2157E+01)	(3.7651E-03,	5.2157E+01)

State vector (unscaled): $\mathbf{x}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2)$

Table 3.8, concluded

```
EIGENVALUES
 5:     -1.8439 + J    9.0998
 6:     -1.8439 - J    9.0998
 7:      -.3589 + J   20.6938
 8:      -.3589 - J   20.6938


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                                5                                  6
(  1.0000E+00,               0)      (  1.0000E+00,               0)
(  8.7551E+00,   5.0591E+01)         (  8.7551E+00,  -5.0591E+01)
(  5.6834E-03,   5.0627E+01)         (  5.6834E-03,  -5.0627E+01)
(  9.4296E-01,  -5.0864E+01)         (  9.4296E-01,   5.0864E+01)
(  1.9359E+01,   1.3417E+02)         (  1.9359E+01,  -1.3417E+02)
(  4.0214E-01,   1.3425E+02)         (  4.0214E-01,  -1.3425E+02)
(  1.7974E+02,  -1.2438E+02)         (  1.7974E+02,   1.2438E+02)
(  3.7338E+00,  -1.2430E+02)         (  3.7338E+00,   1.2430E+02)

                                7                                  8
(  1.0000E+00,               0)      (  1.0000E+00,               0)
(  6.7726E+01,   3.4772E+01)         (  6.7726E+01,  -3.4772E+01)
(  6.1638E-03,   4.2737E+01)         (  6.1638E-03,  -4.2737E+01)
(  3.2723E+00,   3.0378E+02)         (  3.2723E+00,  -3.0378E+02)
(  4.9552E+01,   1.9483E+02)         (  4.9552E+01,  -1.9483E+02)
(  1.2272E+02,   1.5740E+02)         (  1.2272E+02,  -1.5740E+02)
(  1.0256E+03,   2.8582E+02)         (  1.0256E+03,  -2.8582E+02)
(  2.5398E+03,   2.4840E+02)         (  2.5398E+03,  -2.4840E+02)
```

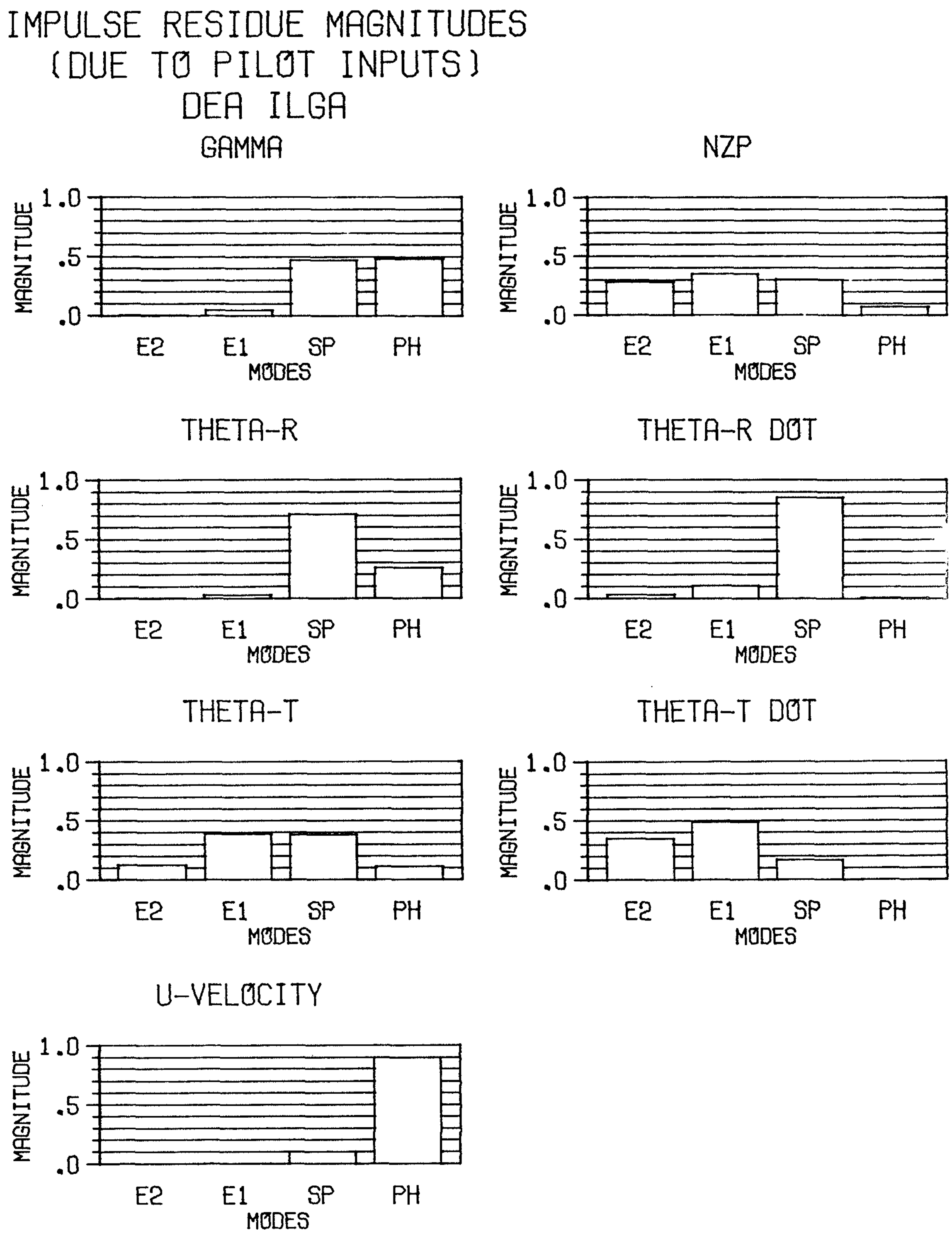
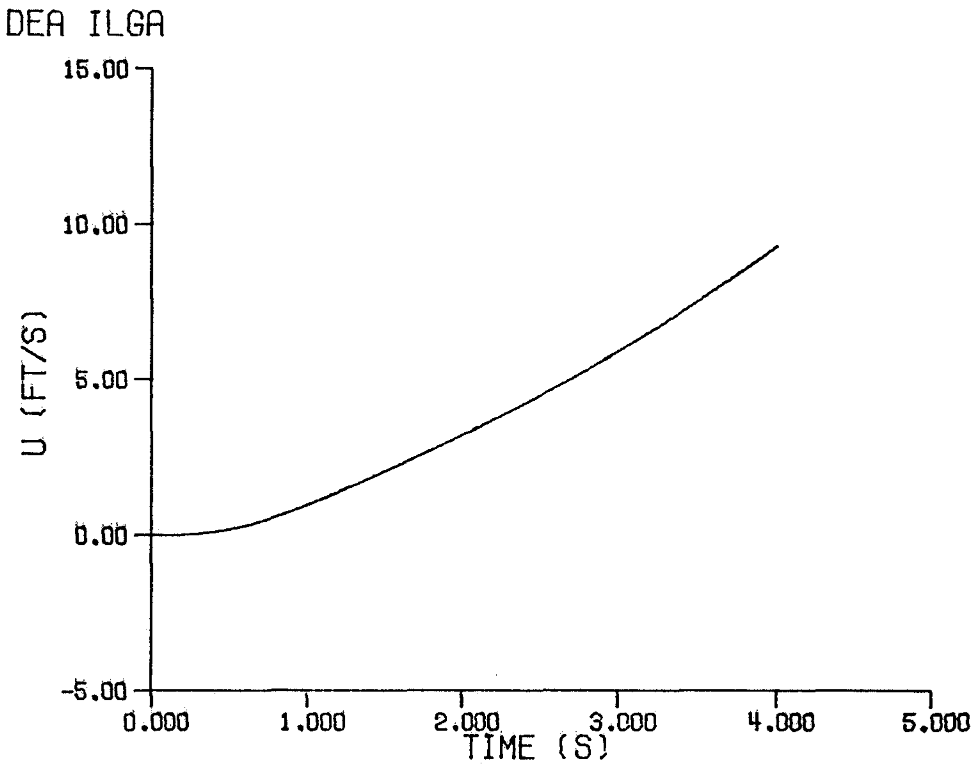


Figure 3.11
Impulse Residue Magnitudes - Example 3.3



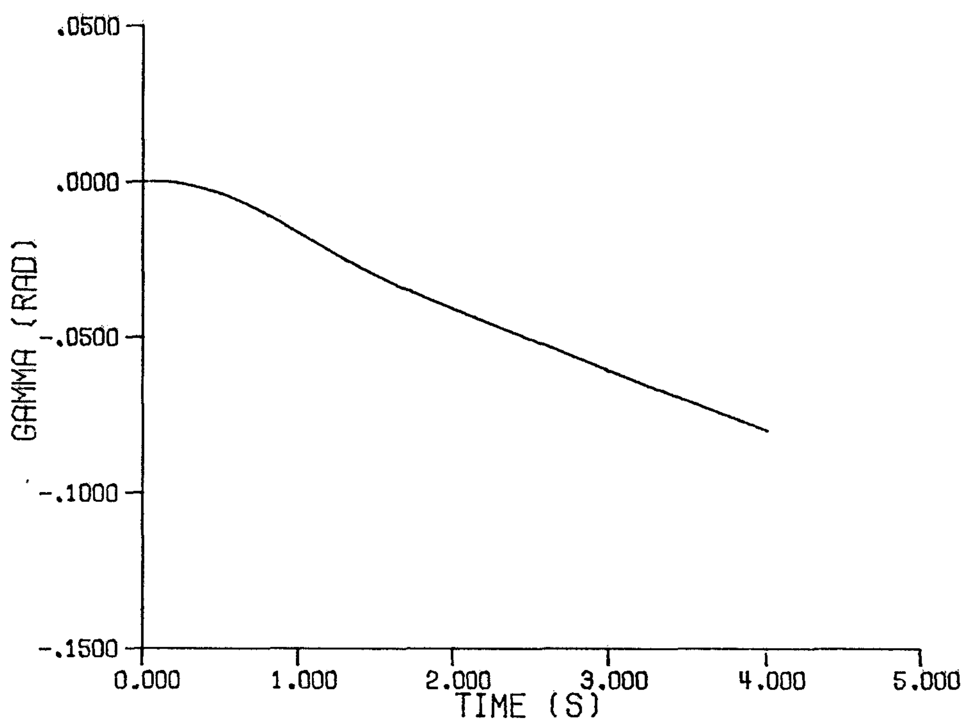
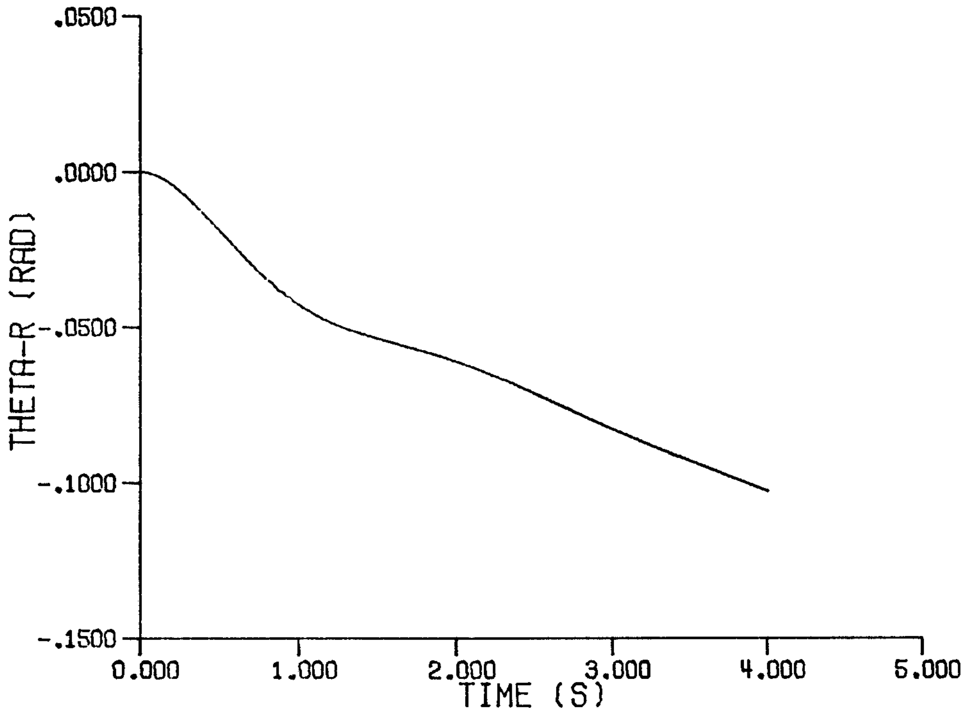


Figure 3.12
Output Time Responses - Example 3.3



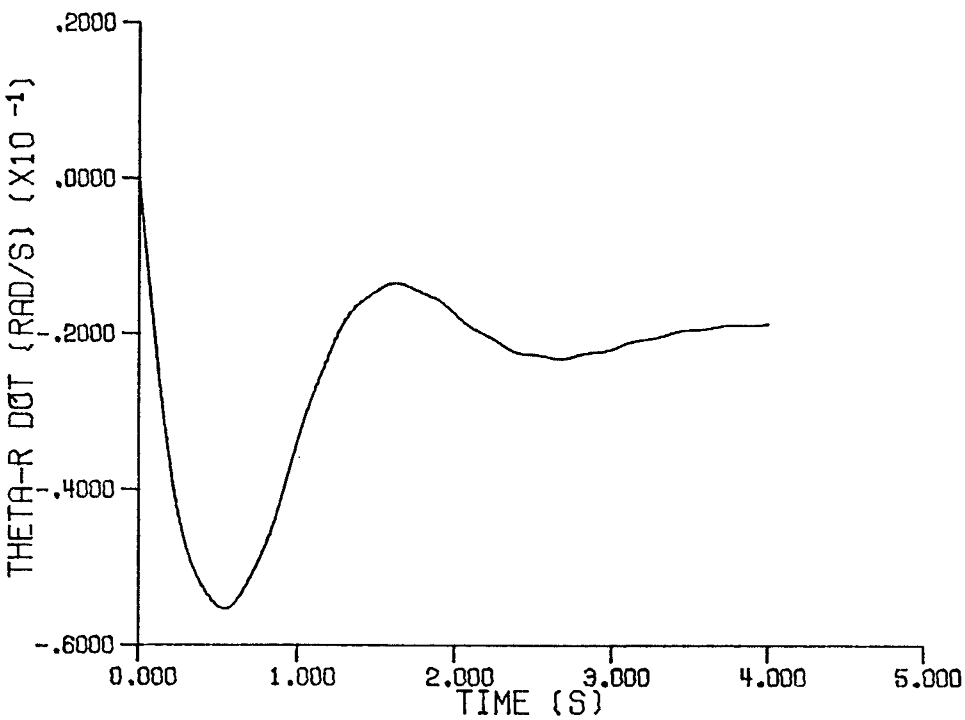
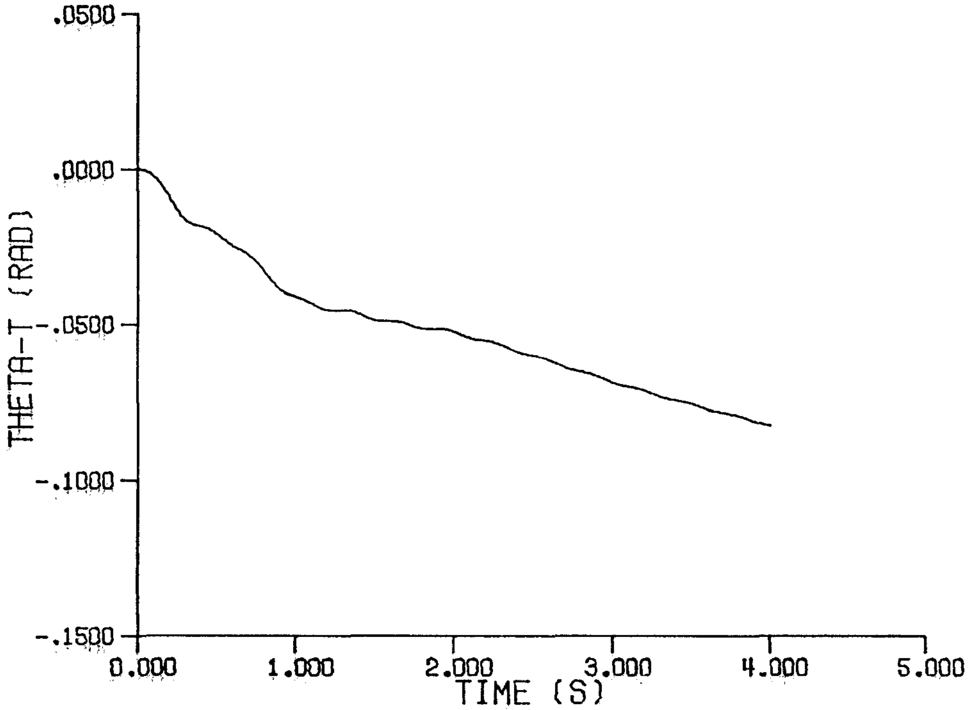


Figure 3.12, continued



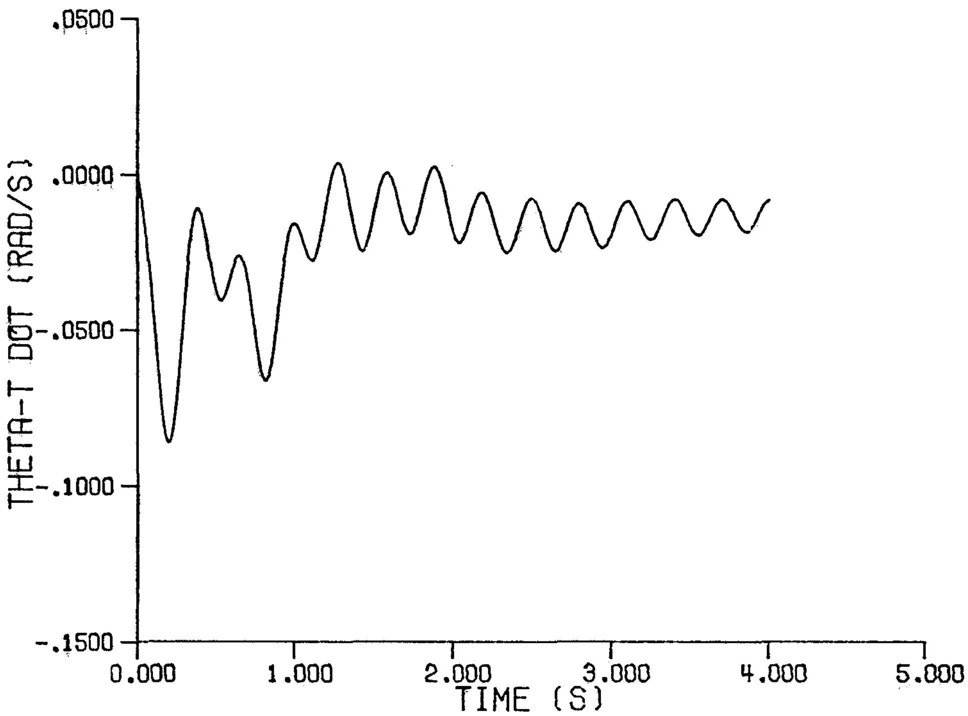


Figure 3.12, continued

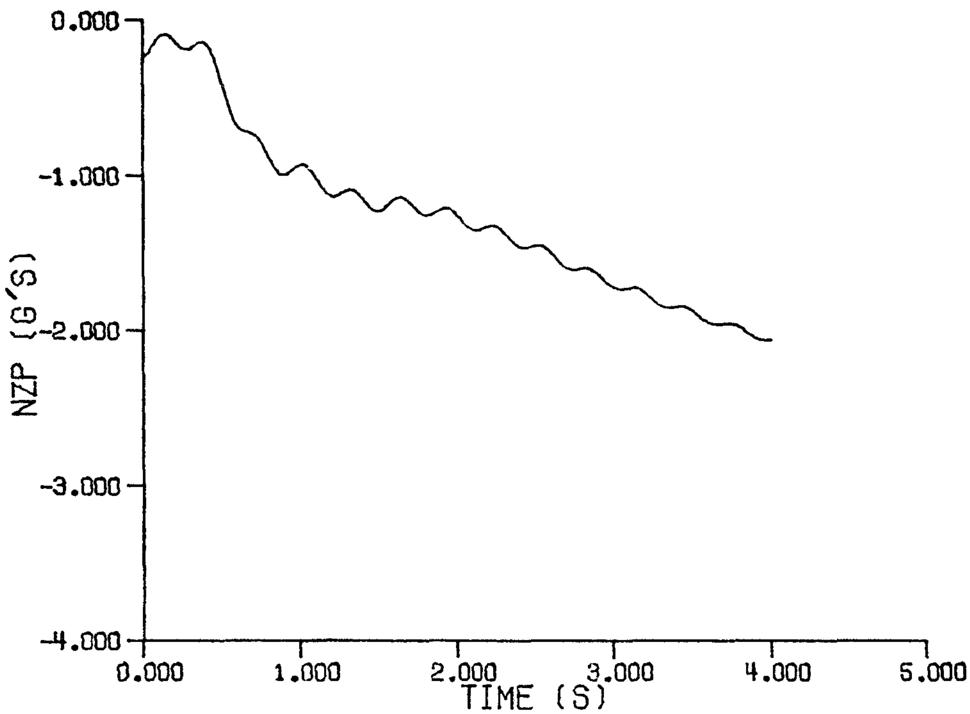


Figure 3.12, concluded

Comments on Example 3.3

By using an individual loop gain adjustment technique on the synthesis result of Example 3.1, one can stabilize the system without increasing the number of measurements. Adjusting the θ_t-to-δ_e gain has stabilized the phugoid eigenvalues and left the short period, first elastic, and second elastic poles relatively close to their original (augmented) locations.

By examining the closed-loop system eigenvectors one can see that the closed-loop system eigenspace obtained using this synthesis has drifted a little, but is still close to the achievable eigenspace (that obtained in Example 3.1). This eigenspace adjustment is reflected in the closed-loop impulse residue magnitudes obtained from this synthesis. The synthesis of Example 3.1 yielded a control law that had practically eliminated the contribution of the first and second elastic modes to the θ_r and $\dot{\theta}_r$ outputs. By comparing the impulse residue magnitudes from Example 3.1 (Figure 3.5) and this example (Figure 3.11), one can see that the eigenspace adjustment has caused a slight increase in the contribution of the first elastic mode to the θ_r and $\dot{\theta}_r$ outputs. This can also be noted in the $\dot{\theta}_r$ time response. The slight ripple on this response is due to the first elastic mode.

As can be seen from the θ_r and $\dot{\theta}_r$ impulse residue magnitudes, the elastic mode's contribution is still small compared to the rigid-body modes contribution. Therefore, this synthesis still tends to achieve the desired synthesis objectives; improving the rigid-body dynamics and making the flexible aircraft respond more like a rigid aircraft.

DEA Conclusions

DEA is a control synthesis technique for directly determining measurement feedback control gains that will yield an achievable closed-loop eigenspace. For an observable, controllable system that has n states, m controls, and l measurements one can determine a gain matrix that will place l eigenvalues to desired locations and their l associated eigenvectors as close as possible in a least squares sense to desired eigenvectors (assuming $l > \mathrm{m}$).

Using DEA the designer chooses l desired eigenvalues and their associated eigenvectors to achieve a desired synthesis objective (such as adjusting residue magnitudes, uncoupling system modes, reducing a certain eigenvalue's contribution to an output, response shaping, etc.) The desired eigenspace is used to determine the system's achievable eigenspace. By comparing the desired and achievable eigenspace, the designer can obtain an estimate of how close the synthesis will come to the desired synthesis objective. This achievable eigenspace is then used to calculate control gains. As the examples demonstrated, DEA has the potential to be a useful control synthesis technique. This technique can be used alone in an iterative fashion as demonstrated in Examples 3.1 and 3.2 or, it can be used in conjunction with other control synthesis techniques (as demonstrated in Example 3.3) to achieve desired results. When using DEA in an iterative fashion, the designer synthesizes a control law using a small number of measurements. If the synthesis results do not meet the desired synthesis objective, the number of measurements is increased to obtain more design freedom and a new control law is synthesized. The number of measurements is increased until an acceptable synthesis is obtained.

The examples also presented the two major disadvantages of this technique. First, this technique does not always yield improved performance. In some cases, it can yield an unstable closed-loop system. Using DEA the designer can only place l of the closed-loop poles. The designer has no information concerning the closed-loop locations of the remaining (n - l) eigenvalues and no way of predicting beforehand their directions of movement. Experience has shown that the choice of the l desired eigenvectors can significantly effect the movement of the unplaced poles.

If all of the states are available for feedback, one could use DEA with full-state feedback and place all of the poles of the closed-loop system. An example of this using a model that includes actuator dynamics is presented in Reference [13]. In this example, by placing all the poles of the closed-loop system, it was possible to obtain acceptable robustness, reasonable actuator bandwidths, and achieve the desired synthesis results. This model with actuator dynamics and the issue of required actuator bandwidths are considered further in Chapter IV.

It is possible for this technique to yield unacceptably high control gains. There is no systematic way of trading control energy for eigenspace assignment goals. In some designs, exactly obtaining an achievable eigenspace may require a large control effort when just getting close would yield satisfactory results. A control system that utilizes large control effort may be very susceptible to sensor noise.

The next chapter presents a possible alternative to the direct eigenspace assignment technique.

CHAPTER IV
LINEAR QUADRATIC EIGENSPACE ASSIGNMENT

As was shown in Chapter III, Direct Eigenspace Assignment is a very useful control synthesis technique but it has two major disadvantages. First, there is no systematic way to trade eigenspace assignment goals versus gain magnitudes. Second, when full state feedback is not used the $(n - l)$ unspecified system poles can move to unpredictable locations, sometimes even destabilizing the system.

A possible alternative is a synthesis method based upon the work of C.A. Harvey and G. Stein [9,10]. This procedure takes advantage of the asymptotic modal characteristics of multi-variable linear quadratic regulators. It is based upon Harvey and Stein's procedure for selecting quadratic weighting matrices that will asymptotically yield an achievable eigenspace in a linear quadratic control synthesis, as the cost function weighting on the control tends toward zero. This method shall be referred to as Linear Quadratic Eigenspace Assignment (LQEA).

Asymptotic Modal Properties

In their paper, "Quadratic Weights for Asymptotic Regulator Properties", Harvey and Stein [9] present the following theorem summarizing the asymptotic modal properties of an LQ regulator as a function of the weights $\mathbf{Q}$ and $\mathbf{R}$.

<u>Theorem 4.1</u>

Given the system

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \qquad \text{(system dynamics)} \tag{4.1a}$$

$$\underline{r} = \mathbf{H}\underline{x} \qquad \text{(system responses)} \tag{4.1b}$$

where $\underline{x} \in R^n$, $\underline{u} \in R^m$, and $\underline{r} \in R^m$.

Consider the linear quadratic regulator

$$\underline{u} = \mathbf{G}\underline{x} \tag{4.2}$$

$$\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}/\rho \tag{4.3}$$

where **P** is a solution to the steady state Ricatti equation

$$0 = \mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{Q} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P}/\rho \tag{4.4}$$

with cost criterion

$$J = \int_0^\infty (\underline{r}^T\underline{r} + \rho\underline{u}^T\mathbf{R}\underline{u})dt \tag{4.5}$$

$$= \int_0^\infty (\underline{x}^T\mathbf{Q}\,\underline{x} + \rho\underline{u}^T\mathbf{R}\underline{u})dt$$

where $\mathbf{Q} = \mathbf{H}^T\mathbf{H}$. Assume further that:

1) Rank $(\mathbf{HB}) = m$
2) the zero's of det $(\mathbf{H}(sI_n - \mathbf{A})^{-1}\mathbf{B})$ are distinct, have negative real parts, and do not belong to the spectrum of **A**.

Then, the optimally controlled system has the following properties:

Asymptotically Finite Modes

As ρ tends toward zero there are (n - m) closed-loop system eigenvalues of the form

$$\lambda_i(\rho) \rightarrow \lambda_i^f \qquad i = 1,2,\ldots,(n-m) \tag{4.6}$$

$$|\lambda_i^f| < \infty$$

with associated eigenvectors

$$\underline{\nu}_i(\rho) \rightarrow (\lambda_i^f \mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\underline{w}_i^f \quad (4.7)$$

where λ_i^f and $\underline{w}_i^f$ are defined by

$$\mathbf{H}(\lambda_i^f \mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\underline{w}_i^f = 0 \quad (4.8)$$

Asymptotically Infinite Modes

As ρ tends toward zero there are also m eigenvalues of the form

$$\lambda_j(\rho) \rightarrow s_j^\infty / \rho^{1/2} \quad (4$$

$$|s_j^\infty| < \infty; \; j = 1,2,\ldots,m$$

with associated eigenvectors

$$\underline{\nu}_j(\rho) \rightarrow \mathbf{B}\underline{w}_j^\infty \quad (4.$$

where $s_j^\infty(\rho)$ and $\underline{w}_j^\infty$ are defined by

$$\mathbf{R} = \mathbf{N}^{-T}\mathbf{S}^{-2}\mathbf{N}^{-1} \quad (4.11)$$

$$\mathbf{Y}^T\mathbf{Y} = (\mathbf{H}_o\mathbf{B})^{-T}\mathbf{N}^{-T}\mathbf{N}^{-1}(\mathbf{H}_o\mathbf{B})^{-1} \quad (4.12)$$

with

$$\mathbf{N} = [\underline{w}_1^\infty \mid \underline{w}_2^\infty \mid \cdots \mid \underline{w}_m^\infty]$$

$$\mathbf{S} = \text{diag}\,[s_1^\infty, s_2^\infty, \ldots, s_m^\infty]$$

$$\mathbf{H} = \mathbf{Y}\mathbf{H}_o$$

This concludes the Theorem.

Discussion of Theorem

As ρ tends toward zero (n - m) closed-loop system eigenvalues tend toward the placed values

$$\lambda_i(\rho) \rightarrow \lambda_i^f \qquad i=1,\ldots,(n-m) \tag{4.6}$$

These desired finite eigenvalues, λ_i^f, are the (n - m) transmission zeros of the system given by Equations (4.1a) and (4.1b) [15]. Their associated eigenvectors tend toward

$$\underline{\nu}_i(\rho) \rightarrow (\lambda_i^f \, \mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\underline{w}_i^f = \underline{\nu}_{a_i} \tag{4.7}$$

By comparing this result with Equation (3.7) of Chapter III, one can see that this equation is just an expression for the achievable eigenvectors of the system.

The remaining m closed-loop system eigenvalues tend toward infinity in m first order Butterworth patterns at a rate of

$$\lambda_j(\rho) \rightarrow s_j^{\infty}/\rho^{1/2} = \lambda_j^{\infty} \qquad j=1, \cdots, m \tag{4.9}$$

and their associated eigenvectors tend towards the vectors

$$\underline{\nu}_j(\rho) \rightarrow \mathbf{B}\underline{w}_j^{\infty} = \underline{\nu}_j^{\infty} \tag{4.10}$$

as ρ tends towards zero. In this development λ_j^{∞} shall be referred to as a desired infinite eigenvalue and $\underline{\nu}_j^{\infty}$ shall be referred to as a desired infinite eigenvector.

Once values for λ_i^f, $\underline{w}_i^f$, s_j^{∞}, and $\underline{w}_j^{\infty}$ are chosen, the $\mathbf{Q}$ and $\mathbf{R}$ matrices can then be determined. The control weighting matrix, $\mathbf{R}$, can be determined easily from the equation

$$\mathbf{R} = \mathbf{N}^{-T}\mathbf{S}^{-2}\mathbf{N}^{-1} \tag{4.11}$$

where

$$\mathbf{S} = \mathrm{diag}[s_1^{\infty}, s_2^{\infty}, \cdots s_m^{\infty}]$$

$$\mathbf{N} = [\,\underline{w}_1^{\infty} \mid \underline{w}_2^{\infty} \mid \cdots \mid \underline{w}_m^{\infty}\,]$$

The state weighting matrix, $\mathbf{Q}$, is calculated by first determining achievable closed-loop system eigenvectors for each λ_i^f and $\underline{w}_i^f$, using Equation (4.7). The achievable eigenvectors can be calculated for a given λ_i^f and desired eigenvector using the method of Chapter III. Concatenate these (n - m) achievable eigenvectors columnwise to form

$$\mathbf{V} = [\underline{\nu}_{a_1}{}^f \mid \underline{\nu}_{a_2}{}^f \mid \cdots \mid \underline{\nu}_{a_{(n-m)}}{}^f]$$

Using this matrix, calculate a system projection matrix $\mathbf{P}_n$

$$\mathbf{P}_n = \mathbf{I}_n - \mathbf{V}[\overline{\mathbf{V}}^T\mathbf{V}]^{-1}\,\overline{\mathbf{V}}^T \tag{4.13}$$

partitioned in the following way

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{22} \end{bmatrix}$$

where $\mathbf{P}_{22}$ is a mxm submatrix.

Calculate

$$\mathbf{H}_o = [\mathbf{P}_{22}^{-1}\,\mathbf{P}_{12}^T \mid \mathbf{I}_m] \tag{4.14}$$

and

$$\mathbf{Y} = (\mathbf{H}_o\mathbf{B}\mathbf{N})^{-1} \tag{4.15}$$

The state weighting matrix, $\mathbf{Q}$, can be determined by using the equation

$$\mathbf{Q} = \mathbf{H}^T\mathbf{H} = (\mathbf{Y}\mathbf{H}_o)^T(\mathbf{Y}\mathbf{H}_o) \tag{4.16}$$

<u>Closed-loop System</u>

Using this method, the augmented (closed-loop) system may be written

$$\dot{\underline{x}} = (\mathbf{A} + \mathbf{B}\mathbf{G})\underline{x} + \mathbf{B}\underline{u}_p \tag{4.17}$$

The closed-loop input to state transfer functions are given by

$$\underline{x}(s) = [s\mathbf{I}_n - (\mathbf{A} + \mathbf{BG})]^{-1}\, \mathbf{B}\underline{u}_p(s) \tag{4.18}$$

The system's physical output responses are given by

$$\underline{y} = \mathbf{C}\underline{x} \tag{4.19}$$

Therefore, the response transfer functions for the pilot input, $\underline{u}_p$, are

$$\underline{y}(s) = \mathbf{C}\, [s\mathbf{I}_n - (\mathbf{A} + \mathbf{BG})]^{-1}\mathbf{B}\underline{u}_p(s) \tag{4.20}$$

By selecting the desired eigenspace, one could say that the designer is placing the poles and zeros of these transfer functions to desirable locations. This control synthesis technique is summarized in the following section.

LQEA Control Synthesis Technique Summary

The control synthesis is a two part procedure:

1) Select values for λ_i^f, $\underline{\nu}_{d_i}$, s_j^∞, and $\underline{w}_j^\infty$. Use Harvey and Stein's method to determine quadratic weighting matrices **Q** and **R** that will asymptotically yield the desired closed-loop eigenspace.
2) Perform an LQ synthesis using various values of control weighting (ρ). Select the synthesis that will yield a closed-loop system as close as needed to the desired eigenspace.

Two examples will be presented in the following section to illustrate the use of this technique.

LQEA Examples

Two examples will be presented to demonstrate some properties of this synthesis technique. The two examples differ only in the desired eigenvalue and eigenvector for the second elastic mode. The results are intended to demonstrate how significantly the desired eigenspace can affect the synthesis results.

The large flexible aircraft model used in Chapter III assumed "infinitely fast" actuators. Both of these examples will also consider this aircraft model, but will include elevator and control vane actuator dynamics. These dynamics will be modeled as:

$$\frac{\delta_i}{\delta_{i_c}} = \frac{(1/\tau)}{s + (1/\tau)}$$

$$\dot{\delta}_i = -\frac{1}{\tau}\delta_i + \frac{1}{\tau}\delta_{i_c}$$

where

δ_i = actual control surface deflection (radians)

δ_{i_c} = commanded control surface deflection (radians)

For the control vane actuator, $(1.0/\tau) = 10.0\ \text{sec}^{-1}$; for the elevator actuator $(1.0/\tau) = 9.0\ \text{sec}^{-1}$.

System

The (unscaled) model that will be used is then of the form:

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \quad \text{(system dynamics)}$$

$$\underline{y} = \mathbf{C}\underline{x} \quad \text{(system responses)}$$

$$\underline{u} = \underline{u}_c + \underline{u}_p \quad \text{(total control input)}$$

$$\underline{u}_c = \mathbf{G}\underline{x} \quad \text{(feedback control law)}$$

$$\underline{u}_p = \text{pilot's input}$$

States

$$\underline{x}^T = (\alpha, \dot{\theta}_r, u_f, \theta_r, \xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2, \delta_e, \delta_{cv})$$

where

α = angle of attack (radians)

$\dot{\theta}_r$ = rigid body pitch rate (radians/sec)

u_f = forward velocity (ft/sec)

θ_r = rigid body pitch angle (radians)

ξ_1 = mode one generalized deflection (dimensionless)

ξ_2 = mode two generalized deflection (dimensionless)

$\dot{\xi}_1$ = mode one generalized deflection rate (1/sec)

$\dot{\xi}_2$ = mode two generalized deflection rate (1/sec)

δ_e = elevator deflection (radians)

δ_{cv} = forward control vane deflection (radians)

Controls

$$\underline{u}_c^T = (\delta_{e_c} , \delta_{cv_c})$$

$$\underline{u}_p^T = (\delta_{e_c} , 0.0)$$

where

δ_{e_c} = commanded elevator deflection (radians)

δ_{cv_c} = commanded forward control vane deflection (radians)

Outputs

$$\underline{y}^T = (u_f , \theta_r , \dot{\theta}_r , \gamma , \theta_t , \dot{\theta}_t , n_{z_p})$$

where

u_f = forward velocity (ft/sec)

θ_r = rigid body pitch angle (radians)

$\dot{\theta}_r$ = rigid body pitch rate (radians/sec)

γ = flight path angle (radians)

θ_t = total pitch angle (radians)

$\dot{\theta}_t$ = total pitch rate (radians/sec)

n_{z_p} = plunge acceleration at the cockpit (g's)

The modified open-loop system matrices can be found in Appendix A.4.

Example 4.1

The design objectives for this example are the same as those of Example 3.1; reducing the contribution of the elastic modes to the rigid-body responses, specifically $\dot{\theta}_r$. For this example dim($\underline{x}$)=n=10 and dim($\underline{u}$)=m=2, therefore one may specify eight desired finite eigenvalues/eigenvectors and two desired infinite eigenvalues/eigenvectors.

Specification of Synthesis Parameters

Desired Finite Eigenvalues. With this technique one has the freedom to specify the short period, phugoid, first elastic, and second elastic eigenvalues. Values for the locations of the desired finite short period, phugoid, and first elastic eigenvalues are chosen to be the same as in Example 3.2 . The second elastic eigenvalue is chosen to be a value close to its open-loop location. The damping is increased slightly, the natural frequency is taken to be the same as the open-loop.

$$\lambda^f_{1,2} = \lambda^f_{sp} = -1.49 \pm j\ 2.37 = (\omega_n = 2.799\ (\text{rad/s})\ ,\ \varsigma = 0.532)$$

$$\lambda^f_{3,4} = \lambda^f_{ph} = -0.0015 \pm j\ 0.0672 = (\omega_n = 0.067\ (\text{rad/s})\ ,\ \varsigma = 0.022)$$

$$\lambda^f_{5,6} = \lambda^f_{\xi_1} = -\,1.758 \pm j\;8.611 = (\omega_n = 8.789\ \text{(rad/s)}\ ,\ \varsigma = 0.200)$$

$$\lambda^f_{7,8} = \lambda^f_{\xi_2} = -\,0.534 \pm j\;21.348 = (\omega_n = 21.35\ \text{(rad/s)}\ ,\ \varsigma = 0.025)$$

Desired Finite Eigenvectors. The desired eigenvectors for the short period, phugoid, and first elastic modes are chosen to be the same as specified in Example 3.2 . The second elastic eigenvector is chosen to be the same as the open-loop second elastic eigenvector. Values chosen for these desired eigenvectors can be found in Table 4.1. In this table, the * denotes elements whose placement is taken as arbitrary. The weighting on each of the specified elements is chosen to be unity.

Achievable Eigenvectors. The achievable eigenvectors are calculated in the same way as for the DEA technique. First Equation (3.10) is used to calculate values of $\underline{w}_i^f$ for i=1,...,(n-m). Then Equation (3.7) is used to calculate the achievable eigenvectors. These achievable eigenvectors are given in Table 4.2. In this table, the eigenvectors have been scaled such that the first element is unity.

Desired Infinite Eigenvalues. Values for s_j^∞, which define the "infinite modes", are chosen to maintain a two-to-one actuator bandwidth ratio. Because the control vane is physically smaller than the elevator and is expected to play a large role in the control of the elastic modes, it is chosen to have the larger bandwidth.

$$s_1^\infty = s_{el}^\infty = 0.5$$

$$s_2^\infty = s_{cv}^\infty = 1.0$$

Desired Infinite Eigenvectors. Values for $\underline{w}_j^\infty$ are chosen to achieve decoupled actuator eigenvectors.

Table 4.1
Desired Eigenvectors - Example 4.1

```
DESIRED EIGENVECTORS

EIGENVALUES
 1:     -1.4900 + J    2.3700
 2:     -1.4900 - J    2.3700
 3:       -.0015 + J    .0672
 4:       -.0015 - J    .0672


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                              1                                  2
(     *       ,       *       )     (     *       ,       *       )
(  1.0000E+00,   1.0000E+02)     (  1.0000E+00,  -1.0000E+02)
(     *       ,       *       )     (     *       ,       *       )
(     *       ,       *       )     (     *       ,       *       )
(  5.0000E-03,   3.0000E+01)     (  5.0000E-03,  -3.0000E+01)
(  5.0000E-04,  -1.7000E+02)     (  5.0000E-04,   1.7000E+02)
(  1.0000E-02,   1.5000E+02)     (  1.0000E-02,  -1.5000E+02)
(  5.0000E-03,  -5.0000E+01)     (  5.0000E-03,   5.0000E+01)
(     *       ,       *       )     (     *       ,       *       )
(     *       ,       *       )     (     *       ,       *       )

                              3                                  4
(     *       ,       *       )     (     *       ,       *       )
(     *       ,       *       )     (     *       ,       *       )
(  1.0000E+00,   1.8000E+02)     (  1.0000E+00,  -1.8000E+02)
(     *       ,       *       )     (     *       ,       *       )
(  1.0000E-02,   1.0000E+00)     (  1.0000E-02,  -1.0000E+00)
(  5.0000E-03,   1.8000E+02)     (  5.0000E-03,  -1.8000E+02)
(  5.0000E-04,   9.1000E+01)     (  5.0000E-04,  -9.1000E+01)
(  1.0000E-04,  -9.0000E+01)     (  1.0000E-04,   9.0000E+01)
(     *       ,       *       )     (     *       ,       *       )
(     *       ,       *       )     (     *       ,       *       )
```

State vector (unscaled): $\mathbf{x}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2 , \delta_e , \delta_{cv})$

Table 4.1, concluded

```
EIGENVALUES
 5:     -1.7577 + J    8.6109
 6:     -1.7577 - J    8.6109
 7:       -.5339 + J   21.3480
 8:       -.5339 - J   21.3480


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                              5                                 6
(  1.0000E-03,               0)    (  1.0000E-03,               0)
(  1.0000E-02,  6.0000E+01)        (  1.0000E-02, -6.0000E+01)
(  1.0000E-04,  7.0000E+01)        (  1.0000E-04, -7.0000E+01)
(  1.0000E-03, -3.0000E+01)        (  1.0000E-03,  3.0000E+01)
(      *     ,       *     )       (      *     ,       *     )
(      *     ,       *     )       (      *     ,       *     )
(  1.0000E+00,  1.4000E+02)        (  1.0000E+00, -1.4000E+02)
(      *     ,       *     )       (      *     ,       *     )
(      *     ,       *     )       (      *     ,       *     )
(      *     ,       *     )       (      *     ,       *     )

                              7                                 8
(  1.0000E+00,               0)    (  1.0000E+00,               0)
(  2.0194E+01,  6.2084E+01)        (  2.0194E+01, -6.2084E+01)
(  2.6518E-03,  7.2826E+01)        (  2.6518E-03, -7.2826E+01)
(  9.4275E-01, -2.9069E+01)        (  9.4275E-01,  2.9069E+01)
(  3.3105E+00, -1.2285E+02)        (  3.3105E+00,  1.2285E+02)
(  6.0998E+01, -7.1512E+01)        (  6.0998E+01,  7.1512E+01)
(  7.0620E+01, -3.1662E+01)        (  7.0620E+01,  3.1662E+01)
(  1.3023E+03,  1.9696E+01)        (  1.3023E+03, -1.9696E+01)
(      *     ,       *     )       (      *     ,       *     )
(      *     ,       *     )       (      *     ,       *     )
```

Table 4.2
Achievable Eigenvectors - Example 4.1

ACHIEVABLE EIGENVECTORS

EIGENVALUES

1:	-1.4900	+	J	2.3700
2:	-1.4900	-	J	2.3700
3:	-.0015	+	J	.0672
4:	-.0015	-	J	.0672

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

1		2	
(1.0000E+00,	0)	(1.0000E+00,	0)
(2.5129E+00,	9.8272E+01)	(2.5129E+00,	-9.8272E+01)
(1.9881E-02,	4.5495E+01)	(1.9881E-02,	-4.5495E+01)
(8.9764E-01,	-2.3886E+01)	(8.9764E-01,	2.3886E+01)
(9.0782E-03,	2.7453E+01)	(9.0782E-03,	-2.7453E+01)
(5.2106E-02,	-1.7034E+02)	(5.2106E-02,	1.7034E+02)
(2.5414E-02,	1.4961E+02)	(2.5414E-02,	-1.4961E+02)
(1.4587E-01,	-4.8185E+01)	(1.4587E-01,	4.8185E+01)
(1.5658E-01,	-1.5405E+02)	(1.5658E-01,	1.5405E+02)
(4.4994E+00,	-1.5586E+02)	(4.4994E+00,	1.5586E+02)

3		4	
(1.0000E+00,	0)	(1.0000E+00,	0)
(3.2734E-01,	1.7157E+02)	(3.2734E-01,	-1.7157E+02)
(2.3967E+00,	-1.7915E+02)	(2.3967E+00,	1.7915E+02)
(4.8700E+00,	8.0312E+01)	(4.8700E+00,	-8.0312E+01)
(2.3473E-02,	2.0170E+00)	(2.3473E-02,	-2.0170E+00)
(5.1546E-02,	1.8022E+02)	(5.1546E-02,	-1.8022E+02)
(1.5778E-03,	9.3279E+01)	(1.5778E-03,	-9.3279E+01)
(3.4647E-03,	-8.8518E+01)	(3.4647E-03,	8.8518E+01)
(1.3566E-01,	1.8008E+02)	(1.3566E-01,	-1.8008E+02)
(3.7290E+00,	1.8057E+02)	(3.7290E+00,	-1.8057E+02)

State vector (unscaled): $\mathbf{x}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2 , \delta_e , \delta_{cv})$

Table 4.2, concluded

```
EIGENVALUES
 5:     -1.7577 + J    8.6109
 6:     -1.7577 - J    8.6109
 7:       -.5339 + J   21.3480
 8:       -.5339 - J   21.3480


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                            5                                   6
(  1.0000E+00,              0)      (  1.0000E+00,              0)
(  1.9826E+00,  9.2299E+01)      (  1.9826E+00, -9.2299E+01)
(  3.8565E-03,  7.6557E+01)      (  3.8565E-03, -7.6557E+01)
(  2.2560E-01, -9.2379E+00)      (  2.2560E-01,  9.2379E+00)
(  2.3537E+01,  7.0103E+01)      (  2.3537E+01, -7.0103E+01)
(  1.0035E-01,  3.6410E+01)      (  1.0035E-01, -3.6410E+01)
(  2.0685E+02,  1.7164E+02)      (  2.0685E+02, -1.7164E+02)
(  8.8194E-01,  1.3795E+02)      (  8.8194E-01, -1.3795E+02)
(  7.1912E+00,  6.1618E+01)      (  7.1912E+00, -6.1618E+01)
(  2.2736E+02, -5.7208E+01)      (  2.2736E+02,  5.7208E+01)

                            7                                   8
(  1.0000E+00,              0)      (  1.0000E+00,              0)
(  1.9760E+01,  6.7263E+01)      (  1.9760E+01, -6.7263E+01)
(  2.6355E-03,  7.5514E+01)      (  2.6355E-03, -7.5514E+01)
(  9.2533E-01, -2.4169E+01)      (  9.2533E-01,  2.4169E+01)
(  2.4966E+00, -1.3127E+02)      (  2.4966E+00,  1.3127E+02)
(  4.5728E+01, -7.9773E+01)      (  4.5728E+01,  7.9773E+01)
(  5.3314E+01, -3.9837E+01)      (  5.3314E+01,  3.9837E+01)
(  9.7650E+02,  1.1659E+01)      (  9.7650E+02, -1.1659E+01)
(  5.5269E+00,  6.5516E+00)      (  5.5269E+00, -6.5516E+00)
(  1.3441E+02, -1.7060E+02)      (  1.3441E+02,  1.7060E+02)
```

$$(\underline{w}_1^{\infty})^T = (1.0,0.0)$$

$$(\underline{w}_2^{\infty})^T = (0.0,1.0)$$

The control weighting matrix, **R**, can now be calculated by using Equation (4.11). The state weighting matrix, **Q**, can be calculated using Equations (4.13) through (4.16). These matrices are given in Table 4.3.

Closed Loop System Analysis

Four LQ control laws will be synthesized for the following values of control weighting, ρ = (1.0 x 10^6 , 1.0 x 10^{-2} , 1.0 x 10^{-4} , 1.0 x 10^{-6}).

Control synthesis for ρ = 1.0 x 10^6. The LQ synthesis for this large value of ρ yields a closed-loop system whose eigenspace is close to that of the open-loop system.

The system closed-loop eigenvalues are:

$$\lambda_{sp} = -1.348 \pm j\ 2.193 = (\omega_n = 2.574\ (rad/s)\ ,\ \varsigma = 0.524)$$

$$\lambda_{ph} = -0.0001 \pm j\ 0.0526 = (\omega_n = 0.053\ (rad/s)\ ,\ \varsigma = 0.002)$$

$$\lambda_{\xi_1} = -0.7264 \pm j\ 8.758 = (\omega_n = 8.788\ (rad/s)\ ,\ \varsigma = 0.083)$$

$$\lambda_{\xi_2} = -0.4564 \pm j\ 21.351 = (\omega_n = 21.36\ (rad/s)\ ,\ \varsigma = 0.021)$$

$$\lambda_{el} = -9.0$$

$$\lambda_{cv} = -10.0$$

The eigenvectors of (**A** + **BG**) are given in Table 4.4. The gain matrix to

Table 4.3
State and Control Weighting Matrices

THE QQ MATRIX (10 BY 10)

	1	2	3	4	5	6	7	8	9
1	1.8328E-01	3.1022E-02	9.3519E-03	2.7917E-05	3.1320E-01	-3.0618E-02	3.8885E-02	4.6157E-03	1.9047E-03
2	3.1022E-02	5.2509E-03	1.5828E-03	4.7212E-06	5.3002E-02	-5.1819E-03	6.5813E-03	7.8105E-04	3.4186E-04
3	9.3519E-03	1.5828E-03	4.7778E-04	1.4422E-06	1.6029E-02	-1.5647E-03	1.9864E-03	2.3644E-04	1.2416E-05
4	2.7917E-05	4.7212E-06	1.4422E-06	4.7889E-09	4.9149E-05	-4.7374E-06	5.9922E-06	7.3087E-07	-2.2818E-06
5	3.1320E-01	5.3002E-02	1.6029E-02	4.9149E-05	5.3911E-01	-5.2521E-02	6.6637E-02	7.9625E-03	-3.6563E-03
6	-3.0618E-02	-5.1819E-03	-1.5647E-03	-4.7374E-06	-5.2521E-02	5.1251E-03	-6.5056E-03	-7.7490E-04	3.4817E-05
7	3.8885E-02	6.5813E-03	1.9864E-03	5.9922E-06	6.6637E-02	-6.5056E-03	8.2590E-03	9.8287E-04	7.2551E-05
8	4.6157E-03	7.8105E-04	2.3644E-04	7.3087E-07	7.9625E-03	-7.7490E-04	9.8287E-04	1.1768E-04	-8.5298E-05
9	1.9047E-03	3.4186E-04	1.2416E-05	-2.2818E-06	-3.6563E-03	3.4817E-05	7.2551E-05	-8.5298E-05	1.2346E-02
10	4.2776E-02	7.2397E-03	2.1858E-03	6.6085E-06	7.3350E-02	-7.1589E-03	9.0877E-03	1.0821E-03	0.

	10
1	4.2776E-02
2	7.2397E-03
3	2.1858E-03
4	6.6085E-06
5	7.3350E-02
6	-7.1589E-03
7	9.0877E-03
8	1.0821E-03
9	0.
10	1.0000E-02

THE RR MATRIX (2 BY 2)

	1	2
1	4.0000E+00	0.
2	0.	1.0000E+00

C-2

Table 4.4
Closed-loop Eigenvectors - Example 4.1, $\rho = 10^6$

```
LQEA RHO=1.0E+06

EIGENVALUES
 1:     -1.3479 + J     2.1929
 2:     -1.3479 - J     2.1929
 3:      -.0001 + J      .0526
 4:      -.0001 - J      .0526


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                             1                                2
(  1.0000E+00,              0)    (  1.0000E+00,              0)
(  2.2948E+00,  9.5837E+01)       (  2.2948E+00, -9.5837E+01)
(  2.1474E-02,  4.5165E+01)       (  2.1474E-02, -4.5165E+01)
(  8.9153E-01, -2.5741E+01)       (  8.9153E-01,  2.5741E+01)
(  2.8846E-01,  2.5619E+01)       (  2.8846E-01, -2.5619E+01)
(  5.1928E-02, -1.7132E+02)       (  5.1928E-02,  1.7132E+02)
(  7.4251E-01,  1.4720E+02)       (  7.4251E-01, -1.4720E+02)
(  1.3366E-01, -4.9747E+01)       (  1.3366E-01,  4.9747E+01)
(  8.0402E-08,  1.3879E+02)       (  8.0402E-08, -1.3879E+02)
(  9.6495E-08, -1.4675E+02)       (  9.6495E-08,  1.4675E+02)

                             3                                4
(  1.0000E+00,              0)    (  1.0000E+00,              0)
(  1.7976E-01, -1.9256E+02)       (  1.7976E-01,  1.9256E+02)
(  2.1292E+00, -1.7933E+02)       (  2.1292E+00,  1.7933E+02)
(  3.4202E+00, -2.8268E+02)       (  3.4202E+00,  2.8268E+02)
(  2.5276E-01,  3.7983E-01)       (  2.5276E-01, -3.7983E-01)
(  5.2150E-02, -1.7984E+02)       (  5.2150E-02,  1.7984E+02)
(  1.3284E-02, -2.6949E+02)       (  1.3284E-02,  2.6949E+02)
(  2.7409E-03, -8.9716E+01)       (  2.7409E-03,  8.9716E+01)
(  8.6888E-05, -2.7297E+02)       (  8.6888E-05,  2.7297E+02)
(  2.0050E-05, -9.3516E+01)       (  2.0050E-05,  9.3516E+01)
```

State vector (unscaled): $x^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2 , \delta_e , \delta_{cv})$

Table 4.4, continued

```
EIGENVALUES
 5:      -.7264 + J    8.7584
 6:      -.7264 - J    8.7584
 7:      -.4564 + J   21.3506
 8:      -.4564 - J   21.3506


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                                 5                                     6
(  1.0000E+00,                  0)       (  1.0000E+00,                 0)
(  7.7404E+00, -2.9187E+02)       (  7.7404E+00,   2.9187E+02)
(  6.2077E-03, -2.8869E+02)       (  6.2077E-03,   2.8869E+02)
(  8.8074E-01, -2.6606E+01)       (  8.8074E-01,   2.6606E+01)
(  8.8783E+00, -2.3329E+02)       (  8.8783E+00,   2.3329E+02)
(  2.4390E-01, -2.1916E+02)       (  2.4390E-01,   2.1916E+02)
(  7.8027E+01, -1.3855E+02)       (  7.8027E+01,   1.3855E+02)
(  2.1435E+00, -1.2442E+02)       (  2.1435E+00,   1.2442E+02)
(  2.7734E-06, -3.3344E+02)       (  2.7734E-06,   3.3344E+02)
(  8.8981E-07, -3.4322E+02)       (  8.8981E-07,   3.4322E+02)

                                 7                                     8
(  1.0000E+00,                  0)       (  1.0000E+00,                 0)
(  2.0127E+01,  6.2180E+01)       (  2.0127E+01, -6.2180E+01)
(  2.6356E-03,  7.2932E+01)       (  2.6356E-03, -7.2932E+01)
(  9.4247E-01, -2.9045E+01)       (  9.4247E-01,  2.9045E+01)
(  3.3014E+00,  2.3713E+02)       (  3.3014E+00, -2.3713E+02)
(  6.0949E+01, -7.1516E+01)       (  6.0949E+01,  7.1516E+01)
(  7.0502E+01, -3.1647E+01)       (  7.0502E+01,  3.1647E+01)
(  1.3016E+03,  1.9709E+01)       (  1.3016E+03, -1.9709E+01)
(  2.4073E-07, -3.1940E+01)       (  2.4073E-07,  3.1940E+01)
(  3.1332E-07,  2.0149E+02)       (  3.1332E-07, -2.0149E+02)
```

Table 4.4, concluded

```
EIGENVALUES
 9:     -9.0000 + J          0
10:    -10.0000 + J          0
```

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

9		10	
(1.0000E+00,	0)	(1.0000E+00,	0)
(9.2712E+00,	-1.8000E+02)	(9.1426E+00,	1.8000E+02)
(6.8055E-03,	0)	(5.7315E-03,	0)
(1.0301E+00,	0)	(9.1426E-01,	0)
(2.2761E+00,	-1.8000E+02)	(2.7597E+00,	0)
(1.5883E-01,	0)	(1.1082E-01,	0)
(2.0485E+01,	0)	(2.7597E+01,	1.8000E+02)
(1.4295E+00,	-1.8000E+02)	(1.1082E+00,	1.8000E+02)
(5.4943E+00,	-1.8000E+02)	(7.3029E-06,	0)
(5.2624E-08,	0)	(1.9544E+02,	0)

obtain this eigenspace in the closed-loop system and the closed-loop system matrices for this synthesis are given in Appendix A.4. The closed-loop system residue magnitudes, for $\underline{u}_p$ = impulse, for the short period, phugoid, first elastic, and second elastic modes for each output are shown in Figure 4.1. Again note that the impulse residues are normalized for plotting such that the sum of the short period, phugoid, first elastic, and second elastic modes residue magnitudes in each response is unity. The output time responses due to a unit step in $\underline{u}_p$, with zero initial conditions, are shown in Figure 4.2.

Control synthesis for $\rho = 1.0 \times 10^{-2}$. The closed-loop system eigenvalues for this value of control weighting are:

$$\lambda_{1,2} = -2.264 \pm j\ 1.137 = (\omega_n = 2.534\ \text{(rad/s)}\ ,\ \varsigma = 0.894)$$

$$\lambda_{3,4} = -0.0490 \pm j\ 0.1464 = (\omega_n = 0.154\ \text{(rad/s)}\ ,\ \varsigma = 0.317)$$

$$\lambda_{5,6} = -7.385 \pm j\ 10.87 = (\omega_n = 13.14\ \text{(rad/s)}\ ,\ \varsigma = 0.562)$$

$$\lambda_{7,8} = -0.5396 \pm j\ 21.45 = (\omega_n = 21.46\ \text{(rad/s)}\ ,\ \varsigma = 0.025)$$

$$\lambda_9 = -10.05$$

$$\lambda_{10} = -13.22$$

The eigenvectors of $(\mathbf{A} + \mathbf{BG})$ are given in Table 4.5. The gain matrix and the closed-loop system matrices for this value of ρ are given in Appendix A.4. The normalized closed-loop system impulse residue magnitudes of the short period, phugoid, first elastic, and second elastic modes for each output are shown in Figure 4.3. The output time responses due to a unit step in $\underline{u}_p$, with zero initial conditions, are shown in Figure 4.4.

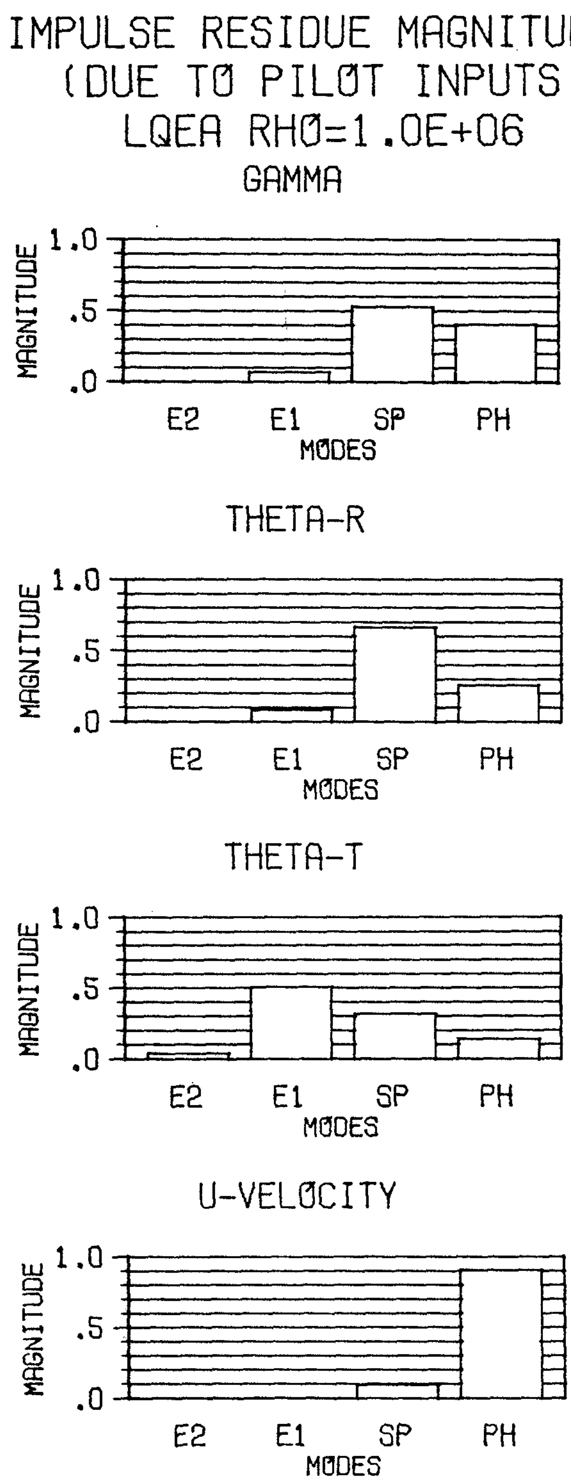


Figure 4.1
Impulse Residue Magnitudes - Example 4.1, $\rho = 10^6$

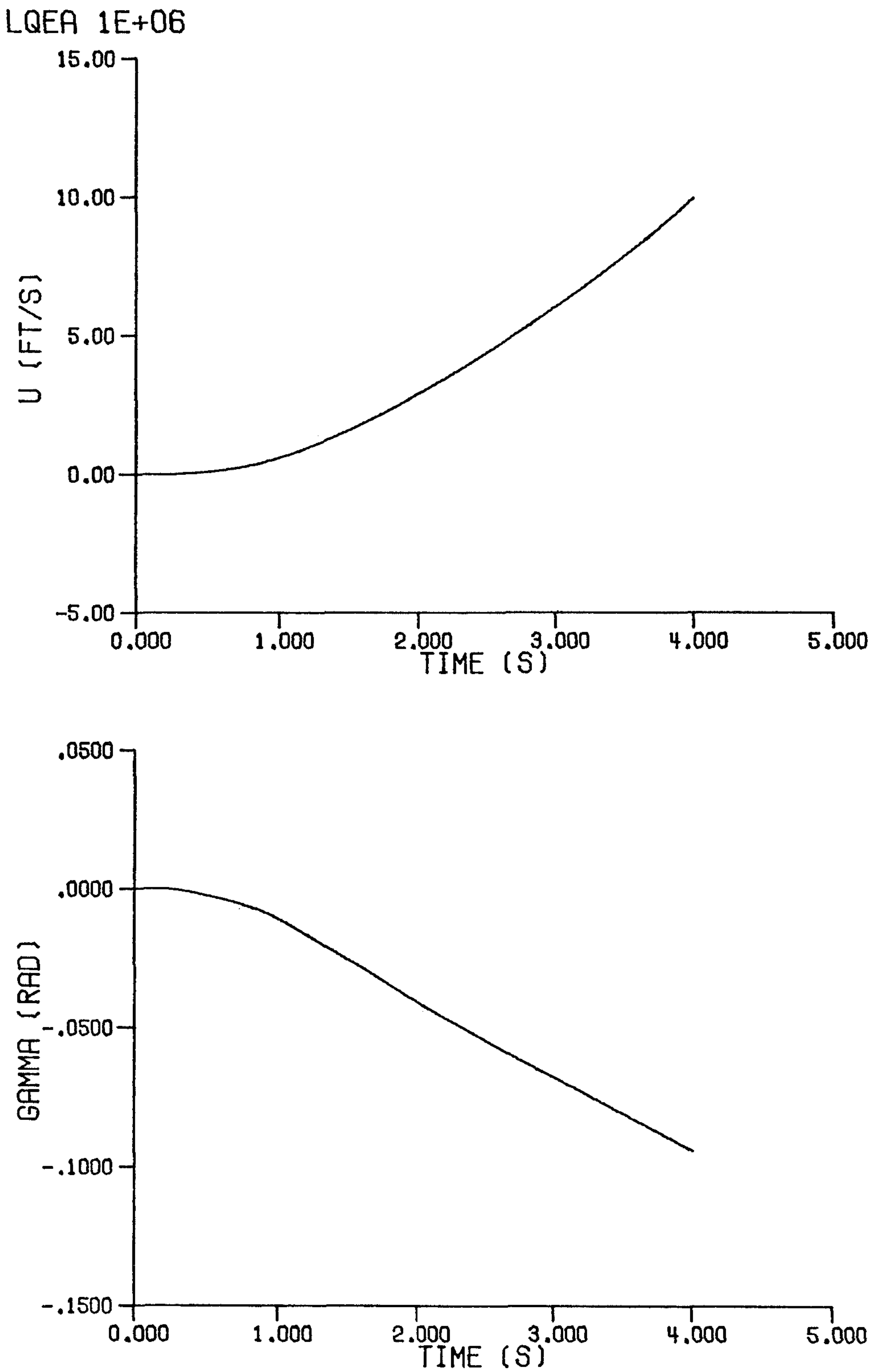


Figure 4.2
Output Time Responses - Example 4.1, $\rho = 10^6$

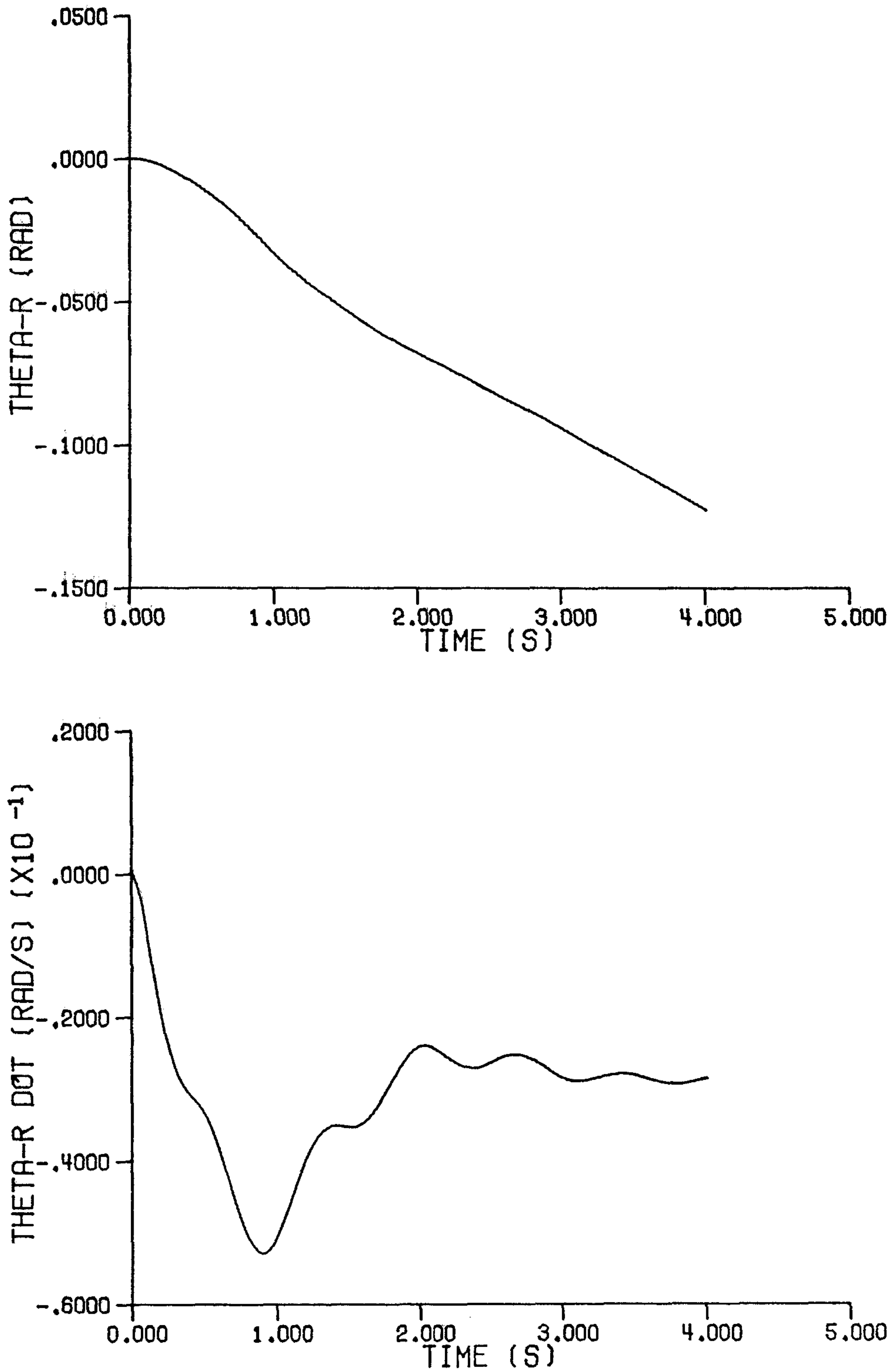


Figure 4.2, continued

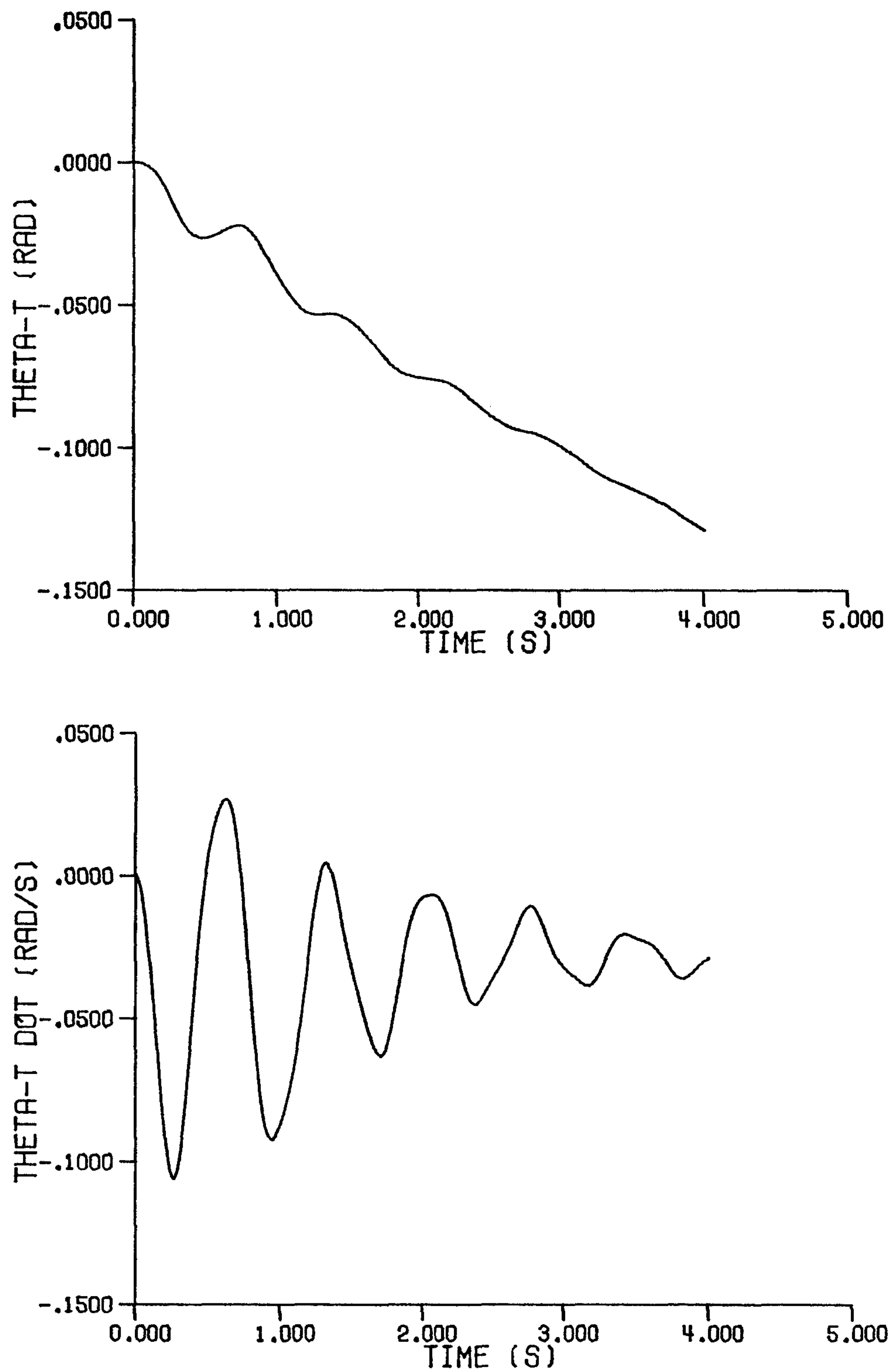


Figure 4.2, continued

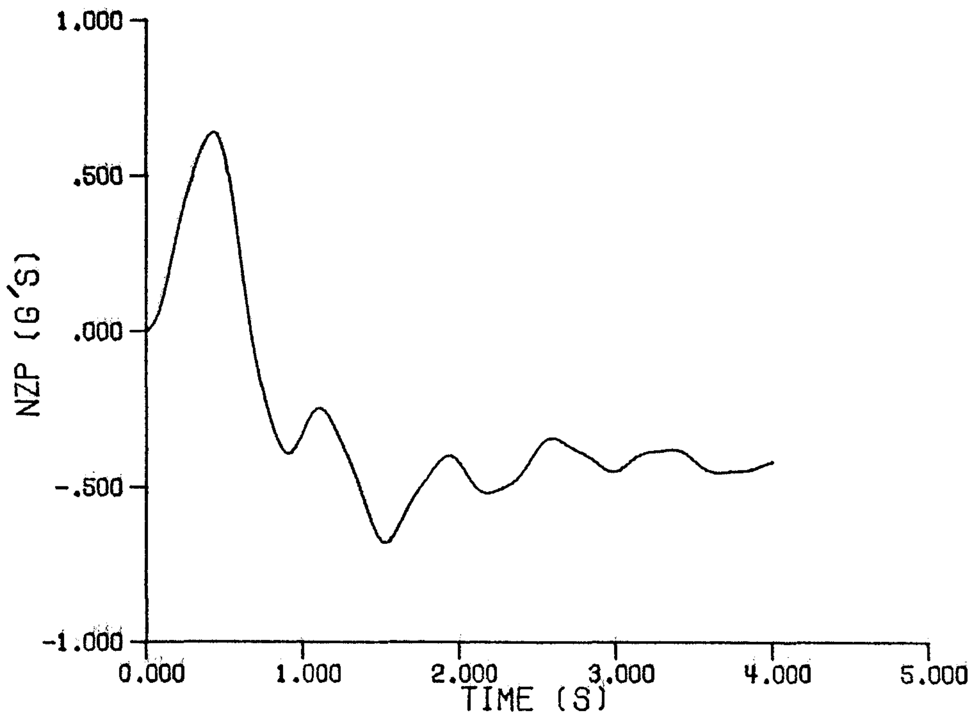


Figure 4.2, concluded

Table 4.5
Closed-loop Eigenvectors - Example 4.1, $\rho = 10^{-2}$

```
LQEA RHO=1.0E-02

EIGENVALUES
 1:     -2.2642 + J    1.1367
 2:     -2.2642 - J    1.1367
 3:       -.0490 + J     .1464
 4:       -.0490 - J     .1464


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                             1                                 2
(  1.0000E+00,             0)       (  1.0000E+00,             0)
(  1.6980E+00,  1.3474E+02)       (  1.6980E+00, -1.3474E+02)
(  1.9235E-02,  1.8329E+01)       (  1.9235E-02, -1.8329E+01)
(  6.7022E-01, -1.8603E+01)       (  6.7022E-01,  1.8603E+01)
(  2.6766E-01,  1.5226E+02)       (  2.6766E-01, -1.5226E+02)
(  3.4672E-02,  1.9613E+02)       (  3.4672E-02, -1.9613E+02)
(  6.7812E-01, -5.4400E+01)       (  6.7812E-01,  5.4400E+01)
(  8.7844E-02, -1.0530E+01)       (  8.7844E-02,  1.0530E+01)
(  5.4025E-01,  1.6477E+02)       (  5.4025E-01, -1.6477E+02)
(  1.5166E+00, -1.4788E+02)       (  1.5166E+00,  1.4788E+02)

                             3                                 4
(  1.0000E+00,             0)       (  1.0000E+00,             0)
(  2.4521E+00, -5.1894E+01)       (  2.4521E+00,  5.1894E+01)
(  3.4442E+00, -7.8686E+01)       (  3.4442E+00,  7.8686E+01)
(  1.5885E+01, -1.6038E+02)       (  1.5885E+01,  1.6038E+02)
(  7.9536E-01,  1.2833E+02)       (  7.9536E-01, -1.2833E+02)
(  3.7454E-02, -1.3425E+02)       (  3.7454E-02,  1.3425E+02)
(  1.2277E-01, -1.2318E+02)       (  1.2277E-01,  1.2318E+02)
(  5.7815E-03, -2.5756E+01)       (  5.7815E-03,  2.5756E+01)
(  1.4265E+00,  1.3614E+02)       (  1.4265E+00, -1.3614E+02)
(  1.8642E+00, -1.2374E+02)       (  1.8642E+00,  1.2374E+02)
```

State vector (unscaled): $\mathbf{x}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2 , \delta_e , \delta_{cv})$

Table 4.5, continued

EIGENVALUES

5:	-7.3849	+ J	10.8696
6:	-7.3849	- J	10.8696
7:	-.5396	+ J	21.4516
8:	-.5396	- J	21.4516

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

5		6	
(1.0000E+00,	0)	(1.0000E+00,	0)
(1.3601E+01,	1.0380E+02)	(1.3601E+01,	-1.0380E+02)
(4.5932E-03,	4.4263E+01)	(4.5932E-03,	-4.4263E+01)
(1.0350E+00,	3.3961E+02)	(1.0350E+00,	-3.3961E+02)
(4.4418E+00,	1.3739E+02)	(4.4418E+00,	-1.3739E+02)
(5.7334E-01,	2.4047E+02)	(5.7334E-01,	-2.4047E+02)
(5.8370E+01,	2.6158E+02)	(5.8370E+01,	-2.6158E+02)
(7.5343E+00,	4.6582E+00)	(7.5343E+00,	-4.6582E+00)
(1.0053E+01,	4.9394E+01)	(1.0053E+01,	-4.9394E+01)
(1.2467E+01,	1.2916E+01)	(1.2467E+01,	-1.2916E+01)

7		8	
(1.0000E+00,	0)	(1.0000E+00,	0)
(2.0132E+01,	6.3106E+01)	(2.0132E+01,	-6.3106E+01)
(2.6205E-03,	7.3138E+01)	(2.6205E-03,	-7.3138E+01)
(9.3821E-01,	-2.8335E+01)	(9.3821E-01,	2.8335E+01)
(2.4657E+00,	-1.5694E+02)	(2.4657E+00,	1.5694E+02)
(3.9175E+01,	-7.0820E+01)	(3.9175E+01,	7.0820E+01)
(5.2909E+01,	-6.5499E+01)	(5.2909E+01,	6.5499E+01)
(8.4062E+02,	2.0621E+01)	(8.4062E+02,	-2.0621E+01)
(9.3608E+00,	-3.5400E+01)	(9.3608E+00,	3.5400E+01)
(1.3773E+01,	-1.6233E+02)	(1.3773E+01,	1.6233E+02)

Table 4.5, concluded

```
EIGENVALUES
 9:    -10.0486 + J          0
10:    -13.2191 + J          0


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                            9                                   10
(  1.0000E+00,             0)      (  1.0000E+00,              0)
(  1.0780E+01,  1.8000E+02)      (  1.6138E+01, -1.8000E+02)
(  6.2370E-03,             0)      (  5.1166E-03,              0)
(  1.0728E+00,             0)      (  1.2208E+00,              0)
(  2.6042E+00,  1.8000E+02)      (  3.9217E+00, -1.8000E+02)
(  2.1026E-01,             0)      (  4.0887E-01,              0)
(  2.6168E+01,             0)      (  5.1841E+01,              0)
(  2.1128E+00,  1.8000E+02)      (  5.4049E+00, -1.8000E+02)
(  7.0229E+00,  1.8000E+02)      (  1.4050E+01, -1.8000E+02)
(  7.9220E-01,  1.8000E+02)      (  2.6095E+01, -1.8000E+02)
```

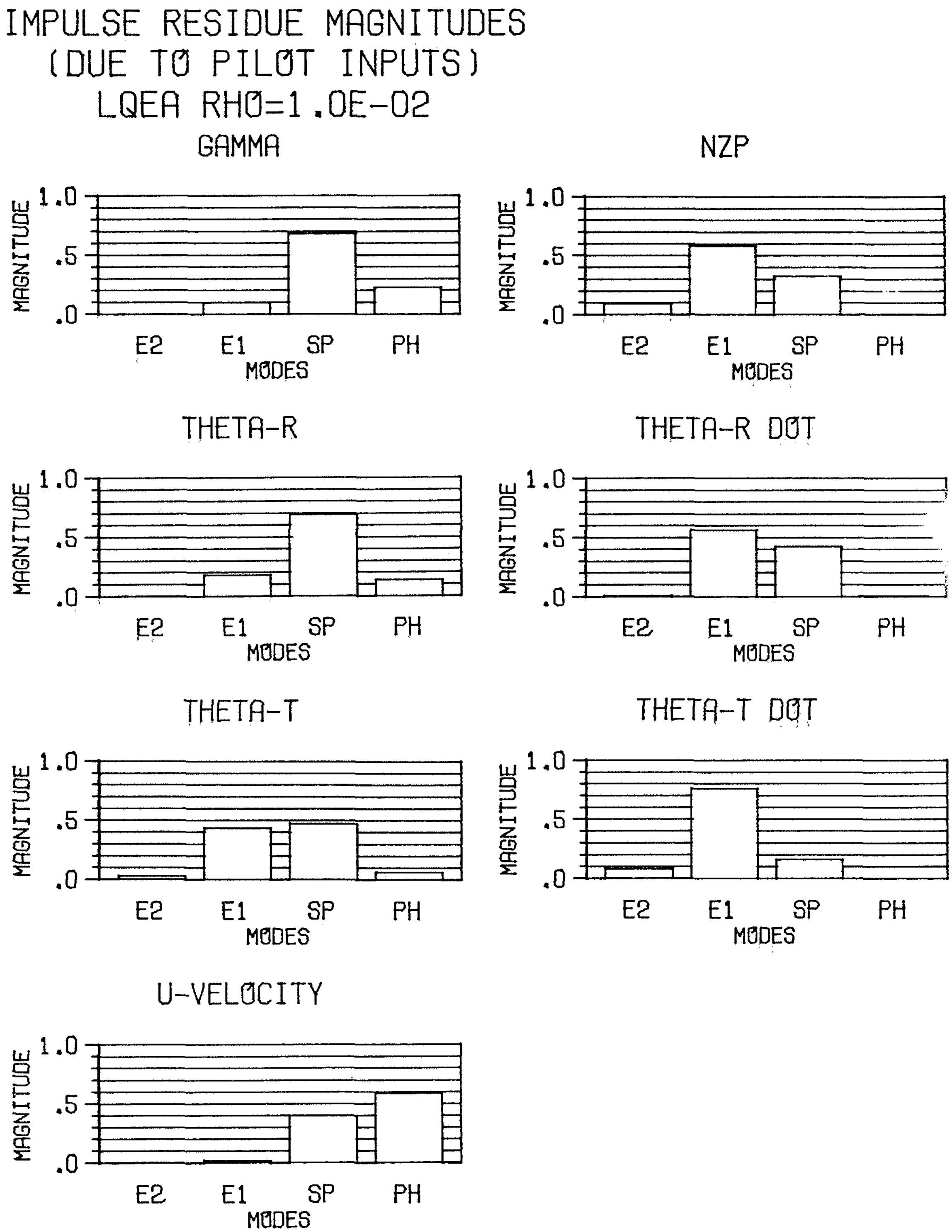


Figure 4.3
Impulse Residue Magnitudes - Example 4.1, $\rho = 10^{-2}$

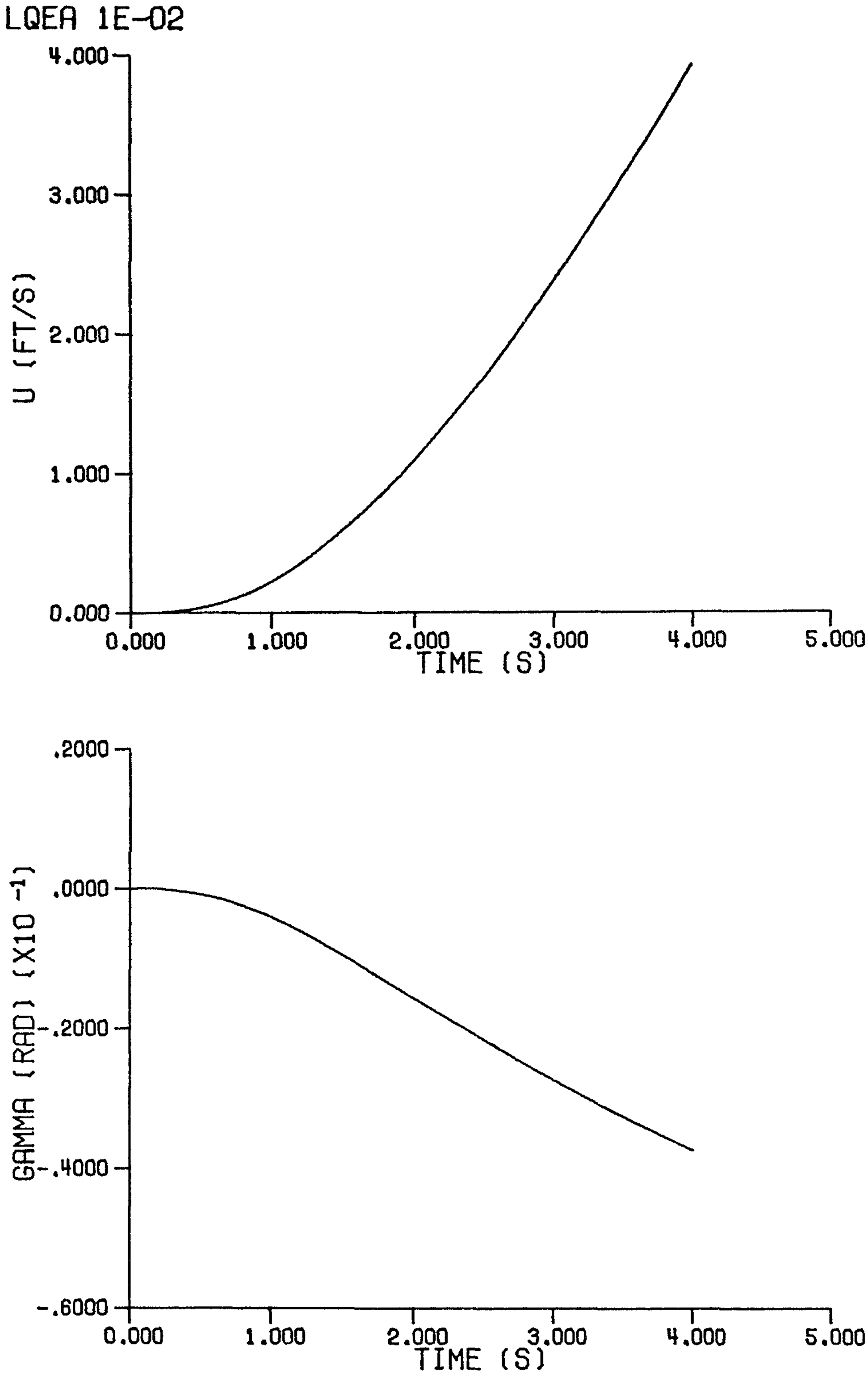
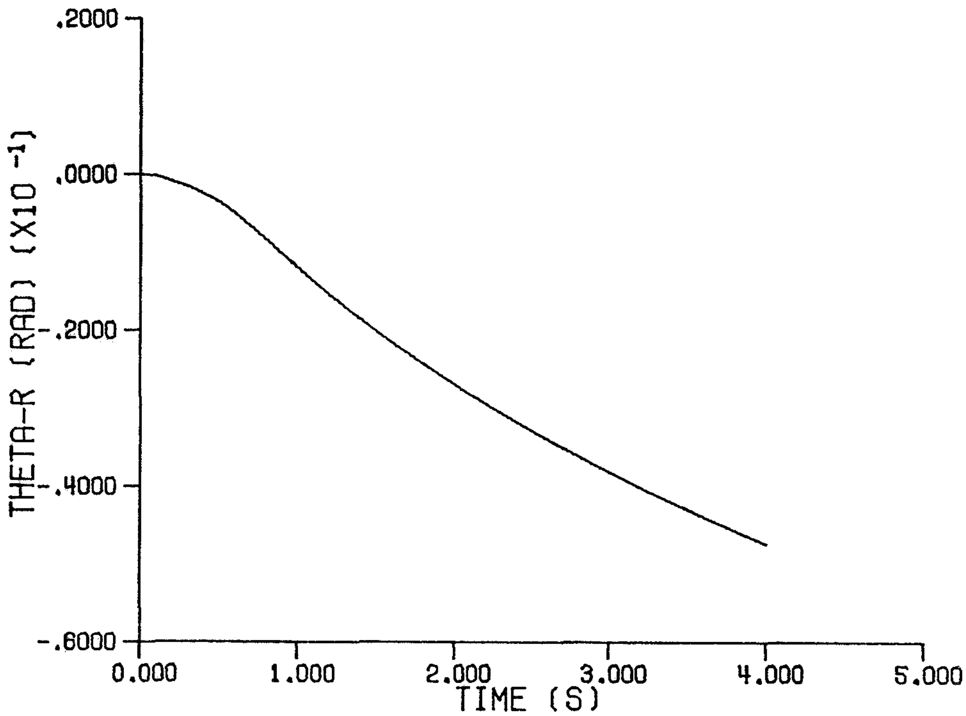


Figure 4.4
Output Time Responses - Example 4.1, $\rho = 10^{-2}$



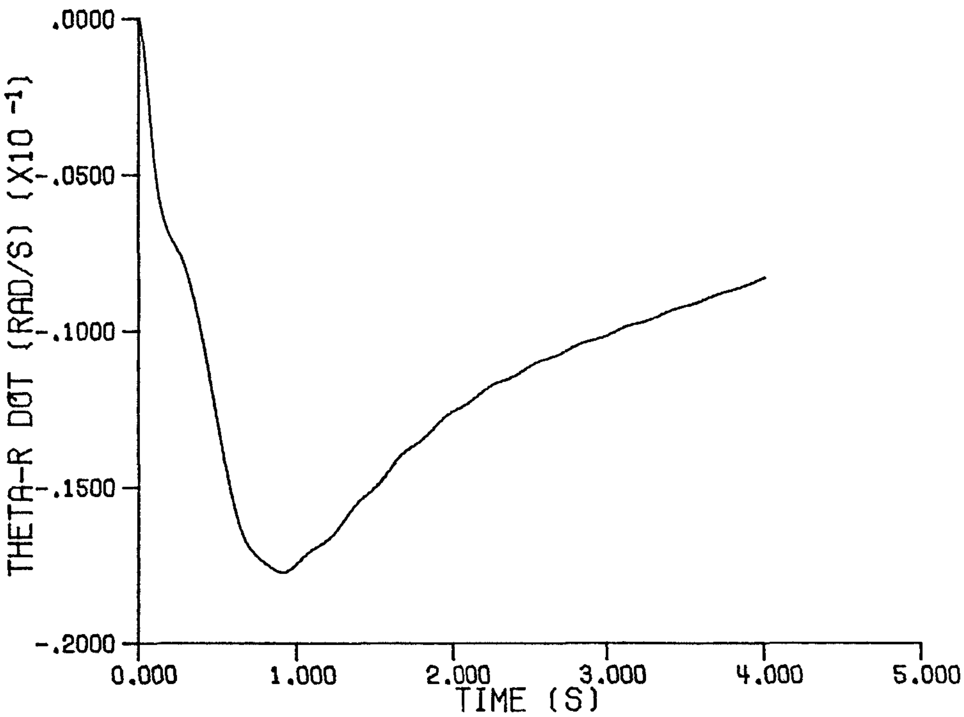


Figure 4.4, continued

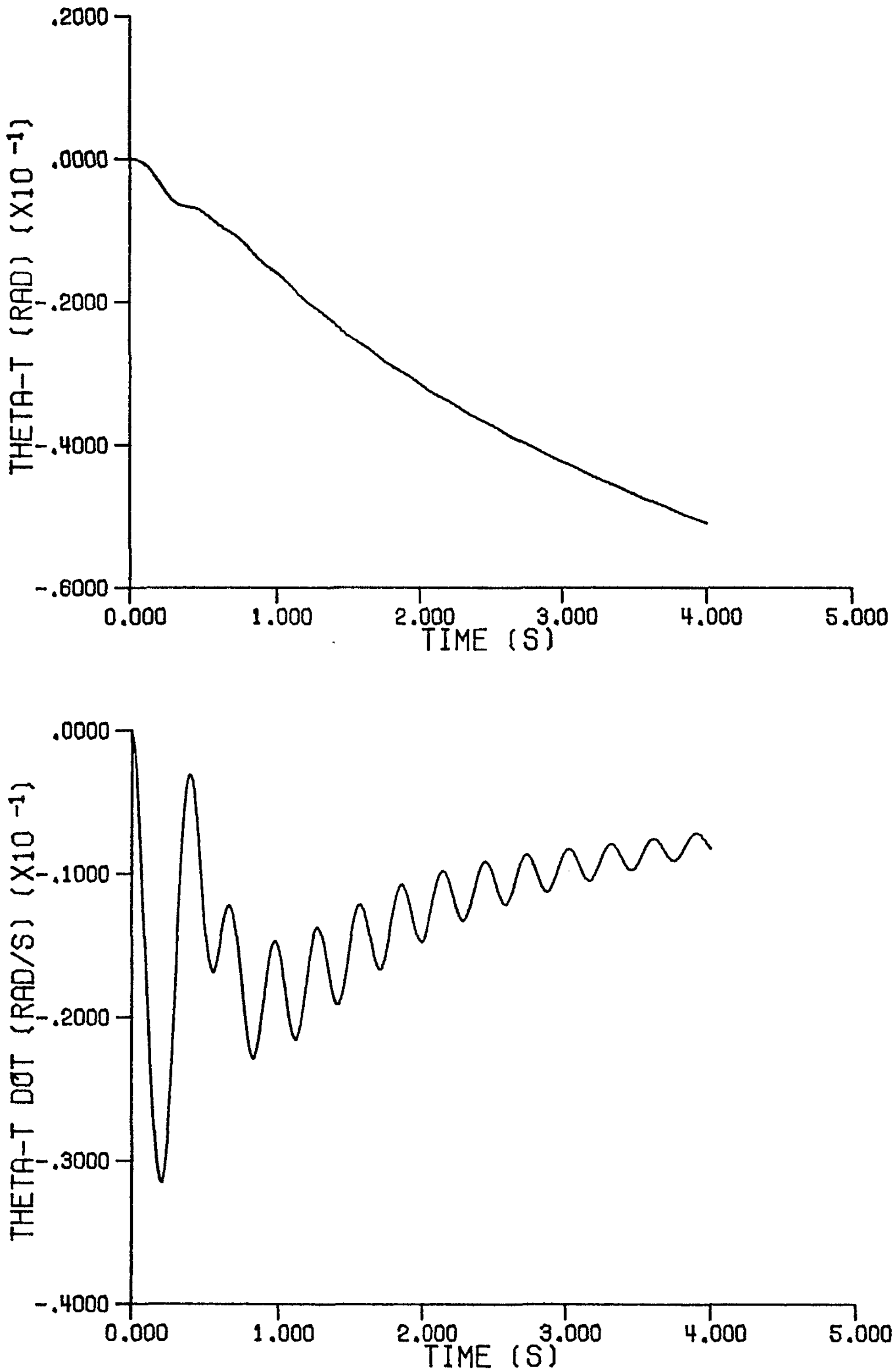


Figure 4.4, continued

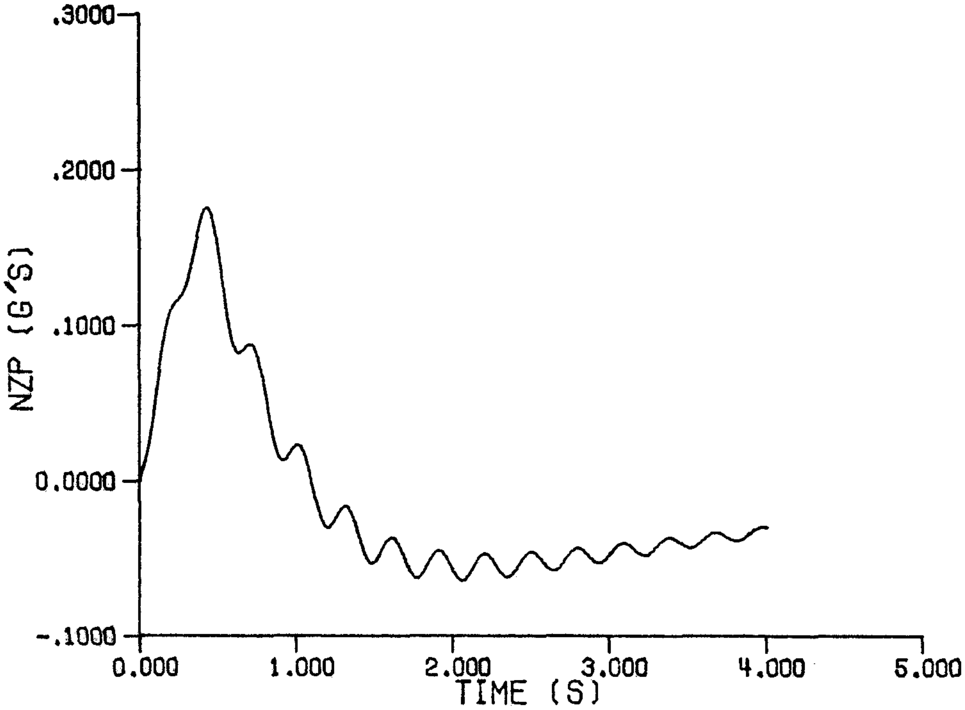


Figure 4.4, concluded

Control synthesis for $\rho = 1.0 \times 10^{-4}$. The system closed-loop eigenvalues, the eigenvalues of $(\mathbf{A} + \mathbf{BG})$ are:

$$\lambda_{1,2} = -1.696 \pm j\ 2.064 = (\omega_n = 2.672\ \text{(rad/s)}\ ,\ \varsigma = 0.635)$$

$$\lambda_{3,4} = -0.0445 \pm j\ 0.0971 = (\omega_n = 0.107\ \text{(rad/s)}\ ,\ \varsigma = 0.417)$$

$$\lambda_{5,6} = -4.079 \pm j\ 8.355 = (\omega_n = 9.297\ \text{(rad/s)}\ ,\ \varsigma = 0.439)$$

$$\lambda_{7,8} = -0.5959 \pm j\ 21.46 = (\omega_n = 21.47\ \text{(rad/s)}\ ,\ \varsigma = 0.028)$$

$$\lambda_9 = -54.11$$

$$\lambda_{10} = -98.55$$

The eigenvectors of $(\mathbf{A} + \mathbf{BG})$ are given in Table 4.6. The gain matrix and the closed-loop system matrices for this value of ρ are given in Appendix A.4. The normalized closed-loop system impulse residue magnitudes of the short period, phugoid, first elastic, and second elastic modes for each output are shown in Figure 4.5. The output time responses due to a unit step in $\underline{u}_p$, with zero initial conditions, are shown in Figure 4.6.

Control synthesis for $\rho = 1.0 \times 10^{-6}$. The system closed-loop eigenvalues, the eigenvalues of $(\mathbf{A} + \mathbf{BG})$ are:

$$\lambda_{sp} = -1.493 \pm j\ 2.365 = (\omega_n = 2.797\ \text{(rad/s)}\ ,\ \varsigma = 0.534)$$

$$\lambda_{ph} = -0.0075 \pm j\ 0.0679 = (\omega_n = 0.068\ \text{(rad/s)}\ ,\ \varsigma = 0.110)$$

$$\lambda_{\xi_1} = -1.807 \pm j\ 8.608 = (\omega_n = 8.796\ \text{(rad/s)}\ ,\ \varsigma = 0.205)$$

Table 4.6
Closed-loop Eigenvectors - Example 4.1, $\rho = 10^{-4}$

```
LQEA RHO=1.0E-04

EIGENVALUES
 1:     -1.6957 + J     2.0645
 2:     -1.6957 - J     2.0645
 3:      -.0445 + J      .0971
 4:      -.0445 - J      .0971


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                              1                                  2
(  1.0000E+00,              0)     (  1.0000E+00,              0)
(  2.2485E+00,   1.0495E+02)     (  2.2485E+00,  -1.0495E+02)
(  2.0139E-02,   3.8333E+01)     (  2.0139E-02,  -3.8333E+01)
(  8.4164E-01,  -2.4447E+01)     (  8.4164E-01,   2.4447E+01)
(  9.2655E-02,   1.4364E+02)     (  9.2655E-02,  -1.4364E+02)
(  4.7214E-02,  -1.6760E+02)     (  4.7214E-02,   1.6760E+02)
(  2.4754E-01,  -8.6960E+01)     (  2.4754E-01,   8.6960E+01)
(  1.2614E-01,  -3.8206E+01)     (  1.2614E-01,   3.8206E+01)
(  2.5761E-01,  -1.8363E+02)     (  2.5761E-01,   1.8363E+02)
(  3.7742E+00,  -1.5251E+02)     (  3.7742E+00,   1.5251E+02)

                              3                                  4
(  1.0000E+00,              0)     (  1.0000E+00,              0)
(  9.0230E-01,   2.5597E+02)     (  9.0230E-01,  -2.5597E+02)
(  2.6739E+00,   2.1640E+02)     (  2.6739E+00,  -2.1640E+02)
(  8.4477E+00,   1.4134E+02)     (  8.4477E+00,  -1.4134E+02)
(  2.7688E-01,   9.3930E+01)     (  2.7688E-01,  -9.3930E+01)
(  5.0967E-02,   1.9449E+02)     (  5.0967E-02,  -1.9449E+02)
(  2.9574E-02,   2.0857E+02)     (  2.9574E-02,  -2.0857E+02)
(  5.4438E-03,   3.0913E+02)     (  5.4438E-03,  -3.0913E+02)
(  4.9927E-01,   1.1451E+02)     (  4.9927E-01,  -1.1451E+02)
(  3.6476E+00,   1.9670E+02)     (  3.6476E+00,  -1.9670E+02)
```

State vector (unscaled): $\mathbf{x}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2 , \delta_e , \delta_{cv})$

Table 4.6, continued

```
EIGENVALUES
 5:     -4.0793 + J    8.3546
 6:     -4.0793 - J    8.3546
 7:      -.5959 + J   21.4568
 8:      -.5959 - J   21.4568


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                              5                                    6
(  1.0000E+00,               0)      (  1.0000E+00,               0)
(  7.9835E+00,  9.7143E+01)      (  7.9835E+00, -9.7143E+01)
(  5.8743E-03,  5.4223E+01)      (  5.8743E-03, -5.4223E+01)
(  8.5869E-01, -1.8882E+01)      (  8.5869E-01,  1.8882E+01)
(  6.0735E+00,  1.0012E+02)      (  6.0735E+00, -1.0012E+02)
(  2.4303E-01, -1.5274E+02)      (  2.4303E-01,  1.5274E+02)
(  5.6467E+01, -1.4386E+02)      (  5.6467E+01,  1.4386E+02)
(  2.2595E+00, -3.6716E+01)      (  2.2595E+00,  3.6716E+01)
(  4.9248E+00,  4.3704E+01)      (  4.9248E+00, -4.3704E+01)
(  4.8660E+01, -2.3108E+01)      (  4.8660E+01,  2.3108E+01)

                              7                                    8
(  1.0000E+00,               0)      (  1.0000E+00,               0)
(  1.9986E+01,  6.5446E+01)      (  1.9986E+01, -6.5446E+01)
(  2.6208E-03,  7.4240E+01)      (  2.6208E-03, -7.4240E+01)
(  9.3108E-01, -2.6144E+01)      (  9.3108E-01,  2.6144E+01)
(  2.2055E+00,  1.9844E+02)      (  2.2055E+00, -1.9844E+02)
(  3.3925E+01, -7.4597E+01)      (  3.3925E+01,  7.4597E+01)
(  4.7342E+01, -6.9968E+01)      (  4.7342E+01,  6.9968E+01)
(  7.2821E+02,  1.6994E+01)      (  7.2821E+02, -1.6994E+01)
(  1.0542E+01, -2.7466E+01)      (  1.0542E+01,  2.7466E+01)
(  7.5550E+01,  1.9043E+02)      (  7.5550E+01, -1.9043E+02)
```

Table 4.6, concluded

```
EIGENVALUES
 9:    -54.1139 + J          0
10:    -98.5507 + J          0


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                             9                                10
(  1.0000E+00,              0)     (  1.0000E+00,              0)
(  6.2340E+02,              0)     (  5.7311E+01,              0)
(  6.6947E-03,  -1.8000E+02)     (  6.8349E-05,              0)
(  1.1520E+01,  -1.8000E+02)     (  5.8154E-01,   1.8000E+02)
(  4.5886E+01,              0)     (  3.8181E+00,              0)
(  1.3822E+01,  -1.8000E+02)     (  8.8285E-01,   1.8000E+02)
(  2.4831E+03,  -1.8000E+02)     (  3.7628E+02,   1.8000E+02)
(  7.4798E+02,              0)     (  8.7006E+01,              0)
(  2.1747E+03,              0)     (  4.5649E+02,              0)
(  1.8788E+03,              0)     (  4.0420E+03,              0)
```

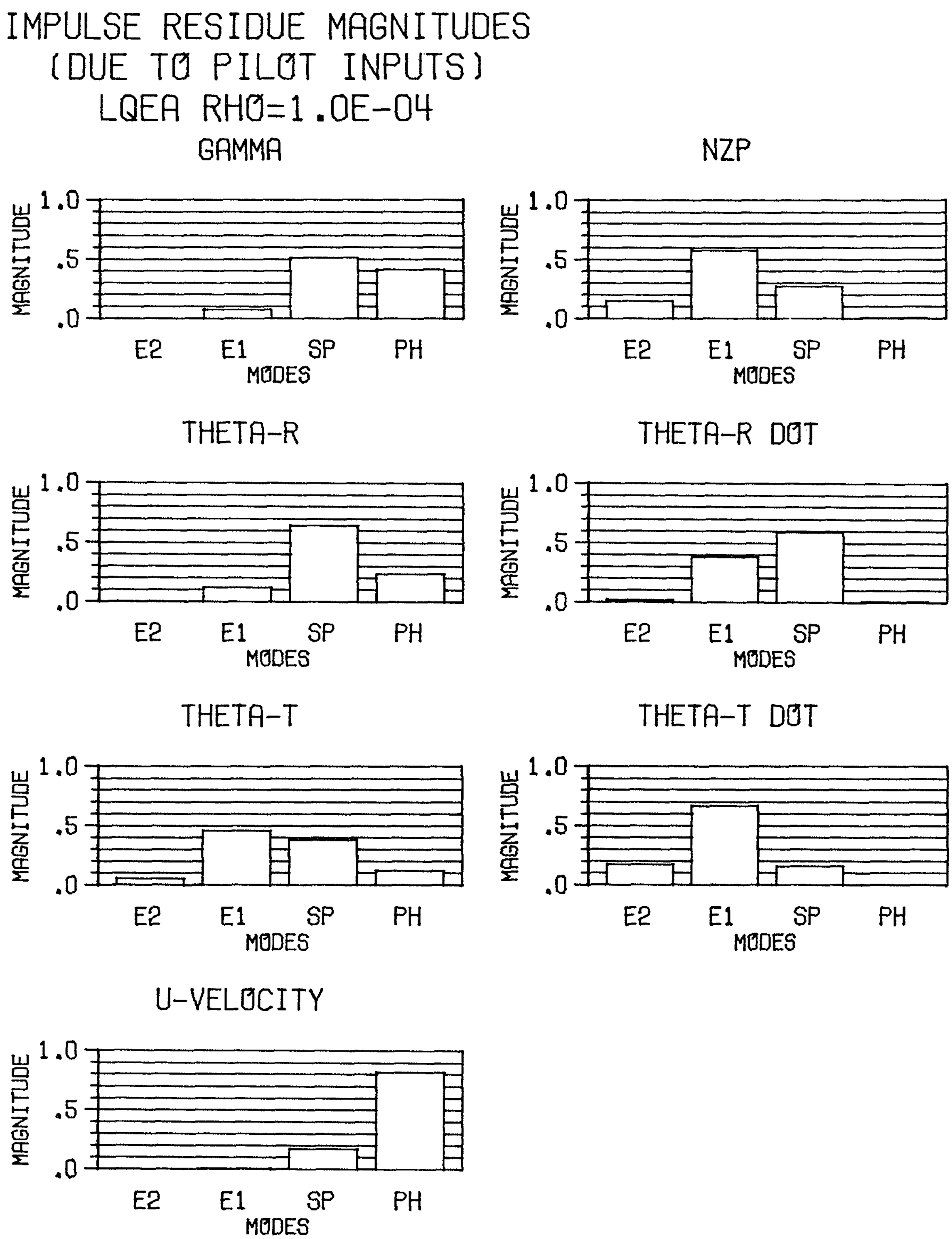


Figure 4.5
Impulse Residue Magnitudes - Example 4.1, $\rho = 10^{-4}$

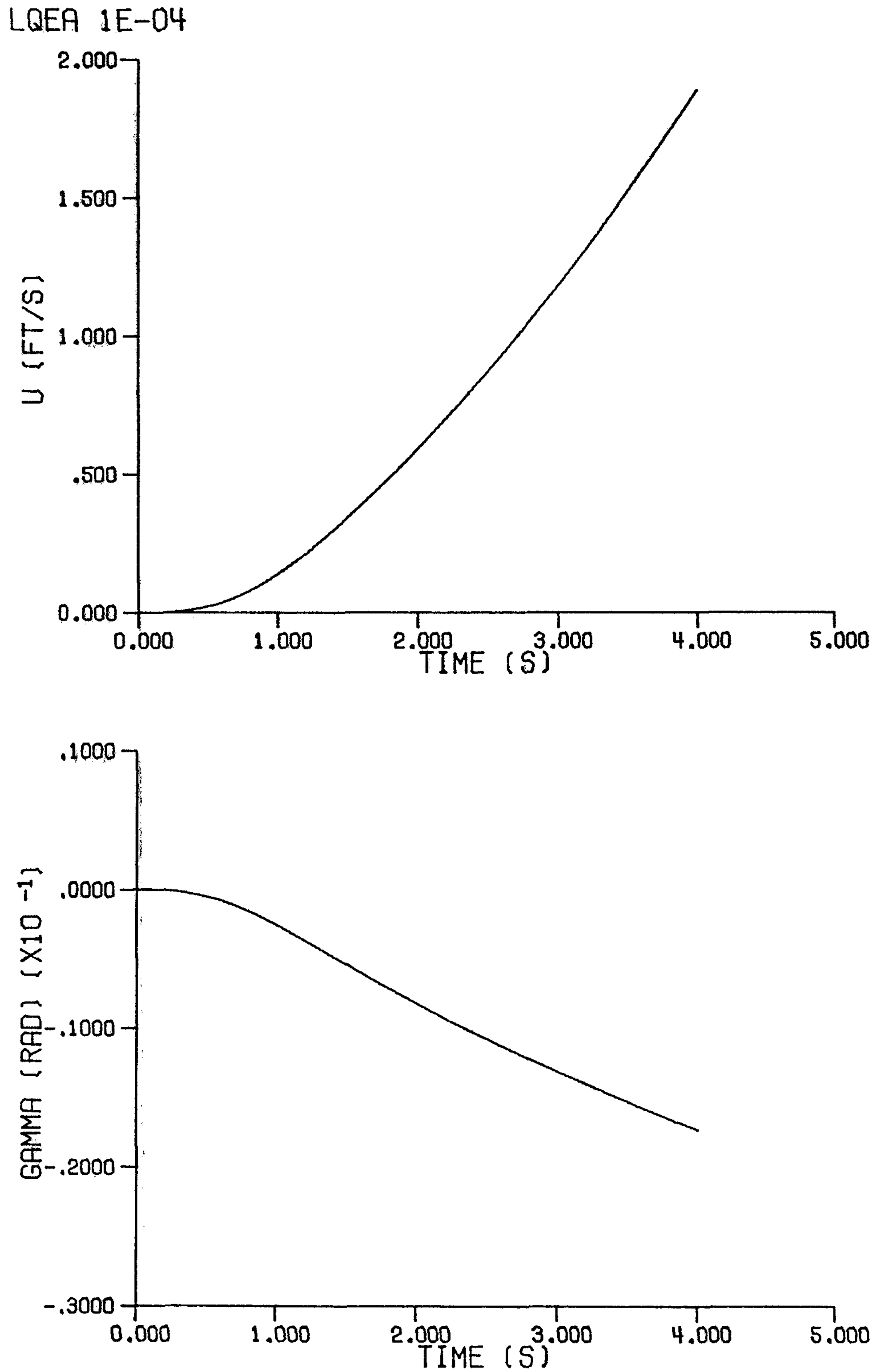
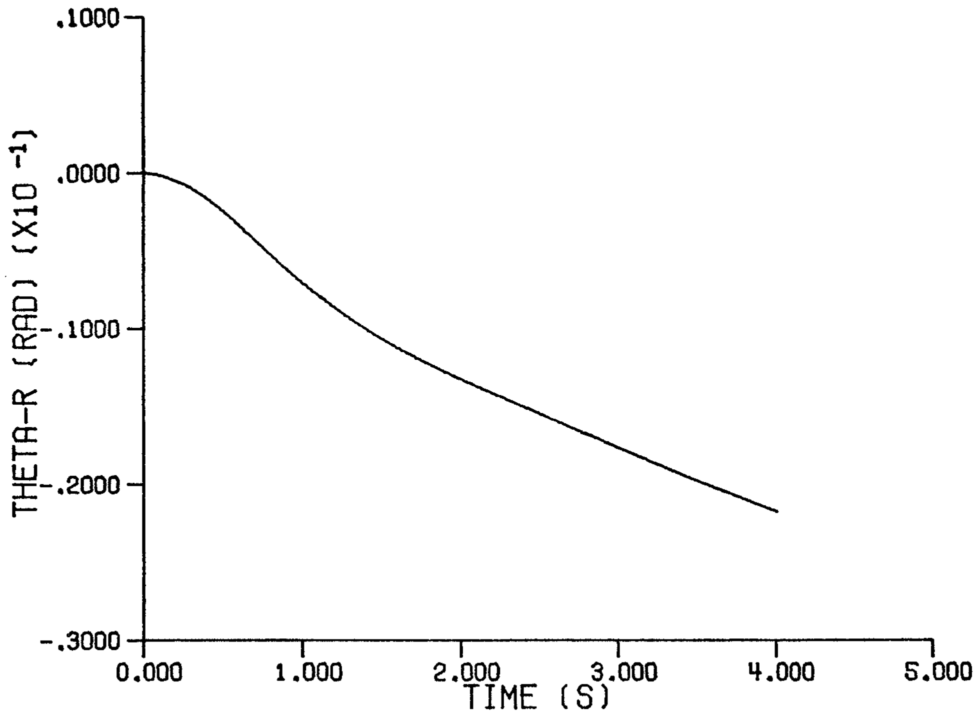


Figure 4.6
Output Time Responses - Example 4.1, $\rho = 10^{-4}$



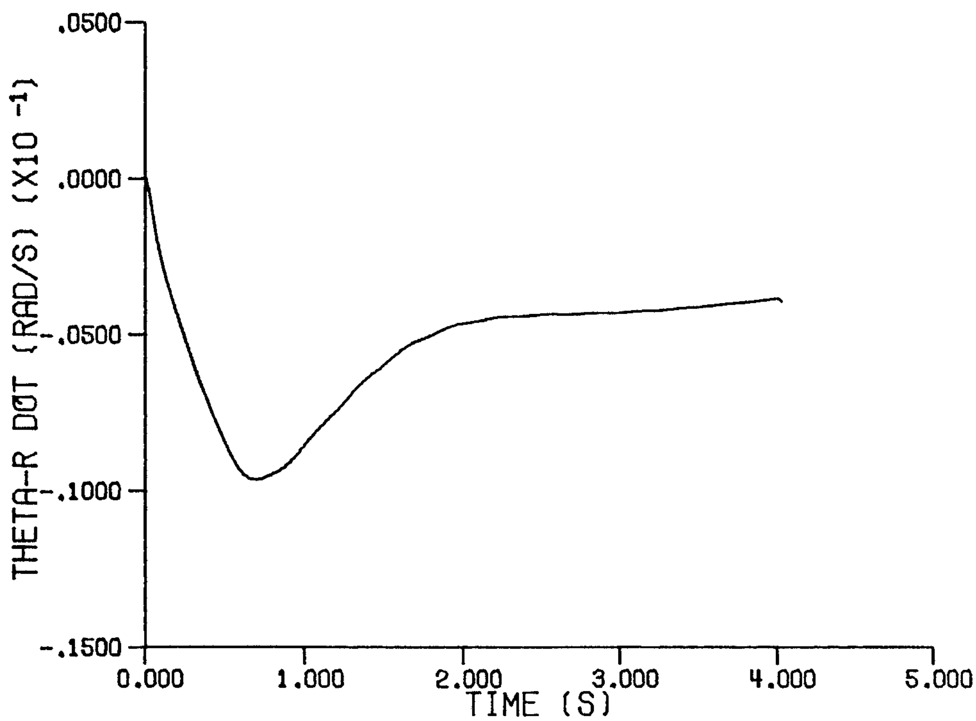
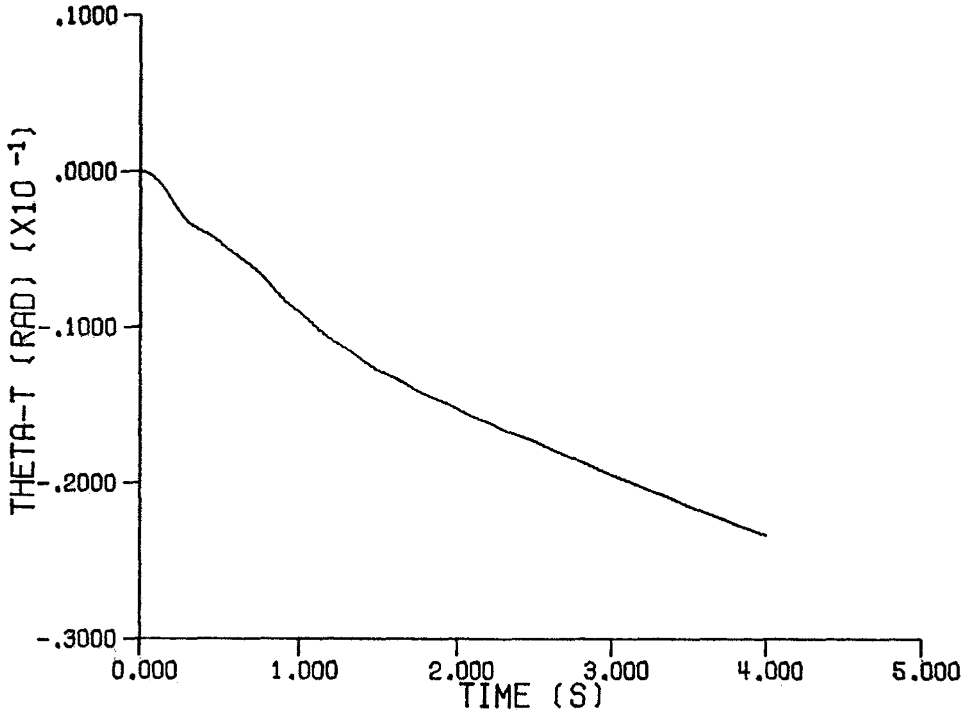


Figure 4.6, continued



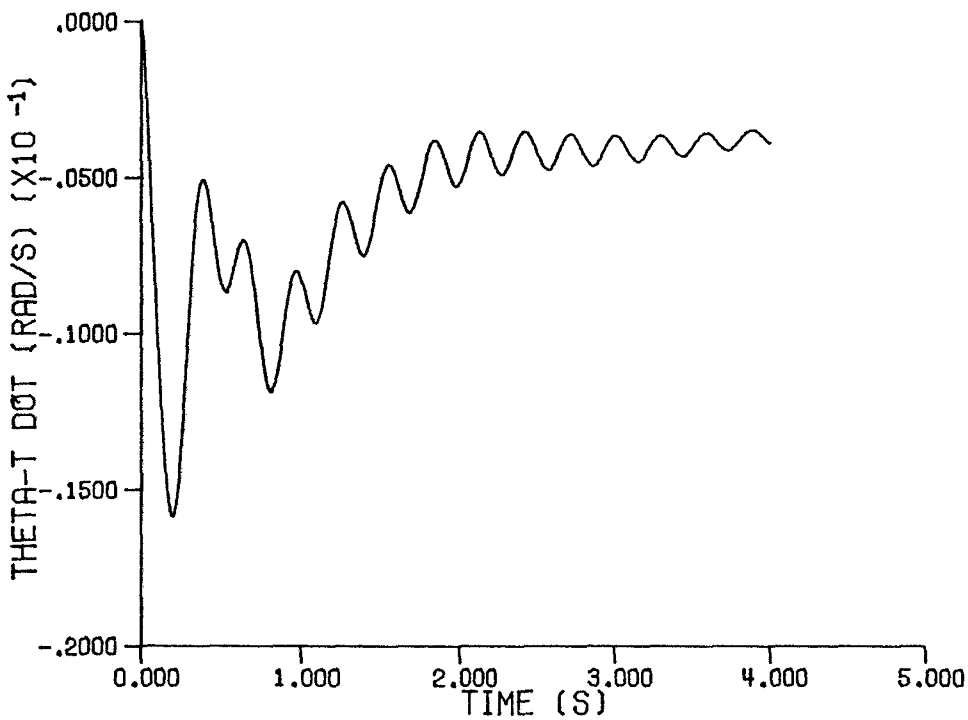


Figure 4.6, continued

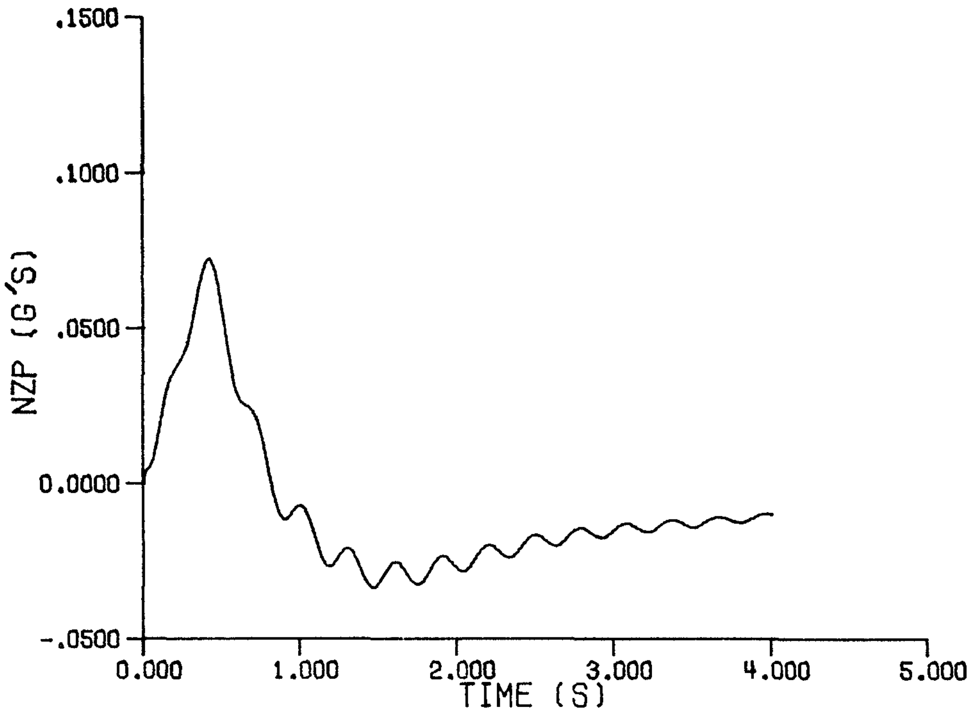


Figure 4.6, concluded

$$\lambda_{\xi_2} = -\,0.5351 \pm j\,21.35 = (\omega_n = 21.36\ \text{(rad/s)}\ ,\ \varsigma = 0.025)$$

$$\lambda_{el} = -\,500.5$$

$$\lambda_{cv} = -\,999.9$$

The eigenvectors of $(\mathbf{A} + \mathbf{BG})$ are given in Table 4.7. The gain matrix and the closed-loop system matrices for this value of ρ are given in Appendix A.4. The normalized closed-loop system impulse residue magnitudes of the short period, phugoid, first elastic, and second elastic modes for each output are shown in Figure 4.7. The output time responses due to a unit step in $\underline{u}_p$, with zero initial conditions, are shown in Figure 4.8.

Comments on Example 4.1

This example illustrates the use of the LQEA control synthesis technique. LQEA yields a family of control laws as a function of LQ cost function control weighting. Each of these control laws will be discussed separately.

The synthesis for $\rho = 1.0 \times 10^6$, as is expected, yields a closed-loop system whose eigenspace is close to that of the open-loop system. This can be seen by comparing the impulse residue magnitudes and time responses obtained from this synthesis with those of the open-loop system presented in Chapter III.

The control synthesis for $\rho = 1.0 \times 10^{-2}$ and for $\rho = 1.0 \times 10^{-4}$ yields interesting results. By comparing the closed-loop system eigenvectors for each value of control weighting, one can see how the closed-loop system eigenvectors change as they tend towards their asymptotic values. For these values of control weighting, the closed-loop eigenspace is not close to the desired eigenspace. The impulse residue magnitudes and time responses show that the first elastic mode is contributing significantly to the rigid-body pitch and rigid-body pitch rate outputs. The control laws for these two values of control weighting are actually increasing the contribution of the first elastic mode in these two outputs. This is in opposition to the desired synthesis result.

A control weighting of $\rho = 1.0 \times 10^{-6}$ is required to achieve the desired synthesis objectives. The short period, first elastic, and second elastic eigenvalues are close to the specified desired finite eigenvalues and the eigenvectors

Table 4.7
Closed-loop Eigenvectors - Example 4.1, $\rho = 10^{-6}$

```
LQEA RHO=1.0E-06

EIGENVALUES
 1:     -1.4930 + J    2.3655
 2:     -1.4930 - J    2.3655
 3:       -.0075 + J     .0679
 4:       -.0075 - J     .0679


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                           1                                 2
(  1.0000E+00,            0)     (  1.0000E+00,             0)
(  2.5088E+00, -2.6164E+02)      (  2.5088E+00,  2.6164E+02)
(  1.9887E-02, -3.1461E+02)      (  1.9887E-02,  3.1461E+02)
(  8.9688E-01, -2.3899E+01)      (  8.9688E-01,  2.3899E+01)
(  8.4733E-03, -3.2331E+02)      (  8.4733E-03,  3.2331E+02)
(  5.2032E-02, -1.7030E+02)      (  5.2032E-02,  1.7030E+02)
(  2.3702E-02, -2.0105E+02)      (  2.3702E-02,  2.0105E+02)
(  1.4555E-01, -4.8041E+01)      (  1.4555E-01,  4.8041E+01)
(  1.5756E-01, -1.5479E+02)      (  1.5756E-01,  1.5479E+02)
(  4.4881E+00, -1.5581E+02)      (  4.4881E+00,  1.5581E+02)

                           3                                 4
(  1.0000E+00,            0)     (  1.0000E+00,             0)
(  3.3618E-01,  1.8462E+02)      (  3.3618E-01, -1.8462E+02)
(  2.4097E+00,  1.8378E+02)      (  2.4097E+00, -1.8378E+02)
(  4.9237E+00,  8.8301E+01)      (  4.9237E+00, -8.8301E+01)
(  3.4129E-02,  4.2316E+01)      (  3.4129E-02, -4.2316E+01)
(  5.1619E-02,  1.8134E+02)      (  5.1619E-02, -1.8134E+02)
(  2.3303E-03,  1.3864E+02)      (  2.3303E-03, -1.3864E+02)
(  3.5245E-03,  2.7767E+02)      (  3.5245E-03, -2.7767E+02)
(  1.3758E-01,  1.6462E+02)      (  1.3758E-01, -1.6462E+02)
(  3.7319E+00,  1.8184E+02)      (  3.7319E+00, -1.8184E+02)
```

State vector (unscaled): $\mathbf{x}^T = (\alpha, \dot{\theta}_r, u_f, \theta_r, \xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2, \delta_e, \delta_{cv})$

Table 4.7, continued

EIGENVALUES

5:	-1.8065	+	J	8.6082
6:	-1.8065	-	J	8.6082
7:	-.5351	+	J	21.3497
8:	-.5351	-	J	21.3497

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

5		6	
(1.0000E+00,	0)	(1.0000E+00,	0)
(2.4172E+00,	-2.7131E+02)	(2.4172E+00,	2.7131E+02)
(4.0304E-03,	-2.8511E+02)	(4.0304E-03,	2.8511E+02)
(2.7482E-01,	-1.3167E+01)	(2.7482E-01,	1.3167E+01)
(2.2073E+01,	-2.8878E+02)	(2.2073E+01,	2.8878E+02)
(8.0568E-02,	-3.1145E+02)	(8.0568E-02,	3.1145E+02)
(1.9415E+02,	-1.8692E+02)	(1.9415E+02,	1.8692E+02)
(7.0865E-01,	-2.0960E+02)	(7.0865E-01,	2.0960E+02)
(6.8016E+00,	-2.9977E+02)	(6.8016E+00,	2.9977E+02)
(2.1259E+02,	-5.6019E+01)	(2.1259E+02,	5.6019E+01)

7		8	
(1.0000E+00,	0)	(1.0000E+00,	0)
(1.9764E+01,	6.7223E+01)	(1.9764E+01,	-6.7223E+01)
(2.6353E-03,	7.5487E+01)	(2.6353E-03,	-7.5487E+01)
(9.2545E-01,	-2.4212E+01)	(9.2545E-01,	2.4212E+01)
(2.4838E+00,	-1.3184E+02)	(2.4838E+00,	1.3184E+02)
(4.5460E+01,	-7.9691E+01)	(4.5460E+01,	7.9691E+01)
(5.3044E+01,	-4.0409E+01)	(5.3044E+01,	4.0409E+01)
(9.7087E+02,	1.1745E+01)	(9.7087E+02,	-1.1745E+01)
(5.5996E+00,	5.2243E+00)	(5.5996E+00,	-5.2243E+00)
(1.3312E+02,	-1.7059E+02)	(1.3312E+02,	1.7059E+02)

Table 4.7, concluded

```
EIGENVALUES
 9:  -500.4525 + J          0
10:  -999.8590 + J          0


EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

                            9                                      10
(  1.0000E+00,             0)     (  1.0000E+00,                   0)
(  5.7613E+01,             0)     (  2.5169E+01,      1.8000E+02)
(  4.4885E-05,             0)     (  2.7197E-05,                   0)
(  1.1512E-01,  1.8000E+02)     (  2.5172E-02,                   0)
(  4.5266E-01,             0)     (  3.9915E-01,                   0)
(  1.6190E-01,  1.8000E+02)     (  2.8332E-03,      1.8000E+02)
(  2.2654E+02,  1.8000E+02)     (  3.9909E+02,      1.8000E+02)
(  8.1025E+01,             0)     (  2.8328E+00,                   0)
(  1.9097E+03,             0)     (  1.3820E+03,                   0)
(  1.9654E+02,             0)     (  1.3028E+05,                   0)
```

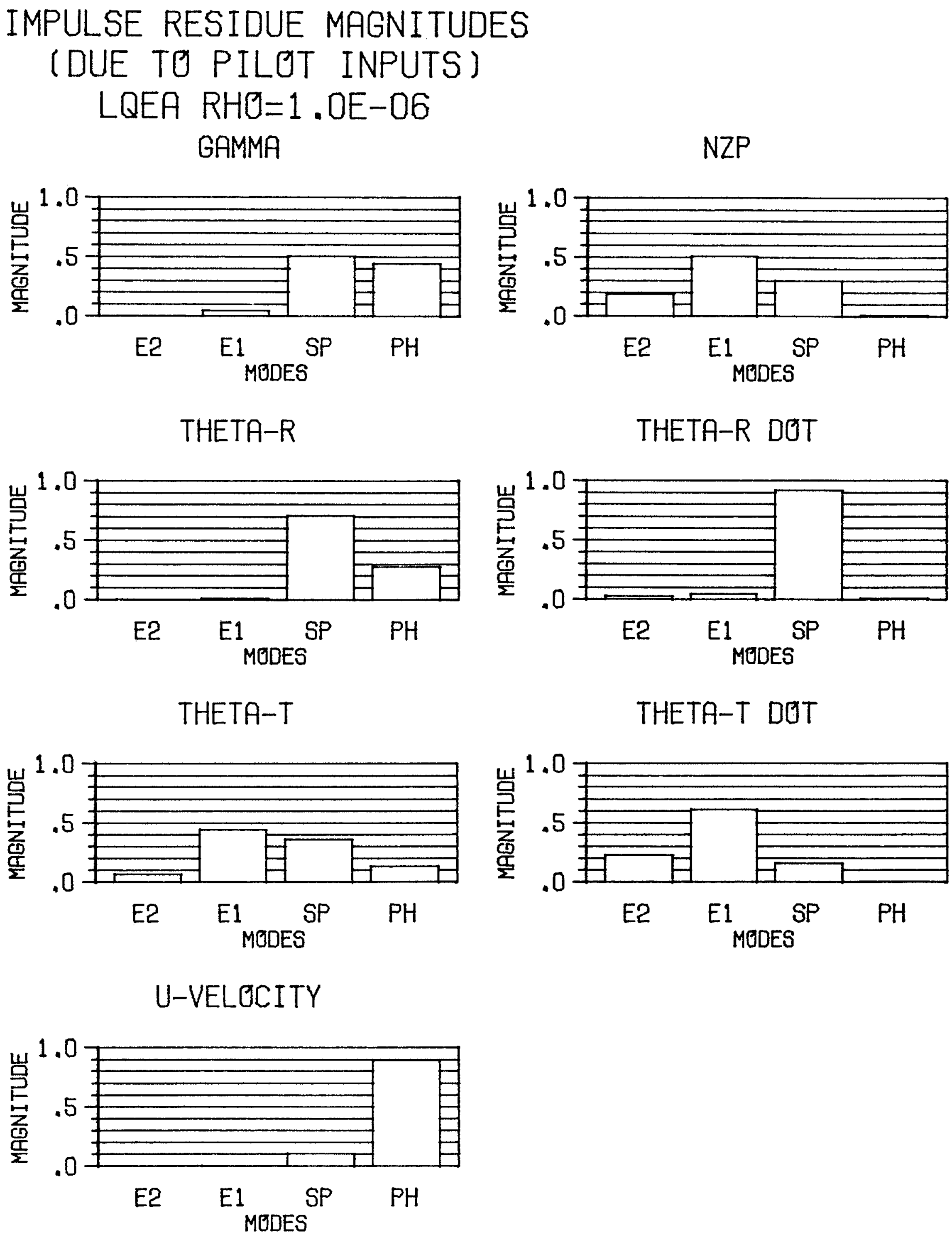


Figure 4.7
Impulse Residue Magnitudes - Example 4.1, $\rho = 10^{-6}$

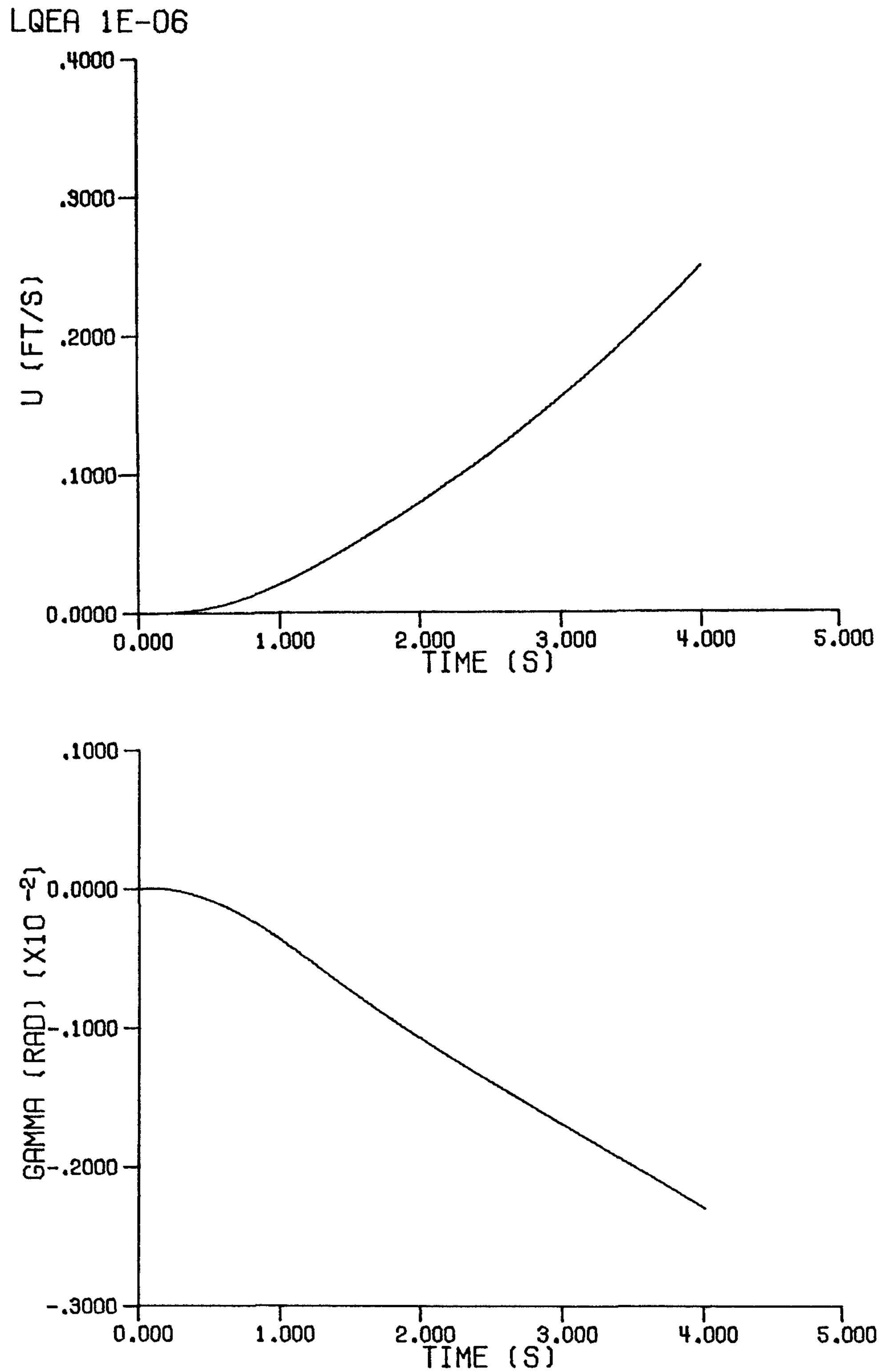
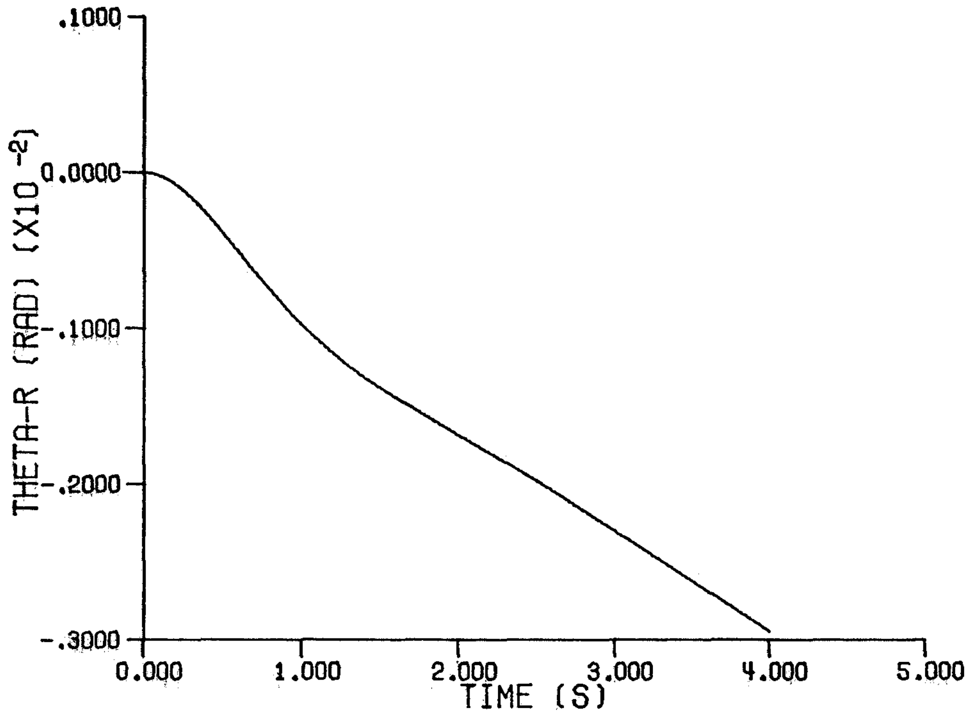


Figure 4.8

Output Time Responses - Example 4.1, $\rho = 10^{-6}$



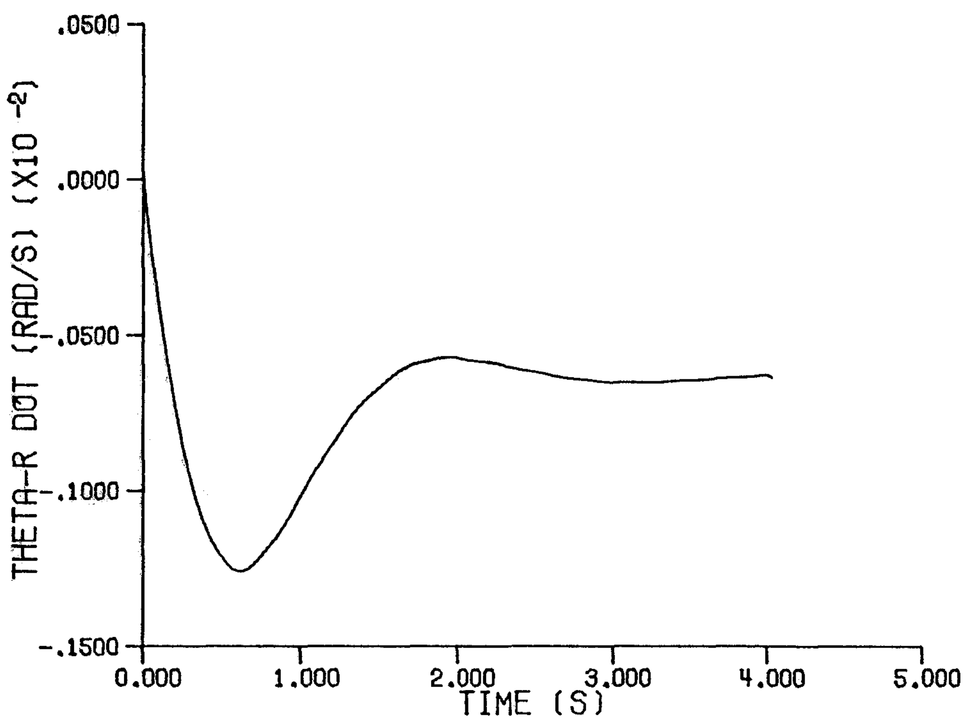


Figure 4.8, continued

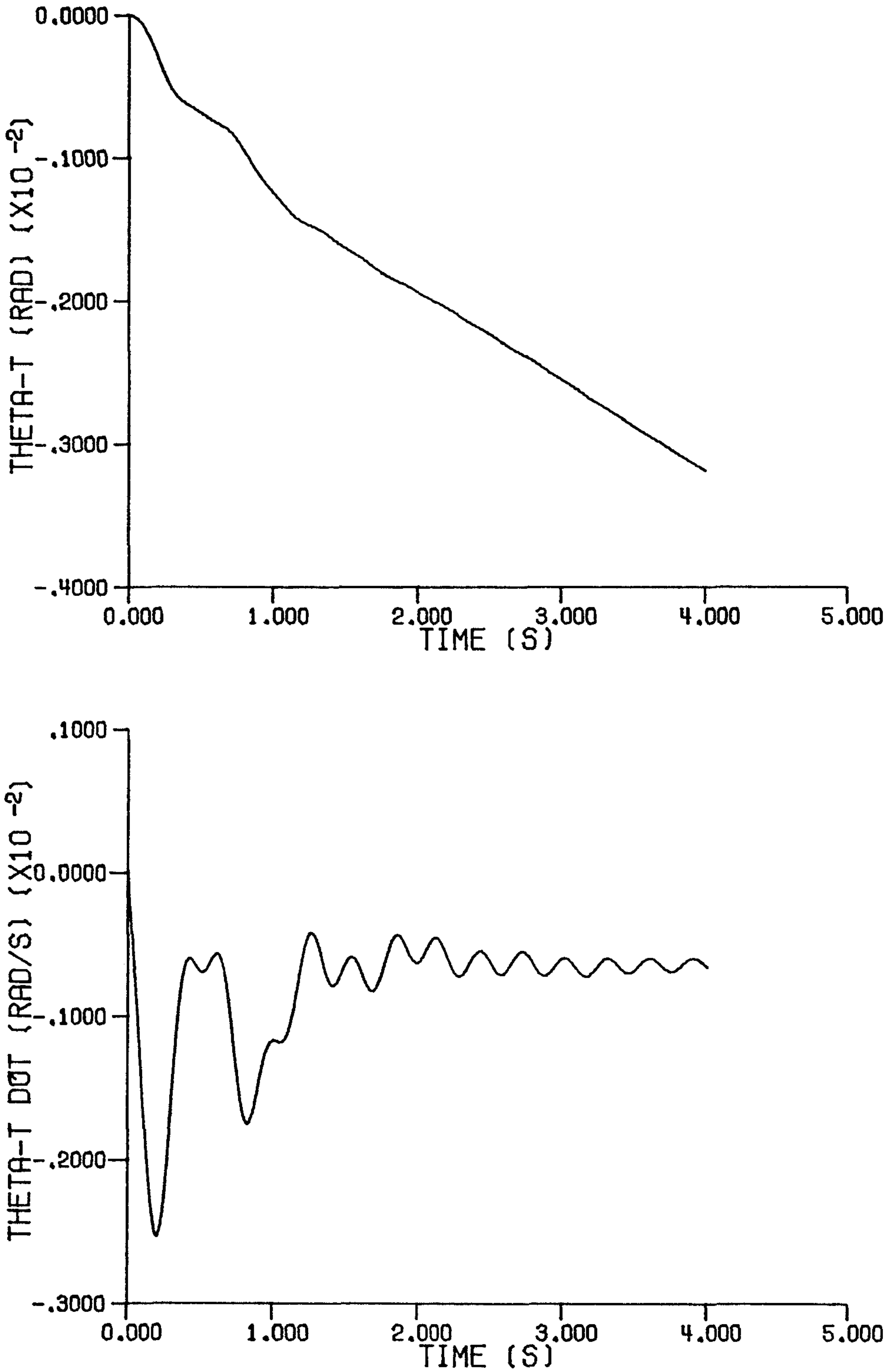


Figure 4.8, continued

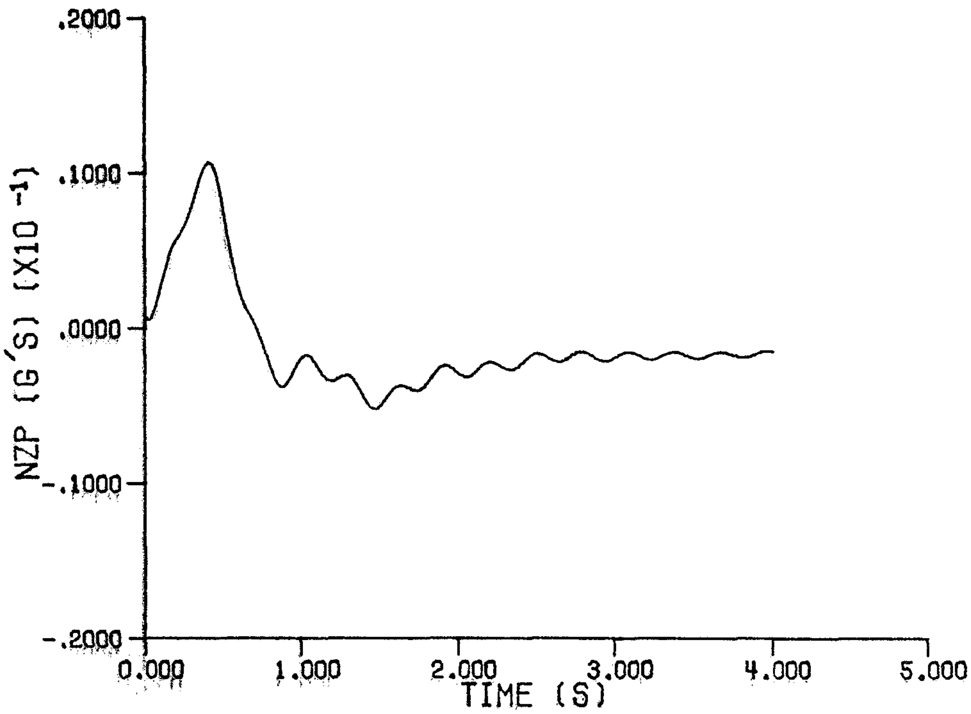


Figure 4.8, concluded

associated with these modes are close to the achievable eigenspace. This is reflected in the closed-loop system impulse residues and time responses. This control law is decreasing the contribution of the elastic modes to the rigid-body dynamics. Also, in this case, one can note that the augmented actuator eigenvalues have achieved the specified two-to-one eigenvalue bandwidth ratio, and their associated eigenvectors have tended towards uncoupling the actuator dynamics.

In this case, very large actuator bandwidths are required. The resulting control vane actuator bandwidth is 1000 (rad/s) and the elevator actuator bandwidth is 500 (rad/s). Though not possible, if the required actuator bandwidth could be achieved, it is clear that filtering would be required to prevent exciting higher order, unmodeled elastic modes.

It should be noted that a low value of control weighting was required obtain satisfactory results. By comparing the $\dot{\theta}_r$ time responses for each val of control weighting, one can see that the amplitude of these responses tends t decrease as ρ decreases. It appears that as ρ is decreased, the pilot's input having less effect upon the system's dynamic response. This result will explored in a later section.

A different choice of the desired eigenspace can lead to significantly different results. This is illustrated in the next example.

Example 4.2

This example illustrates how a change in one mode of the desired eigenspace can significantly affect the synthesis results. For this example, the desired infinite eigenvalues and eigenvectors are chosen to be the same values as in Example 4.1. The desired finite eigenvectors and eigenvalues are also chosen to be the same as in Example 4.1, with the exception of the second elastic mode.

The second elastic eigenvalue is chosen to be placed at a location such that the mode's natural frequency is the same as the open-loop natural frequency and the mode has increased damping. This value was chosen to be:

$$\lambda^{f}_{\xi_2} = -4.271 \pm j\ 20.92 = (\omega_n = 21.35\ \text{(rad/s)}\ ,\ \zeta = 0.200)$$

The desired eigenvector for the second elastic eigenvector is chosen using the same strategy as used in Example 3.1 and 3.2. The elements of the eigenvector that are associated with the (scaled) rigid-body states are chosen to have small magnitudes. The element associated with the state $\dot{\xi}_2$ is chosen to be unit magnitude as a reference value. The chosen desired second elastic mode's eigenvector is given in Table 4.8. The achievable second elastic eigenvalue is given in Table 4.9. The achievable eigenvectors for the short period, phugoid, and first elastic modes are identical to those given in Table 4.2.

Control laws were synthesized for various values of control weighting, ρ. At a control weighting of $\rho = 1.0 \times 10^{-6}$, the LQEA synthesis yields a closed-loop system eigenspace that is close to the desired eigenspace. The eigenvalues for this synthesis are:

$$\lambda_{sp} = -1.495 \pm j\ 2.366 = (\omega_n = 2.799\ (\text{rad/s})\ ,\ \varsigma = 0.534)$$

$$\lambda_{ph} = -0.0075 \pm j\ 0.0680 = (\omega_n = 0.068\ (\text{rad/s})\ ,\ \varsigma = 0.109)$$

$$\lambda_{\xi_1} = -1.774 \pm j\ 8.638 = (\omega_n = 8.818\ (\text{rad/s})\ ,\ \varsigma = 0.201)$$

$$\lambda_{\xi_2} = -6.401 \pm j\ 19.02 = (\omega_n = 20.07\ (\text{rad/s})\ ,\ \varsigma = 0.319)$$

$$\lambda_{el} = -595.9$$

$$\lambda_{cv} = -945.9$$

The normalized closed-loop system impulse residue magnitudes for each output are shown in Figure 4.9.

Comments on Example 4.2

In this example, the eigenvector selection strategy that worked well with the short period, phugoid, and first elastic modes has yielded an achievable second elastic eigenvector, calculated from $\underline{\nu}^f_{\xi_2}$ and $\lambda^f_{\xi_2}$, that is a "mixed mode". As can be seen in Table 4.9, in this "second elastic" eigenvector the element associated with the $\dot{\xi}_1$ element has a large magnitude.

This control law yields closed-loop system eigenvalues that are close to the desired locations, but unacceptable impulse residue magnitudes. By comparing

Table 4.8
Desired Second Elastic Eigenvector - Example 4.2

DESIRED EIGENVECTORS

EIGENVALUES
1: -4.2709 + J 20.9230
2: -4.2709 - J 20.9230

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

1		2	
(1.0000E-05,	0)	(1.0000E-05,	0)
(1.0000E-04,	6.0000E+01)	(1.0000E-04,	-6.0000E+01)
(1.0000E-08,	7.0000E+01)	(1.0000E-08,	-7.0000E+01)
(1.0000E-05,	-3.0000E+01)	(1.0000E-05,	3.0000E+01)
(* ,	*)	(* ,	*)
(* ,	*)	(* ,	*)
(* ,	*)	(* ,	*)
(1.0000E+00,	2.0000E+01)	(1.0000E+00,	-2.0000E+01)
(* ,	*)	(* ,	*)
(* ,	*)	(* ,	*)

Table 4.9
Achievable Second Elastic Eigenvector - Example 4.2

ACHIEVABLE EIGENVECTORS

EIGENVALUES
1: -4.2709 + J 20.9230
2: -4.2709 - J 20.9230

EIGENVECTORS: (MAGNITUDE,PHASE(DEGREES))

1		2	
(1.0000E+00,	0)	(1.0000E+00,	0)
(8.5440E-02,	-5.2034E+01)	(8.5440E-02,	5.2034E+01)
(1.2283E-03,	-2.8160E+02)	(1.2283E-03,	2.8160E+02)
(4.0010E-03,	-1.5357E+02)	(4.0010E-03,	1.5357E+02)
(1.5204E+01,	-2.8515E+02)	(1.5204E+01,	2.8515E+02)
(2.7229E+00,	-1.9847E+02)	(2.7229E+00,	1.9847E+02)
(3.2467E+02,	-1.8361E+02)	(3.2467E+02,	1.8361E+02)
(5.8146E+01,	-9.6934E+01)	(5.8146E+01,	9.6934E+01)
(3.0235E+01,	-8.3876E+01)	(3.0235E+01,	8.3876E+01)
(1.6273E+03,	-7.9623E+01)	(1.6273E+03,	7.9623E+01)

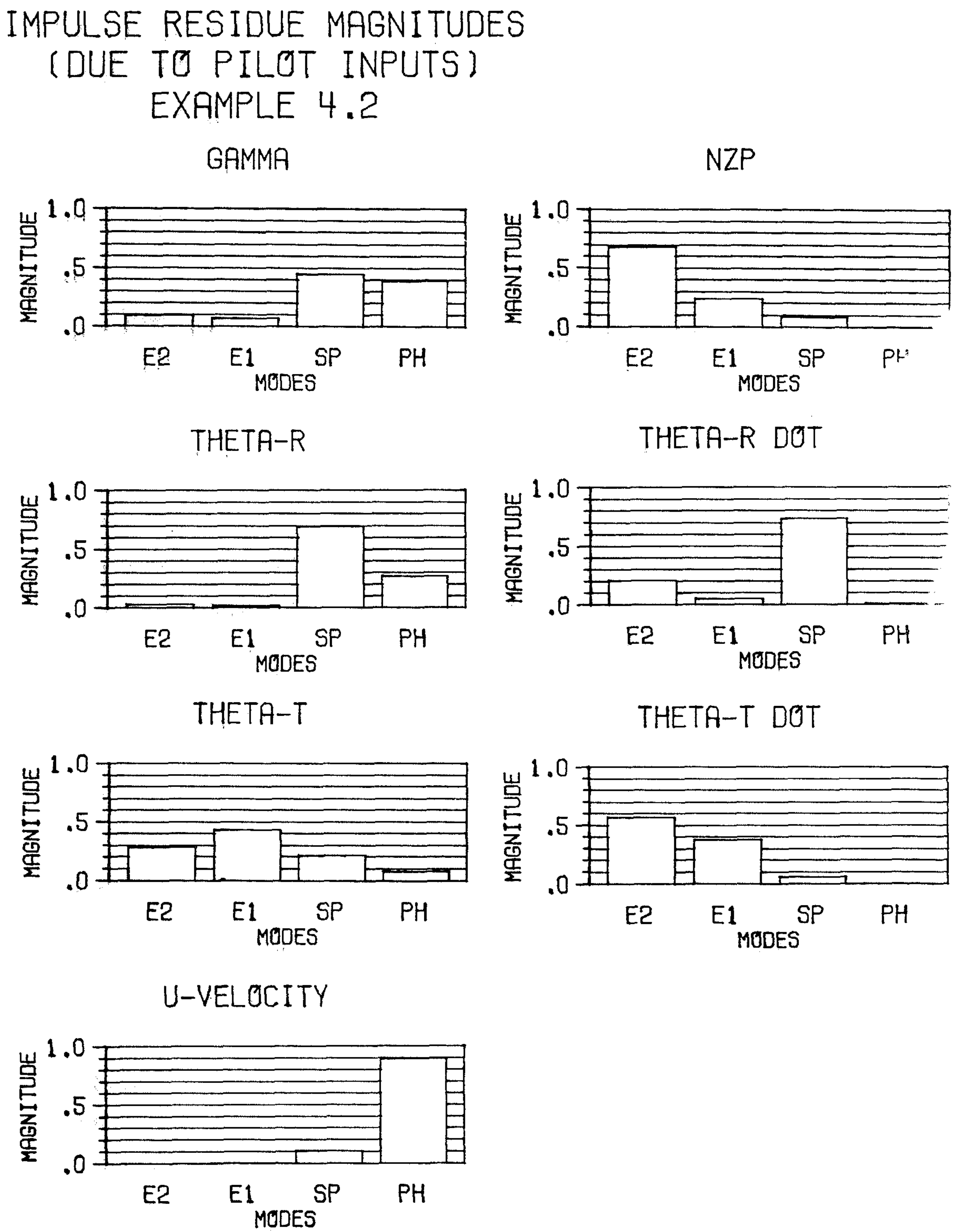


Figure 4.9
Impulse Residue Magnitudes - Example 4.2

the open-loop system impulse residue magnitudes presented in Chapter III and those of Figure 4.9, one can see that this control law has decreased the contribution of the first elastic mode to the θ_r and $\dot{\theta}_r$ outputs, but it has increased the contribution of the second elastic mode to these outputs. This is in opposition to the desired result. This is an interesting result because this desired eigenvector selection procedure "worked" for the other modes. In this example, attempting to change a mode that was not contributing much to the rigid-body outputs, has yielded a control law that has caused this mode to contribute significantly to the $\dot{\theta}_r$ output.

As can be seen from Example 4.1, selecting the desired second elastic eigenvector to be the open-loop eigenvector and the desired second elastic eigenvalue to be close to its open-loop location yielded a much better result.

Comments upon Response Amplitudes

As noted earlier, when comparing the time responses obtained in Example 4.1, the amplitudes of the responses decrease as the control weighting, ρ, decreases or different control laws result in different steady-state responses (or D.C. gains). The D.C. gain of a response due to a unit step can be determined in the following way.

As was shown earlier, a system's input-output transfer function relationship is given by

$$\underline{y}(s) = \mathbf{C}[s\mathbf{I}_n - \mathbf{A}]^{-1}\,\mathbf{B}u_p(s) \tag{3.15}$$

For a stable system, the steady-state response due to a unit step input ($u_p(t)$ = one degree , $u_p(s) = 1/s$) is given by

$$\underline{y}_{ss} = -\,[\mathbf{C}\mathbf{A}^{-1}\mathbf{B}] \tag{4.21}$$

For example, the steady-state angle of attack response due to a unit step pilot input is given by

$$\alpha_{ss} = -\,[\ c\mathbf{A}^{-1}b\ ] = C_{\alpha,\delta_e}$$

where

c = row of $\mathbf{C}$ matrix associated with α

b = column of $\mathbf{B}$ matrix associated with δ_e

This equation shows that a commanded elevator deflection input of, for example, δ_e = (one degree) yields a steady-state angle of attack of C_{α,δ_e} degrees.

The larger the magnitude of C_{α,δ_e}, the larger the effect δ_e has upon the steady-state angle of attack. Values obtained for α_{ss} for the open-loop large flexible aircraft (Configuration Two) and several of the stable augmented systems presented in examples in Chapters III and IV are given in Table 4.10.

Table 4.10
Steady-State D.C. Gain

System	C_{α,δ_e}
Open-loop	-10.38
Example 4.1, $\rho=1 \times 10^{6}$	-10.38
Example 3.2 (DEA, six meas.)	-6.893
Example 4.1, $\rho=1 \times 10^{-2}$	-0.6293
Example 4.1, $\rho=1 \times 10^{-4}$	-0.5388
Example 4.1, $\rho=1 \times 10^{-6}$	-0.1460

One may gain insight into how this occurs by calculating the steady-state value of elevator deflection due to unit step pilot input. Recall, the control law is given by

$$\underline{u} = \underline{u}_c + \underline{u}_p = \mathbf{G}\underline{x} + \underline{u}_p \tag{3.1e}$$

where

$\underline{u}_c$ = feedback control law input

$\underline{u}_p$ = pilot's input

$\mathbf{G}$ = matrix of state feedback control law gains

This equation can be rewritten in the following way

$$\begin{Bmatrix} \delta_{e_c} \\ \delta_{cv_c} \end{Bmatrix} = \left[\begin{array}{c|c} G_{e,\chi} & G_{e,\delta_e} \\ \hline G_{cv,\chi} & G_{cv,\delta_e} \end{array}\right] \begin{Bmatrix} \underline{\chi} \\ \delta_e \end{Bmatrix} + \begin{Bmatrix} \delta_{e_p} \\ 0 \end{Bmatrix} \tag{4.22}$$

where

$\underline{\chi}^T = (\alpha , \dot{\theta}_r , u_f , \theta_r , \xi_1 , \xi_2 , \dot{\xi}_1 , \dot{\xi}_2 , \delta_{cv})$

δ_e = elevator deflection (scalar)

δ_{e_c} = commanded elevator deflection (scalar)

δ_{e_p} = pilot's commanded elevator deflection (scalar)

The equation for δ_{e_c} is, for example,

$$\delta_{e_c} = G_{e,\chi}\underline{\chi} + G_{e,\delta_e}\delta_e + \delta_{e_p} \tag{4.23}$$

$$= u'_c + G_{e,\delta_e}\delta_e + \delta_{e_p}$$

Now, the elevator actuator dynamics are described by

$$\dot{\delta}_e = -\frac{1}{\tau}\delta_e + \frac{1}{\tau}\delta_{e_c} \tag{4.24}$$

Substituting Equation (4.23) into Equation (4.24) one obtains

$$\dot{\delta}_e = (-\frac{1}{\tau} + \frac{G_{e,\delta_e}}{\tau})\delta_e + \frac{1}{\tau}(u'_c + \delta_{e_p}) \tag{4.25}$$

Taking the Laplace transform of this equation and finding the transfer function between δ_e and pilot's input, δ_{e_p}, (with $u'_c = 0$) yields

$$\frac{\delta_e}{\delta_{e_p}} = \frac{1}{\tau s + (1 - G_{e,\delta_e})} \tag{4.26}$$

The steady-state value of elevator deflection due to a unit step in δ_{e_p} is

$$\delta_{e_{ss}} = \frac{1}{(1 - G_{e,\delta_e})} \tag{4.27}$$

This equation shows how the elevator deflection is reduced as the magnitude of G_{e,δ_e} increases. If it is desired to hold the D.C. gain of a steady-state response constant or meet some desired system output specification, one can simply multiply the pilot's input by the constant ($1 - G_{e,\delta_e}$).

LQEA Conclusions

LQEA is a technique for selecting quadratic weighting matrices that when used in an LQ synthesis will asymptotically place (n-m) finite eigenvalues and eigenvectors and m infinite eigenvalues and eigenvectors in the closed-loop system. With LQEA the designer has information concerning the asymptotic location of all of the eigenvalues and eigenvectors of the closed-loop system. As ρ tends toward zero in the LQ synthesis, (n-m) of the closed-loop system eigenvalues tend toward desired locations. Their (n-m) associated eigenvectors tend toward the (n-m) system achievable eigenvectors. The remaining m closed-loop system eigenvalues tend toward infinity in first order Butterworth patterns. Their m associated eigenvectors tend toward the vectors $\underline{\nu}_i^\infty$.

Using LQEA, the designer determines which (n-m) of the system's eigenvalues are to be finite eigenvalues/eigenvectors. The remaining m are infinite eigenvalues/eigenvectors. One then chooses the desired eigenspace to obtain the desired synthesis objective. The desired eigenspace is used to determine the state weighting **Q**, and the control weighting matrix, **R**. These weighting matrices are used in an LQ synthesis with various values of control weighting to yield a family of control laws.

The examples presented in this chapter illustrate the flexibility and potential usefulness of this control synthesis technique. As with DEA, the proper choice of the desired eigenspace is an important part of the LQEA synthesis. The examples showed that the choice of the desired eigenspace can significantly effect the synthesis results.

One potential disadvantage to this technique is that it yields a full state feedback control law. In many physical systems, the complete state can not be measured. Even in the cases where it can be, this type of control system may require too many sensors and may therefore be too complex and costly to implement.

Also, in some cases, when it is necessary to use low values of control weighting to obtain the desired synthesis results, the LQEA technique may yield augmented systems with very high actuator bandwidth requirements. This could lead to problems with the practical application of this method.

The next chapter will explore how some of the disadvantages of both the direct and LQ techniques can be overcome by using these techniques with state estimation.

CHAPTER V
USING EIGENSPACE ASSIGNMENT TECHNIQUES WITH STATE ESTIMATION

In chapter III, the Direct Eigenspace Assignment technique was presented. This technique was shown to be an attractive method for eigenspace assignment because it could be used to improve system dynamics and it yielded a relatively simple measurement feedback design. But as discussed, DEA had a number of disadvantages. In Chapter IV, it was shown that these disadvantages could be partially overcome by using the LQEA control synthesis technique. But, overcoming these limitations comes at a cost of increased system complexity with full state feedback.

This chapter explores how some of the disadvantages of both the direct and LQ techniques can be overcome by using state estimation. By using state estimation with the direct technique (full-state feedback) one has the freedom to place all of the modes of the system and one obtains the practicality of a measurement feedback control law. Using state estimation with the LQ technique one can obtain a measurement feedback control law.

Effect of State Estimation on System Dynamics

How the state estimator dynamics affect the system transient responses, what method should be used to synthesize the state estimator, and how the state estimator affects the controller dynamics are three questions that will be addressed. The estimator dynamics could significantly affect the system's transient response and thereby nullify any effort expended improving plant dynamics.

The effect of the state estimator dynamics on the system transient response can be examined in terms of their effect upon the augmented system transfer functions. This is done in the following way.

Given the system

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} + \mathbf{E}\underline{w}_x \quad \text{(system dynamics)} \tag{5.1a}$$

$$\underline{y} = \mathbf{C}\underline{x} \quad \text{(system responses)} \tag{5.1b}$$

$$\underline{z} = \mathbf{M}\underline{x} + \underline{v}_z \quad \text{(system measurements)} \tag{5.1c}$$

$$\underline{u}_c = \mathbf{G}\hat{\underline{x}} \quad \text{(control law)} \tag{5.1d}$$

$$\underline{u} = \underline{u}_c + \underline{u}_p \quad \text{(total control input)} \tag{5.1e}$$

where

$$\hat{\underline{x}} = \text{estimated state}$$

$$\underline{u}_p = \text{pilot's input}$$

with $\underline{x} \in R^n$, $\underline{u} \in R^l$, $\underline{y} \in R^p$, and $\hat{\underline{x}} \in R^n$. The vectors, $\underline{w}_x$ and $\underline{v}_z$, are random processes reflecting external disturbances and sensor noises, respectively.

Define the state estimator dynamics by

$$\dot{\hat{\underline{x}}} = \mathbf{A}\hat{\underline{x}} + \mathbf{B}\underline{u} + \mathbf{F}(\underline{z} - \mathbf{M}\hat{\underline{x}}) \tag{5.2}$$

where $\mathbf{F}$ is a matrix of chosen estimator gains, and the state estimation error is defined by

$$\tilde{\underline{x}} = \underline{x} - \hat{\underline{x}} \tag{5.3}$$

The system equations as functions of $\underline{x}$ and $\tilde{\underline{x}}$ become

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} + \mathbf{E}\underline{w}_x$$

$$= \mathbf{A}\underline{x} + \mathbf{B}(\underline{u}_c + \underline{u}_p) + \mathbf{E}\underline{w}_x$$

$$= \mathbf{A}\underline{x} + \mathbf{BG}\hat{\underline{x}} + \mathbf{B}\underline{u}_p + \mathbf{E}\underline{w}_x$$

Therefore,

$$\dot{\underline{x}} = (\mathbf{A} + \mathbf{BG})\underline{x} - \mathbf{BG}\tilde{\underline{x}} + \mathbf{B}\underline{u}_p + \mathbf{E}\underline{w}_x \tag{5.4}$$

The error dynamics are obtained from Equation (5.1a) and (5.2), or

$$\dot{\tilde{\underline{x}}} = \dot{\underline{x}} - \dot{\hat{\underline{x}}}$$

$$= (\mathbf{A}\underline{x} + \mathbf{B}\underline{u} + \mathbf{E}\underline{w}_x) - (\mathbf{A}\hat{\underline{x}} + \mathbf{B}\underline{u} + \mathbf{F}(\underline{z} - \mathbf{M}\hat{\underline{x}}))$$

therefore,

$$\dot{\tilde{\underline{x}}} = (\mathbf{A} - \mathbf{FM})\tilde{\underline{x}} - \mathbf{F}\underline{v}_z + \mathbf{E}\underline{w}_x \tag{5.5}$$

The augmented system then can be written

$$\begin{Bmatrix} \dot{\underline{x}} \\ \dot{\tilde{\underline{x}}} \end{Bmatrix} = \left[\begin{array}{c|c} \mathbf{A} + \mathbf{BG} & -\mathbf{BG} \\ \hline 0 & \mathbf{A} - \mathbf{FM} \end{array}\right] \begin{Bmatrix} \underline{x} \\ \tilde{\underline{x}} \end{Bmatrix} + \left[\begin{array}{c} \mathbf{B} \\ \hline 0 \end{array}\right] \underline{u}_p + \left[\begin{array}{c|c} \mathbf{E} & 0 \\ \hline \mathbf{E} & -\mathbf{F} \end{array}\right] \begin{Bmatrix} \underline{w}_x \\ \underline{v}_z \end{Bmatrix} \tag{5.6}$$

One can now calculate the $\underline{u}_p$-to-$\underline{x}$ transfer functions after taking the Laplace transform of Equation (5.6). One obtains

$$\begin{Bmatrix} \underline{x}(s) \\ \tilde{\underline{x}}(s) \end{Bmatrix} = \left[\begin{array}{c|c} (s\mathbf{I}_n - (\mathbf{A}+\mathbf{BG})) & -\mathbf{BG} \\ \hline 0 & (s\mathbf{I}_n - (\mathbf{A}-\mathbf{FM})) \end{array}\right]^{-1} \left[\begin{array}{c} \mathbf{B} \\ \hline 0 \end{array}\right] \underline{u}_p(s)$$

$$+ \left[\begin{array}{c|c} \mathbf{E} & 0 \\ \hline \mathbf{E} & -\mathbf{F} \end{array}\right] \begin{Bmatrix} \underline{w}_x \\ \underline{v}_z \end{Bmatrix} \tag{5.7}$$

$$= \left[\begin{array}{c|c} (s\mathbf{I}_n - (\mathbf{A}+\mathbf{BG}))^{-1} & (s\mathbf{I}_n - (\mathbf{A}+\mathbf{BG}))^{-1}\mathbf{BG}(s\mathbf{I}_n - (\mathbf{A}-\mathbf{FM}))^{-1} \\ \hline 0 & (s\mathbf{I}_n - (\mathbf{A}-\mathbf{FM}))^{-1} \end{array}\right] \left[\begin{array}{c} \mathbf{B} \\ \hline 0 \end{array}\right] \underline{u}_p(s)$$

$$+ \left[\begin{array}{c|c} \mathbf{E} & \mathbf{0} \\ \hline \mathbf{E} & -\mathbf{F} \end{array}\right] \begin{Bmatrix} \underline{w}_x \\ \underline{v}_z \end{Bmatrix}$$

Since

$$\underline{y} = \mathbf{C}\underline{x}(s)$$

and assuming $\underline{w}_x = 0.0$, $\underline{v}_z = 0.0$; $\underline{x}(s)$ is given by

$$\underline{x}(s) = [s\mathbf{I}_n - (\mathbf{A}+\mathbf{BG})]^{-1}\mathbf{B}\underline{u}_p(s) \tag{5.8}$$

then,

$$\underline{y}(s) = \mathbf{C}[s\mathbf{I}_n - (\mathbf{A} + \mathbf{BG})]^{-1}\mathbf{B}\underline{u}_p(s) \tag{5.9}$$

By referring to Equation (4.20) of chapter IV, one can see that the $\underline{u}_p$-to-$\underline{y}$ transfer functions of this system with state estimation (Equation (5.9)) are the same as the $\underline{u}_p$-to-$\underline{y}$ transfer functions without state estimation. Therefore, as implemented here, the closed-loop system transient response due to a pilot's input is independent of the state estimator dynamics. In this formulation, the estimator dynamics are uncontrollable from the pilot's input, which is obvious from Equation (5.9). In the closed-loop system, the estimator poles (or the eigenvalues of $(\mathbf{A} - \mathbf{FM})$), will always be canceled by zeros in each of the individual y_i-to-u_p transfer functions. A numerical example of this estimator pole/zero cancellation will be presented in Example 5.1 . Because of this independence, the state estimator can, in fact, be synthesized using any appropriate method. This yields design freedom to meet additional objectives such as robustness. For example, by using a Kalman filter as a state estimator and the method of Doyle and Stein [16], one could obtain enhanced closed-loop robustness properties.

Unlike the closed-loop system transient response due to a pilot's input, the choice of state estimator does affect the controller dynamics. The effect of state estimation on the controller dynamics can be explored by determining the controller transfer functions. These transfer functions can be determined in the following way. Recall, the state estimator dynamics are given by

$$\dot{\underline{\hat{x}}} = \mathbf{A}\hat{x} + \mathbf{B}\underline{u} + \mathbf{F}(\underline{z} - \mathbf{M}\underline{\hat{x}}) \tag{5.2}$$

$$= (\mathbf{A} + \mathbf{BG} - \mathbf{FM})\underline{\hat{x}} + \mathbf{B}\underline{u}_p + \mathbf{F}\underline{z}$$

$$= \mathbf{A}_{con}\underline{\hat{x}} + \mathbf{B}\underline{u}_p + \mathbf{F}\underline{z}$$

where $\mathbf{A}_{con} = \mathbf{A} + \mathbf{BG} - \mathbf{FM}$.

Taking the Laplace transform of this equation yields

$$\underline{\hat{x}}(s) = (s\mathbf{I}_n - \mathbf{A}_{con})^{-1}\mathbf{B}\underline{u}_p(s) + (s\mathbf{I}_n - \mathbf{A}_{con})^{-1}\mathbf{F}\underline{z}(s) \tag{5.10}$$

The control, $\underline{u}_c$, is related to $\underline{\hat{x}}$ by the relation

$$\underline{u}_c = \mathbf{G}\underline{\hat{x}}$$

Therefore,

$$\underline{u}_c(s) = \mathbf{G}(s\mathbf{I}_n - \mathbf{A}_{con})^{-1}\mathbf{B}\underline{u}_p(s) + \mathbf{G}(s\mathbf{I}_n - \mathbf{A}_{con})^{-1}\mathbf{F}\underline{z}(s) \tag{5.11}$$

The total control is given by

$$\underline{u}(s) = \underline{u}_c(s) + \underline{u}_p(s)$$

So, the controller transfer functions are given by

$$\underline{u}(s) = [\mathbf{I}_n + \mathbf{G}(s\mathbf{I}_n - \mathbf{A}_{con})^{-1}\mathbf{B}]\underline{u}_p(s) + \mathbf{G}(s\mathbf{I}_n - \mathbf{A}_{con})^{-1}\mathbf{F}\underline{z}(s) \tag{5.12}$$

From this equation one can see that the controller output is affected by both the measurements and the pilot's inputs. The controller poles, the poles of $\mathbf{A}_{con}$, are directly dependent upon the state estimator. In the case of using a Kalman filter as a state estimator, the controller poles are dependent upon the choice of system noise intensity. A numerical example of this dependence of the controller dynamics upon the state estimator will be presented in Example 5.1.

Control Synthesis Technique Summary

In summary then, the solution method for this technique is a two part process:

1) Use either the direct technique (full-state feedback) or the LQ technique to achieve the desired closed-loop eigenspace.
2) Synthesize a state estimator using any appropriate method.

An example will be presented in the next section to demonstrate the use of the LQ assignment technique with state estimation.

LQEA with State Estimation Example

How the LQEA synthesis technique can be used with state estimation will be demonstrated by an example. This example will extend the LQEA synthesis results obtained in Example 4.1 for $\rho = 1.0 \times 10^{-6}$. A Kalman filter will be used as a state estimator. The model used is the same as that presented in chapter IV, Example 4.1 . The Kalman filter will be synthesized for the measurement vector

$$\underline{z}^T = (\theta_t , \dot{\theta}_t , n_{z_p} , n_{z_t})$$

where

θ_t = total pitch angle (radians)

$\dot{\theta}_t$ = total pitch rate (radians/sec)

n_{z_p} = plunge acceleration at the cockpit (g' s)

n_{z_t} = plunge acceleration at a tail station (g' s)

These are the same four measurements considered in Chapter III, Example 3.1.

<u>Example 5.1</u>

This example will start with the synthesis results of the LQEA example for $\rho = 1.0 \times 10^{-6}$.

The augmented system eigenvalues, the eigenvalues of $(\mathbf{A} + \mathbf{BG})$ were:

$$\lambda_{sp} = -\,1.493\ \pm\ j\ 2.365\ = (\ \omega_n = 2.797\ (\text{rad/s})\ ,\ \varsigma = 0.534\ )$$

$$\lambda_{ph} = -\,0.0075\ \pm\ j\ 0.0679\ = (\ \omega_n = 0.068\ (\text{rad/s})\ ,\ \varsigma = 0.110\ )$$

$$\lambda_{\xi_1} = -\,1.807\ \pm\ j\ 8.608\ = (\ \omega_n = 8.796\ (\text{rad/s})\ ,\ \varsigma = 0.205\ )$$

$$\lambda_{\xi_2} = -\,0.5351\ \pm\ j\ 21.35\ = (\ \omega_n = 21.36\ (\text{rad/s})\ ,\ \varsigma = 0.025\ )$$

$$\lambda_{el} = -\,500.5$$

$$\lambda_{cv} = -\,999.9$$

The eigenvectors of $(\mathbf{A} + \mathbf{BG})$ can be found in Chapter IV, Table 4.7.

The Kalman estimator will be synthesized for a fictitious state noise intensity (actually a design variable selected to yield an acceptable design) of

$$\mathbf{W}_x = \mathbf{I}_n$$

and a measurement noise intensity (also, in effect, a design parameter) of

$$\mathbf{V}_z = \kappa\, \mathbf{I}_m$$

where κ is a positive scalar quantity.

The estimator will be synthesized for two values of κ (1.0×10^6 and 1.0×10^{-6}) to demonstrate how the choice of state estimator can significantly effect the controller dynamics.

Control synthesis for $\kappa = 1.0 \times 10^6$. The filter gain matrix and the augmented system matrices for this value of ρ and κ can be found in Appendix A.5. The system estimator eigenvalues, the eigenvalues of $(\mathbf{A} - \mathbf{FM})$ are:

$$\lambda_{1,2} = -\,0.4594\ \pm\ j\ 21.35\ = (\ \omega_n = 21.36\ (\text{rad/s})\ ,\ \varsigma = 0.022\ )$$

$$\lambda_{3,4} = -\ 0.7268\ \pm j\ 8.758 = (\ \omega_n = 8.789\ (\text{rad/s})\ ,\ \varsigma = 0.083\ )$$

$$\lambda_{5,6} = -\ 1.349\ \pm j\ 2.193 = (\ \omega_n = 2.574\ (\text{rad/s})\ ,\ \varsigma = 0.524\ )$$

$$\lambda_{7,8} = -\ 0.0042\ \pm j\ 0.0525 = (\ \omega_n = 0.053\ (\text{rad/s})\ ,\ \varsigma = 0.079\ )$$

$$\lambda_9 = -\ 10.0 \qquad \lambda_{10} = -\ 9.0$$

The eigenvectors of the complete augmented system are given in Appendix A.5. Note that the eigenvectors associated with the poles of $(\mathbf{A} + \mathbf{BG})$ are the same as before.

The poles of the controller, the poles of $\mathbf{A}_{con}$ are:

$$\lambda_{1,2} = -0.538\ \pm j\ 21.35$$

$$\lambda_{3,4} = -1.809\ \pm j\ 8.610$$

$$\lambda_{5,6} = -1.495\ \pm j\ 2.366$$

$$\lambda_{7,8} = -0.0052\ \pm j\ 0.0679$$

$$\lambda_9 = -999.9 \qquad \lambda_{10} = -500.5$$

To demonstrate the estimator pole/zero cancellation in the $\underline{y}$-to-$\underline{u}_p$ transfer functions, the θ_r-to-δ_e transfer function is determined (recall, in this model $\underline{u}_p^T = (\ \delta_e\ ,\ 0.0\ )$). The poles of this transfer function are the eigenvalues of $(\mathbf{A} + \mathbf{BG})$ and $(\mathbf{A} - \mathbf{FM})$. The zero's of the θ_r-to-δ_e transfer function were calculated using the method of Sandburg and So [14]. This transfer function is given in Figure 5.1. As can be seen, the poles of $(\mathbf{A} - \mathbf{FM})$ are a subset of the transfer function zeros and will therefore cancel in the transfer function.

$$\frac{\theta_r}{\delta_e} = \frac{k(0.277 \pm j21.16)(1.61 \pm j8.67)(999.7)(1.12)(0.0108)(0.459 \pm j21.35)(0.727 \pm j8.76)(1.35 \pm j2.19)(0.0042 \pm j0.053)(9.0)(10.0)}{(1.49 \pm j2.37)(0.0075 \pm j0.068)(1.807 \pm j8.61)(0.535 \pm j21.35)(500.0)(999.9)(0.459 \pm j21.35)(0.727 \pm j8.76)(1.35 \pm j2.19)(0.0042 \pm j0.053)(10.0)(9.0)}$$

Notation: $(s + \sigma \pm j\omega) = (\sigma \pm j\omega)$

Figure 5.1
θ_r/δ_e Transfer Function

The output time responses due to a unit step in $\underline{u}_p$, with zero initial conditions, are shown in Figure 5.2. As is expected, these time responses are identical to those obtained in Example 4.1 for $\rho = 1.0 \times 10^{-6}$ (See Figure 4.8).

Control synthesis for $\kappa = 1.0 \times 10^{-6}$. The filter gain matrix and the augmented system matrices for this value of ρ and κ can be found in Appendix A.5. The system estimator eigenvalues, the eigenvalues of $(\mathbf{A} - \mathbf{FM})$ are:

$$\lambda_{1,2} = -2.244 \pm j\,2.035 = (\,\omega_n = 3.030 \text{ (rad/s)}\,,\ \varsigma = 0.741\,)$$

$$\lambda_3 = -1.106 \times 10^5 \qquad \lambda_4 = -7.326 \times 10^4$$

$$\lambda_5 = -1705.7 \qquad \lambda_6 = -1606.1$$

$$\lambda_7 = -96.95 \qquad \lambda_8 = -9.974$$

$$\lambda_9 = -6.209 \qquad \lambda_{10} = -1.566$$

The eigenvectors of the complete augmented system are given in Appendix A.5.

The poles of the controller are:

$$\lambda_{1,2} = -0.0017 \pm j\,75.98$$

$$\lambda_3 = -1.106 \times 10^5 \qquad \lambda_4 = -7.35 \times 10^4$$

$$\lambda_5 = -781.5 \qquad \lambda_6 = -411.2$$

$$\lambda_7 = -54.35 \qquad \lambda_8 = -8.249$$

$$\lambda_9 = -2.497 \qquad \lambda_{10} = 0.0728$$

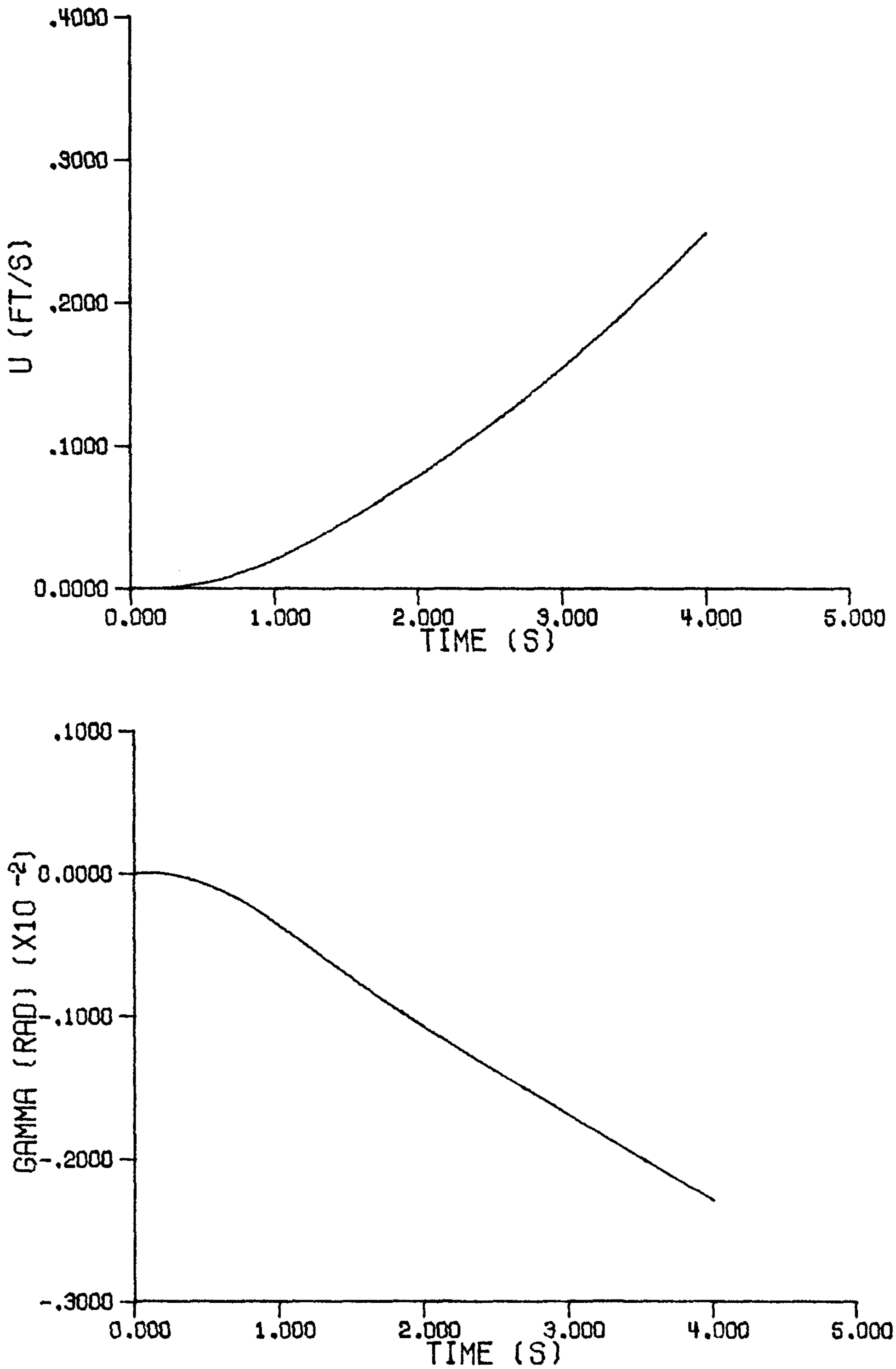


Figure 5.2
Output Time Responses - Example 5.1

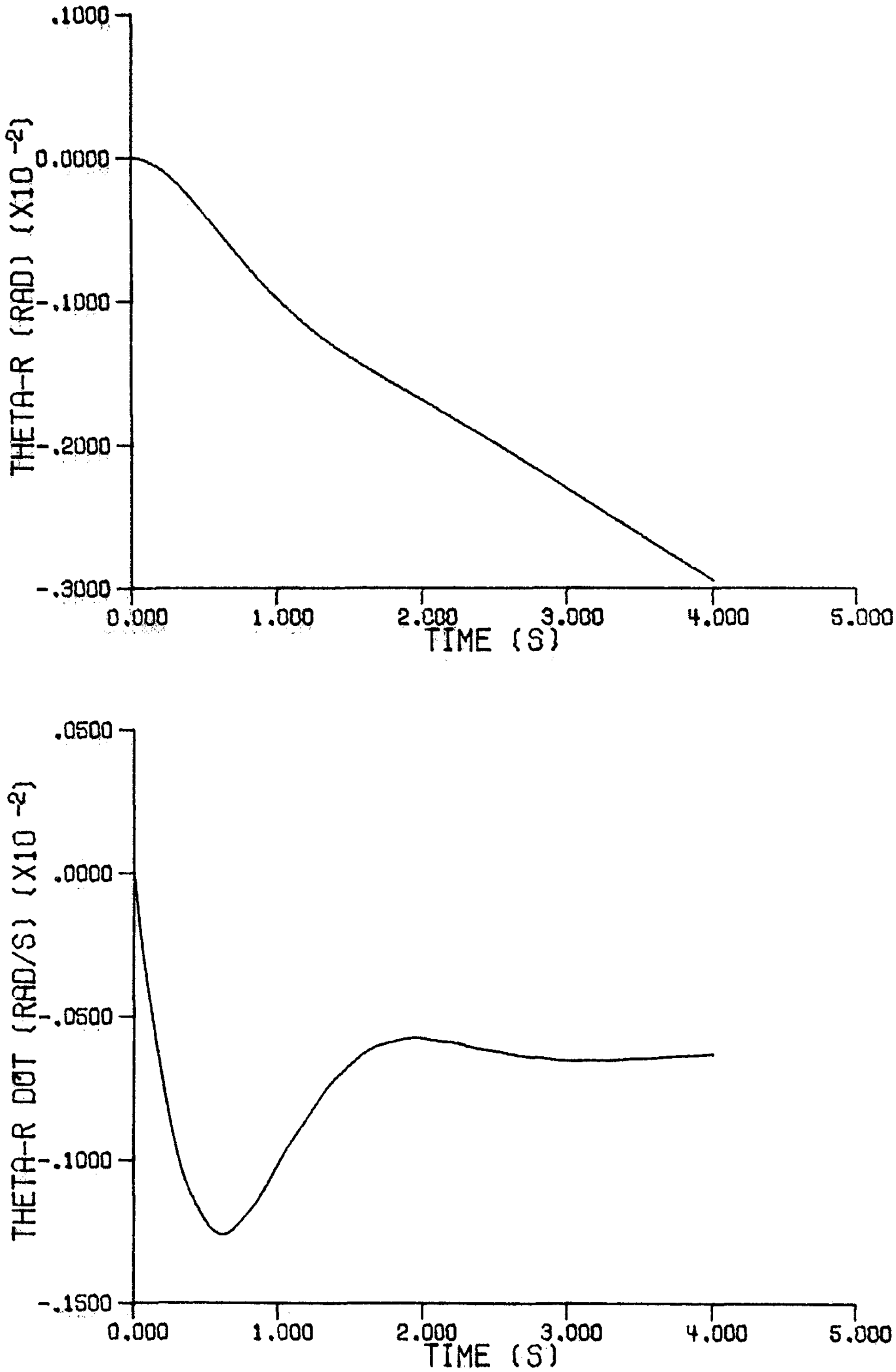
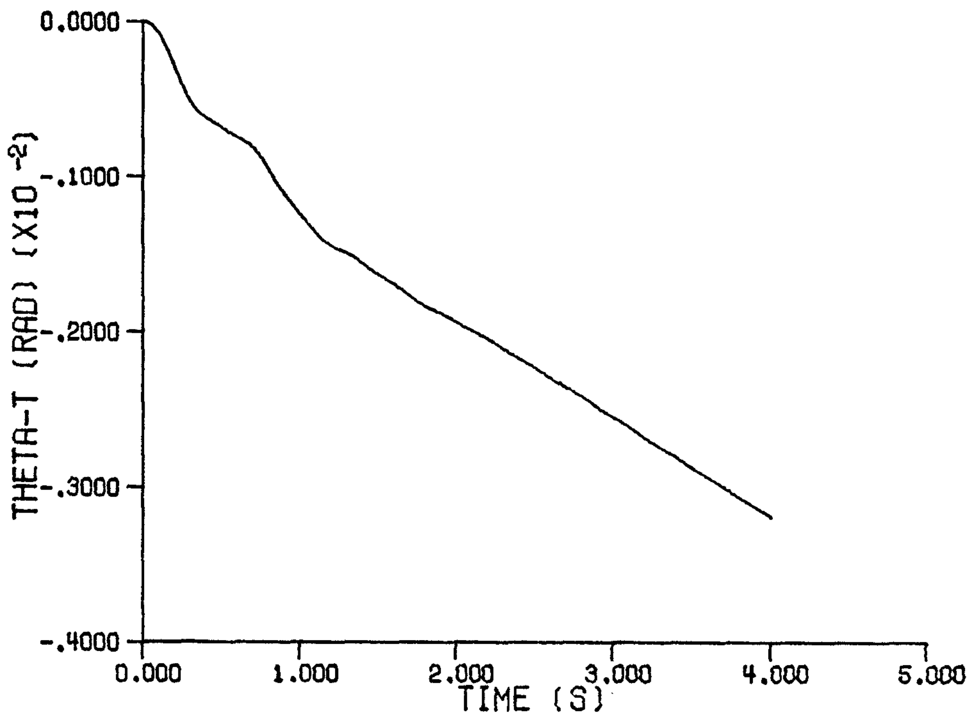


Figure 5.2, continued



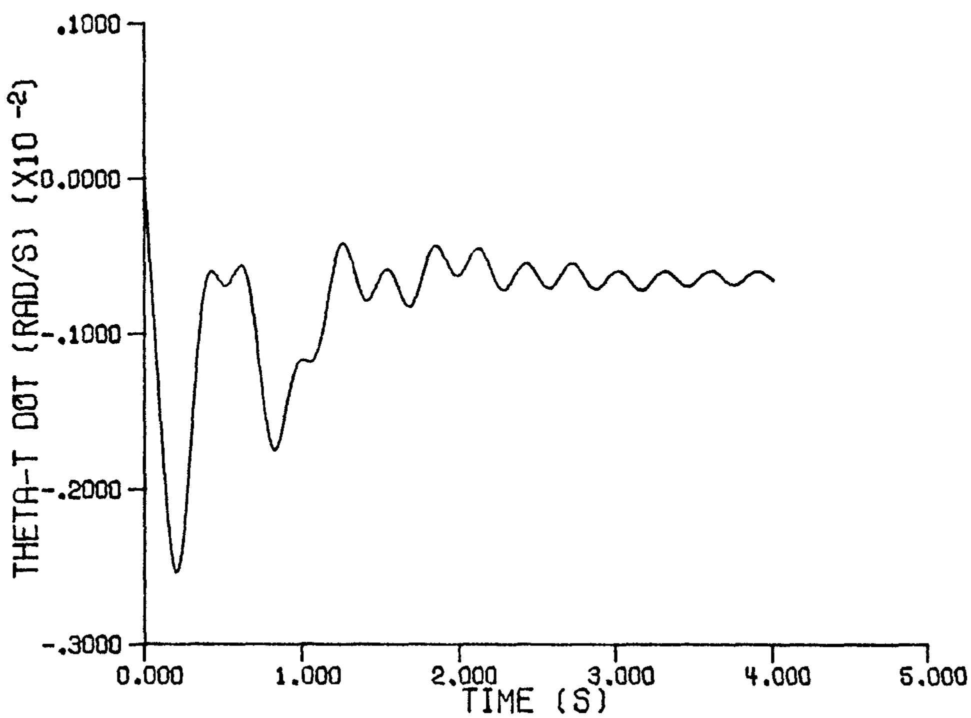


Figure 5.2, continued

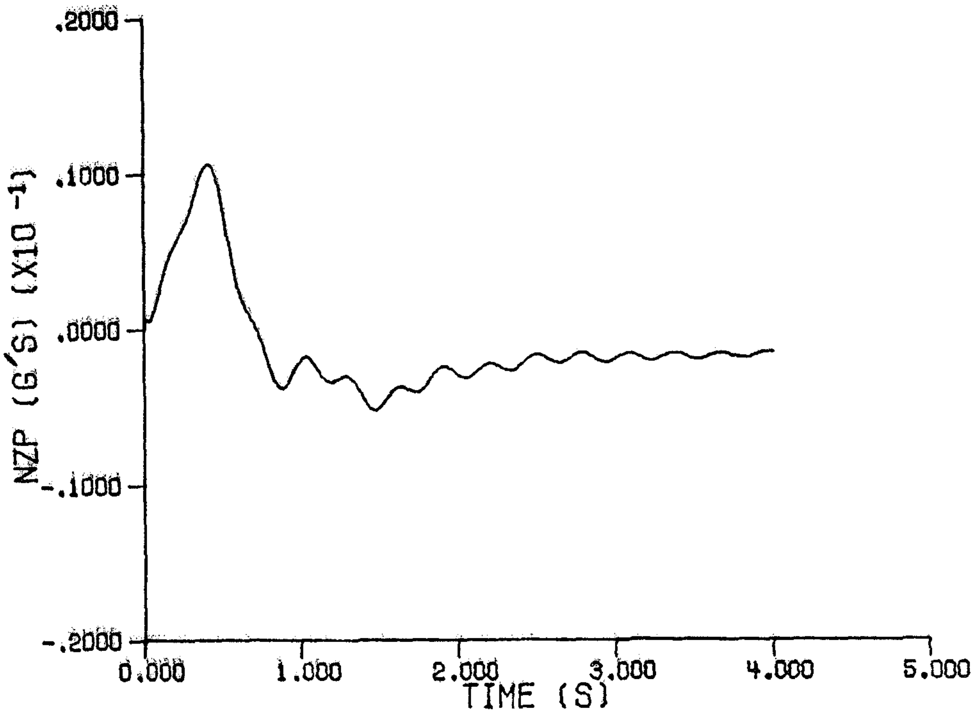


Figure 5.2, concluded

The output time responses due to a unit step in $\underline{u}_p$, with zero initial conditions, are identical to those of Figure 5.2.

Comments on Example 5.1

This example illustrates how the LQEA control synthesis technique can be used with state estimation. It is significant that the regulator gain synthesis and state estimator synthesis are independent. Therefore, the impulse residues and time responses for each case here are the same as the impulse residues and time responses for the $\rho = 1.0 \times 10^{-6}$ LQEA synthesis.

The Kalman filter synthesis for $\kappa = 1.0 \times 10^{6}$ is a synthesis for a large measurement noise intensity. This synthesis produces small filter gains and an estimator with a small bandwidth (poles of $(\mathbf{A} - \mathbf{FM})$). For this synthesis, the controller poles are stable. Because of the small filter gains, the poles of the controller are very close to those of $(\mathbf{A} + \mathbf{BG})$.

The Kalman filter synthesis for $\kappa = 1.0 \times 10^{-6}$ is a synthesis for a small measurement noise intensity. Therefore, this synthesis produces large filter gains and an estimator with a large bandwidth. As can be seen from the controller poles, the controller is unstable for this synthesis. From a practical point of view this is very undesirable.

Conclusions

This chapter explored the advantages of using the direct and LQ eigenspace assignment methods with state estimation. By using either the direct or LQ methods with state estimation one can control the placement of all of the poles of the closed-loop system and obtain a perhaps more practical measurement feedback control law.

As was shown in the development and in the example, the closed-loop system transient response due to a pilot's input is independent of the state estimator dynamics. This is because in this formulation, the estimator dynamics are uncontrollable from the pilot's input. Because of this independence, the state estimator can be synthesized using any appropriate method.

Although the choice of state estimator does not effect the $\underline{u}_p$-to-$\underline{y}$ dynamics, it does effect the controller dynamics. As was shown in the examples, some choices of state estimator can even lead to an unstable controller. Therefore, the controller dynamics must be considered when choosing and synthesizing the state estimator.

CHAPTER VI
CONCLUSIONS

This report has presented two eigenspace assignment techniques: Direct Eigenspace Assignment and Linear Quadratic Eigenspace Assignment; and has explored the advantages of using these techniques with state estimation. These techniques have been applied in several cases to the problem of control law synthesis for a flexible aircraft.

Direct Eigenspace Assignment (DEA), presented in Chapter III, is a control synthesis technique for directly determining measurement feedback control gains that will yield an achievable eigenspace in the closed-loop. For an observable, controllable system that has n states, m controls, and l measurements one can determine a gain matrix that will place l eigenvalues to desired locations and their l associated eigenvectors as close as possible in a least squares sense to desired eigenvectors (this assumes $l > m$). This technique was shown to have two major disadvantages. First, there is no systematic way to trade eigenspace assignment goals versus gain magnitudes. Second, the (n-l) unspecified system poles can move to unpredictable locations, sometimes even destabilizing the system.

These disadvantages were partially overcome in the Linear Quadratic Eigenspace Assignment (LQEA) control synthesis technique. LQEA is a synthesis technique for selecting quadratic weighting matrices that when used in an LQ synthesis will asymptotically place (n-m) finite eigenvalues/eigenvectors and m infinite eigenvalues/eigenvectors in the closed-loop system. It yields a guaranteed stable closed-loop system. The designer has information about the asymptotic behavior of all of the modes of the system and has the ability to trade eigenspace assignment goals versus gain magnitudes. But, using LQEA comes at the cost of increased system complexity due to the necessity of full state feedback. This is a problem because many times the complete state can not be measured. Also, LQEA may yield augmented systems with high actuator bandwidth requirements.

Chapter V explored the advantages of using the direct and LQ eigenspace assignment techniques with state estimation. By using state estimation with either technique the designer has control over the placement of all of the poles of the closed-loop system and can obtain the practicality of a measurement feedback control law.

As was shown in Chapter V, by using the proper formulation the closed-loop system transient response due to a pilot's input is independent of the estimator dynamics. Therefore, the state estimator can be synthesized using any appropriate method. But, the choice of state estimator does affect the controller dynamics. Unlike the closed-loop system, the controller is not guaranteed to be stable. The controller poles are directly dependent upon the state estimator. Some choices of state estimator can result in an unstable controller. This is very undesirable from a practical point of view. Therefore, the controller dynamics must be considered when choosing the estimator.

There are many areas for future work. Some of these include:

1) The art of choosing the desired eigenspace. As was pointed out in the examples, the choice of the desired eigenspace is very important and can significantly effect the synthesis results. Much work can be done in the area of exploring what desired eigenspace is needed to achieve a certain desired dynamics. For example, what desired eigenspaces would yield good handling qualities?
2) Extending DEA to be able to trade gain magnitudes versus eigenspace assignment. DEA would be much more useful if one knew how a change in an element of the desired eigenspace would affect an individual gain magnitude or the location of one of the unplaced poles.
3) Exploring how the phase relationships between desired eigenvector elements would effect the synthesis results. Choosing proper phase relationships could make the achievable eigenspace closer to the desired eigenspace.
4) Determining under what conditions an LQG control synthesis yields an unstable controller.
5) Investigating how the choice of measurements (sensor locations) effect the synthesis results.
6) Determining how well each of these synthesis methods rejects disturbances.
7) Examining the sensitivity of each synthesis technique due to variations in the model parameters.

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Appendix A.1
DEA Development

Case of Control Feed-Through in the Measurements

Given the system

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \quad \text{(system dynamics)} \tag{A.1.1a}$$

$$\underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u} \quad \text{(system responses)} \tag{A.1.1b}$$

$$\underline{z} = \mathbf{M}\underline{x} + \mathbf{N}\underline{u} \quad \text{(system measurements)} \tag{A.1.1c}$$

$$\underline{u}_c = \mathbf{G}\underline{z} \quad \text{(feedback control law)} \tag{A.1.1d}$$

$$\underline{u} = \underline{u}_c + \underline{u}_p \quad \text{(total control law)} \tag{A.1.1e}$$

$$\underline{u}_p = \text{pilot's input} \tag{A.1.1f}$$

where $\underline{x}\epsilon R^n$, $\underline{u}\epsilon R^m$, $\underline{y}\epsilon R^p$, $\underline{z}\epsilon R^l$.

Solving for $\underline{u}_c$ as a function of $\underline{x}$ yields

$$\underline{u}_c = \mathbf{G}\underline{z}$$

$$= \mathbf{G}[\mathbf{M}\underline{x} + \mathbf{N}\underline{u}_c]$$

$$= \mathbf{GM}\underline{x} + \mathbf{GN}\underline{u}_c$$

$$\underline{u}_c - \mathbf{GN}\underline{u}_c = \mathbf{GM}\underline{x}$$

$$\underline{u}_c = [\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM}\underline{x} \tag{A.1.2}$$

The closed-loop system is given by

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u}$$

$$= \mathbf{A}\underline{x} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM}\underline{x} + \mathbf{B}\underline{u}_p$$

$$\dot{\underline{x}} = (\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})\underline{x} + \mathbf{B}\underline{u}_p \tag{A.1.3}$$

The spectral decomposition of the closed-loop system is given by

$$(\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})\underline{\nu}_i = \lambda_i\underline{\nu}_i \quad i = 1, \ldots, n \tag{A.1.4}$$

Let $\underline{w}_i$ be defined by

$$\underline{w}_i = [\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM}\underline{\nu}_i \tag{A.1.5}$$

Substituting Equation (A.1.5) into Equation (A.1.4) and solving for $\underline{\nu}_i$ one obtains

$$\underline{\nu}_i = (\lambda_i\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\underline{w}_i \tag{A.1.6}$$

Equation (A.1.6) is an expression for the achievable eigenvectors of the system given λ_i and $\underline{w}_i$. Note that this is the same result as obtained in the $\mathbf{N} = [0]$ case. An expression for $\underline{w}_i$ can be obtained by substituting (A.1.6) into the cost function

$$J_i = \frac{1}{2}\,(\underline{\nu}_{a_i} - \underline{\nu}_{d_i})^T\mathbf{Q}_{d_i}(\underline{\nu}_{a_i} - \underline{\nu}_{d_i}) \tag{A.1.7}$$

where

$\underline{\nu}_{a_i}$ = i^{th} achievable eigenvector
$\underline{\nu}_{d_i}$ = i^{th} desired eigenvector
$\mathbf{Q}_{d_i}$ = i^{th} weighting on eigenvector elements.

Taking the gradient of J_i with respect to $\underline{w}_i$, setting this equal to zero, and solving for $\underline{w}_i$ yields

$$\underline{w}_i^T = \nu_{d_i}^T\mathbf{Q}_{d_i}(\lambda_{d_i}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}[\mathbf{B}^T(\lambda_{d_i}\mathbf{I}_n - \mathbf{A})^{-T}\mathbf{Q}_{d_i}(\lambda_{d_i}\mathbf{I}_n - \mathbf{A})\mathbf{B}]^{-1} \tag{A.1.8}$$

Concatenate the individual $\underline{w}_i$'s and $\underline{\nu}_i$'s columnwise to form $\mathbf{W}$ and $\mathbf{V}$ respectively. Solve for the gain matrix, $\mathbf{G}$, using Equation (A.1.5)

$$\mathbf{W} = [\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GMV} \tag{A.1.5}$$

$$[\mathbf{I}_m - \mathbf{GN}]\mathbf{W} = \mathbf{GMV}$$

$$\mathbf{W} = \mathbf{GMV} + \mathbf{GNW}$$

$$\mathbf{G} = \mathbf{W}[\mathbf{MV} + \mathbf{NW}]^{-1} \tag{A.1.9}$$

Complex Eigenvalues and Eigenvectors

System

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \quad \text{(system dynamics)}$$

$$\underline{z} = \mathbf{M}\underline{x} + \mathbf{N}\underline{u} \quad \text{(systemmeasurements)}$$

$$\underline{u} = \underline{u}_c + \underline{u}_p \quad \text{(total control input)}$$

$$\underline{u}_c = \mathbf{G}\underline{z} \quad \text{(feedback control law)}$$

For this case the cost function is given by

$$J_i = \frac{1}{2}\,(\underline{\nu}_{a_i} - \underline{\nu}_{d_i})^*\mathbf{Q}_{d_i}(\underline{\nu}_{a_i} - \underline{\nu}_{d_i}) \tag{A.1.10}$$

where superscript $*$ denotes complex conjugate transpose. Again, the spectral decomposition of the closed-loop system is given by Equation (A.1.4)

$$(\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})\underline{\nu}_i = \lambda_i\underline{\nu}_i \tag{A.1.4}$$

Let

$$\underline{\nu}_i = \underline{\nu}_{R_i} + j\underline{\nu}_{I_i} \tag{A.1.11}$$

$$\lambda_i = \sigma_i + j\omega_i \tag{A.1.12}$$

Equation (A.1.4) can then be written as

$$(\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})(\underline{\nu}_{R_i} + j\underline{\nu}_{I_i}) = (\sigma_i + j\omega_i)(\underline{\nu}_{R_i} + j\underline{\nu}_{I_i}) \tag{A.1.13}$$

and its complex conjugate

$$(\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})(\underline{\nu}_{R_i} - j\underline{\nu}_{I_i}) = (\sigma_i - j\omega_i)(\underline{\nu}_{R_i} - j\underline{\nu}_{I_i}) \tag{A.1.14}$$

Multipling out either of these equations one obtains

$$(\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})\underline{\nu}_{R_i} - j(\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})\underline{\nu}_{I_i}$$

$$= \sigma_i\underline{\nu}_{R_i} - j\underline{\nu}_{R_i}\omega_i - j\underline{\nu}_{I_i}\sigma_i - \omega_i\underline{\nu}_{I_i} \tag{A.1.15}$$

Separating the real and imaginary parts one obtains

$$(\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})\underline{\nu}_{R_i} = \sigma_i\underline{\nu}_{R_i} - \omega_i\underline{\nu}_{I_i} \tag{A.1.16R}$$

$$(\mathbf{A} + \mathbf{B}[\mathbf{I}_m - \mathbf{GN}]^{-1}\mathbf{GM})\underline{\nu}_{I_i} = \underline{\nu}_{R_i}\omega_i + \underline{\nu}_{I_i}\sigma_i \tag{A.1.16I}$$

In these equations let

$$\underline{w}_{R_i} = (\mathbf{I}_m - \mathbf{GN})^{-1}\mathbf{GM}\underline{\nu}_{R_i} \tag{A.1.17R}$$

$$\underline{w}_{I_i} = (\mathbf{I}_m - \mathbf{GN})^{-1}\mathbf{GM}\underline{\nu}_{I_i} \tag{A.1.17I}$$

Substituting Equations (A.1.17R) and (A.1.17I) into Equations (A.1.16R) and (A.1.16I) yields

$$\mathbf{A}\underline{\nu}_{R_i} + \mathbf{B}\underline{w}_{R_i} = \sigma_i\underline{\nu}_{R_i} - \omega_i\underline{\nu}_{I_i}$$

$$\mathbf{A}\underline{\nu}_{I_i} + \mathbf{B}\underline{w}_{I_i} = \underline{\nu}_{R_i}\omega_i + \sigma_i\underline{\nu}_{I_i}$$

In matrix form this becomes

$$\left[\begin{array}{c|c} \mathbf{B} & 0 \\ \hline 0 & \mathbf{B} \end{array}\right] \begin{Bmatrix} \underline{w}_{R_i} \\ \underline{w}_{I_i} \end{Bmatrix} = \left[\begin{array}{c|c} (\sigma_i\mathbf{I}_n - \mathbf{A}) & -\omega_i\mathbf{I}_n \\ \hline \omega_i\mathbf{I}_n & (\sigma_i\mathbf{I}_n - \mathbf{A}) \end{array}\right] \begin{Bmatrix} \underline{\nu}_{R_i} \\ \underline{\nu}_{I_i} \end{Bmatrix}$$

$$B\underline{w}_i = M(\lambda_i)\underline{\nu}_i \tag{A.1.18}$$

The expression for the achievable eigenvectors of the system is given by

$$\underline{\nu}_i = M^{-1}(\lambda_i)B\underline{w}_i \tag{A.1.19}$$

where

$$\underline{\nu}_i = \begin{Bmatrix} \underline{\nu}_{R_i} \\ \underline{\nu}_{I_i} \end{Bmatrix} , \quad \underline{w}_i = \begin{Bmatrix} \underline{w}_{R_i} \\ \underline{w}_{I_i} \end{Bmatrix}$$

The equation for $\underline{w}_i$ is determined in the same way as before. For this case the equation becomes

$$\underline{w}_i^T = \underline{\nu}_{d_i}^T Q_{d_i} M^{-1}(\lambda) B [\, B^T M^{-T}(\lambda) Q_{d_i} M^{-1}(\lambda) B \,]^{-1} \tag{A.1.20}$$

where

$$\underline{\nu}_{d_i} = \begin{Bmatrix} \underline{\nu}_{d_{R_i}} \\ \underline{\nu}_{d_{I_i}} \end{Bmatrix} \;, \quad Q_{d_i} = \left[\begin{array}{c|c} \mathbf{Q}_{d_{R_i}} & 0 \\ \hline 0 & \mathbf{Q}_{d_{I_i}} \end{array}\right]$$

The gain matrix is solved for using Equations (A.1.17R) and (A.1.17I).

$$\mathbf{W} = (\mathbf{I}_m - \mathbf{GN})^{-1}\mathbf{GMV}$$

$$\mathbf{G} = \mathbf{W}[\mathbf{MV} + \mathbf{NW}]^{-1}$$

where

$$\mathbf{W} = [\mathrm{w}_{R_1} \mid \mathrm{w}_{I_1} \mid \mathrm{w}_{R_2} \mid \mathrm{w}_{I_2} \mid \dots]$$

$$\mathbf{V} = [\mathrm{v}_{R_1} \mid \mathrm{v}_{I_1} \mid \mathrm{v}_{R_2} \mid \mathrm{v}_{I_2} \mid \dots]$$

Appendix A.2
Open-loop System Data

8 STATES
2 CONTROLS
4 MEASUREMENTS
7 OUTPUTS

THE AA MATRIX (8 BY 8) (unscaled)

	1	2	3	4	5	6	7	8
1	-1.2080E+00	9.4400E-01	-6.3400E-04	0.	-8.9400E-03	4.6180E-03	-2.1300E-04	1.5260E-04
2	-7.0590E+00	-2.0180E+00	-2.2900E-03	0.	-1.8420E-01	2.0720E-01	-7.4380E-03	6.9630E-03
3	-2.5000E+01	0.	-2.5000E-02	-3.2200E+01	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00
7	-7.3830E+02	-1.3330E+02	-1.4350E-03	0.	-7.6800E+01	-1.5400E+01	-9.6220E-01	9.1590E-01
8	7.5770E+02	5.5950E+01	-3.9000E-03	0.	6.9900E+00	-4.5650E+02	-1.2190E-01	-8.4850E-01

THE BB MATRIX (8 BY 2) (unscaled)

	1	2
1	-2.8840E-01	-4.3890E-03
2	-1.5030E+01	3.5310E-01
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	-2.2300E+03	-9.2280E+01
8	6.1360E+02	-5.8600E+00

THE BBP VECTOR (8 BY 1) (unscaled)

	1
1	-2.8840E-01
2	-1.5030E+01
3	0.
4	0.
5	0.
6	0.
7	-2.2300E+03
8	6.1360E+02

Figure A.[illegible]
Open-loop System Matri[illegible]

THE CC MATRIX (7 BY 8) (unscaled)

	1	2	3	4	5	6	7	8
1	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	2.6400E-02	3.4560E-02	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	2.6400E-02	3.4560E-02
7	2.6494E+01	-1.0067E+00	1.4230E-02	0.	-1.0770E+00	3.0120E+00	4.1000E-03	2.8000E-03

THE DD MATRIX (7 BY 2) (unscaled)

	1	2
1	0.	0.
2	0.	0.
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	2.7472E+00	1.9869E+00

THE DDP VECTOR (7 BY 1) (unscaled)

	1
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	2.7472E+00

Figure A.2.1, continued

THE MM MATRIX (4 BY 8) (unscaled)

	1	2	3	4	5	6	7	8
1	0.	0.	0.	1.0000E+00	2.6400E-02	3.4560E-02	0.	0.
2	0.	1.0000E+00	0.	0.	0.	0.	2.6400E-02	3.4560E-02
3	2.6494E+01	-1.0067E+00	1.4230E-02	0.	-1.0770E+00	3.0120E+00	4.1000E-03	2.8000E-03
4	4.9330E+01	4.4780E+00	2.0700E-02	0.	-3.7700E-02	-1.9340E+00	1.8500E-02	-1.9600E-02

THE NN MATRIX (4 BY 2) (unscaled)

	1	2
1	0.	0.
2	0.	0.
3	2.7472E+00	1.9869E+00
4	3.8181E+01	3.8520E-01

THE NNP VECTOR (4 BY 1) (unscaled)

	1
1	0.
2	0.
3	2.7472E+00
4	3.8181E+01

THE SCALING MATRIX (8 BY 8)

	1	2	3	4	5	6	7	8
1	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
2	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
3	0.	0.	1.0537E-03	0.	0.	0.	0.	0.
4	0.	0.	0.	1.0000E+00	0.	0.	0.	0.
5	0.	0.	0.	0.	-2.6400E-02	0.	0.	0.
6	0.	0.	0.	0.	0.	-3.4560E-02	0.	0.
7	0.	0.	0.	0.	0.	0.	-2.6400E-02	0.
8	0.	0.	0.	0.	0.	0.	0.	-3.4560E-02

Figure A.2.1, concluded

SPECTRAL DECOMPOSITION OF MATRIX AA

	REAL PART	IMAGINARY PART	FREQUENCY (HERTZ)	DAMPING RATIO	TIME CONST. (SEC/RAD)
1	-4.56441E-01	2.13506E+01	3.39883E+00	.02137348	2.19086E+00
3	-7.26413E-01	8.75840E+00	1.39873E+00	.08265522	1.37663E+00
5	-1.34794E+00	2.19291E+00	4.09674E-01	.52366280	7.41874E-01
7	-5.71890E-05	5.27199E-02	8.39064E-03	.00108477	1.74859E+04

THE EE MATRIX (8 BY 8)

	1	2	3	4	5
1	(3.36E-04, 6.88E-04)	(3.36E-04,-6.88E-04)	(5.75E-03,-1.13E-02)	(5.75E-03, 1.13E-02)	(-3.47E-01, 9.49E-02)
2	(-9.09E-03, 1.25E-02)	(-9.09E-03,-1.25E-02)	(9.76E-02, 8.80E-03)	(9.76E-02,-8.80E-03)	(-1.36E-01,-8.15E-01)
3	(-1.48E-06, 1.39E-06)	(-1.48E-06,-1.39E-06)	(7.81E-05, 1.14E-05)	(7.81E-05,-1.14E-05)	(-6.73E-03,-3.86E-03)
4	(5.92E-04, 4.13E-04)	(5.92E-04,-4.13E-04)	(7.94E-05,-1.12E-02)	(7.94E-05, 1.12E-02)	(-2.42E-01, 2.11E-01)
5	(1.31E-03,-2.17E-03)	(1.31E-03, 2.17E-03)	(4.98E-02, 1.01E-01)	(4.98E-02,-1.01E-01)	(-1.02E-01,-1.86E-02)
6	(4.63E-02,-6.15E-03)	(4.63E-02, 6.15E-03)	(6.50E-04, 3.02E-03)	(6.50E-04,-3.02E-03)	(1.86E-02,-2.15E-03)
7	(4.57E-02, 2.89E-02)	(4.57E-02,-2.89E-02)	(-9.19E-01, 3.63E-01)	(-9.19E-01,-3.63E-01)	(1.79E-01,-1.99E-01)
8	(1.10E-01, 9.91E-01)	(1.10E-01,-9.91E-01)	(-2.69E-02, 3.50E-03)	(-2.69E-02,-3.50E-03)	(-2.03E-02, 4.36E-02)

	6	7	8
1	(-3.47E-01,-9.49E-02)	(-2.29E-01, 7.49E-02)	(-2.29E-01,-7.49E-02)
2	(-1.36E-01, 8.15E-01)	(3.72E-02,-2.21E-02)	(3.72E-02, 2.21E-02)
3	(-6.73E-03, 3.86E-03)	(4.89E-01,-1.54E-01)	(4.89E-01, 1.54E-01)
4	(-2.42E-01,-2.11E-01)	(-4.21E-01,-7.05E-01)	(-4.21E-01, 7.05E-01)
5	(-1.02E-01, 1.86E-02)	(-5.79E-02, 1.86E-02)	(-5.79E-02,-1.86E-02)
6	(1.86E-02, 2.15E-03)	(1.19E-02,-3.87E-03)	(1.19E-02, 3.87E-03)
7	(1.79E-01, 1.99E-01)	(-9.75E-04,-3.06E-03)	(-9.75E-04, 3.06E-03)
8	(-2.03E-02,-4.36E-02)	(2.04E-04, 6.30E-04)	(2.04E-04,-6.30E-04)

Figure A.2.2
Spectral Decomposition - Co

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THE EI         MATRIX (  8 BY  8 )
------------------------------------

                        1                         2                         3                         4                         5
 1 ( 6.23E-01, 1.18E-01) ( 5.35E-02,-1.80E-02) (-3.99E-03, 2.27E-02) (-3.62E-05,-5.56E-06) (-2.61E-01,-4.54E-02)
 2 ( 6.23E-01,-1.18E-01) ( 5.35E-02, 1.80E-02) (-3.99E-03,-2.27E-02) (-3.62E-05, 5.56E-06) (-2.61E-01, 4.54E-02)
 3 (-6.81E-01, 1.05E+00) (-1.51E-03, 2.64E-01) (-1.41E-01,-3.42E-02) ( 8.66E-05,-5.54E-04) ( 1.58E+00,-4.19E+00)
 4 (-6.81E-01,-1.05E+00) (-1.51E-03,-2.64E-01) (-1.41E-01, 3.42E-02) ( 8.66E-05, 5.54E-04) ( 1.58E+00, 4.19E+00)
 5 (-1.37E+00,-1.20E-01) (-1.49E-01, 5.91E-01) (-6.37E-01,-1.39E-01) (-2.83E-03,-8.11E-03) (-4.36E-02,-2.45E-01)
 6 (-1.37E+00, 1.20E-01) (-1.49E-01,-5.91E-01) (-6.37E-01, 1.39E-01) (-2.83E-03, 8.11E-03) (-4.36E-02, 2.45E-01)
 7 ( 1.24E-01,-4.49E-01) (-3.77E-02, 9.86E-02) ( 9.23E-01, 2.92E-01) (-1.87E-01, 5.94E-01) ( 6.64E-03,-1.80E-02)
 8 ( 1.24E-01, 4.49E-01) (-3.77E-02,-9.86E-02) ( 9.23E-01,-2.92E-01) (-1.87E-01,-5.94E-01) ( 6.64E-03, 1.80E-02)

                        6                         7                         8
 1 ( 1.07E+01, 1.17E+00) ( 1.12E-03, 1.18E-02) ( 6.44E-02,-4.97E-01)
 2 ( 1.07E+01,-1.17E+00) ( 1.12E-03,-1.18E-02) ( 6.44E-02, 4.97E-01)
 3 ( 3.70E-01, 3.64E-01) (-4.63E-01,-1.92E-01) ( 1.97E-02,-5.35E-03)
 4 ( 3.70E-01,-3.64E-01) (-4.63E-01, 1.92E-01) ( 1.97E-02, 5.35E-03)
 5 ( 5.75E-05, 9.63E-02) (-2.69E-02, 4.91E-02) ( 3.52E-03,-8.71E-03)
 6 ( 5.75E-05,-9.63E-02) (-2.69E-02,-4.91E-02) ( 3.52E-03, 8.71E-03)
 7 (-4.55E-03, 1.20E-02) (-2.83E-03, 6.82E-03) ( 5.34E-04,-1.34E-03)
 8 (-4.55E-03,-1.20E-02) (-2.83E-03,-6.82E-03) ( 5.34E-04, 1.34E-03)


--------------------------------------------------
O.L. EIGENVECTORS    MATRIX ( 8 BY  8)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
--------------------------------------------------

                         1                           2                           3                           4
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.013E+01, 6.218E+01) ( 2.013E+01,-6.218E+01) ( 7.740E+00, 6.813E+01) ( 7.740E+00,-6.813E+01)
 3 ( 2.645E-03, 7.289E+01) ( 2.645E-03,-7.289E+01) ( 6.228E-03, 7.127E+01) ( 6.228E-03,-7.127E+01)
 4 ( 9.425E-01,-2.905E+01) ( 9.425E-01, 2.905E+01) ( 8.807E-01,-2.661E+01) ( 8.807E-01, 2.661E+01)
 5 ( 3.301E+00,-1.229E+02) ( 3.301E+00, 1.229E+02) ( 8.878E+00, 1.267E+02) ( 8.878E+00,-1.267E+02)
 6 ( 6.095E+01,-7.152E+01) ( 6.095E+01, 7.152E+01) ( 2.439E-01, 1.408E+02) ( 2.439E-01,-1.408E+02)
 7 ( 7.050E+01,-3.165E+01) ( 7.050E+01, 3.165E+01) ( 7.803E+01, 2.215E+02) ( 7.803E+01,-2.215E+02)
 8 ( 1.302E+03, 1.971E+01) ( 1.302E+03,-1.971E+01) ( 2.143E+00, 2.356E+02) ( 2.143E+00,-2.356E+02)

                         5                           6                           7                           8
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.295E+00,-2.642E+02) ( 2.295E+00, 2.642E+02) ( 1.798E-01,-1.926E+02) ( 1.798E-01, 1.926E+02)
 3 ( 2.155E-02,-3.149E+02) ( 2.155E-02, 3.149E+02) ( 2.129E+00,-1.793E+02) ( 2.129E+00, 1.793E+02)
 4 ( 8.915E-01,-2.574E+01) ( 8.915E-01, 2.574E+01) ( 3.410E+00,-2.827E+02) ( 3.410E+00, 2.827E+02)
 5 ( 2.885E-01,-3.344E+02) ( 2.885E-01, 3.344E+02) ( 2.528E-01, 3.676E-01) ( 2.528E-01,-3.676E-01)
 6 ( 5.193E-02,-1.713E+02) ( 5.193E-02, 1.713E+02) ( 5.215E-02,-1.798E+02) ( 5.215E-02, 1.798E+02)
 7 ( 7.425E-01,-2.128E+02) ( 7.425E-01, 2.128E+02) ( 1.333E-02,-2.696E+02) ( 1.333E-02, 2.696E+02)
 8 ( 1.337E-01,-4.975E+01) ( 1.337E-01, 4.975E+01) ( 2.749E-03,-8.978E+01) ( 2.749E-03, 8.978E+01)
```

Figure A.2.2, concluded

```
*** IMPULSE RESIDUE MATRIX :IMPULSE IN CONTROL  1

 ---------------------------------
 THE IMP RESID  MATRIX (  7 BY  8 )
 ---------------------------------

                            1                          2                          3                          4
   1 (-1.1897E-02,-1.9105E-02) (-1.1897E-02, 1.9105E-02) (-1.8685E+00,-1.4395E+00) (-1.8685E+00, 1.4395E+00)
   2 (-6.0957E-03, 5.8542E-03) (-6.0957E-03,-5.8542E-03) (-1.7435E-01, 3.0517E-01) (-1.7435E-01,-3.0517E-01)
   3 (-1.2221E-01,-1.3282E-01) (-1.2221E-01, 1.3282E-01) (-2.5462E+00,-1.7487E+00) (-2.5462E+00, 1.7487E+00)
   4 ( 2.5744E-03, 3.5639E-03) ( 2.5744E-03,-3.5639E-03) ( 1.5783E-01, 8.4027E-02) ( 1.5783E-01,-8.4027E-02)
   5 ( 7.1636E-03,-5.5951E-01) ( 7.1636E-03, 5.5951E-01) (-3.9211E-01, 3.9360E+00) (-3.9211E-01,-3.9360E+00)
   6 ( 1.1943E+01, 4.0833E-01) ( 1.1943E+01,-4.0833E-01) (-3.4189E+01,-6.2934E+00) (-3.4189E+01, 6.2934E+00)
   7 ( 4.8589E+00,-4.6518E+01) ( 4.8589E+00, 4.6518E+01) (-5.9152E+00,-1.2931E+02) (-5.9152E+00, 1.2931E+02)

                            5                          6                          7                          8
   1 (-2.7399E+01, 3.3292E+01) (-2.7399E+01,-3.3292E+01) ( 2.9279E+01,-4.7997E+02) ( 2.9279E+01, 4.7997E+02)
   2 ( 9.7986E-01, 1.6045E+00) ( 9.7986E-01,-1.6045E+00) (-7.9941E-01, 1.3931E-01) (-7.9941E-01,-1.3931E-01)
   3 (-4.8393E+00,-1.4040E-02) (-4.8393E+00, 1.4040E-02) (-7.2989E-03,-4.2153E-02) (-7.2989E-03, 4.2153E-02)
   4 ( 7.7148E-01,-4.9395E-01) ( 7.7148E-01, 4.9395E-01) (-7.8768E-01,-9.8380E-02) (-7.8768E-01, 9.8380E-02)
   5 ( 1.1817E+00, 1.1420E+00) ( 1.1817E+00,-1.1420E+00) (-7.9671E-01, 9.1649E-02) (-7.9671E-01,-9.1649E-02)
   6 (-4.0972E+00, 1.0518E+00) (-4.0972E+00,-1.0518E+00) (-4.7861E-03,-4.2008E-02) (-4.7861E-03, 4.2008E-02)
   7 ( 1.1681E+00, 8.9119E+01) ( 1.1681E+00,-8.9119E+01) (-7.9320E-02, 3.0403E+00) (-7.9320E-02,-3.0403E+00)


--------------------------------------------------
IMPULSE   RESIDUE    MATRIX ( 7 BY  8)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
--------------------------------------------------

                          1                        2                        3                        4
   1 ( 2.251E-02,-1.219E+02) ( 2.251E-02, 1.219E+02) ( 2.359E+00,-1.424E+02) ( 2.359E+00, 1.424E+02)
   2 ( 8.452E-03, 1.362E+02) ( 8.452E-03,-1.362E+02) ( 3.515E-01, 1.197E+02) ( 3.515E-01,-1.197E+02)
   3 ( 1.805E-01,-1.326E+02) ( 1.805E-01, 1.326E+02) ( 3.089E+00,-1.455E+02) ( 3.089E+00, 1.455E+02)
   4 ( 4.396E-03, 5.416E+01) ( 4.396E-03,-5.416E+01) ( 1.788E-01, 2.803E+01) ( 1.788E-01,-2.803E+01)
   5 ( 5.596E-01,-8.927E+01) ( 5.596E-01, 8.927E+01) ( 3.956E+00, 9.569E+01) ( 3.956E+00,-9.569E+01)
   6 ( 1.195E+01, 1.958E+00) ( 1.195E+01,-1.958E+00) ( 3.476E+01,-1.696E+02) ( 3.476E+01, 1.696E+02)
   7 ( 4.677E+01,-8.404E+01) ( 4.677E+01, 8.404E+01) ( 1.294E+02,-9.262E+01) ( 1.294E+02, 9.262E+01)

                          5                        6                        7                        8
   1 ( 4.312E+01, 1.295E+02) ( 4.312E+01,-1.295E+02) ( 4.809E+02,-8.651E+01) ( 4.809E+02, 8.651E+01)
   2 ( 1.880E+00, 5.859E+01) ( 1.880E+00,-5.859E+01) ( 8.115E-01, 1.701E+02) ( 8.115E-01,-1.701E+02)
   3 ( 4.839E+00,-1.798E+02) ( 4.839E+00, 1.798E+02) ( 4.278E-02,-9.982E+01) ( 4.278E-02, 9.982E+01)
   4 ( 9.161E-01,-3.263E+01) ( 9.161E-01, 3.263E+01) ( 7.938E-01,-1.729E+02) ( 7.938E-01, 1.729E+02)
   5 ( 1.643E+00, 4.402E+01) ( 1.643E+00,-4.402E+01) ( 8.020E-01, 1.734E+02) ( 8.020E-01,-1.734E+02)
   6 ( 4.230E+00, 1.656E+02) ( 4.230E+00,-1.656E+02) ( 4.228E-02,-9.650E+01) ( 4.228E-02, 9.650E+01)
   7 ( 8.913E+01, 8.925E+01) ( 8.913E+01,-8.925E+01) ( 3.041E+00, 9.149E+01) ( 3.041E+00,-9.149E+01)
```

Figure A.2.3
Impulse Residues - Configuration Two

Appendix A.3
DEA Examples Data

THE GG MATRIX (2 BY 4)

	1	2	3	4
1	-1.1566E-01	-3.7997E-03	9.3630E-03	-6.8182E-03
2	6.4103E+00	1.1539E+00	7.3118E-02	-2.8363E-01

THE AAUG MATRIX (8 BY 8)

	1	2	3	4	5	6	7	8
1	-1.1041E+00	9.5284E-01	-5.6659E-01	-2.7584E-02	2.8389E-01	2.6170E-01	1.3982E-02	5.4617E-03
2	-7.6104E+00	-1.4584E+00	-2.7327E+00	2.4744E+00	-1.1886E-01	4.3909E+00	4.7039E-02	-3.8055E-01
3	-2.6344E-02	0.	-2.5000E-02	-3.3930E-02	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00
7	-1.7509E+01	1.9751E+00	-1.3444E+01	1.5987E+01	-7.6856E+01	-1.1951E+02	-3.6065E+00	-3.1533E+00
8	-2.3933E+01	-1.1942E+00	5.5175E-01	1.5032E+00	1.2946E+00	-4.3665E+02	-2.1484E-01	-7.4857E-01

THE BBAUG VECTOR (8 BY 1)

	1
1	-1.7013E-01
2	-1.3413E+01
3	0.
4	0.
5	0.
6	0.
7	2.2691E+01
8	-1.6614E+01

Figure A.3.1
Closed-loop System Matrices - Example 3.1

THE CCAUG MATRIX (7 BY 8)

	1	2	3	4	5	6	7	8
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00
7	5.5608E+00	-7.6096E-01	5.1892E+00	1.3341E+01	2.7133E+01	-1.2507E+02	-2.0265E+00	-2.4946E+00

THE DDAUG VECTOR (7 BY 1)

	1
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	-1.3581E+01

THE MMAUG MATRIX (4 BY 8)

	1	2	3	4	5	6	7	8
1	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.
2	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00
3	5.5608E+00	-7.6096E-01	5.1892E+00	1.3341E+01	2.7133E+01	-1.2507E+02	-2.0265E+00	-2.4946E+00
4	3.7584E+01	3.2739E+00	1.5805E+01	2.3350E+00	1.0104E+01	7.0631E+00	-1.3009E+00	-5.0695E-01

THE NNAUG VECTOR (4 BY 1)

	1
1	0.
2	0.
3	-1.3581E+01
4	2.4054E+01

Figure A.3.1, co

SPECTRAL DECOMPOSITION OF AAUG

	REAL PART	IMAGINARY PART	FREQUENCY (HERTZ)	DAMPING RATIO	TIME CONST. (SEC/RAD)
1	-3.93081E-01	2.08499E+01	3.31896E+00	.01884953	2.54400E+00
3	-1.75770E+00	8.61090E+00	1.39873E+00	.20000081	5.68925E-01
5	-1.49000E+00	2.37000E+00	4.45549E-01	.53224470	6.71141E-01
7	3.27311E-01	0.	0.	-1.00000000	-3.05520E+00
8	1.16551E-02	0.	0.	-1.00000000	-8.57997E+01

THE EE MATRIX (8 BY 8)

	1	2	3	4	5
1	(1.56E-04, 9.15E-05)	(1.56E-04,-9.15E-05)	(3.73E-03, 3.02E-03)	(3.73E-03,-3.02E-03)	(-3.30E-01,-1.18E-01)
2	(-1.85E-02,-3.08E-03)	(-1.85E-02, 3.08E-03)	(-6.29E-03, 7.15E-03)	(-6.29E-03,-7.15E-03)	(4.13E-01,-7.78E-01)
3	(-1.57E-06, 1.16E-08)	(-1.57E-06,-1.16E-08)	(-8.00E-06, 1.67E-05)	(-8.00E-06,-1.67E-05)	(-2.94E-03,-6.34E-03)
4	(-1.31E-04, 8.92E-04)	(-1.31E-04,-8.92E-04)	(9.40E-04, 5.38E-04)	(9.40E-04,-5.38E-04)	(-3.14E-01, 2.31E-02)
5	(1.61E-02, 4.59E-03)	(1.61E-02,-4.59E-03)	(-3.70E-02, 1.07E-01)	(-3.70E-02,-1.07E-01)	(-2.16E-03,-2.33E-03)
6	(4.15E-02,-1.71E-02)	(4.15E-02, 1.71E-02)	(1.21E-04, 4.67E-04)	(1.21E-04,-4.67E-04)	(1.59E-02, 8.95E-03)
7	(-1.02E-01, 3.35E-01)	(-1.02E-01,-3.35E-01)	(-8.55E-01,-5.07E-01)	(-8.55E-01, 5.07E-01)	(8.75E-03,-1.65E-03)
8	(3.40E-01, 8.71E-01)	(3.40E-01,-8.71E-01)	(-4.23E-03, 2.24E-04)	(-4.23E-03,-2.24E-04)	(-4.49E-02, 2.44E-02)

	6	7	8
1	(-3.30E-01, 1.18E-01)	(2.52E-01, 0.)	(-4.12E-01, 0.)
2	(4.13E-01, 7.78E-01)	(2.94E-01, 0.)	(-5.70E-03, 0.)
3	(-2.94E-03, 6.34E-03)	(-1.05E-01, 0.)	(7.49E-01, 0.)
4	(-3.14E-01,-2.31E-02)	(8.98E-01, 0.)	(-4.89E-01, 0.)
5	(-2.16E-03, 2.33E-03)	(1.70E-01, 0.)	(-1.72E-01, 0.)
6	(1.59E-02,-8.95E-03)	(-1.12E-02, 0.)	(2.13E-02, 0.)
7	(8.75E-03, 1.65E-03)	(5.57E-02, 0.)	(-2.01E-03, 0.)
8	(-4.49E-02,-2.44E-02)	(-3.66E-03, 0.)	(2.49E-04, 0.)

Figure A.3.2
Spectral Decomposition -

```
---------------------------------
THE EI          MATRIX (  8 BY  8 )
---------------------------------

                       1                       2                       3                       4                       5
 1 ( 5.60E-01, 2.71E-01) ( 4.07E-02,-1.06E-02) (-2.16E-02, 1.84E-02) (-3.44E-02,-2.16E-02) (-4.33E-02,-2.78E-03)
 2 ( 5.60E-01,-2.71E-01) ( 4.07E-02, 1.06E-02) (-2.16E-02,-1.84E-02) (-3.44E-02, 2.16E-02) (-4.33E-02, 2.78E-03)
 3 (-4.69E-01,-4.19E-01) ( 8.38E-02, 1.26E-01) (-4.13E-01,-6.94E-01) ( 5.03E-01, 7.48E-01) (-2.30E+00,-3.90E+00)
 4 (-4.69E-01, 4.19E-01) ( 8.38E-02,-1.26E-01) (-4.13E-01, 6.94E-01) ( 5.03E-01,-7.48E-01) (-2.30E+00, 3.90E+00)
 5 (-1.46E+00, 9.82E-01) ( 2.36E-01, 4.83E-01) (-6.35E-01, 2.72E-01) ( 2.62E-01,-3.90E-01) (-1.08E-02,-4.78E-02)
 6 (-1.46E+00,-9.82E-01) ( 2.36E-01,-4.83E-01) (-6.35E-01,-2.72E-01) ( 2.62E-01, 3.90E-01) (-1.08E-02, 4.78E-02)
 7 (-1.07E+00, 3.16E-29) ( 2.01E-01,-8.30E-31) ( 3.17E-01,-5.94E-30) ( 1.39E+00,-1.26E-29) (-1.04E-02,-7.10E-30)
 8 (-1.78E-01, 7.89E-30) ( 2.20E-02,-4.88E-33) ( 1.37E+00,-8.21E-31) ( 2.04E-01,-2.37E-30) (-9.65E-04,-1.38E-30)

                       6                       7                       8
 1 ( 1.04E+01, 4.07E+00) ( 4.46E-03, 4.39E-03) ( 2.03E-01,-4.96E-01)
 2 ( 1.04E+01,-4.07E+00) ( 4.46E-03,-4.39E-03) ( 2.03E-01, 4.96E-01)
 3 (-1.64E+00, 2.02E+00) (-4.88E-01, 1.67E-01) ( 1.68E-01,-4.32E-03)
 4 (-1.64E+00,-2.02E+00) (-4.88E-01,-1.67E-01) ( 1.68E-01, 4.32E-03)
 5 ( 5.88E-02, 1.93E-01) (-7.37E-03, 2.37E-03) ( 4.76E-03, 5.13E-03)
 6 ( 5.88E-02,-1.93E-01) (-7.37E-03,-2.37E-03) ( 4.76E-03,-5.13E-03)
 7 ( 7.20E-02, 2.64E-30) (-4.17E-03, 3.93E-33) ( 2.47E-03, 8.97E-32)
 8 ( 7.42E-03, 3.54E-31) (-6.87E-04,-3.28E-34) ( 3.02E-04, 1.91E-32)


------------------------------------------------
C.L. EIGENVECTORS    MATRIX ( 8 BY  8)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
------------------------------------------------

                         1                         2                         3                         4
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 1.041E+02,-2.010E+02) ( 1.041E+02, 2.010E+02) ( 1.983E+00, 9.231E+01) ( 1.983E+00,-9.231E+01)
 3 ( 8.673E-03, 1.491E+02) ( 8.673E-03,-1.491E+02) ( 3.862E-03, 7.655E+01) ( 3.862E-03,-7.655E+01)
 4 ( 4.990E+00, 6.792E+01) ( 4.990E+00,-6.792E+01) ( 2.257E-01,-9.230E+00) ( 2.257E-01, 9.230E+00)
 5 ( 9.293E+01,-1.457E+01) ( 9.293E+01, 1.457E+01) ( 2.354E+01, 7.011E+01) ( 2.354E+01,-7.011E+01)
 6 ( 2.483E+02,-5.283E+01) ( 2.483E+02, 5.283E+01) ( 1.004E-01, 3.642E+01) ( 1.004E-01,-3.642E+01)
 7 ( 1.938E+03, 7.651E+01) ( 1.938E+03,-7.651E+01) ( 2.069E+02,-1.884E+02) ( 2.069E+02, 1.884E+02)
 8 ( 5.178E+03, 3.825E+01) ( 5.178E+03,-3.825E+01) ( 8.823E-01, 1.380E+02) ( 8.823E-01,-1.380E+02)

                         5                         6                         7                         8
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.513E+00, 9.827E+01) ( 2.513E+00,-9.827E+01) ( 1.165E+00, 0.        ) ( 1.384E-02, 0.        )
 3 ( 1.995E-02, 4.546E+01) ( 1.995E-02,-4.546E+01) ( 4.176E-01, 1.800E+02) ( 1.818E+00,-1.800E+02)
 4 ( 8.976E-01, 3.361E+02) ( 8.976E-01,-3.361E+02) ( 3.560E+00, 0.        ) ( 1.188E+00, 0.        )
 5 ( 9.078E-03, 2.745E+01) ( 9.078E-03,-2.745E+01) ( 6.739E-01, 0.        ) ( 4.180E-01, 0.        )
 6 ( 5.211E-02, 1.897E+02) ( 5.211E-02,-1.897E+02) ( 4.434E-02, 1.800E+02) ( 5.182E-02,-1.800E+02)
 7 ( 2.541E-02, 1.496E+02) ( 2.541E-02,-1.496E+02) ( 2.206E-01, 0.        ) ( 4.872E-03, 0.        )
 8 ( 1.459E-01, 3.118E+02) ( 1.459E-01,-3.118E+02) ( 1.451E-02, 1.800E+02) ( 6.039E-04,-1.800E+02)
```

Figure A.3.2, concluded

*** IMPULSE RESIDUE MATRIX :IMPULSE IN CONTROL 1

THE IMP RESID MATRIX (7 BY 8)

	1	2	3	4
1	(5.7335E-03,-1.2580E-02)	(5.7335E-03, 1.2580E-02)	(7.7496E-02,-2.5380E-01)	(7.7496E-02, 2.5380E-01)
2	(-7.0072E-03,-4.5977E-03)	(-7.0072E-03, 4.5977E-03)	(-1.5232E-02,-5.9088E-03)	(-1.5232E-02, 5.9088E-03)
3	(9.8616E-02,-1.4429E-01)	(9.8616E-02, 1.4429E-01)	(7.7653E-02,-1.2078E-01)	(7.7653E-02, 1.2078E-01)
4	(-5.6254E-03,-5.5526E-03)	(-5.6254E-03, 5.5526E-03)	(4.7196E-02, 3.0764E-02)	(4.7196E-02,-3.0764E-02)
5	(1.1331E-01,-5.3940E-01)	(1.1331E-01, 5.3940E-01)	(-3.2423E-01, 1.6766E+00)	(-3.2423E-01,-1.6766E+00)
6	(1.1202E+01, 2.5746E+00)	(1.1202E+01,-2.5746E+00)	(-1.3867E+01,-5.7388E+00)	(-1.3867E+01, 5.7388E+00)
7	(2.5917E+01,-4.3081E+01)	(2.5917E+01, 4.3081E+01)	(-2.0079E+01,-5.6205E+01)	(-2.0079E+01, 5.6205E+01)

	5	6	7	8
1	(-3.1344E+01, 3.7619E+01)	(-3.1344E+01,-3.7619E+01)	(2.6553E+02,-4.3657E-28)	(-2.0301E+02,-1.1384E-27)
2	(1.1449E+00, 2.0201E+00)	(1.1449E+00,-2.0201E+00)	(-2.3851E+00, 3.9214E-30)	(1.3975E-01, 7.8363E-31)
3	(-6.4935E+00,-2.9648E-01)	(-6.4935E+00, 2.9648E-01)	(-7.8066E-01, 1.2835E-30)	(1.6288E-03, 9.1333E-33)
4	(8.9000E-01,-5.5405E-01)	(8.9000E-01, 5.5405E-01)	(-1.7151E+00, 2.8199E-30)	(2.2093E-02, 1.2389E-31)
5	(1.1442E+00, 2.1327E+00)	(1.1442E+00,-2.1327E+00)	(-1.9633E+00, 3.2279E-30)	(9.6663E-02, 5.4204E-31)
6	(-6.7594E+00,-4.6596E-01)	(-6.7594E+00, 4.6596E-01)	(-6.4260E-01, 1.0565E-30)	(1.1266E-03, 6.3175E-33)
7	(1.9366E+01, 5.8655E+01)	(1.9366E+01,-5.8655E+01)	(-4.9189E+01, 8.0874E-29)	(3.5032E+00, 1.9644E-29)

IMPULSE RESIDUE MATRIX (7 BY 8)
OUTPUT FORM: (MAGNITUDE , PHASE(DEG.))

	1	2	3	4
1	(1.382E-02,-6.550E+01)	(1.382E-02, 6.550E+01)	(2.654E-01,-7.302E+01)	(2.654E-01, 7.302E+01)
2	(8.381E-03,-1.467E+02)	(8.381E-03, 1.467E+02)	(1.634E-02,-1.588E+02)	(1.634E-02, 1.588E+02)
3	(1.748E-01,-5.565E+01)	(1.748E-01, 5.565E+01)	(1.436E-01,-5.726E+01)	(1.436E-01, 5.726E+01)
4	(7.904E-03,-1.354E+02)	(7.904E-03, 1.354E+02)	(5.634E-02, 3.310E+01)	(5.634E-02,-3.310E+01)
5	(5.512E-01,-7.814E+01)	(5.512E-01, 7.814E+01)	(1.708E+00, 1.009E+02)	(1.708E+00,-1.009E+02)
6	(1.149E+01, 1.294E+01)	(1.149E+01,-1.294E+01)	(1.501E+01,-1.575E+02)	(1.501E+01, 1.575E+02)
7	(5.028E+01,-5.897E+01)	(5.028E+01, 5.897E+01)	(5.968E+01,-1.097E+02)	(5.968E+01, 1.097E+02)

	5	6	7	8
1	(4.897E+01, 1.298E+02)	(4.897E+01,-1.298E+02)	(2.655E+02,-9.420E-29)	(2.030E+02,-1.800E+02)
2	(2.322E+00, 6.046E+01)	(2.322E+00,-6.046E+01)	(2.385E+00, 1.800E+02)	(1.397E-01, 3.213E-28)
3	(6.500E+00,-1.774E+02)	(6.500E+00, 1.774E+02)	(7.807E-01, 1.800E+02)	(1.629E-03, 3.213E-28)
4	(1.048E+00,-3.190E+01)	(1.048E+00, 3.190E+01)	(1.715E+00, 1.800E+02)	(2.209E-02, 3.213E-28)
5	(2.420E+00, 6.179E+01)	(2.420E+00,-6.179E+01)	(1.963E+00, 1.800E+02)	(9.666E-02, 3.213E-28)
6	(6.775E+00,-1.761E+02)	(6.775E+00, 1.761E+02)	(6.426E-01, 1.800E+02)	(1.127E-03, 3.213E-28)
7	(6.177E+01, 7.173E+01)	(6.177E+01,-7.173E+01)	(4.919E+01, 1.800E+02)	(3.503E+00, 3.213E-28)

Figure A.3.3
Impulse Residues - Example 3.1

THE GG   MATRIX (2 BY 6)

	1	2	3	4	5	6
1	-1.1055E-01	-1.9128E-03	9.3023E-03	-6.2336E-03	5.7665E-03	-2.0062E-03
2	-6.1421E+00	1.3132E+00	-6.7032E-01	4.3888E-01	-1.6390E+00	1.8472E+00

THE AAUG   MATRIX (8 BY 8)

	1	2	3	4	5	6	7	8
1	-1.1489E+00	9.5528E-01	-6.0078E-01	-5.7183E-05	2.8547E-01	-2.0272E-01	1.3681E-02	2.1107E-03
2	-5.2696E+00	-1.7899E+00	-2.2431E+00	-3.6113E-03	-1.3400E-01	1.2972E+01	5.8455E-02	-3.0301E-01
3	-2.6344E-02	0.	-2.5000E-02	-3.3930E-02	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00
7	4.0074E+00	2.6318E-01	-4.5170E-01	1.0002E-02	-7.7437E+01	6.2084E+01	-3.4735E+00	-1.8017E+00
8	-2.3005E+01	-1.4289E+00	8.8237E-02	-4.7744E-03	1.3228E+00	-4.4121E+02	-2.1250E-01	-7.6733E-01

THE BBAUG   VECTOR (8 BY 1)

	1
1	-2.8929E-01
2	-1.1434E+01
3	0.
4	0.
5	0.
6	0.
7	6.8697E+01
8	-1.7983E+01

Figure A.3.4
Closed-loop System Matrices

THE CCAUG MATRIX (7 BY 8)

	1	2	3	4	5	6	7	8
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00
7	2.1615E+01	-2.3173E+00	1.3108E+01	-1.5654E-03	2.6792E+01	-1.1048E+01	-1.9331E+00	-1.6196E+00

THE DDAUG VECTOR (7 BY 1)

	1
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	1.4933E+01

THE MMAUG MATRIX (6 BY 8)

	1	2	3	4	5	6	7	8
1	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.
2	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00
3	2.1615E+01	-2.3173E+00	1.3108E+01	-1.5654E-03	2.6792E+01	-1.1048E+01	-1.9331E+00	-1.6196E+00
4	4.1939E+01	3.1054E+00	1.9566E+01	7.7828E-03	9.9280E+00	5.7511E+01	-1.2702E+00	-1.4820E-01
5	-2.9471E+01	1.9699E+00	0.	2.9472E+01	0.	0.	4.6212E-01	1.6204E-01
6	-2.9471E+01	-8.7670E-01	0.	2.9472E+01	0.	0.	2.4129E-01	-1.0880E-01

THE NNAUG VECTOR (6 BY 1)

	1
1	0.
2	0.
3	1.4933E+01
4	3.7073E+01
5	0.
6	0.

Figure A.3.4, concluded

SPECTRAL DECOMPOSITION OF AAUG

	REAL PART	IMAGINARY PART	FREQUENCY (HERTZ)	DAMPING RATIO	TIME CONST. (SEC/RAD)
1	-3.53102E-01	2.09739E+01	3.33857E+00	.01683290	2.83205E+00
3	-1.75770E+00	8.61090E+00	1.39873E+00	.20000081	5.68925E-01
5	-1.49000E+00	2.37000E+00	4.45549E-01	.53224470	6.71141E-01
7	-1.48000E-03	6.72000E-02	1.06978E-02	.02201847	6.75676E+02

THE EE MATRIX (8 BY 8)

	1	2	3	4	5
1	(-7.89E-04, 1.62E-03)	(-7.89E-04,-1.62E-03)	(-3.66E-03,-3.11E-03)	(-3.66E-03, 3.11E-03)	(-3.01E-01,-1.79E-01)
2	(-2.43E-02,-2.21E-02)	(-2.43E-02, 2.21E-02)	(6.46E-03,-7.00E-03)	(6.46E-03, 7.00E-03)	(5.54E-01,-6.84E-01)
3	(-3.98E-06,-2.60E-06)	(-3.98E-06, 2.60E-06)	(8.40E-06,-1.65E-05)	(8.40E-06, 1.65E-05)	(-1.67E-03,-6.79E-03)
4	(-1.04E-03, 1.17E-03)	(-1.04E-03,-1.17E-03)	(-9.27E-04,-5.61E-04)	(-9.27E-04, 5.61E-04)	(-3.12E-01,-3.75E-02)
5	(-6.37E-03, 6.74E-03)	(-6.37E-03,-6.74E-03)	(3.96E-02,-1.06E-01)	(3.96E-02, 1.06E-01)	(-1.68E-03,-2.70E-03)
6	(4.25E-02,-1.94E-02)	(4.25E-02, 1.94E-02)	(-1.10E-04,-4.69E-04)	(-1.10E-04, 4.69E-04)	(1.39E-02, 1.18E-02)
7	(-1.39E-01,-1.36E-01)	(-1.39E-01, 1.36E-01)	(8.42E-01, 5.27E-01)	(8.42E-01,-5.27E-01)	(8.90E-03, 5.24E-05)
8	(3.91E-01, 8.98E-01)	(3.91E-01,-8.98E-01)	(4.23E-03,-1.22E-04)	(4.23E-03, 1.22E-04)	(-4.88E-02, 1.53E-02)

	6	7	8
1	(-3.01E-01, 1.79E-01)	(-4.05E-02,-1.77E-01)	(-4.05E-02, 1.77E-01)
2	(5.54E-01, 6.84E-01)	(2.14E-02, 5.51E-02)	(2.14E-02,-5.51E-02)
3	(-1.67E-03, 6.79E-03)	(9.03E-02, 4.26E-01)	(9.03E-02,-4.26E-01)
4	(-3.12E-01, 3.75E-02)	(8.13E-01,-3.36E-01)	(8.13E-01, 3.36E-01)
5	(-1.68E-03, 2.70E-03)	(-7.99E-04,-4.19E-03)	(-7.99E-04, 4.19E-03)
6	(1.39E-02,-1.18E-02)	(2.05E-03, 9.15E-03)	(2.05E-03,-9.15E-03)
7	(8.90E-03,-5.24E-05)	(2.83E-04,-4.75E-05)	(2.83E-04, 4.75E-05)
8	(-4.88E-02,-1.53E-02)	(-6.18E-04, 1.24E-04)	(-6.18E-04,-1.24E-04)

Figure A.3.5
Spectral Decomposition -

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THE EI          MATRIX (  8 BY  8 )
----------------------------------

                          1                         2                         3                         4                         5
  1 ( 5.02E-01, 2.63E-01) ( 4.41E-02,-5.34E-03) (-8.73E-03, 1.84E-02) ( 7.89E-05, 4.02E-05) (-4.16E-02,-4.50E-03)
  2 ( 5.02E-01,-2.63E-01) ( 4.41E-02, 5.34E-03) (-8.73E-03,-1.84E-02) ( 7.89E-05,-4.02E-05) (-4.16E-02, 4.50E-03)
  3 ( 4.52E-02,-1.12E-02) ( 1.06E-03,-6.31E-03) ( 1.63E-02, 2.46E-02) (-3.54E-04,-3.77E-04) ( 2.39E+00, 3.83E+00)
  4 ( 4.52E-02, 1.12E-02) ( 1.06E-03, 6.31E-03) ( 1.63E-02,-2.46E-02) (-3.54E-04, 3.77E-04) ( 2.39E+00,-3.83E+00)
  5 (-1.11E+00, 9.12E-01) ( 2.96E-01, 4.92E-01) (-5.05E-01, 3.10E-01) (-6.80E-03,-2.49E-03) (-1.83E-02,-4.82E-02)
  6 (-1.11E+00,-9.12E-01) ( 2.96E-01,-4.92E-01) (-5.05E-01,-3.10E-01) (-6.80E-03, 2.49E-03) (-1.83E-02, 4.82E-02)
  7 (-4.36E-01,-7.35E-02) ( 8.11E-02, 2.39E-02) ( 2.52E-01,-1.11E+00) ( 5.63E-01, 1.19E-01) (-4.19E-03,-1.39E-03)
  8 (-4.36E-01, 7.35E-02) ( 8.11E-02,-2.39E-02) ( 2.52E-01, 1.11E+00) ( 5.63E-01,-1.19E-01) (-4.19E-03, 1.39E-03)

                          6                         7                         8
  1 ( 9.85E+00, 4.29E+00) ( 4.01E-03, 4.29E-03) ( 2.13E-01,-4.64E-01)
  2 ( 9.85E+00,-4.29E+00) ( 4.01E-03,-4.29E-03) ( 2.13E-01, 4.64E-01)
  3 ( 1.19E+00, 4.15E-01) ( 4.81E-01,-1.80E-01) ( 8.06E-02,-4.71E-02)
  4 ( 1.19E+00,-4.15E-01) ( 4.81E-01, 1.80E-01) ( 8.06E-02, 4.71E-02)
  5 ( 3.90E-02, 1.63E-01) (-6.26E-03, 2.39E-03) ( 9.33E-03, 1.47E-02)
  6 ( 3.90E-02,-1.63E-01) (-6.26E-03,-2.39E-03) ( 9.33E-03,-1.47E-02)
  7 ( 2.42E-02, 7.54E-03) (-1.71E-03,-2.97E-04) ( 2.35E-03, 6.90E-04)
  8 ( 2.42E-02,-7.54E-03) (-1.71E-03, 2.97E-04) ( 2.35E-03,-6.90E-04)


-------------------------------------------------
C.L. EIGENVECTORS    MATRIX ( 8 BY  8)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
-------------------------------------------------

                              1                             2                             3                             4
  1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
  2 ( 1.819E+01,-2.535E+02) ( 1.819E+01, 2.535E+02) ( 1.983E+00, 9.231E+01) ( 1.983E+00,-9.231E+01)
  3 ( 2.635E-03,-2.627E+02) ( 2.635E-03, 2.627E+02) ( 3.862E-03, 7.655E+01) ( 3.862E-03,-7.655E+01)
  4 ( 8.673E-01, 1.549E+01) ( 8.673E-01,-1.549E+01) ( 2.257E-01,-9.230E+00) ( 2.257E-01, 9.230E+00)
  5 ( 5.135E+00, 1.745E+01) ( 5.135E+00,-1.745E+01) ( 2.354E+01, 7.011E+01) ( 2.354E+01,-7.011E+01)
  6 ( 2.586E+01,-1.404E+02) ( 2.586E+01, 1.404E+02) ( 1.004E-01, 3.642E+01) ( 1.004E-01,-3.642E+01)
  7 ( 1.077E+02,-2.516E+02) ( 1.077E+02, 2.516E+02) ( 2.069E+02, 1.716E+02) ( 2.069E+02,-1.716E+02)
  8 ( 5.425E+02,-4.947E+01) ( 5.425E+02, 4.947E+01) ( 8.823E-01, 1.380E+02) ( 8.823E-01,-1.380E+02)

                              5                             6                             7                             8
  1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
  2 ( 2.513E+00, 9.827E+01) ( 2.513E+00,-9.827E+01) ( 3.248E-01, 1.716E+02) ( 3.248E-01,-1.716E+02)
  3 ( 1.995E-02, 4.546E+01) ( 1.995E-02,-4.546E+01) ( 2.393E+00, 1.809E+02) ( 2.393E+00,-1.809E+02)
  4 ( 8.976E-01,-2.389E+01) ( 8.976E-01, 2.389E+01) ( 4.832E+00, 8.035E+01) ( 4.832E+00,-8.035E+01)
  5 ( 9.078E-03, 2.745E+01) ( 9.078E-03,-2.745E+01) ( 2.343E-02, 2.043E+00) ( 2.343E-02,-2.043E+00)
  6 ( 5.211E-02, 1.897E+02) ( 5.211E-02,-1.897E+02) ( 5.152E-02, 1.802E+02) ( 5.152E-02,-1.802E+02)
  7 ( 2.541E-02, 1.496E+02) ( 2.541E-02,-1.496E+02) ( 1.575E-03, 9.330E+01) ( 1.575E-03,-9.330E+01)
  8 ( 1.459E-01, 3.118E+02) ( 1.459E-01,-3.118E+02) ( 3.463E-03, 2.715E+02) ( 3.463E-03,-2.715E+02)
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Figure A.3.5, concluded

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*** IMPULSE RESIDUE MATRIX :IMPULSE IN CONTROL  1

 ----------------------------------
 THE IMP RESID  MATRIX (  7 BY  8 )
 ----------------------------------

                         1                         2                         3                         4
 1 ( 3.7234E-02,-2.2176E-02) ( 3.7234E-02, 2.2176E-02) ( 7.2751E-02,-5.8667E-01) ( 7.2751E-02, 5.8667E-01)
 2 (-5.7695E-03,-1.3883E-02) (-5.7695E-03, 1.3883E-02) (-3.5692E-02,-7.1263E-03) (-3.5692E-02, 7.1263E-03)
 3 ( 2.9321E-01,-1.1611E-01) ( 2.9321E-01, 1.1611E-01) ( 1.2410E-01,-2.9482E-01) ( 1.2410E-01, 2.9482E-01)
 4 ( 4.9157E-03,-2.3426E-04) ( 4.9157E-03, 2.3426E-04) ( 1.1537E-01, 4.9414E-02) ( 1.1537E-01,-4.9414E-02)
 5 ( 3.7398E-02,-3.7871E-01) ( 3.7398E-02, 3.7871E-01) (-6.8681E-02, 3.8035E+00) (-6.8681E-02,-3.8035E+00)
 6 ( 7.9298E+00, 9.1811E-01) ( 7.9298E+00,-9.1811E-01) (-3.2631E+01,-7.2768E+00) (-3.2631E+01, 7.2768E+00)
 7 ( 1.0199E+01,-5.3373E+00) ( 1.0199E+01, 5.3373E+00) (-6.5612E+01,-1.1564E+02) (-6.5612E+01, 1.1564E+02)

                         5                         6                         7                         8
 1 (-3.2816E+01, 3.3060E+01) (-3.2816E+01,-3.3060E+01) ( 3.2706E+01,-4.1288E+02) ( 3.2706E+01, 4.1288E+02)
 2 ( 9.1799E-01, 2.0091E+00) ( 9.1799E-01,-2.0091E+00) (-8.7653E-01, 9.1993E-02) (-8.7653E-01,-9.1993E-02)
 3 (-6.1293E+00,-8.1790E-01) (-6.1293E+00, 8.1790E-01) (-4.8846E-03,-5.9039E-02) (-4.8846E-03, 5.9039E-02)
 4 ( 8.8927E-01,-4.5148E-01) ( 8.8927E-01, 4.5148E-01) (-8.6490E-01,-9.0047E-02) (-8.6490E-01, 9.0047E-02)
 5 ( 9.0802E-01, 2.1158E+00) ( 9.0802E-01,-2.1158E+00) (-8.7674E-01, 9.7116E-02) (-8.7674E-01,-9.7116E-02)
 6 (-6.3674E+00,-1.0005E+00) (-6.3674E+00, 1.0005E+00) (-5.2286E-03,-5.9061E-02) (-5.2286E-03, 5.9061E-02)
 7 ( 1.3503E+01, 5.7188E+01) ( 1.3503E+01,-5.7188E+01) ( 1.9426E-01,-1.4140E+00) ( 1.9426E-01, 1.4140E+00)


 -------------------------------------------------
 IMPULSE   RESIDUE    MATRIX ( 7 BY  8)
      OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
 -------------------------------------------------

                       1                       2                       3                       4
 1 ( 4.334E-02,-3.078E+01) ( 4.334E-02, 3.078E+01) ( 5.912E-01,-8.293E+01) ( 5.912E-01, 8.293E+01)
 2 ( 1.503E-02,-1.126E+02) ( 1.503E-02, 1.126E+02) ( 3.640E-02,-1.687E+02) ( 3.640E-02, 1.687E+02)
 3 ( 3.154E-01,-2.160E+01) ( 3.154E-01, 2.160E+01) ( 3.199E-01,-6.717E+01) ( 3.199E-01, 6.717E+01)
 4 ( 4.921E-03,-2.728E+00) ( 4.921E-03, 2.728E+00) ( 1.255E-01, 2.319E+01) ( 1.255E-01,-2.319E+01)
 5 ( 3.806E-01,-8.436E+01) ( 3.806E-01, 8.436E+01) ( 3.804E+00, 9.103E+01) ( 3.804E+00,-9.103E+01)
 6 ( 7.983E+00, 6.604E+00) ( 7.983E+00,-6.604E+00) ( 3.343E+01,-1.674E+02) ( 3.343E+01, 1.674E+02)
 7 ( 1.151E+01,-2.762E+01) ( 1.151E+01, 2.762E+01) ( 1.330E+02,-1.196E+02) ( 1.330E+02, 1.196E+02)

                       5                       6                       7                       8
 1 ( 4.658E+01, 1.348E+02) ( 4.658E+01,-1.348E+02) ( 4.142E+02,-8.547E+01) ( 4.142E+02, 8.547E+01)
 2 ( 2.209E+00, 6.544E+01) ( 2.209E+00,-6.544E+01) ( 8.813E-01, 1.740E+02) ( 8.813E-01,-1.740E+02)
 3 ( 6.184E+00,-1.724E+02) ( 6.184E+00, 1.724E+02) ( 5.924E-02,-9.473E+01) ( 5.924E-02, 9.473E+01)
 4 ( 9.973E-01,-2.692E+01) ( 9.973E-01, 2.692E+01) ( 8.696E-01,-1.741E+02) ( 8.696E-01, 1.741E+02)
 5 ( 2.302E+00, 6.677E+01) ( 2.302E+00,-6.677E+01) ( 8.821E-01, 1.737E+02) ( 8.821E-01,-1.737E+02)
 6 ( 6.446E+00,-1.711E+02) ( 6.446E+00, 1.711E+02) ( 5.929E-02,-9.506E+01) ( 5.929E-02, 9.506E+01)
 7 ( 5.876E+01, 7.671E+01) ( 5.876E+01,-7.671E+01) ( 1.427E+00,-8.218E+01) ( 1.427E+00, 8.218E+01)
```

Figure A.3.6
Impulse Residues - Example 3.2

ILGA ABJUSTMENT: GG(1,1)=GG(1,1)+0.28

THE GGPRIME MATRIX (2 BY 4)

	1	2	3	4
1	1.6434E-01	-3.7997E-03	9.3630E-03	-6.8182E-03
2	6.4103E+00	1.1539E+00	7.3118E-02	-2.8363E-01

THE AAUG MATRIX (8 BY 8)

	1	2	3	4	5	6	7	8
1	-1.1041E+00	9.5284E-01	-5.6659E-01	-7.5220E-02	3.3152E-01	3.0934E-01	1.3982E-02	5.4617E-03
2	-7.6104E+00	-1.4584E+00	-2.7327E+00	-1.2813E+00	3.6369E+00	8.1466E+00	4.7039E-02	-3.8055E-01
3	-2.6344E-02	0.	-2.5000E-02	-3.3930E-02	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00
7	-1.7509E+01	1.9751E+00	-1.3444E+01	2.2340E+01	-8.3209E+01	-1.2586E+02	-3.6065E+00	-3.1533E+00
8	-2.3933E+01	-1.1942E+00	5.5175E-01	-3.1488E+00	5.9467E+00	-4.3200E+02	-2.1484E-01	-7.4857E-01

THE BBAUG VECTOR (8 BY 1)

	1
1	-1.7013E-01
2	-1.3413E+01
3	0.
4	0.
5	0.
6	0.
7	2.2691E+01
8	-1.6614E+01

Figure A.3.7
Closed-loop System Matrices - Example 3.3

THE CCAUG MATRIX (7 BY 8)

	1	2	3	4	5	6	7	8
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00
7	5.5608E+00	-7.6096E-01	5.1892E+00	9.5379E+00	3.0935E+01	-1.2127E+02	-2.0265E+00	-2.4946E+00

THE DDAUG VECTOR (7 BY 1)

	1
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	-1.3581E+01

THE MMAUG MATRIX (4 BY 8)

	1	2	3	4	5	6	7	8
1	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.
2	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00
3	5.5608E+00	-7.6096E-01	5.1892E+00	9.5379E+00	3.0935E+01	-1.2127E+02	-2.0265E+00	-2.4946E+00
4	3.7584E+01	3.2739E+00	1.5805E+01	9.0700E+00	3.3694E+00	3.2800E-01	-1.3009E+00	-5.0695E-01

THE NNAUG VECTOR (4 BY 1)

	1
1	0.
2	0.
3	-1.3581E+01
4	2.4054E+01

Figure A.3.7, concluded

SPECTRAL DECOMPOSITION OF AAUG

	REAL PART	IMAGINARY PART	FREQUENCY (HERTZ)	DAMPING RATIO	TIME CONST. (SEC/RAD)
1	-3.58918E-01	2.06938E+01	3.29402E+00	.01734160	2.78615E+00
3	-1.84387E+00	9.09981E+00	1.47771E+00	.19859185	5.42336E-01
5	-1.24322E+00	2.90734E+00	5.03248E-01	.39317435	8.04365E-01
7	-2.52892E-02	6.88167E-02	1.16687E-02	.34493240	3.95426E+01

THE EE MATRIX (8 BY 8)

	1	2	3	4	5
1	(-3.58E-04, 6.99E-05)	(-3.58E-04,-6.99E-05)	(-6.73E-04, 5.48E-03)	(-6.73E-04,-5.48E-03)	(-2.83E-01,-8.88E-02)
2	(-2.26E-02,-9.93E-03)	(-2.26E-02, 9.93E-03)	(-4.08E-02, 2.59E-02)	(-4.08E-02,-2.59E-02)	(2.96E-01,-8.50E-01)
3	(-1.91E-06,-1.18E-06)	(-1.91E-06, 1.18E-06)	(-2.65E-05, 1.68E-05)	(-2.65E-05,-1.68E-05)	(-1.61E-03,-5.21E-03)
4	(-4.61E-04, 1.10E-03)	(-4.61E-04,-1.10E-03)	(3.61E-03, 3.76E-03)	(3.61E-03,-3.76E-03)	(-2.84E-01, 1.96E-02)
5	(1.80E-02, 1.19E-03)	(1.80E-02,-1.19E-03)	(-6.71E-02,-8.33E-02)	(-6.71E-02, 8.33E-02)	(-3.85E-02,-7.58E-03)
6	(3.72E-02,-2.48E-02)	(3.72E-02, 2.48E-02)	(-1.39E-03,-1.73E-03)	(-1.39E-03, 1.73E-03)	(1.66E-02, 7.39E-03)
7	(-3.11E-02, 3.73E-01)	(-3.11E-02,-3.73E-01)	(8.82E-01,-4.57E-01)	(8.82E-01, 4.57E-01)	(6.99E-02,-1.02E-01)
8	(5.00E-01, 7.79E-01)	(5.00E-01,-7.79E-01)	(1.83E-02,-9.46E-03)	(1.83E-02, 9.46E-03)	(-4.21E-02, 3.90E-02)

	6	7	8
1	(-2.83E-01, 8.88E-02)	(-1.08E-01, 1.38E-01)	(-1.08E-01,-1.38E-01)
2	(2.96E-01, 8.50E-01)	(6.13E-02,-1.51E-02)	(6.13E-02, 1.51E-02)
3	(-1.61E-03, 5.21E-03)	(2.98E-01,-2.81E-01)	(2.98E-01, 2.81E-01)
4	(-2.84E-01,-1.96E-02)	(-4.82E-01,-7.14E-01)	(-4.82E-01, 7.14E-01)
5	(-3.85E-02, 7.58E-03)	(-1.65E-01,-1.68E-01)	(-1.65E-01, 1.68E-01)
6	(1.66E-02,-7.39E-03)	(7.45E-03,-5.04E-03)	(7.45E-03, 5.04E-03)
7	(6.99E-02, 1.02E-01)	(1.57E-02,-7.14E-03)	(1.57E-02, 7.14E-03)
8	(-4.21E-02,-3.90E-02)	(1.59E-04, 6.40E-04)	(1.59E-04,-6.40E-04)

Figure A.3.8
Spectral Decomposition - Ex...ple 3.3

```
-----------------------------------------
THE EI         MATRIX (  8 BY  8 )
-----------------------------------------

                       1                      2                      3                      4                      5
1 ( 5.02E-01, 3.84E-01) ( 4.76E-02,-6.78E-03) (-2.84E-02, 1.11E-02) ( 8.00E-02, 5.16E-02) (-1.69E-01,-1.08E-01)
2 ( 5.02E-01,-3.84E-01) ( 4.76E-02, 6.78E-03) (-2.84E-02,-1.11E-02) ( 8.00E-02,-5.16E-02) (-1.69E-01, 1.08E-01)
3 (-3.19E-02, 5.19E-01) ( 1.06E-02,-1.66E-01) (-3.31E-01, 6.68E-01) ( 6.22E-01,-1.15E+00) (-2.22E+00, 4.19E+00)
4 (-3.19E-02,-5.19E-01) ( 1.06E-02, 1.66E-01) (-3.31E-01,-6.68E-01) ( 6.22E-01, 1.15E+00) (-2.22E+00,-4.19E+00)
5 (-1.55E+00, 5.47E-01) ( 1.67E-01, 5.30E-01) (-6.74E-01, 1.45E-01) (-2.35E-02, 2.55E-02) (-6.34E-02, 4.02E-02)
6 (-1.55E+00,-5.47E-01) ( 1.67E-01,-5.30E-01) (-6.74E-01,-1.45E-01) (-2.35E-02,-2.55E-02) (-6.34E-02,-4.02E-02)
7 ( 3.34E-01,-3.74E-01) (-5.12E-02, 4.55E-02) ( 1.17E+00, 5.32E-01) (-3.93E-01, 4.17E-01) (-5.75E-03, 5.04E-03)
8 ( 3.34E-01, 3.74E-01) (-5.12E-02,-4.55E-02) ( 1.17E+00,-5.32E-01) (-3.93E-01,-4.17E-01) (-5.75E-03,-5.04E-03)

                       6                      7                      8
1 ( 9.48E+00, 6.12E+00) (-1.97E-03, 1.06E-02) ( 3.03E-01,-4.52E-01)
2 ( 9.48E+00,-6.12E+00) (-1.97E-03,-1.06E-02) ( 3.03E-01, 4.52E-01)
3 ( 2.78E+00, 1.90E-01) ( 4.02E-01, 3.20E-01) (-1.01E-01,-1.54E-01)
4 ( 2.78E+00,-1.90E-01) ( 4.02E-01,-3.20E-01) (-1.01E-01, 1.54E-01)
5 ( 6.86E-02, 2.97E-01) ( 1.83E-03, 2.83E-02) ( 3.70E-03, 2.52E-03)
6 ( 6.86E-02,-2.97E-01) ( 1.83E-03,-2.83E-02) ( 3.70E-03,-2.52E-03)
7 (-2.46E-02, 2.13E-02) (-9.37E-04, 5.37E-04) (-4.51E-04, 4.39E-04)
8 (-2.46E-02,-2.13E-02) (-9.37E-04,-5.37E-04) (-4.51E-04,-4.39E-04)


-------------------------------------------------------
C.L. EIGENVECTORS    MATRIX ( 8 BY  8)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
-------------------------------------------------------

                          1                         2                         3                         4
1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
2 ( 6.773E+01,-3.252E+02) ( 6.773E+01, 3.252E+02) ( 8.755E+00, 5.059E+01) ( 8.755E+00,-5.059E+01)
3 ( 6.164E-03,-3.173E+02) ( 6.164E-03, 3.173E+02) ( 5.683E-03, 5.063E+01) ( 5.683E-03,-5.063E+01)
4 ( 3.272E+00,-5.622E+01) ( 3.272E+00, 5.622E+01) ( 9.430E-01,-5.086E+01) ( 9.430E-01, 5.086E+01)
5 ( 4.955E+01,-1.652E+02) ( 4.955E+01, 1.652E+02) ( 1.936E+01,-2.258E+02) ( 1.936E+01, 2.258E+02)
6 ( 1.227E+02,-2.026E+02) ( 1.227E+02, 2.026E+02) ( 4.021E-01,-2.258E+02) ( 4.021E-01, 2.258E+02)
7 ( 1.026E+03,-7.418E+01) ( 1.026E+03, 7.418E+01) ( 1.797E+02,-1.244E+02) ( 1.797E+02, 1.244E+02)
8 ( 2.540E+03,-1.116E+02) ( 2.540E+03, 1.116E+02) ( 3.734E+00,-1.243E+02) ( 3.734E+00, 1.243E+02)

                          5                         6                         7                         8
1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
2 ( 3.036E+00, 9.177E+01) ( 3.036E+00,-9.177E+01) ( 3.605E-01,-1.421E+02) ( 3.605E-01, 1.421E+02)
3 ( 1.837E-02, 5.543E+01) ( 1.837E-02,-5.543E+01) ( 2.337E+00,-1.715E+02) ( 2.337E+00, 1.715E+02)
4 ( 9.603E-01, 3.386E+02) ( 9.603E-01,-3.386E+02) ( 4.917E+00,-2.523E+02) ( 4.917E+00, 2.523E+02)
5 ( 1.323E-01,-6.282E+00) ( 1.323E-01, 6.282E+00) ( 1.344E+00,-2.628E+02) ( 1.344E+00, 2.628E+02)
6 ( 6.120E-02, 1.866E+02) ( 6.120E-02,-1.866E+02) ( 5.135E-02,-1.623E+02) ( 5.135E-02, 1.623E+02)
7 ( 4.183E-01, 1.069E+02) ( 4.183E-01,-1.069E+02) ( 9.855E-02,-1.527E+02) ( 9.855E-02, 1.527E+02)
8 ( 1.935E-01, 2.998E+02) ( 1.935E-01,-2.998E+02) ( 3.765E-03,-5.216E+01) ( 3.765E-03, 5.216E+01)
```

Figure A.3.8, concluded

```
*** IMPULSE RESIDUE MATRIX :IMPULSE IN CONTROL  1

 ----------------------------------
 THE IMP RESID  MATRIX (  7 BY  8 )
 ----------------------------------

                           1                          2                          3                          4
  1 ( 1.9234E-02,-7.6030E-03) ( 1.9234E-02, 7.6030E-03) (-4.5931E-01,-1.3046E-01) (-4.5931E-01, 1.3046E-01)
  2 (-5.8769E-03,-9.9664E-03) (-5.8769E-03, 9.9664E-03) (-6.3533E-03, 8.3236E-02) (-6.3533E-03,-8.3236E-02)
  3 ( 2.0835E-01,-1.1804E-01) ( 2.0835E-01, 1.1804E-01) (-7.4571E-01,-2.1129E-01) (-7.4571E-01, 2.1129E-01)
  4 (-7.4100E-03,-6.7802E-03) (-7.4100E-03, 6.7802E-03) ( 6.6366E-02, 3.2748E-02) ( 6.6366E-02,-3.2748E-02)
  5 ( 1.3142E-01,-5.7641E-01) ( 1.3142E-01, 5.7641E-01) (-2.9199E-01, 1.8092E+00) (-2.9199E-01,-1.8092E+00)
  6 ( 1.1881E+01, 2.9264E+00) ( 1.1881E+01,-2.9264E+00) (-1.5925E+01,-5.9929E+00) (-1.5925E+01, 5.9929E+00)
  7 ( 2.6964E+01,-4.1975E+01) ( 2.6964E+01, 4.1975E+01) (-2.2861E+01,-5.8585E+01) (-2.2861E+01, 5.8585E+01)

                           5                          6                          7                          8
  1 (-2.9548E+01, 1.9902E+01) (-2.9548E+01,-1.9902E+01) ( 2.9988E+01,-3.1720E+02) ( 2.9988E+01, 3.1720E+02)
  2 ( 6.9588E-01, 1.8346E+00) ( 6.9588E-01,-1.8346E+00) (-6.8365E-01,-1.7834E-01) (-6.8365E-01, 1.7834E-01)
  3 (-6.1989E+00,-2.5762E-01) (-6.1989E+00, 2.5762E-01) ( 2.9562E-02,-4.2537E-02) ( 2.9562E-02, 4.2537E-02)
  4 ( 7.1750E-01,-2.0855E-01) ( 7.1750E-01, 2.0855E-01) (-6.9139E-01,-3.2182E-01) (-6.9139E-01, 3.2182E-01)
  5 ( 6.5342E-01, 1.6897E+00) ( 6.5342E-01,-1.6897E+00) (-4.9284E-01,-1.5746E-01) (-4.9284E-01, 1.5746E-01)
  6 (-5.7248E+00,-2.0092E-01) (-5.7248E+00, 2.0092E-01) ( 2.3299E-02,-2.9934E-02) ( 2.3299E-02, 2.9934E-02)
  7 ( 1.0793E+01, 5.2551E+01) ( 1.0793E+01,-5.2551E+01) (-1.2534E+01,-2.1387E+00) (-1.2534E+01, 2.1387E+00)
```

```
 -------------------------------------------------
 IMPULSE   RESIDUE    MATRIX ( 7 BY  8)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
 -------------------------------------------------

                          1                        2                        3                        4
  1 ( 2.068E-02,-2.157E+01) ( 2.068E-02, 2.157E+01) ( 4.775E-01,-1.641E+02) ( 4.775E-01, 1.641E+02)
  2 ( 1.157E-02,-1.205E+02) ( 1.157E-02, 1.205E+02) ( 8.348E-02, 9.436E+01) ( 8.348E-02,-9.436E+01)
  3 ( 2.395E-01,-2.953E+01) ( 2.395E-01, 2.953E+01) ( 7.751E-01,-1.642E+02) ( 7.751E-01, 1.642E+02)
  4 ( 1.004E-02,-1.375E+02) ( 1.004E-02, 1.375E+02) ( 7.401E-02, 2.626E+01) ( 7.401E-02,-2.626E+01)
  5 ( 5.912E-01,-7.716E+01) ( 5.912E-01, 7.716E+01) ( 1.833E+00, 9.917E+01) ( 1.833E+00,-9.917E+01)
  6 ( 1.224E+01, 1.384E+01) ( 1.224E+01,-1.384E+01) ( 1.701E+01,-1.594E+02) ( 1.701E+01, 1.594E+02)
  7 ( 4.989E+01,-5.728E+01) ( 4.989E+01, 5.728E+01) ( 6.289E+01,-1.113E+02) ( 6.289E+01, 1.113E+02)

                          5                        6                        7                        8
  1 ( 3.563E+01, 1.460E+02) ( 3.563E+01,-1.460E+02) ( 3.186E+02,-8.460E+01) ( 3.186E+02, 8.460E+01)
  2 ( 1.962E+00, 6.923E+01) ( 1.962E+00,-6.923E+01) ( 7.065E-01,-1.654E+02) ( 7.065E-01, 1.654E+02)
  3 ( 6.204E+00,-1.776E+02) ( 6.204E+00, 1.776E+02) ( 5.180E-02,-5.520E+01) ( 5.180E-02, 5.520E+01)
  4 ( 7.472E-01,-1.621E+01) ( 7.472E-01, 1.621E+01) ( 7.626E-01,-1.550E+02) ( 7.626E-01, 1.550E+02)
  5 ( 1.812E+00, 6.886E+01) ( 1.812E+00,-6.886E+01) ( 5.174E-01,-1.623E+02) ( 5.174E-01, 1.623E+02)
  6 ( 5.728E+00,-1.780E+02) ( 5.728E+00, 1.780E+02) ( 3.793E-02,-5.210E+01) ( 3.793E-02, 5.210E+01)
  7 ( 5.365E+01, 7.839E+01) ( 5.365E+01,-7.839E+01) ( 1.272E+01,-1.703E+02) ( 1.272E+01, 1.703E+02)
```

Figure A.3.9
Impulse Residues - Example 3.3

Appendix A.4
LQEA Examples Data

NUMBER OF STATES = 10
NUMBER OF CONTROLS = 2
NUMBER OF OUTPUTS = 7

THE AA MATRIX (10 BY 10)

	1	2	3	4	5	6	7	8	9
1	-1.2080E+00	9.4400E-01	-6.0167E-01	0.	3.3864E-01	-1.3362E-01	8.0682E-03	-4.4155E-03	-2.8840E-01
2	-7.0590E+00	-2.0180E+00	-2.1732E+00	0.	6.9773E+00	-5.9954E+00	2.8174E-01	-2.0148E-01	-1.5030E+01
3	-2.6344E-02	0.	-2.5000E-02	-3.3720E-02	0.	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
7	1.9491E+01	3.5191E+00	3.5952E-02	0.	-7.6800E+01	-1.1764E+01	-9.6220E-01	6.9965E-01	5.8872E+01
8	-2.6186E+01	-1.9336E+00	1.2791E-01	0.	9.1505E+00	-4.5650E+02	-1.5958E-01	-8.4850E-01	-2.1206E+01
9	0.	0.	0.	0.	0.	0.	0.	0.	-9.0000E+00
10	0.	0.	0.	0.	0.	0.	0.	0.	0.

	10
1	-4.4000E-03
2	3.5310E-01
3	0.
4	0.
5	0.
6	0.
7	2.4362E+00
8	2.0252E-01
9	0.
10	-1.0000E+01

THE BB MATRIX (10 BY 2)

	1	2
1	0.	0.
2	0.	0.
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	0.	0.
8	0.	0.
9	9.0000E+00	0.
10	0.	1.0000E+01

Figure A.4.1
Open-loop System Matrices - Modified Configuration Two

THE CC MATRIX (7 BY 10)

	1	2	3	4	5	6	7	8	9
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00	0.
7	2.6494E+01	-1.0067E+00	1.3504E+01	0.	4.0795E+01	-8.7153E+01	-1.5530E-01	-8.1019E-02	2.7472E+00

	10
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	1.9869E+00

THE DD MATRIX (7 BY 2)

	1	2
1	0.	0.
2	0.	0.
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	0.	0.

Figure A.4.1, concluded

```
SPECTRAL DECOMPOSITION OF AA

     REAL PART        IMAGINARY     FREQUENCY       DAMPING      TIME CONST.
                        PART         (HERTZ)         RATIO        (SEC/RAD)

 1  -4.56441E-01    2.13506E+01   3.39883E+00     .02137348    2.19086E+00
 3  -7.26413E-01    8.75840E+00   1.39873E+00     .08265523    1.37663E+00
 5  -1.34790E+00    2.19289E+00   4.09668E-01     .52365557    7.41895E-01
 7  -9.57681E-05    5.25567E-02   8.36467E-03     .00182218    1.04419E+04
 9  -9.00000E+00    0.            0.             1.00000000    1.11111E-01
10  -1.00000E+01    0.            0.             1.00000000    1.00000E-01


THE EE        MATRIX ( 10 BY 10 )

                          1                        2                        3                        4                        5
 1 ( 3.36E-04, 6.88E-04) ( 3.36E-04,-6.88E-04) ( 5.75E-03,-1.13E-02) ( 5.75E-03, 1.13E-02) (-3.48E-01, 9.35E-02)
 2 (-9.09E-03, 1.25E-02) (-9.09E-03,-1.25E-02) ( 9.76E-02, 8.80E-03) ( 9.76E-02,-8.80E-03) (-1.32E-01,-8.15E-01)
 3 (-1.47E-06, 1.38E-06) (-1.47E-06,-1.38E-06) ( 7.78E-05, 1.14E-05) ( 7.78E-05,-1.14E-05) (-6.69E-03,-3.88E-03)
 4 ( 5.92E-04, 4.13E-04) ( 5.92E-04,-4.13E-04) ( 7.94E-05,-1.12E-02) ( 7.94E-05, 1.12E-02) (-2.43E-01, 2.10E-01)
 5 ( 1.31E-03,-2.17E-03) ( 1.31E-03, 2.17E-03) ( 4.98E-02, 1.01E-01) ( 4.98E-02,-1.01E-01) (-1.02E-01,-1.90E-02)
 6 ( 4.63E-02,-6.15E-03) ( 4.63E-02, 6.15E-03) ( 6.50E-04, 3.02E-03) ( 6.50E-04,-3.02E-03) ( 1.86E-02,-2.08E-03)
 7 ( 4.57E-02, 2.89E-02) ( 4.57E-02,-2.89E-02) (-9.19E-01, 3.63E-01) (-9.19E-01,-3.63E-01) ( 1.79E-01,-1.98E-01)
 8 ( 1.10E-01, 9.91E-01) ( 1.10E-01,-9.91E-01) (-2.69E-02, 3.50E-03) (-2.69E-02,-3.50E-03) (-2.05E-02, 4.35E-02)
 9 ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      )
10 ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      )

                          6                        7                        8                        9                       10
 1 (-3.48E-01,-9.35E-02) (-2.29E-01, 7.17E-02) (-2.29E-01,-7.17E-02) (-4.28E-02, 0.      ) ( 5.06E-03, 0.      )
 2 (-1.32E-01, 8.15E-01) ( 3.74E-02,-2.16E-02) ( 3.74E-02, 2.16E-02) ( 3.97E-01, 0.      ) (-4.63E-02, 0.      )
 3 (-6.69E-03, 3.88E-03) ( 4.90E-01,-1.47E-01) ( 4.90E-01, 1.47E-01) (-2.91E-04, 0.      ) ( 2.90E-05, 0.      )
 4 (-2.43E-01,-2.10E-01) (-4.12E-01,-7.11E-01) (-4.12E-01, 7.11E-01) (-4.41E-02, 0.      ) ( 4.63E-03, 0.      )
 5 (-1.02E-01, 1.90E-02) (-5.81E-02, 1.78E-02) (-5.81E-02,-1.78E-02) ( 9.75E-02, 0.      ) ( 1.40E-02, 0.      )
 6 ( 1.86E-02, 2.08E-03) ( 1.20E-02,-3.71E-03) ( 1.20E-02, 3.71E-03) (-6.80E-03, 0.      ) ( 5.61E-04, 0.      )
 7 ( 1.79E-01, 1.98E-01) (-9.27E-04,-3.05E-03) (-9.27E-04, 3.05E-03) (-8.77E-01, 0.      ) (-1.40E-01, 0.      )
 8 (-2.05E-02,-4.35E-02) ( 1.94E-04, 6.29E-04) ( 1.94E-04,-6.29E-04) ( 6.12E-02, 0.      ) (-5.61E-03, 0.      )
 9 ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      ) ( 2.35E-01, 0.      ) ( 0.      , 0.      )
10 ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      ) ( 0.      , 0.      ) ( 9.89E-01, 0.      )
```

Figure A.4.2
Spectral Decomposition - Modified Configuration Two

```
------------------------------------------------
    EIGENVECTORS     MATRIX (10 BY 10)
    OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
------------------------------------------------

                         1                        2                        3                        4
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.013E+01, 6.218E+01) ( 2.013E+01,-6.218E+01) ( 7.740E+00, 6.813E+01) ( 7.740E+00,-6.813E+01)
 3 ( 2.636E-03, 7.293E+01) ( 2.636E-03,-7.293E+01) ( 6.208E-03, 7.131E+01) ( 6.208E-03,-7.131E+01)
 4 ( 9.425E-01,-2.905E+01) ( 9.425E-01, 2.905E+01) ( 8.807E-01,-2.661E+01) ( 8.807E-01, 2.661E+01)
 5 ( 3.301E+00,-1.229E+02) ( 3.301E+00, 1.229E+02) ( 8.878E+00, 1.267E+02) ( 8.878E+00,-1.267E+02)
 6 ( 6.095E+01,-7.152E+01) ( 6.095E+01, 7.152E+01) ( 2.439E-01, 1.408E+02) ( 2.439E-01,-1.408E+02)
 7 ( 7.050E+01,-3.165E+01) ( 7.050E+01, 3.165E+01) ( 7.803E+01, 2.215E+02) ( 7.803E+01,-2.215E+02)
 8 ( 1.302E+03, 1.971E+01) ( 1.302E+03,-1.971E+01) ( 2.143E+00, 2.356E+02) ( 2.143E+00,-2.356E+02)
 9 ( 0.       ,-6.395E+01) ( 0.       , 6.395E+01) ( 0.       , 6.299E+01) ( 0.       ,-6.299E+01)
10 ( 0.       ,-6.395E+01) ( 0.       , 6.395E+01) ( 0.       , 6.299E+01) ( 0.       ,-6.299E+01)

                         5                        6                        7                        8
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.295E+00,-2.642E+02) ( 2.295E+00, 2.642E+02) ( 1.798E-01,-1.926E+02) ( 1.798E-01, 1.926E+02)
 3 ( 2.147E-02,-3.148E+02) ( 2.147E-02, 3.148E+02) ( 2.129E+00,-1.793E+02) ( 2.129E+00, 1.793E+02)
 4 ( 8.915E-01,-2.574E+01) ( 8.915E-01, 2.574E+01) ( 3.420E+00,-2.827E+02) ( 3.420E+00, 2.827E+02)
 5 ( 2.885E-01,-3.344E+02) ( 2.885E-01, 3.344E+02) ( 2.528E-01, 3.665E-01) ( 2.528E-01,-3.665E-01)
 6 ( 5.193E-02,-1.713E+02) ( 5.193E-02, 1.713E+02) ( 5.215E-02,-1.798E+02) ( 5.215E-02, 1.798E+02)
 7 ( 7.425E-01,-2.128E+02) ( 7.425E-01, 2.128E+02) ( 1.328E-02,-2.695E+02) ( 1.328E-02, 2.695E+02)
 8 ( 1.337E-01,-4.975E+01) ( 1.337E-01, 4.975E+01) ( 2.741E-03,-8.974E+01) ( 2.741E-03, 8.974E+01)
 9 ( 0.       ,-1.649E+02) ( 0.       , 1.649E+02) ( 0.       ,-1.626E+02) ( 0.       , 1.626E+02)
10 ( 0.       ,-1.649E+02) ( 0.       , 1.649E+02) ( 0.       ,-1.626E+02) ( 0.       , 1.626E+02)

                         9                       10
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 9.271E+00,-1.800E+02) ( 9.143E+00, 1.800E+02)
 3 ( 6.806E-03, 0.        ) ( 5.732E-03, 0.        )
 4 ( 1.030E+00, 0.        ) ( 9.143E-01, 0.        )
 5 ( 2.276E+00,-1.800E+02) ( 2.760E+00, 0.        )
 6 ( 1.588E-01, 0.        ) ( 1.108E-01, 0.        )
 7 ( 2.049E+01, 0.        ) ( 2.760E+01, 1.800E+02)
 8 ( 1.429E+00,-1.800E+02) ( 1.108E+00, 1.800E+02)
 9 ( 5.494E+00,-1.800E+02) ( 0.       , 0.        )
10 ( 0.       ,-1.800E+02) ( 1.954E+02, 0.        )
```

Figure A.4.2, con

```
------------------------------------------------------
IMPULSE   RESIDUE    MATRIX ( 7 BY 10)
    OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
------------------------------------------------------

                        1                       2                       3                       4
 1 ( 8.778E-03, 1.699E+02) ( 8.778E-03,-1.699E+02) ( 1.756E+00, 1.710E+02) ( 1.756E+00,-1.710E+02)
 2 ( 3.308E-03, 6.797E+01) ( 3.308E-03,-6.797E+01) ( 2.625E-01, 7.311E+01) ( 2.625E-01,-7.311E+01)
 3 ( 7.064E-02, 1.592E+02) ( 7.064E-02,-1.592E+02) ( 2.307E+00, 1.679E+02) ( 2.307E+00,-1.679E+02)
 4 ( 1.721E-03,-1.403E+01) ( 1.721E-03, 1.403E+01) ( 1.336E-01,-1.860E+01) ( 1.336E-01, 1.860E+01)
 5 ( 2.190E-01,-1.575E+02) ( 2.190E-01, 1.575E+02) ( 2.955E+00, 4.906E+01) ( 2.955E+00,-4.906E+01)
 6 ( 4.677E+00,-6.623E+01) ( 4.677E+00, 6.623E+01) ( 2.597E+01, 1.438E+02) ( 2.597E+01,-1.438E+02)
 7 ( 1.830E+01,-1.522E+02) ( 1.830E+01, 1.522E+02) ( 9.669E+01,-1.392E+02) ( 9.669E+01, 1.392E+02)

                        5                       6                       7                       8
 1 ( 4.859E+01, 1.135E+02) ( 4.859E+01,-1.135E+02) ( 4.793E+02,-8.684E+01) ( 4.793E+02, 8.684E+01)
 2 ( 2.126E+00, 4.260E+01) ( 2.126E+00,-4.260E+01) ( 8.114E-01, 1.698E+02) ( 8.114E-01,-1.698E+02)
 3 ( 5.472E+00, 1.642E+02) ( 5.472E+00,-1.642E+02) ( 4.265E-02,-1.001E+02) ( 4.265E-02, 1.001E+02)
 4 ( 1.036E+00,-4.862E+01) ( 1.036E+00, 4.862E+01) ( 7.937E-01,-1.733E+02) ( 7.937E-01, 1.733E+02)
 5 ( 1.858E+00, 2.803E+01) ( 1.858E+00,-2.803E+01) ( 8.020E-01, 1.731E+02) ( 8.020E-01,-1.731E+02)
 6 ( 4.783E+00, 1.496E+02) ( 4.783E+00,-1.496E+02) ( 4.215E-02,-9.680E+01) ( 4.215E-02, 9.680E+01)
 7 ( 1.008E+02, 7.326E+01) ( 1.008E+02,-7.326E+01) ( 3.032E+00, 9.117E+01) ( 3.032E+00,-9.117E+01)

                        9                      10
 1 ( 1.058E+01, 1.800E+02) ( 0.        , 0.       )
 2 ( 1.687E+00, 1.800E+02) ( 0.        , 0.       )
 3 ( 1.519E+01, 0.       ) ( 0.        , 0.       )
 4 ( 4.937E-02, 1.800E+02) ( 0.        , 0.       )
 5 ( 5.156E+00, 1.800E+02) ( 0.        , 0.       )
 6 ( 4.640E+01, 0.       ) ( 0.        , 0.       )
 7 ( 1.457E+02, 0.       ) ( 0.        , 0.       )
```

C-3

Figure A.4.3
Impulse Residues - Modified Configuration Two

**********LOOP FOR RHO =   .100000E+07

THE GG   MATRIX (2 BY 10)

	1	2	3	4	5	6	7	8	9
1	-1.9477E-05	4.4131E-06	-2.3199E-06	2.5943E-05	-5.9274E-07	5.6421E-07	2.6855E-07	-5.7891E-08	-4.8543E-06
2	4.3729E-06	-1.0432E-06	5.8737E-07	-5.9826E-06	2.2918E-07	-1.0450E-07	-8.7614E-08	1.4698E-08	1.0058E-06

	10
1	2.2630E-07
2	-6.4805E-08

THE A+BG   MATRIX (10 BY 10)

	1	2	3	4	5	6	7	8	9
1	-1.2080E+00	9.4400E-01	-6.0167E-01	0.	3.3864E-01	-1.3362E-01	8.0682E-03	-4.4155E-03	-2.8840E-01
2	-7.0590E+00	-2.0180E+00	-2.1732E+00	0.	6.9773E+00	-5.9954E+00	2.8174E-01	-2.0148E-01	-1.5030E+01
3	-2.6344E-02	0.	-2.5000E-02	-3.3720E-02	0.	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
7	1.9491E+01	3.5191E+00	3.5952E-02	0.	-7.6800E+01	-1.1764E+01	-9.6220E-01	6.9965E-01	5.8872E+01
8	-2.6186E+01	-1.9336E+00	1.2791E-01	0.	9.1505E+00	-4.5650E+02	-1.5958E-01	-8.4850E-01	-2.1206E+01
9	-1.7529E-04	3.9718E-05	-2.0879E-05	2.3348E-04	-5.3347E-06	5.0779E-06	2.4169E-06	-5.2102E-07	-9.0000E+00
10	4.3729E-05	-1.0432E-05	5.8737E-06	-5.9826E-05	2.2918E-06	-1.0450E-06	-8.7614E-07	1.4698E-07	1.0058E-05

	10
1	-4.4000E-03
2	3.5310E-01
3	0.
4	0.
5	0.
6	0.
7	2.4362E+00
8	2.0252E-01
9	2.0367E-06
10	-1.0000E+01

Figure A.4.4
Closed-loop System Matrices - Ex

THE C+DG MATRIX (7 BY 10)

	1	2	3	4	5	6	7	8	9
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00	0.
7	2.6494E+01	-1.0067E+00	1.3504E+01	0.	4.0795E+01	-8.7153E+01	-1.5530E-01	-8.1019E-02	2.7472E+00

	10
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	1.9869E+00

Figure A.4.4, concluded

```
-----------------------------------
SPECTRAL DECOMPOSITION OF AAUG
-----------------------------------

     REAL PART       IMAGINARY    FREQUENCY        DAMPING        TIME CONST.
                       PART        (HERTZ)          RATIO          (SEC/RAD)

 1  -4.56441E-01   2.13506E+01  3.39883E+00     .02137348     2.19086E+00
 3  -7.26414E-01   8.75840E+00  1.39873E+00     .08265535     1.37662E+00
 5  -1.34790E+00   2.19289E+00  4.09668E-01     .52365562     7.41895E-01
 7  -9.00000E+00   0.           0.             1.00000000     1.11111E-01
 8  -1.16655E-04   5.25567E-02  8.36468E-03     .00221959     8.57230E+03
10  -1.00000E+01   0.           0.             1.00000000     1.00000E-01

---------------------------------
THE EE         MATRIX ( 10 BY 10 )
---------------------------------

                           1                         2                         3                         4                         5
 1 ( 4.48E-04,-6.22E-04) ( 4.48E-04, 6.22E-04) (-1.74E-03, 1.25E-02) (-1.74E-03,-1.25E-02) ( 1.79E-01, 3.12E-01)
 2 ( 1.53E-02, 2.13E-03) ( 1.53E-02,-2.13E-03) (-9.51E-02, 2.36E-02) (-9.51E-02,-2.36E-02) (-7.54E-01, 3.37E-01)
 3 ( 1.91E-06, 6.47E-07) ( 1.91E-06,-6.47E-07) (-7.72E-05, 1.47E-05) (-7.72E-05,-1.47E-05) (-2.04E-03, 7.46E-03)
 4 ( 8.45E-05,-7.17E-04) ( 8.45E-05, 7.17E-04) ( 3.58E-03, 1.06E-02) ( 3.58E-03,-1.06E-02) ( 2.65E-01, 1.81E-01)
 5 (-2.53E-03,-1.28E-04) (-2.53E-03, 1.28E-04) (-8.00E-02,-7.90E-02) (-8.00E-02, 7.90E-02) ( 7.71E-03, 1.04E-01)
 6 (-2.73E-02,-3.79E-02) (-2.73E-02, 3.79E-02) (-1.60E-03,-2.64E-03) (-1.60E-03, 2.64E-03) (-6.76E-03,-1.74E-02)
 7 ( 3.88E-03,-5.39E-02) ( 3.88E-03, 5.39E-02) ( 7.50E-01,-6.44E-01) ( 7.50E-01, 6.44E-01) (-2.37E-01,-1.23E-01)
 8 ( 8.22E-01,-5.65E-01) ( 8.22E-01, 5.65E-01) ( 2.43E-02,-1.21E-02) ( 2.43E-02, 1.21E-02) ( 4.73E-02, 8.67E-03)
 9 ( 1.23E-11,-1.84E-10) ( 1.23E-11, 1.84E-10) (-1.99E-08, 2.90E-08) (-1.99E-08,-2.90E-08) (-2.74E-08,-9.38E-09)
10 (-2.02E-10, 1.30E-10) (-2.02E-10,-1.30E-10) (-4.71E-09, 1.02E-08) (-4.71E-09,-1.02E-08) ( 2.04E-09,-3.47E-08)

                           6                         7                         8                         9                        10
 1 ( 1.79E-01,-3.12E-01) (-4.28E-02, 0.       ) (-2.36E-01,-4.60E-02) (-2.36E-01, 4.60E-02) ( 5.06E-03, 0.       )
 2 (-7.54E-01,-3.37E-01) ( 3.97E-01, 0.       ) ( 4.32E-02,-1.14E-03) ( 4.32E-02, 1.14E-03) (-4.63E-02, 0.       )
 3 (-2.04E-03,-7.46E-03) (-2.91E-04, 0.       ) ( 5.01E-01, 1.04E-01) ( 5.01E-01,-1.04E-01) ( 2.90E-05, 0.       )
 4 ( 2.65E-01,-1.81E-01) (-4.41E-02, 0.       ) (-2.34E-02,-8.21E-01) (-2.34E-02, 8.21E-01) ( 4.63E-03, 0.       )
 5 ( 7.71E-03,-1.04E-01) ( 9.75E-02, 0.       ) (-5.95E-02,-1.20E-02) (-5.95E-02, 1.20E-02) ( 1.40E-02, 0.       )
 6 (-6.76E-03, 1.74E-02) (-6.80E-03, 0.       ) ( 1.23E-02, 2.43E-03) ( 1.23E-02,-2.43E-03) ( 5.61E-04, 0.       )
 7 (-2.37E-01, 1.23E-01) (-8.77E-01, 0.       ) ( 6.39E-04,-3.13E-03) ( 6.39E-04, 3.13E-03) (-1.40E-01, 0.       )
 8 ( 4.73E-02,-8.67E-03) ( 6.12E-02, 0.       ) (-1.29E-04, 6.46E-04) (-1.29E-04,-6.46E-04) (-5.61E-03, 0.       )
 9 (-2.74E-08, 9.38E-09) ( 2.35E-01, 0.       ) ( 2.93E-06,-2.07E-05) ( 2.93E-06, 2.07E-05) ( 3.70E-08, 0.       )
10 ( 2.04E-09, 3.47E-08) (-2.25E-09, 0.       ) (-6.31E-07, 4.77E-06) (-6.31E-07,-4.77E-06) ( 9.89E-01, 0.       )
```

Figure A.4.5
Spectral Decomposition - Example 4.1, $\rho = 10^6$

```
----------------------------------------------------
C.L. EIGENVECTORS     MATRIX (10 BY 10)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
----------------------------------------------------

                           1                          2                          3                          4
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.013E+01, 6.218E+01) ( 2.013E+01,-6.218E+01) ( 7.740E+00, 6.813E+01) ( 7.740E+00,-6.813E+01)
 3 ( 2.636E-03, 7.293E+01) ( 2.636E-03,-7.293E+01) ( 6.208E-03, 7.131E+01) ( 6.208E-03,-7.131E+01)
 4 ( 9.425E-01,-2.905E+01) ( 9.425E-01, 2.905E+01) ( 8.807E-01,-2.661E+01) ( 8.807E-01, 2.661E+01)
 5 ( 3.301E+00,-1.229E+02) ( 3.301E+00, 1.229E+02) ( 8.878E+00,-2.333E+02) ( 8.878E+00, 2.333E+02)
 6 ( 6.095E+01,-7.152E+01) ( 6.095E+01, 7.152E+01) ( 2.439E-01,-2.192E+02) ( 2.439E-01, 2.192E+02)
 7 ( 7.050E+01,-3.165E+01) ( 7.050E+01, 3.165E+01) ( 7.803E+01,-1.385E+02) ( 7.803E+01, 1.385E+02)
 8 ( 1.302E+03, 1.971E+01) ( 1.302E+03,-1.971E+01) ( 2.143E+00,-1.244E+02) ( 2.143E+00, 1.244E+02)
 9 ( 2.407E-07,-3.194E+01) ( 2.407E-07, 3.194E+01) ( 2.773E-06, 2.656E+01) ( 2.773E-06,-2.656E+01)
10 ( 3.133E-07, 2.015E+02) ( 3.133E-07,-2.015E+02) ( 8.898E-07, 1.678E+01) ( 8.898E-07,-1.678E+01)

                           5                          6                          7                          8
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.295E+00, 9.584E+01) ( 2.295E+00,-9.584E+01) ( 9.271E+00,-1.800E+02) ( 1.798E-01, 1.674E+02)
 3 ( 2.147E-02, 4.517E+01) ( 2.147E-02,-4.517E+01) ( 6.806E-03, 0.        ) ( 2.129E+00, 1.807E+02)
 4 ( 8.915E-01,-2.574E+01) ( 8.915E-01, 2.574E+01) ( 1.030E+00, 0.        ) ( 3.420E+00, 7.732E+01)
 5 ( 2.885E-01, 2.562E+01) ( 2.885E-01,-2.562E+01) ( 2.276E+00,-1.800E+02) ( 2.528E-01, 3.798E-01)
 6 ( 5.193E-02,-1.713E+02) ( 5.193E-02, 1.713E+02) ( 1.588E-01, 0.        ) ( 5.215E-02, 1.802E+02)
 7 ( 7.425E-01,-2.128E+02) ( 7.425E-01, 2.128E+02) ( 2.049E+01, 0.        ) ( 1.328E-02, 9.051E+01)
 8 ( 1.337E-01,-4.975E+01) ( 1.337E-01, 4.975E+01) ( 1.429E+00,-1.800E+02) ( 2.741E-03, 2.703E+02)
 9 ( 8.040E-08,-2.212E+02) ( 8.040E-08, 2.212E+02) ( 5.494E+00,-1.800E+02) ( 8.689E-05, 8.703E+01)
10 ( 9.649E-08,-1.468E+02) ( 9.649E-08, 1.468E+02) ( 5.262E-08, 0.        ) ( 2.005E-05, 2.665E+02)

                           9                         10
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 1.798E-01,-1.674E+02) ( 9.143E+00, 1.800E+02)
 3 ( 2.129E+00,-1.807E+02) ( 5.732E-03, 0.        )
 4 ( 3.420E+00,-7.732E+01) ( 9.143E-01, 0.        )
 5 ( 2.528E-01,-3.798E-01) ( 2.760E+00, 0.        )
 6 ( 5.215E-02,-1.802E+02) ( 1.108E-01, 0.        )
 7 ( 1.328E-02,-9.051E+01) ( 2.760E+01, 1.800E+02)
 8 ( 2.741E-03,-2.703E+02) ( 1.108E+00, 1.800E+02)
 9 ( 8.689E-05,-8.703E+01) ( 7.303E-06, 0.        )
10 ( 2.005E-05,-2.665E+02) ( 1.954E+02, 0.        )
```

Figure A.4.5, concluded

```
--------------------------------------------------
IMPULSE   RESIDUE    MATRIX ( 7 BY 10)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
--------------------------------------------------

                         1                        2                        3                        4
 1 ( 8.778E-03, 1.699E+02) ( 8.778E-03,-1.699E+02) ( 1.756E+00, 1.710E+02) ( 1.756E+00,-1.710E+02)
 2 ( 3.308E-03, 6.797E+01) ( 3.308E-03,-6.797E+01) ( 2.625E-01, 7.311E+01) ( 2.625E-01,-7.311E+01)
 3 ( 7.064E-02, 1.592E+02) ( 7.064E-02,-1.592E+02) ( 2.307E+00, 1.679E+02) ( 2.307E+00,-1.679E+02)
 4 ( 1.721E-03,-1.403E+01) ( 1.721E-03, 1.403E+01) ( 1.336E-01,-1.860E+01) ( 1.336E-01, 1.860E+01)
 5 ( 2.190E-01,-1.575E+02) ( 2.190E-01, 1.575E+02) ( 2.955E+00, 4.906E+01) ( 2.955E+00,-4.906E+01)
 6 ( 4.677E+00,-6.623E+01) ( 4.677E+00, 6.623E+01) ( 2.597E+01, 1.438E+02) ( 2.597E+01,-1.438E+02)
 7 ( 1.830E+01,-1.522E+02) ( 1.830E+01, 1.522E+02) ( 9.669E+01,-1.392E+02) ( 9.669E+01, 1.392E+02)

                         5                        6                        7                        8
 1 ( 4.859E+01, 1.135E+02) ( 4.859E+01,-1.135E+02) ( 1.058E+01, 1.800E+02) ( 4.793E+02,-8.684E+01)
 2 ( 2.126E+00, 4.260E+01) ( 2.126E+00,-4.260E+01) ( 1.687E+00, 1.800E+02) ( 8.114E-01, 1.698E+02)
 3 ( 5.472E+00, 1.642E+02) ( 5.472E+00,-1.642E+02) ( 1.519E+01,-5.048E-33) ( 4.264E-02,-1.001E+02)
 4 ( 1.036E+00,-4.862E+01) ( 1.036E+00, 4.862E+01) ( 4.937E-02, 1.800E+02) ( 7.938E-01,-1.732E+02)
 5 ( 1.858E+00, 2.803E+01) ( 1.858E+00,-2.803E+01) ( 5.156E+00, 1.800E+02) ( 8.019E-01, 1.731E+02)
 6 ( 4.783E+00, 1.496E+02) ( 4.783E+00,-1.496E+02) ( 4.640E+01,-5.048E-33) ( 4.215E-02,-9.676E+01)
 7 ( 1.008E+02, 7.326E+01) ( 1.008E+02,-7.326E+01) ( 1.457E+02,-5.048E-33) ( 3.032E+00, 9.116E+01)

                         9                       10
 1 ( 4.793E+02, 8.684E+01) ( 2.583E-07, 1.800E+02)
 2 ( 8.114E-01,-1.698E+02) ( 4.342E-08, 1.800E+02)
 3 ( 4.264E-02, 1.001E+02) ( 4.342E-07,-1.193E-27)
 4 ( 7.938E-01, 1.732E+02) ( 4.072E-09,-1.193E-27)
 5 ( 8.019E-01,-1.731E+02) ( 9.291E-08,-1.193E-27)
 6 ( 4.215E-02, 9.676E+01) ( 9.291E-07, 1.800E+02)
 7 ( 3.032E+00,-9.116E+01) ( 2.524E-05, 1.800E+02)
```

Figure A.4.6
Impulse Residues - Example 4.

**********LOOP FOR RHO =   .100000E-01

THE GG   MATRIX (2 BY 10)

	1	2	3	4	5	6	7	8	9
1	-1.4696E+00	1.4381E-02	-9.3814E-01	7.7838E-02	-7.2891E-01	7.7467E-01	-4.2380E-01	3.6977E-02	-1.6141E+00
2	-3.2656E+00	-1.0399E+00	8.6182E-01	-7.0014E-02	-1.7602E+00	1.5108E+00	-7.1002E-01	7.5001E-04	-7.6909E-01

	10
1	-1.7304E-01
2	-5.1549E-01

THE A+BG   MATRIX (10 BY 10)

	1	2	3	4	5	6	7	8	9
1	-1.2080E+00	9.4400E-01	-6.0167E-01	0.	3.3864E-01	-1.3362E-01	8.0682E-03	-4.4155E-03	-2.8840E-01
2	-7.0590E+00	-2.0180E+00	-2.1732E+00	0.	6.9773E+00	-5.9954E+00	2.8174E-01	-2.0148E-01	-1.5030E+01
3	-2.6344E-02	0.	-2.5000E-02	-3.3720E-02	0.	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
7	1.9491E+01	3.5191E+00	3.5952E-02	0.	-7.6800E+01	-1.1764E+01	-9.6220E-01	6.9965E-01	5.8872E+01
8	-2.6186E+01	-1.9336E+00	1.2791E-01	0.	9.1505E+00	-4.5650E+02	-1.5958E-01	-8.4850E-01	-2.1206E+01
9	-1.3226E+01	1.2943E-01	-8.4433E+00	7.0054E-01	-6.5602E+00	6.9720E+00	-3.8142E+00	3.3279E-01	-2.3527E+01
10	-3.2656E+01	-1.0399E+01	8.6182E+00	-7.0014E-01	-1.7602E+01	1.5108E+01	-7.1002E+00	7.5001E-03	-7.6909E+00

	10
1	-4.4000E-03
2	3.5310E-01
3	0.
4	0.
5	0.
6	0.
7	2.4362E+00
8	2.0252E-01
9	-1.5574E+00
10	-1.5155E+01

Figure A.4.7
Closed-loop System Matrices - Example 4.1, $\rho = 10^{-2}$

THE C+DG MATRIX (7 BY 10)

	1	2	3	4	5	6	7	8	9
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00	0.
7	2.6494E+01	-1.0067E+00	1.3504E+01	0.	4.0795E+01	-8.7153E+01	-1.5530E-01	-8.1019E-02	2.7472E+00

	10
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	1.9869E+00

Figure A.4.7, concl.

SPECTRAL DECOMPOSITION OF AAUG

	REAL PART	IMAGINARY PART	FREQUENCY (HERTZ)	DAMPING RATIO	TIME CONST. (SEC/RAD)
1	-5.39616E-01	2.14516E+01	3.41521E+00	.02514713	1.85317E+00
3	-7.38491E+00	1.08696E+01	2.09146E+00	.56197416	1.35411E-01
5	-1.32191E+01	0.	0.	1.00000000	7.56482E-02
6	-1.00486E+01	0.	0.	1.00000000	9.95161E-02
7	-2.26424E+00	1.13667E+00	4.03224E-01	.89370795	4.41650E-01
9	-4.89553E-02	1.46393E-01	2.45674E-02	.31714693	2.04268E+01

THE EE MATRIX (10 BY 10)

	1	2	3	4	5
1	(3.41E-04,-1.14E-03)	(3.41E-04, 1.14E-03)	(8.63E-03,-1.34E-02)	(8.63E-03, 1.34E-02)	(1.61E-02, 0.)
2	(2.35E-02,-4.22E-03)	(2.35E-02, 4.22E-03)	(1.49E-01, 1.58E-01)	(1.49E-01,-1.58E-01)	(-2.59E-01, 0.)
3	(3.11E-06,-7.80E-09)	(3.11E-06, 7.80E-09)	(7.14E-05,-1.65E-05)	(7.14E-05, 1.65E-05)	(8.22E-05, 0.)
4	(-2.24E-04,-1.09E-03)	(-2.24E-04, 1.09E-03)	(3.53E-03,-1.61E-02)	(3.53E-03, 1.61E-02)	(1.96E-02, 0.)
5	(-1.87E-03, 2.25E-03)	(-1.87E-03,-2.25E-03)	(1.22E-02, 6.98E-02)	(1.22E-02,-6.98E-02)	(-6.30E-02, 0.)
6	(-3.76E-02,-2.72E-02)	(-3.76E-02, 2.72E-02)	(-9.13E-03,-5.10E-04)	(-9.13E-03, 5.10E-04)	(6.57E-03, 0.)
7	(-4.72E-02,-4.13E-02)	(-4.72E-02, 4.13E-02)	(-8.49E-01,-3.83E-01)	(-8.49E-01, 3.83E-01)	(8.33E-01, 0.)
8	(6.04E-01,-7.92E-01)	(6.04E-01, 7.92E-01)	(7.30E-02,-9.55E-02)	(7.30E-02, 9.55E-02)	(-8.68E-02, 0.)
9	(-3.55E-03,-1.05E-02)	(-3.55E-03, 1.05E-02)	(1.59E-01,-2.20E-02)	(1.59E-01, 2.20E-02)	(-2.26E-01, 0.)
10	(-9.22E-03, 1.35E-02)	(-9.22E-03,-1.35E-02)	(1.42E-01,-1.39E-01)	(1.42E-01, 1.39E-01)	(-4.19E-01, 0.)

	6	7	8	9	10
1	(-3.40E-02, 0.)	(-3.22E-01,-1.74E-01)	(-3.22E-01, 1.74E-01)	(-5.80E-02, 1.56E-02)	(-5.80E-02,-1.56E-02)
2	(3.67E-01, 0.)	(5.95E-01,-1.80E-01)	(5.95E-01, 1.80E-01)	(-5.77E-02, 1.35E-01)	(-5.77E-02,-1.35E-01)
3	(-2.12E-04, 0.)	(-4.82E-03,-5.13E-03)	(-4.82E-03, 5.13E-03)	(1.34E-02, 2.06E-01)	(1.34E-02,-2.06E-01)
4	(-3.65E-02, 0.)	(-2.42E-01,-4.20E-02)	(-2.42E-01, 4.20E-02)	(9.51E-01, 7.63E-02)	(9.51E-01,-7.63E-02)
5	(8.86E-02, 0.)	(9.80E-02, 1.22E-03)	(9.80E-02,-1.22E-03)	(1.89E-02,-4.39E-02)	(1.89E-02, 4.39E-02)
6	(-7.15E-03, 0.)	(9.04E-03, 8.91E-03)	(9.04E-03,-8.91E-03)	(1.93E-03, 1.15E-03)	(1.93E-03,-1.15E-03)
7	(-8.90E-01, 0.)	(-2.23E-01, 1.09E-01)	(-2.23E-01,-1.09E-01)	(5.50E-03, 4.91E-03)	(5.50E-03,-4.91E-03)
8	(7.19E-02, 0.)	(-3.06E-02,-9.89E-03)	(-3.06E-02, 9.89E-03)	(-2.63E-04, 2.27E-04)	(-2.63E-04,-2.27E-04)
9	(2.39E-01, 0.)	(1.92E-01, 4.52E-02)	(1.92E-01,-4.52E-02)	(4.43E-02,-7.33E-02)	(4.43E-02, 7.33E-02)
10	(2.69E-02, 0.)	(2.73E-01, 4.83E-01)	(2.73E-01,-4.83E-01)	(8.42E-02, 7.38E-02)	(8.42E-02,-7.38E-02)

Figure A.4.8

Spectral Decomposition - Example 4.1 $= 10^{-2}$

```
-------------------------------------------------
C.L. EIGENVECTORS    MATRIX (10 BY 10)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
-------------------------------------------------

                     1                       2                       3                       4
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.013E+01, 6.311E+01) ( 2.013E+01,-6.311E+01) ( 1.360E+01, 1.038E+02) ( 1.360E+01,-1.038E+02)
 3 ( 2.621E-03, 7.314E+01) ( 2.621E-03,-7.314E+01) ( 4.593E-03, 4.426E+01) ( 4.593E-03,-4.426E+01)
 4 ( 9.382E-01,-2.833E+01) ( 9.382E-01, 2.833E+01) ( 1.035E+00,-2.039E+01) ( 1.035E+00, 2.039E+01)
 5 ( 2.466E+00, 2.031E+02) ( 2.466E+00,-2.031E+02) ( 4.442E+00, 1.374E+02) ( 4.442E+00,-1.374E+02)
 6 ( 3.917E+01,-7.082E+01) ( 3.917E+01, 7.082E+01) ( 5.733E-01,-1.195E+02) ( 5.733E-01, 1.195E+02)
 7 ( 5.291E+01,-6.550E+01) ( 5.291E+01, 6.550E+01) ( 5.837E+01,-9.842E+01) ( 5.837E+01, 9.842E+01)
 8 ( 8.406E+02, 2.062E+01) ( 8.406E+02,-2.062E+01) ( 7.534E+00, 4.658E+00) ( 7.534E+00,-4.658E+00)
 9 ( 9.361E+00,-3.540E+01) ( 9.361E+00, 3.540E+01) ( 1.005E+01, 4.939E+01) ( 1.005E+01,-4.939E+01)
10 ( 1.377E+01, 1.977E+02) ( 1.377E+01,-1.977E+02) ( 1.247E+01, 1.292E+01) ( 1.247E+01,-1.292E+01)

                     5                       6                       7                       8
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 1.614E+01, 1.800E+02) ( 1.078E+01,-1.800E+02) ( 1.698E+00, 1.347E+02) ( 1.698E+00,-1.347E+02)
 3 ( 5.117E-03, 0.        ) ( 6.237E-03, 0.        ) ( 1.924E-02, 1.833E+01) ( 1.924E-02,-1.833E+01)
 4 ( 1.221E+00, 0.        ) ( 1.073E+00, 0.        ) ( 6.702E-01,-1.860E+01) ( 6.702E-01, 1.860E+01)
 5 ( 3.922E+00, 1.800E+02) ( 2.604E+00,-1.800E+02) ( 2.677E-01, 1.523E+02) ( 2.677E-01,-1.523E+02)
 6 ( 4.089E-01, 0.        ) ( 2.103E-01, 0.        ) ( 3.467E-02, 1.961E+02) ( 3.467E-02,-1.961E+02)
 7 ( 5.184E+01, 0.        ) ( 2.617E+01, 0.        ) ( 6.781E-01, 3.056E+02) ( 6.781E-01,-3.056E+02)
 8 ( 5.405E+00, 1.800E+02) ( 2.113E+00,-1.800E+02) ( 8.784E-02,-1.053E+01) ( 8.784E-02, 1.053E+01)
 9 ( 1.405E+01, 1.800E+02) ( 7.023E+00,-1.800E+02) ( 5.402E-01, 1.648E+02) ( 5.402E-01,-1.648E+02)
10 ( 2.609E+01, 1.800E+02) ( 7.922E-01,-1.800E+02) ( 1.517E+00, 2.121E+02) ( 1.517E+00,-2.121E+02)

                     9                      10
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.452E+00,-5.189E+01) ( 2.452E+00, 5.189E+01)
 3 ( 3.444E+00,-7.869E+01) ( 3.444E+00, 7.869E+01)
 4 ( 1.589E+01,-1.604E+02) ( 1.589E+01, 1.604E+02)
 5 ( 7.954E-01,-2.317E+02) ( 7.954E-01, 2.317E+02)
 6 ( 3.745E-02,-1.342E+02) ( 3.745E-02, 1.342E+02)
 7 ( 1.228E-01,-1.232E+02) ( 1.228E-01, 1.232E+02)
 8 ( 5.781E-03,-2.576E+01) ( 5.781E-03, 2.576E+01)
 9 ( 1.427E+00,-2.239E+02) ( 1.427E+00, 2.239E+02)
10 ( 1.864E+00,-1.237E+02) ( 1.864E+00, 1.237E+02)
```

Figure A.4.8, concluded

```
-----------------------------------------------
IMPULSE   RESIDUE    MATRIX ( 7 BY 10)
    OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
-----------------------------------------------

                         1                      2                      3                      4
1 ( 1.205E-02,-1.448E+02) ( 1.205E-02, 1.448E+02) ( 2.395E+00, 1.046E+02) ( 2.395E+00,-1.046E+02)
2 ( 4.544E-03, 1.137E+02) ( 4.544E-03,-1.137E+02) ( 5.687E-01, 3.992E+01) ( 5.687E-01,-3.992E+01)
3 ( 9.751E-02,-1.549E+02) ( 9.751E-02, 1.549E+02) ( 7.474E+00, 1.641E+02) ( 7.474E+00,-1.641E+02)
4 ( 2.316E-03; 3.067E+01) ( 2.316E-03,-3.067E+01) ( 1.988E-01,-3.442E+01) ( 1.988E-01, 3.442E+01)
5 ( 1.878E-01,-1.134E+02) ( 1.878E-01, 1.134E+02) ( 2.943E+00, 2.792E+01) ( 2.943E+00,-2.792E+01)
6 ( 4.030E+00,-2.192E+01) ( 4.030E+00, 2.192E+01) ( 3.867E+01, 1.521E+02) ( 3.867E+01,-1.521E+02)
7 ( 1.635E+01,-1.059E+02) ( 1.635E+01, 1.059E+02) ( 1.054E+02, 1.612E+02) ( 1.054E+02,-1.612E+02)

                         5                      6                      7                      8
1 ( 1.676E+00,-1.004E-27) ( 1.645E+01,-1.800E+02) ( 6.074E+01, 9.225E+01) ( 6.074E+01,-9.225E+01)
2 ( 4.213E-01,-1.004E-27) ( 2.982E+00,-1.800E+02) ( 2.230E+00, 5.532E+01) ( 2.230E+00,-5.532E+01)
3 ( 5.570E+00, 1.800E+02) ( 2.996E+01, 1.801E-27) ( 5.650E+00,-1.513E+02) ( 5.650E+00, 1.513E+02)
4 ( 7.621E-02,-1.004E-27) ( 2.023E-01,-1.800E+02) ( 1.407E+00,-7.570E+01) ( 1.407E+00, 7.570E+01)
5 ( 1.634E+00,-1.004E-27) ( 9.636E+00,-1.800E+02) ( 3.205E+00, 5.397E+01) ( 3.205E+00,-5.397E+01)
6 ( 2.160E+01, 1.800E+02) ( 9.683E+01, 1.801E-27) ( 8.120E+00,-1.527E+02) ( 8.120E+00, 1.527E+02)
7 ( 8.658E+01, 1.800E+02) ( 3.110E+02, 1.801E-27) ( 5.829E+01, 8.600E+01) ( 5.829E+01,-8.600E+01)

                         9                     10
1 ( 9.006E+01,-8.338E+01) ( 9.006E+01, 8.338E+01)
2 ( 4.377E-01,-1.651E+02) ( 4.377E-01, 1.651E+02)
3 ( 6.757E-02,-5.659E+01) ( 6.757E-02, 5.659E+01)
4 ( 4.638E-01,-1.662E+02) ( 4.638E-01, 1.662E+02)
5 ( 4.302E-01,-1.624E+02) ( 4.302E-01, 1.624E+02)
6 ( 6.641E-02,-5.388E+01) ( 6.641E-02, 5.388E+01)
7 ( 5.460E-01,-5.965E+01) ( 5.460E-01, 5.965E+01)
```

Figure A.4.9
Impulse Residues - Example 4.1, $\rho = 10^{-2}$

**********LOOP FOR RHO =   .100000E-03

THE GG   MATRIX (2 BY 10)

	1	2	3	4	5	6	7	8	9
1	-4.1186E+00	-4.4451E-01	-9.7192E-01	2.2142E-01	-2.8332E+00	1.9412E+00	-9.9750E-01	3.9875E-02	-5.4519E+00
2	-4.2566E+01	-7.7541E+00	-1.5075E+00	-7.3437E-02	-6.6368E+01	1.3054E+01	-9.5167E+00	-1.0014E+00	-2.6278E+00

	10
1	-5.9127E-01
2	-9.2367E+00

THE A+BG   MATRIX (10 BY 10)

	1	2	3	4	5	6	7	8	9
1	-1.2080E+00	9.4400E-01	-6.0167E-01	0.	3.3864E-01	-1.3362E-01	8.0682E-03	-4.4155E-03	-2.8840E-01
2	-7.0590E+00	-2.0180E+00	-2.1732E+00	0.	6.9773E+00	-5.9954E+00	2.8174E-01	-2.0148E-01	-1.5030E+01
3	-2.6344E-02	0.	-2.5000E-02	-3.3720E-02	0.	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
7	1.9491E+01	3.5191E+00	3.5952E-02	0.	-7.6800E+01	-1.1764E+01	-9.6220E-01	6.9965E-01	5.8872E+01
8	-2.6186E+01	-1.9336E+00	1.2791E-01	0.	9.1505E+00	-4.5650E+02	-1.5958E-01	-8.4850E-01	-2.1206E+01
9	-3.7067E+01	-4.0006E+00	-8.7473E+00	1.9928E+00	-2.5499E+01	1.7471E+01	-8.9775E+00	3.5887E-01	-5.8067E+01
10	-4.2566E+02	-7.7541E+01	-1.5075E+01	-7.3437E-01	-6.6368E+02	1.3054E+02	-9.5167E+01	-1.0014E+01	-2.6278E+01

	10
1	-4.4000E-03
2	3.5310E-01
3	0.
4	0.
5	0.
6	0.
7	2.4362E+00
8	2.0252E-01
9	-5.3214E+00
10	-1.0237E+02

Figure A.4.10
Closed-loop System Matrices - Example 4.1, $\rho =$ [illegible]

THE C+DG MATRIX (7 BY 10)

	1	2	3	4	5	6	7	8	9
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00	0.
7	2.6494E+01	-1.0067E+00	1.3504E+01	0.	4.0795E+01	-8.7153E+01	-1.5530E-01	-8.1019E-02	2.7472E+00

	10
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	1.9869E+00

ORIGINAL PAGE IS
OF POOR QUALITY

Figure A.4.10, conclu

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SPECTRAL DECOMPOSITION OF AAUG
-------------------------------------

     REAL PART      IMAGINARY     FREQUENCY       DAMPING       TIME CONST.
                      PART         (HERTZ)         RATIO         (SEC/RAD)

 1  -9.85507E+01   0.             0.              1.00000000    1.01471E-02
 2  -5.41139E+01   0.             0.              1.00000000    1.84795E-02
 3  -5.95908E-01   2.14568E+01   3.41627E+00       .02776175    1.67811E+00
 5  -4.07931E+00   8.35460E+00   1.47971E+00       .43876203    2.45140E-01
 7  -1.69569E+00   2.06451E+00   4.25202E-01       .63470435    5.89730E-01
 9  -4.45254E-02   9.70872E-02   1.69994E-02       .41686452    2.24591E+01

-----------------------------------
THE EE         MATRIX ( 10 BY 10 )
-----------------------------------

                          1                       2                       3                       4                       5
  1 ( 2.45E-04, 0.        ) (-2.55E-04, 0.        ) (-1.30E-03, 4.16E-04) (-1.30E-03,-4.16E-04) ( 3.54E-03, 1.28E-02)
  2 ( 1.40E-02, 0.        ) (-1.59E-01, 0.        ) (-1.83E-02,-2.01E-02) (-1.83E-02, 2.01E-02) (-1.05E-01, 1.54E-02)
  3 ( 1.67E-08, 0.        ) ( 1.71E-06, 0.        ) (-1.97E-06,-2.97E-06) (-1.97E-06, 2.97E-06) (-4.87E-05, 6.08E-05)
  4 (-1.42E-04, 0.        ) ( 2.94E-03, 0.        ) (-9.12E-04, 8.80E-04) (-9.12E-04,-8.80E-04) ( 6.43E-03, 9.40E-03)
  5 ( 9.34E-04, 0.        ) (-1.17E-02, 0.        ) ( 3.00E-03, 3.26E-05) ( 3.00E-03,-3.26E-05) (-8.02E-02, 7.55E-03)
  6 (-2.16E-04, 0.        ) ( 3.53E-03, 0.        ) ( 1.95E-03, 4.61E-02) ( 1.95E-03,-4.61E-02) ( 6.57E-04,-3.15E-03)
  7 (-9.21E-02, 0.        ) ( 6.33E-01, 0.        ) (-2.49E-03, 6.44E-02) (-2.49E-03,-6.44E-02) ( 2.64E-01,-7.01E-01)
  8 ( 2.13E-02, 0.        ) (-1.91E-01, 0.        ) (-9.91E-01, 1.43E-02) (-9.91E-01,-1.43E-02) ( 2.37E-02, 1.84E-02)
  9 ( 1.12E-01, 0.        ) (-5.55E-01, 0.        ) (-1.01E-02, 1.02E-02) (-1.01E-02,-1.02E-02) (-3.09E-02, 5.75E-02)
 10 ( 9.89E-01, 0.        ) (-4.79E-01, 0.        ) ( 1.02E-01,-1.32E-02) ( 1.02E-01, 1.32E-02) ( 4.03E-01, 5.04E-01)

                          6                       7                       8                       9                      10
  1 ( 3.54E-03,-1.28E-02) (-1.18E-01,-1.83E-01) (-1.18E-01, 1.83E-01) ( 9.44E-02, 4.16E-02) ( 9.44E-02,-4.16E-02)
  2 (-1.05E-01,-1.54E-02) ( 4.65E-01,-1.50E-01) ( 4.65E-01, 1.50E-01) ( 1.58E-02,-9.17E-02) ( 1.58E-02, 9.17E-02)
  3 (-4.87E-05,-6.08E-05) ( 4.19E-04,-4.36E-03) ( 4.19E-04, 4.36E-03) (-1.37E-01,-2.39E-01) (-1.37E-01, 2.39E-01)
  4 ( 6.43E-03,-9.40E-03) (-1.54E-01,-9.89E-02) (-1.54E-01, 9.89E-02) (-8.42E-01, 2.23E-01) (-8.42E-01,-2.23E-01)
  5 (-8.02E-02,-7.55E-03) ( 1.88E-02, 7.15E-03) ( 1.88E-02,-7.15E-03) (-1.33E-02, 2.53E-02) (-1.33E-02,-2.53E-02)
  6 ( 6.57E-04, 3.15E-03) ( 3.59E-03, 9.62E-03) ( 3.59E-03,-9.62E-03) (-4.13E-03,-3.26E-03) (-4.13E-03, 3.26E-03)
  7 ( 2.64E-01, 7.01E-01) (-4.67E-02, 2.67E-02) (-4.67E-02,-2.67E-02) (-1.86E-03,-2.42E-03) (-1.86E-03, 2.42E-03)
  8 ( 2.37E-02,-1.84E-02) (-2.59E-02,-8.90E-03) (-2.59E-02, 8.90E-03) ( 5.00E-04,-2.55E-04) ( 5.00E-04, 2.55E-04)
  9 (-3.09E-02,-5.75E-02) ( 3.33E-02, 4.50E-02) ( 3.33E-02,-4.50E-02) (-3.85E-02, 3.42E-02) (-3.85E-02,-3.42E-02)
 10 ( 4.03E-01,-5.04E-01) ( 7.66E-02, 8.17E-01) ( 7.66E-02,-8.17E-01) (-2.86E-01,-2.44E-01) (-2.86E-01, 2.44E-01)
```

Figure A.4.11
Spectral Decomposition - Examp[illegible]

```
-------------------------------------------------------
C.L. EIGENVECTORS     MATRIX (10 BY 10)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
-------------------------------------------------------

                             1                        2                        3                        4
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 5.731E+01, 0.        ) ( 6.234E+02, 0.        ) ( 1.999E+01,-2.946E+02) ( 1.999E+01, 2.946E+02)
 3 ( 6.835E-05, 0.        ) ( 6.695E-03,-1.800E+02) ( 2.621E-03,-2.858E+02) ( 2.621E-03, 2.858E+02)
 4 ( 5.815E-01, 1.800E+02) ( 1.152E+01,-1.800E+02) ( 9.311E-01,-2.614E+01) ( 9.311E-01, 2.614E+01)
 5 ( 3.818E+00, 0.        ) ( 4.589E+01, 0.        ) ( 2.206E+00,-1.616E+02) ( 2.206E+00, 1.616E+02)
 6 ( 8.829E-01, 1.800E+02) ( 1.382E+01,-1.800E+02) ( 3.393E+01,-7.460E+01) ( 3.393E+01, 7.460E+01)
 7 ( 3.763E+02, 1.800E+02) ( 2.483E+03,-1.800E+02) ( 4.734E+01,-6.997E+01) ( 4.734E+01, 6.997E+01)
 8 ( 8.701E+01, 0.        ) ( 7.480E+02, 0.        ) ( 7.282E+02, 1.699E+01) ( 7.282E+02,-1.699E+01)
 9 ( 4.565E+02, 0.        ) ( 2.175E+03, 0.        ) ( 1.054E+01,-2.747E+01) ( 1.054E+01, 2.747E+01)
10 ( 4.042E+03, 0.        ) ( 1.879E+03, 0.        ) ( 7.555E+01,-1.696E+02) ( 7.555E+01, 1.696E+02)

                             5                        6                        7                        8
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 7.984E+00, 9.714E+01) ( 7.984E+00,-9.714E+01) ( 2.249E+00, 1.050E+02) ( 2.249E+00,-1.050E+02)
 3 ( 5.874E-03, 5.422E+01) ( 5.874E-03,-5.422E+01) ( 2.014E-02, 3.833E+01) ( 2.014E-02,-3.833E+01)
 4 ( 8.587E-01,-1.888E+01) ( 8.587E-01, 1.888E+01) ( 8.416E-01,-2.445E+01) ( 8.416E-01, 2.445E+01)
 5 ( 6.073E+00, 1.001E+02) ( 6.073E+00,-1.001E+02) ( 9.266E-02, 1.436E+02) ( 9.266E-02,-1.436E+02)
 6 ( 2.430E-01,-1.527E+02) ( 2.430E-01, 1.527E+02) ( 4.721E-02, 1.924E+02) ( 4.721E-02,-1.924E+02)
 7 ( 5.647E+01,-1.439E+02) ( 5.647E+01, 1.439E+02) ( 2.475E-01, 2.730E+02) ( 2.475E-01,-2.730E+02)
 8 ( 2.259E+00,-3.672E+01) ( 2.259E+00, 3.672E+01) ( 1.261E-01,-3.821E+01) ( 1.261E-01, 3.821E+01)
 9 ( 4.925E+00, 4.370E+01) ( 4.925E+00,-4.370E+01) ( 2.576E-01, 1.764E+02) ( 2.576E-01,-1.764E+02)
10 ( 4.866E+01,-2.311E+01) ( 4.866E+01, 2.311E+01) ( 3.774E+00, 2.075E+02) ( 3.774E+00,-2.075E+02)

                             9                       10
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 9.023E-01,-1.040E+02) ( 9.023E-01, 1.040E+02)
 3 ( 2.674E+00,-1.436E+02) ( 2.674E+00, 1.436E+02)
 4 ( 8.448E+00, 1.413E+02) ( 8.448E+00,-1.413E+02)
 5 ( 2.769E-01, 9.393E+01) ( 2.769E-01,-9.393E+01)
 6 ( 5.097E-02,-1.655E+02) ( 5.097E-02, 1.655E+02)
 7 ( 2.957E-02,-1.514E+02) ( 2.957E-02, 1.514E+02)
 8 ( 5.444E-03,-5.087E+01) ( 5.444E-03, 5.087E+01)
 9 ( 4.993E-01, 1.145E+02) ( 4.993E-01,-1.145E+02)
10 ( 3.648E+00,-1.633E+02) ( 3.648E+00, 1.633E+02)
```

Figure A.4.11, concluded

```
---------------------------------------------------
IMPULSE   RESIDUE    MATRIX ( 7 BY 10)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
---------------------------------------------------

                         1                        2                        3                        4
 1 ( 4.269E-05, 1.800E+02) ( 3.023E-02, 1.800E+02) ( 5.680E-03,-1.255E+02) ( 5.680E-03, 1.255E+02)
 2 ( 3.828E-04,-2.117E-27) ( 5.482E-02, 1.800E+02) ( 2.126E-03, 1.341E+02) ( 2.126E-03,-1.341E+02)
 3 ( 3.772E-02, 1.800E+02) ( 2.967E+00,-4.360E-29) ( 4.564E-02,-1.343E+02) ( 4.564E-02, 1.343E+02)
 4 ( 1.041E-03,-2.117E-27) ( 5.958E-02, 1.800E+02) ( 1.009E-03, 4.843E+01) ( 1.009E-03,-4.843E+01)
 5 ( 2.315E-03,-2.117E-27) ( 2.074E-01, 1.800E+02) ( 7.661E-02,-9.932E+01) ( 7.661E-02, 9.932E+01)
 6 ( 2.281E-01, 1.800E+02) ( 1.122E+01,-4.360E-29) ( 1.645E+00,-7.726E+00) ( 1.645E+00, 7.726E+00)
 7 ( 6.278E+00, 1.800E+02) ( 5.952E+01,-4.360E-29) ( 6.715E+00,-8.919E+01) ( 6.715E+00, 8.919E+01)

                         5                        6                        7                        8
 1 ( 5.922E-01,-1.420E+02) ( 5.922E-01, 1.420E+02) ( 1.121E+01, 1.194E+02) ( 1.121E+01,-1.194E+02)
 2 ( 9.121E-02, 1.449E+02) ( 9.121E-02,-1.449E+02) ( 4.938E-01, 5.660E+01) ( 4.938E-01,-5.660E+01)
 3 ( 8.480E-01,-9.911E+01) ( 8.480E-01, 9.911E+01) ( 1.319E+00,-1.740E+02) ( 1.319E+00, 1.740E+02)
 4 ( 3.561E-02, 3.973E+01) ( 3.561E-02,-3.973E+01) ( 2.461E-01,-4.283E+01) ( 2.461E-01, 4.283E+01)
 5 ( 6.897E-01, 9.257E+01) ( 6.897E-01,-9.257E+01) ( 5.692E-01, 5.714E+01) ( 5.692E-01,-5.714E+01)
 6 ( 6.412E+00,-1.514E+02) ( 6.412E+00, 1.514E+02) ( 1.521E+00,-1.735E+02) ( 1.521E+00, 1.735E+02)
 7 ( 2.628E+01,-1.323E+02) ( 2.628E+01, 1.323E+02) ( 1.233E+01, 7.486E+01) ( 1.233E+01,-7.486E+01)

                         9                       10
 1 ( 5.433E+01,-8.367E+01) ( 5.433E+01, 8.367E+01)
 2 ( 1.809E-01,-1.587E+02) ( 1.809E-01, 1.587E+02)
 3 ( 1.932E-02,-4.410E+01) ( 1.932E-02, 4.410E+01)
 4 ( 1.981E-01,-1.549E+02) ( 1.981E-01, 1.549E+02)
 5 ( 1.763E-01,-1.576E+02) ( 1.763E-01, 1.576E+02)
 6 ( 1.883E-02,-4.297E+01) ( 1.883E-02, 4.297E+01)
 7 ( 2.356E-01,-6.520E+01) ( 2.356E-01, 6.520E+01)
```

Figure A.4.12
Impulse Residues - Example 4.1, $\rho = 10^{-4}$

```
**********LOOP FOR RHO =    .100000E-05
```

THE GG MATRIX (2 BY 10)

	1	2	3	4	5	6	7	8	9
1	-1.1385E+01	-1.9098E+00	-3.4121E-01	3.2031E-01	1.2064E+01	2.3738E-01	-9.2180E-01	3.2808E-01	-5.4607E+01
2	-4.2798E+02	-7.2918E+01	-2.1433E+01	-8.8438E-02	-7.2753E+02	7.8094E+01	-9.1527E+01	-1.0789E+01	-2.6299E+00

	10
1	-5.9174E-01
2	-9.9247E+01

THE A+BG MATRIX (10 BY 10)

	1	2	3	4	5	6	7	8	9
1	-1.2080E+00	9.4400E-01	-6.0167E-01	0.	3.3864E-01	-1.3362E-01	8.0682E-03	-4.4155E-03	-2.8840E-01
2	-7.0590E+00	-2.0180E+00	-2.1732E+00	0.	6.9773E+00	-5.9954E+00	2.8174E-01	-2.0148E-01	-1.5030E+01
3	-2.6344E-02	0.	-2.5000E-02	-3.3720E-02	0.	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
7	1.9491E+01	3.5191E+00	3.5952E-02	0.	-7.6800E+01	-1.1764E+01	-9.6220E-01	6.9965E-01	5.8872E+01
8	-2.6186E+01	-1.9336E+00	1.2791E-01	0.	9.1505E+00	-4.5650E+02	-1.5958E-01	-8.4850E-01	-2.1206E+01
9	-1.0247E+02	-1.7188E+01	-3.0709E+00	2.8828E+00	1.0857E+02	2.1364E+00	-8.2962E+00	2.9527E+00	-5.0047E+02
10	-4.2798E+03	-7.2918E+02	-2.1433E+02	-8.8438E-01	-7.2753E+03	7.8094E+02	-9.1527E+02	-1.0789E+02	-2.6299E+01

	10
1	-4.4000E-03
2	3.5310E-01
3	0.
4	0.
5	0.
6	0.
7	2.4362E+00
8	2.0252E-01
9	-5.3256E+00
10	-1.0025E+03

Figure A.4.13
Closed-loop System Matrices - Example 4.1, $\rho = 10^{-6}$

THE C+DG MATRIX (7 BY 10)

	1	2	3	4	5	6	7	8	9
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00	0.
7	2.6494E+01	-1.0067E+00	1.3504E+01	0.	4.0795E+01	-8.7153E+01	-1.5530E-01	-8.1019E-02	2.7472E+00

	10
1	0.
2	0.
3	0.
4	0.
5	0.
6	0.
7	1.9869E+00

Figure A.4.13, concluded

SPECTRAL DECOMPOSITION OF AAUG

	REAL PART	IMAGINARY PART	FREQUENCY (HERTZ)	DAMPING RATIO	TIME CONST. (SEC/RAD)
1	-9.99859E+02	0.	0.	1.00000000	1.00014E-03
2	-5.00453E+02	0.	0.	1.00000000	1.99819E-03
3	-5.35097E-01	2.13497E+01	3.39898E+00	.02505558	1.86882E+00
5	-1.80654E+00	8.60820E+00	1.39988E+00	.20538894	5.53543E-01
7	-1.49301E+00	2.36555E+00	4.45204E-01	.53373164	6.69789E-01
9	-7.51900E-03	6.78634E-02	1.08669E-02	.11012221	1.32996E+02

THE EE MATRIX (10 BY 10)

	1	2	3	4	5
1	(-7.68E-06, 0.)	(-5.17E-04, 0.)	(2.94E-04, 9.74E-04)	(2.94E-04,-9.74E-04)	(-3.36E-03,-8.17E-04)
2	(1.93E-04, 0.)	(-2.98E-02, 0.)	(-1.55E-02, 1.28E-02)	(-1.55E-02,-1.28E-02)	(1.79E-03,-8.18E-03)
3	(-2.09E-10, 0.)	(-2.32E-08, 0.)	(-2.29E-06, 1.39E-06)	(-2.29E-06,-1.39E-06)	(-3.55E-07,-1.39E-05)
4	(-1.93E-07, 0.)	(5.95E-05, 0.)	(6.18E-04, 7.11E-04)	(6.18E-04,-7.11E-04)	(-9.51E-04,-8.06E-06)
5	(-3.06E-06, 0.)	(-2.34E-04, 0.)	(1.32E-03,-2.16E-03)	(1.32E-03, 2.16E-03)	(-6.82E-03,-7.61E-02)
6	(2.17E-08, 0.)	(8.36E-05, 0.)	(4.60E-02,-5.22E-03)	(4.60E-02, 5.22E-03)	(-1.30E-04,-2.47E-04)
7	(3.06E-03, 0.)	(1.17E-01, 0.)	(4.54E-02, 2.92E-02)	(4.54E-02,-2.92E-02)	(6.68E-01, 7.88E-02)
8	(-2.17E-05, 0.)	(-4.19E-02, 0.)	(8.68E-02, 9.84E-01)	(8.68E-02,-9.84E-01)	(2.36E-03,-6.74E-04)
9	(-1.06E-02, 0.)	(-9.87E-01, 0.)	(1.14E-03, 5.58E-03)	(1.14E-03,-5.58E-03)	(-6.54E-03,-2.26E-02)
10	(-1.00E+00, 0.)	(-1.02E-01, 0.)	(-1.74E-02,-1.34E-01)	(-1.74E-02, 1.34E-01)	(-5.44E-01, 4.96E-01)

	6	7	8	9	10
1	(-3.36E-03, 8.17E-04)	(-1.80E-01, 5.50E-02)	(-1.80E-01,-5.50E-02)	(-8.10E-02,-1.25E-01)	(-8.10E-02, 1.25E-01)
2	(1.79E-03, 8.18E-03)	(-7.08E-02,-4.66E-01)	(-7.08E-02, 4.66E-01)	(2.38E-02, 4.41E-02)	(2.38E-02,-4.41E-02)
3	(-3.55E-07, 1.39E-05)	(-3.29E-03,-1.78E-03)	(-3.29E-03, 1.78E-03)	(1.75E-01, 3.13E-01)	(1.75E-01,-3.13E-01)
4	(-9.51E-04, 8.06E-06)	(-1.27E-01, 1.10E-01)	(-1.27E-01,-1.10E-01)	(6.03E-01,-4.17E-01)	(6.03E-01, 4.17E-01)
5	(-6.82E-03, 7.61E-02)	(-1.50E-03,-5.37E-04)	(-1.50E-03, 5.37E-04)	(8.26E-04,-5.01E-03)	(8.26E-04, 5.01E-03)
6	(-1.30E-04, 2.47E-04)	(9.70E-03,-1.24E-03)	(9.70E-03, 1.24E-03)	(4.03E-03, 6.54E-03)	(4.03E-03,-6.54E-03)
7	(6.68E-01,-7.88E-02)	(3.51E-03,-2.75E-03)	(3.51E-03, 2.75E-03)	(3.34E-04, 9.38E-05)	(3.34E-04,-9.38E-05)
8	(2.36E-03, 6.74E-04)	(-1.15E-02, 2.48E-02)	(-1.15E-02,-2.48E-02)	(-4.74E-04, 2.24E-04)	(-4.74E-04,-2.24E-04)
9	(-6.54E-03, 2.26E-02)	(2.93E-02, 4.23E-03)	(2.93E-02,-4.23E-03)	(1.53E-02, 1.36E-02)	(1.53E-02,-1.36E-02)
10	(-5.44E-01,-4.96E-01)	(8.37E-01, 1.06E-01)	(8.37E-01,-1.06E-01)	(2.87E-01, 4.76E-01)	(2.87E-01,-4.76E-01)

Figure A.4.14
Spectral Decomposition - Example 4.1, $\rho = 10^{-6}$

```
-----------------------------------------------------------
C.L. EIGENVECTORS    MATRIX (10 BY 10)
     OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
-----------------------------------------------------------

                        1                        2                        3                        4
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.517E+01,-1.800E+02) ( 5.761E+01, 0.        ) ( 1.976E+01, 6.722E+01) ( 1.976E+01,-6.722E+01)
 3 ( 2.720E-05, 0.        ) ( 4.488E-05, 0.        ) ( 2.635E-03, 7.549E+01) ( 2.635E-03,-7.549E+01)
 4 ( 2.517E-02, 0.        ) ( 1.151E-01,-1.800E+02) ( 9.255E-01,-2.421E+01) ( 9.255E-01, 2.421E+01)
 5 ( 3.991E-01, 0.        ) ( 4.527E-01, 0.        ) ( 2.484E+00,-1.318E+02) ( 2.484E+00, 1.318E+02)
 6 ( 2.833E-03,-1.800E+02) ( 1.619E-01,-1.800E+02) ( 4.546E+01,-7.969E+01) ( 4.546E+01, 7.969E+01)
 7 ( 3.991E+02,-1.800E+02) ( 2.265E+02,-1.800E+02) ( 5.304E+01,-4.041E+01) ( 5.304E+01, 4.041E+01)
 8 ( 2.833E+00, 0.        ) ( 8.103E+01, 0.        ) ( 9.709E+02, 1.174E+01) ( 9.709E+02,-1.174E+01)
 9 ( 1.382E+03, 0.        ) ( 1.910E+03, 0.        ) ( 5.600E+00, 5.224E+00) ( 5.600E+00,-5.224E+00)
10 ( 1.303E+05, 0.        ) ( 1.965E+02, 0.        ) ( 1.331E+02,-1.706E+02) ( 1.331E+02, 1.706E+02)

                        5                        6                        7                        8
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 2.417E+00, 8.869E+01) ( 2.417E+00,-8.869E+01) ( 2.509E+00,-2.616E+02) ( 2.509E+00, 2.616E+02)
 3 ( 4.030E-03, 7.489E+01) ( 4.030E-03,-7.489E+01) ( 1.989E-02,-3.146E+02) ( 1.989E-02, 3.146E+02)
 4 ( 2.748E-01,-1.317E+01) ( 2.748E-01, 1.317E+01) ( 8.969E-01,-2.390E+01) ( 8.969E-01, 2.390E+01)
 5 ( 2.207E+01, 7.122E+01) ( 2.207E+01,-7.122E+01) ( 8.473E-03,-3.233E+02) ( 8.473E-03, 3.233E+02)
 6 ( 8.057E-02, 4.855E+01) ( 8.057E-02,-4.855E+01) ( 5.203E-02,-1.703E+02) ( 5.203E-02, 1.703E+02)
 7 ( 1.942E+02, 1.731E+02) ( 1.942E+02,-1.731E+02) ( 2.370E-02,-2.010E+02) ( 2.370E-02, 2.010E+02)
 8 ( 7.087E-01, 1.504E+02) ( 7.087E-01,-1.504E+02) ( 1.455E-01,-4.804E+01) ( 1.455E-01, 4.804E+01)
 9 ( 6.802E+00, 6.023E+01) ( 6.802E+00,-6.023E+01) ( 1.576E-01,-1.548E+02) ( 1.576E-01, 1.548E+02)
10 ( 2.126E+02, 3.040E+02) ( 2.126E+02,-3.040E+02) ( 4.488E+00,-1.558E+02) ( 4.488E+00, 1.558E+02)

                        9                       10
 1 ( 1.000E+00, 0.        ) ( 1.000E+00, 0.        )
 2 ( 3.362E-01, 1.846E+02) ( 3.362E-01,-1.846E+02)
 3 ( 2.410E+00, 1.838E+02) ( 2.410E+00,-1.838E+02)
 4 ( 4.924E+00, 8.830E+01) ( 4.924E+00,-8.830E+01)
 5 ( 3.413E-02, 4.232E+01) ( 3.413E-02,-4.232E+01)
 6 ( 5.162E-02, 1.813E+02) ( 5.162E-02,-1.813E+02)
 7 ( 2.330E-03, 1.386E+02) ( 2.330E-03,-1.386E+02)
 8 ( 3.524E-03, 2.777E+02) ( 3.524E-03,-2.777E+02)
 9 ( 1.376E-01, 1.646E+02) ( 1.376E-01,-1.646E+02)
10 ( 3.732E+00, 1.818E+02) ( 3.732E+00,-1.818E+02)
```

Figure A.4.14, concluded

```
-------------------------------------------------
IMPULSE   RESIDUE    MATRIX ( 7 BY 10)
    OUTPUT FORM: ( MAGNITUDE , PHASE(DEG.) )
-------------------------------------------------

                               1                          2                          3                          4
 1 ( 4.579E-08, 1.800E+02) ( 2.009E-04,-1.671E-29) ( 5.127E-04,-1.149E+02) ( 5.127E-04, 1.149E+02)
 2 ( 4.466E-08, 1.800E+02) ( 5.429E-04, 1.800E+02) ( 1.897E-04, 1.454E+02) ( 1.897E-04,-1.454E+02)
 3 ( 4.465E-05,-1.408E-26) ( 2.717E-01,-1.671E-29) ( 4.052E-03,-1.232E+02) ( 4.052E-03, 1.232E+02)
 4 ( 1.729E-06,-1.408E-26) ( 5.259E-03, 1.800E+02) ( 8.413E-05, 5.724E+01) ( 8.413E-05,-5.724E+01)
 5 ( 6.585E-07,-1.408E-26) ( 1.914E-03, 1.800E+02) ( 9.542E-03,-9.347E+01) ( 9.542E-03, 9.347E+01)
 6 ( 6.584E-04, 1.800E+02) ( 9.579E-01,-1.671E-29) ( 2.038E-01,-2.032E+00) ( 2.038E-01, 2.032E+00)
 7 ( 4.662E-01, 1.800E+02) ( 2.672E+01,-1.671E-29) ( 8.008E-01,-8.427E+01) ( 8.008E-01, 8.427E+01)

                               5                          6                          7                          8
 1 ( 1.085E-02,-8.734E+01) ( 1.085E-02, 8.734E+01) ( 1.030E+00, 1.348E+02) ( 1.030E+00,-1.348E+02)
 2 ( 7.799E-04,-1.754E+02) ( 7.799E-04, 1.754E+02) ( 4.896E-02, 6.546E+01) ( 4.896E-02,-6.546E+01)
 3 ( 6.860E-03,-7.355E+01) ( 6.860E-03, 7.355E+01) ( 1.370E-01,-1.723E+02) ( 1.370E-01, 1.723E+02)
 4 ( 2.086E-03, 2.265E+01) ( 2.086E-03,-2.265E+01) ( 2.214E-02,-2.699E+01) ( 2.214E-02, 2.699E+01)
 5 ( 6.278E-02, 8.962E+01) ( 6.278E-02,-8.962E+01) ( 5.111E-02, 6.677E+01) ( 5.111E-02,-6.677E+01)
 6 ( 5.522E-01,-1.685E+02) ( 5.522E-01, 1.685E+02) ( 1.430E-01,-1.710E+02) ( 1.430E-01, 1.710E+02)
 7 ( 2.202E+00,-1.214E+02) ( 2.202E+00, 1.214E+02) ( 1.301E+00, 7.684E+01) ( 1.301E+00,-7.684E+01)

                               9                         10
 1 ( 8.885E+00,-8.532E+01) ( 8.885E+00, 8.532E+01)
 2 ( 1.913E-02, 1.792E+02) ( 1.913E-02,-1.792E+02)
 3 ( 1.306E-03,-8.447E+01) ( 1.306E-03, 8.447E+01)
 4 ( 1.941E-02,-1.693E+02) ( 1.941E-02, 1.693E+02)
 5 ( 1.905E-02, 1.789E+02) ( 1.905E-02,-1.789E+02)
 6 ( 1.301E-03,-8.479E+01) ( 1.301E-03, 8.479E+01)
 7 ( 3.101E-02,-8.037E+01) ( 3.101E-02, 8.037E+01)
```

Figure A.4.15
Impulse Residues - Example 4.1, $\rho = 10^{-6}$

Appendix A.5
Chapter V Examples Data

```
**********LOOP FOR RHO =   .1000000E-05
++++++++++         KAY =   .1000000E+07
```

THE FF MATRIX (10 BY 4)

	1	2	3	4
1	2.7235E-05	-7.3515E-06	5.5527E-04	1.9177E-04
2	1.1394E-06	2.7897E-06	-8.3778E-05	-3.7862E-05
3	-6.1292E-05	1.5506E-05	-1.1538E-03	-3.7326E-04
4	4.9684E-04	-1.7502E-06	4.1765E-04	1.1010E-04
5	6.5488E-06	-7.4130E-07	1.5761E-04	5.1162E-05
6	-2.0227E-06	3.1276E-07	-7.5731E-05	1.9955E-05
7	1.7994E-06	-5.0917E-05	-7.0622E-05	-1.4734E-05
8	6.6159E-07	-2.5969E-04	4.1405E-06	1.2533E-04
9	-2.2608E-08	-2.0313E-07	8.7718E-07	1.4042E-06
10	-5.3053E-10	-5.2612E-09	1.2839E-07	3.3101E-08

THE AAUG MATRIX (20 BY 20)

	1	2	3	4	5	6	7	8	9
1	-1.2080E+00	9.4400E-01	-6.0167E-01	0.	3.3864E-01	-1.3362E-01	8.0682E-03	-4.4155E-03	-2.8840E-01
2	-7.0590E+00	-2.0180E+00	-2.1732E+00	0.	6.9773E+00	-5.9954E+00	2.8174E-01	-2.0148E-01	-1.5030E+01
3	-2.6344E-02	0.	-2.5000E-02	-3.3720E-02	0.	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
7	1.9491E+01	3.5191E+00	3.5952E-02	0.	-7.6800E+01	-1.1764E+01	-9.6220E-01	6.9965E-01	5.8872E+01
8	-2.6186E+01	-1.9336E+00	1.2791E-01	0.	9.1505E+00	-4.5650E+02	-1.5958E-01	-8.4850E-01	-2.1206E+01
9	-1.0247E+02	-1.7188E+01	-3.0709E+00	2.8828E+00	1.0857E+02	2.1364E+00	-8.2962E+00	2.9527E+00	-5.0047E+02
10	-4.2798E+03	-7.2918E+02	-2.1433E+02	-8.8438E-01	-7.2753E+03	7.8094E+02	-9.1527E+02	-1.0789E+02	-2.6299E+01
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	0.	0.	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	0.	0.	0.	0.	0.	0.	0.
17	0.	0.	0.	0.	0.	0.	0.	0.	0.
18	0.	0.	0.	0.	0.	0.	0.	0.	0.
19	0.	0.	0.	0.	0.	0.	0.	0.	0.
20	0.	0.	0.	0.	0.	0.	0.	0.	0.

Figure A.5.1
Augmented System Matrices - Example 5.1, $\kappa = 10^6$

	10	11	12	13	14	15	16	17	18
1	-4.4000E-03	0.	0.	0.	0.	0.	0.	0.	0.
2	3.5310E-01	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	2.4362E+00	0.	0.	0.	0.	0.	0.	0.	0.
8	2.0252E-01	0.	0.	0.	0.	0.	0.	0.	0.
9	-5.3256E+00	1.0247E+02	1.7188E+01	3.0709E+00	-2.8828E+00	-1.0857E+02	-2.1364E+00	8.2962E+00	-2.9527E+00
10	-1.0025E+03	4.2798E+03	7.2918E+02	2.1433E+02	8.8438E-01	7.2753E+03	-7.8094E+02	9.1527E+02	1.0789E+02
11	0.	-1.2322E+00	9.4371E-01	-6.1293E-01	-2.7235E-05	3.1574E-01	-9.5934E-02	8.2814E-03	-4.4866E-03
12	0.	-7.0549E+00	-2.0179E+00	-2.1713E+00	-1.1394E-06	6.9807E+00	-6.0006E+00	2.8171E-01	-2.0146E-01
13	0.	2.2639E-02	4.9438E-04	-2.0860E-03	-3.3658E-02	4.7543E-02	-7.9733E-02	-4.2525E-04	1.3371E-04
14	0.	-1.6497E-02	9.9993E-01	-7.8029E-03	-4.9684E-04	-1.6699E-02	3.0735E-02	1.4027E-04	-3.0356E-05
15	0.	-6.6995E-03	-6.9696E-05	-3.1334E-03	-6.5488E-06	-6.4963E-03	1.0880E-02	1.0001E+00	-1.6987E-05
16	0.	1.0220E-03	-1.6591E-04	6.3068E-04	2.0227E-06	3.0590E-03	-7.7189E-03	2.5354E-06	9.9998E-01
17	0.	1.9494E+01	3.5192E+00	3.7195E-02	-1.7994E-06	-7.6797E+01	-1.1769E+01	-9.6227E-01	6.9960E-01
18	0.	-2.6192E+01	-1.9339E+00	1.2539E-01	-6.6159E-07	9.1502E+00	-4.5651E+02	-1.5975E-01	-8.4883E-01
19	0.	-9.2508E-05	-5.2017E-06	-3.9430E-05	2.2608E-08	-3.7813E-05	-2.1523E-06	9.1708E-07	-9.2841E-07
20	0.	-5.0345E-06	-1.3711E-08	-2.3841E-06	5.3053E-10	-5.2856E-06	9.3368E-06	3.7874E-08	-1.3631E-08

	19	20
1	0.	0.
2	0.	0.
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	0.	0.
8	0.	0.
9	4.9147E+02	5.3256E+00
10	2.6299E+01	9.9247E+02
11	-2.9725E-01	-5.5771E-03
12	-1.5028E+01	3.5328E-01
13	1.7421E-02	2.4363E-03
14	-5.3512E-03	-8.7224E-04
15	-2.3864E-03	-3.3286E-04
16	-5.5387E-04	1.4278E-04
17	5.8873E+01	2.4363E+00
18	-2.1211E+01	2.0247E-01
19	-9.0001E+00	-2.2838E-06
20	-1.6165E-06	-1.0000E+01

Figure A.5.1, continued

THE BBAUG MATRIX (20 BY 20)

	1	2
1	0.	0.
2	0.	0.
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	0.	0.
8	0.	0.
9	9.0000E+00	0.
10	0.	1.0000E+01
11	0.	0.
12	0.	0.
13	0.	0.
14	0.	0.
15	0.	0.
16	0.	0.
17	0.	0.
18	0.	0.
19	0.	0.
20	0.	0.

Figure A.5.1, continued

```
THE CCAUG      MATRIX (  7 BY 20 )
```

	1	2	3	4	5	6	7	8	9
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00	0.
7	2.6494E+01	-1.0067E+00	1.3504E+01	0.	4.0795E+01	-8.7153E+01	-1.5530E-01	-8.1019E-02	2.7472E+00

	10	11	12	13	14	15	16	17	18
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	1.9869E+00	0.	0.	0.	0.	0.	0.	0.	0.

	19	20
1	0.	0.
2	0.	0.
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	0.	0.

Figure A.5.1, concluded

SPECTRAL DECOMPOSITION OF AAUG

	REAL PART	IMAGINARY PART	FREQUENCY (HERTZ)	DAMPING RATIO	TIME CONST. (SEC/RAD)
1	-9.99859E+02	0.	0.	1.00000000	1.00014E-03
2	-5.00453E+02	0.	0.	1.00000000	1.99819E-03
3	-5.35097E-01	2.13497E+01	3.39898E+00	.02505558	1.86882E+00
5	-1.80654E+00	8.60820E+00	1.39988E+00	.20538894	5.53543E-01
7	-1.49301E+00	2.36555E+00	4.45204E-01	.53373164	6.69789E-01
9	-7.51900E-03	6.78634E-02	1.08669E-02	.11012221	1.32996E+02
11	-4.59449E-01	2.13505E+01	3.39883E+00	.02151434	2.17652E+00
13	-7.26793E-01	8.75844E+00	1.39874E+00	.08269785	1.37591E+00
15	-1.34858E+00	2.19269E+00	4.09698E-01	.52387907	7.41523E-01
17	-9.00007E+00	0.	0.	1.00000000	1.11110E-01
18	-4.17146E-03	5.25125E-02	8.38396E-03	.07918792	2.39724E+02
20	-1.00000E+01	0.	0.	1.00000000	1.00000E-01

Figure A.5.2
Spectral Decomposition - Examp

THE EE MATRIX (20 BY 20)

	1	2	3	4	5
1	(1.20E-04, 0.)	(-4.13E-04, 0.)	(5.55E-04, 1.13E-03)	(5.55E-04,-1.13E-03)	(-3.64E-03, 3.84E-03)
2	(-3.01E-03, 0.)	(-2.38E-02, 0.)	(-1.64E-02, 1.88E-02)	(-1.64E-02,-1.88E-02)	(-9.49E-03,-8.59E-03)
3	(3.25E-09, 0.)	(-1.86E-08, 0.)	(-2.52E-06, 2.16E-06)	(-2.52E-06,-2.16E-06)	(-1.88E-05,-1.01E-05)
4	(3.01E-06, 0.)	(4.76E-05, 0.)	(8.97E-04, 7.43E-04)	(8.97E-04,-7.43E-04)	(-7.34E-04, 1.26E-03)
5	(4.78E-05, 0.)	(-1.87E-04, 0.)	(1.17E-03,-2.90E-03)	(1.17E-03, 2.90E-03)	(-1.06E-01,-4.88E-02)
6	(-3.39E-07, 0.)	(6.69E-05, 0.)	(5.51E-02,-1.56E-02)	(5.51E-02, 1.56E-02)	(-4.26E-04,-1.49E-05)
7	(-4.78E-02, 0.)	(9.37E-02, 0.)	(6.13E-02, 2.66E-02)	(6.13E-02,-2.66E-02)	(6.12E-01,-8.26E-01)
8	(3.39E-04, 0.)	(-3.35E-02, 0.)	(3.04E-01, 1.18E+00)	(3.04E-01,-1.18E+00)	(8.98E-04,-3.64E-03)
9	(1.65E-01, 0.)	(-7.90E-01, 0.)	(2.52E-03, 6.59E-03)	(2.52E-03,-6.59E-03)	(-3.50E-02,-8.52E-03)
10	(1.56E+01, 0.)	(-8.13E-02, 0.)	(-4.83E-02,-1.61E-01)	(-4.83E-02, 1.61E-01)	(2.45E-01, 1.10E+00)
11	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
12	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
13	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
14	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
15	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
16	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
17	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
18	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
19	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
20	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)

	6	7	8	9	10
1	(-3.64E-03,-3.84E-03)	(-1.91E-01,-2.11E-01)	(-1.91E-01, 2.11E-01)	(-7.84E-02,-9.24E-02)	(-7.84E-02, 9.24E-02)
2	(-9.49E-03, 8.59E-03)	(5.94E-01,-3.97E-01)	(5.94E-01, 3.97E-01)	(2.38E-02, 3.31E-02)	(2.38E-02,-3.31E-02)
3	(-1.88E-05, 1.01E-05)	(3.25E-04,-5.65E-03)	(3.25E-04, 5.65E-03)	(1.74E-01, 2.35E-01)	(1.74E-01,-2.35E-01)
4	(-7.34E-04,-1.26E-03)	(-2.33E-01,-1.04E-01)	(-2.33E-01, 1.04E-01)	(4.43E-01,-3.99E-01)	(4.43E-01, 3.99E-01)
5	(-1.06E-01, 4.88E-02)	(-2.28E-04,-2.40E-03)	(-2.28E-04, 2.40E-03)	(1.44E-04,-4.13E-03)	(1.44E-04, 4.13E-03)
6	(-4.26E-04, 1.49E-05)	(7.94E-03, 1.25E-02)	(7.94E-03,-1.25E-02)	(3.94E-03, 4.86E-03)	(3.94E-03,-4.86E-03)
7	(6.12E-01, 8.26E-01)	(6.02E-03, 3.05E-03)	(6.02E-03,-3.05E-03)	(2.79E-04, 4.09E-05)	(2.79E-04,-4.09E-05)
8	(8.98E-04, 3.64E-03)	(-4.15E-02, 1.05E-04)	(-4.15E-02,-1.05E-04)	(-3.60E-04, 2.30E-04)	(-3.60E-04,-2.30E-04)
9	(-3.50E-02, 8.52E-03)	(1.30E-02, 4.29E-02)	(1.30E-02,-4.29E-02)	(1.38E-02, 9.40E-03)	(1.38E-02,-9.40E-03)
10	(2.45E-01,-1.10E+00)	(3.93E-01, 1.22E+00)	(3.93E-01,-1.22E+00)	(2.81E-01, 3.54E-01)	(2.81E-01,-3.54E-01)
11	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
12	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
13	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
14	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
15	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
16	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
17	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
18	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
19	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)
20	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)	(0. , 0.)

Figure A.5.2, continued

	11	12	13	14	15
1	(2.90E-04,-8.15E-04)	(2.90E-04, 8.15E-04)	(5.52E-03,-1.95E-03)	(5.52E-03, 1.95E-03)	(-2.45E-04,-1.28E-02)
2	(1.72E-02,-2.41E-03)	(1.72E-02, 2.41E-03)	(2.99E-02, 3.40E-02)	(2.99E-02,-3.40E-02)	(2.93E-02, 2.44E-03)
3	(2.28E-06, 1.06E-07)	(2.28E-06,-1.06E-07)	(2.25E-05, 2.86E-05)	(2.25E-05,-2.86E-05)	(1.91E-04,-1.97E-04)
4	(-1.30E-04,-8.03E-04)	(-1.30E-04, 8.03E-04)	(3.58E-03,-3.72E-03)	(3.58E-03, 3.72E-03)	(-5.15E-03,-1.02E-02)
5	(-2.74E-03, 6.66E-04)	(-2.74E-03,-6.66E-04)	(-1.54E-02, 4.98E-02)	(-1.54E-02,-4.98E-02)	(1.53E-03,-3.35E-03)
6	(-4.12E-02,-3.18E-02)	(-4.12E-02, 3.18E-02)	(-7.42E-04, 1.22E-03)	(-7.42E-04,-1.22E-03)	(-8.77E-05, 6.58E-04)
7	(-1.30E-02,-5.88E-02)	(-1.30E-02, 5.88E-02)	(-4.25E-01,-1.71E-01)	(-4.25E-01, 1.71E-01)	(5.30E-03, 7.87E-03)
8	(6.98E-01,-8.65E-01)	(6.98E-01, 8.65E-01)	(-1.01E-02,-7.39E-03)	(-1.01E-02, 7.39E-03)	(-1.33E-03,-1.08E-03)
9	(1.31E-04,-2.03E-04)	(1.31E-04, 2.03E-04)	(2.04E-06, 3.93E-05)	(2.04E-06,-3.93E-05)	(-3.01E-06, 2.42E-06)
10	(-3.89E-03, 5.04E-03)	(-3.89E-03,-5.04E-03)	(2.81E-04,-9.88E-04)	(2.81E-04, 9.88E-04)	(-1.15E-04, 3.46E-05)
11	(2.20E-04,-7.05E-04)	(2.20E-04, 7.05E-04)	(5.39E-03,-1.96E-03)	(5.39E-03, 1.96E-03)	(-2.04E-04,-1.27E-02)
12	(1.64E-02,-2.50E-03)	(1.64E-02, 2.50E-03)	(3.00E-02, 3.40E-02)	(3.00E-02,-3.40E-02)	(2.89E-02, 2.18E-03)
13	(1.09E-04,-1.50E-04)	(1.09E-04, 1.50E-04)	(2.80E-04, 3.11E-05)	(2.80E-04,-3.11E-05)	(-1.06E-04,-5.21E-05)
14	(-1.75E-04,-7.06E-04)	(-1.75E-04, 7.06E-04)	(3.48E-03,-3.72E-03)	(3.48E-03, 3.72E-03)	(-5.07E-03,-1.01E-02)
15	(-2.65E-03, 6.63E-04)	(-2.65E-03,-6.63E-04)	(-1.54E-02, 4.96E-02)	(-1.54E-02,-4.96E-02)	(1.52E-03,-3.35E-03)
16	(-3.96E-02,-3.06E-02)	(-3.96E-02, 3.06E-02)	(-7.38E-04, 1.22E-03)	(-7.38E-04,-1.22E-03)	(-8.87E-05, 6.56E-04)
17	(-1.25E-02,-5.66E-02)	(-1.25E-02, 5.66E-02)	(-4.24E-01,-1.71E-01)	(-4.24E-01, 1.71E-01)	(5.31E-03, 7.75E-03)
18	(6.71E-01,-8.32E-01)	(6.71E-01, 8.32E-01)	(-1.01E-02,-7.48E-03)	(-1.01E-02, 7.48E-03)	(-1.32E-03,-1.05E-03)
19	(2.51E-08, 3.62E-08)	(2.51E-08,-3.62E-08)	(-1.49E-07,-8.77E-08)	(-1.49E-07, 8.77E-08)	(2.34E-08, 1.63E-07)
20	(-1.72E-08, 9.52E-09)	(-1.72E-08,-9.52E-09)	(-1.16E-08,-1.58E-08)	(-1.16E-08, 1.58E-08)	(1.59E-09, 9.80E-09)

	16	17	18	19	20
1	(-2.45E-04, 1.28E-02)	(3.65E-05, 0.)	(-2.07E-02, 5.23E-03)	(-2.07E-02,-5.23E-03)	(1.25E-06, 0.)
2	(2.93E-02,-2.44E-03)	(-3.39E-04, 0.)	(3.66E-03,-1.09E-03)	(3.66E-03, 1.09E-03)	(-1.15E-05, 0.)
3	(1.91E-04, 1.97E-04)	(2.49E-07, 0.)	(4.46E-02,-9.46E-03)	(4.46E-02, 9.46E-03)	(7.19E-09, 0.)
4	(-5.15E-03, 1.02E-02)	(3.76E-05, 0.)	(-2.61E-02,-6.77E-02)	(-2.61E-02, 6.77E-02)	(1.15E-06, 0.)
5	(1.53E-03, 3.35E-03)	(-8.32E-05, 0.)	(-4.50E-03, 1.82E-03)	(-4.50E-03,-1.82E-03)	(3.47E-06, 0.)
6	(-8.77E-05,-6.58E-04)	(5.80E-06, 0.)	(1.08E-03,-2.68E-04)	(1.08E-03, 2.68E-04)	(1.39E-07, 0.)
7	(5.30E-03,-7.87E-03)	(7.49E-04, 0.)	(-7.68E-05,-2.44E-04)	(-7.68E-05, 2.44E-04)	(-3.47E-05, 0.)
8	(-1.33E-03, 1.08E-03)	(-5.22E-05, 0.)	(9.57E-06, 5.77E-05)	(9.57E-06,-5.77E-05)	(-1.39E-06, 0.)
9	(-3.01E-06,-2.42E-06)	(-2.01E-04, 0.)	(5.24E-04, 2.33E-04)	(5.24E-04,-2.33E-04)	(5.34E-09, 0.)
10	(-1.15E-04,-3.46E-05)	(-2.20E-06, 0.)	(1.05E-02, 1.02E-02)	(1.05E-02,-1.02E-02)	(2.45E-04, 0.)
11	(-2.04E-04, 1.27E-02)	(3.62E-05, 0.)	(-1.78E-02, 7.87E-03)	(-1.78E-02,-7.87E-03)	(1.29E-06, 0.)
12	(2.89E-02,-2.18E-03)	(-3.39E-04, 0.)	(2.76E-03,-2.05E-03)	(2.76E-03, 2.05E-03)	(-1.15E-05, 0.)
13	(-1.06E-04, 5.21E-05)	(9.84E-07, 0.)	(3.81E-02,-1.63E-02)	(3.81E-02, 1.63E-02)	(-7.50E-08, 0.)
14	(-5.07E-03, 1.01E-02)	(3.74E-05, 0.)	(-4.36E-02,-5.15E-02)	(-4.36E-02, 5.15E-02)	(1.18E-06, 0.)
15	(1.52E-03, 3.35E-03)	(-8.33E-05, 0.)	(-4.51E-03, 1.96E-03)	(-4.51E-03,-1.96E-03)	(3.47E-06, 0.)
16	(-8.87E-05,-6.56E-04)	(5.82E-06, 0.)	(9.29E-04,-4.08E-04)	(9.29E-04, 4.08E-04)	(1.36E-07, 0.)
17	(5.31E-03,-7.75E-03)	(7.49E-04, 0.)	(-1.23E-04,-2.27E-04)	(-1.23E-04, 2.27E-04)	(-3.46E-05, 0.)
18	(-1.32E-03, 1.05E-03)	(-5.23E-05, 0.)	(3.32E-05, 4.34E-05)	(3.32E-05,-4.34E-05)	(-1.41E-06, 0.)
19	(2.34E-08,-1.63E-07)	(-2.01E-04, 0.)	(3.29E-08,-1.67E-08)	(3.29E-08, 1.67E-08)	(7.79E-10, 0.)
20	(1.59E-09,-9.80E-09)	(6.68E-10, 0.)	(3.11E-09,-1.50E-09)	(3.11E-09, 1.50E-09)	(2.45E-04, 0.)

Figure A.5.2, concluded

```
**********LOOP FOR RHO =    .1000000E-05
++++++++++         KAY =    .1000000E-05
```

THE FF MATRIX (10 BY 4)

	1	2	3	4
1	9.1390E+01	8.8448E+01	7.7909E+02	7.0446E+02
2	2.5098E+02	2.6253E+02	-1.7616E+01	6.4984E+02
3	3.2568E+02	-3.8346E+02	-7.1592E+02	4.6602E+02
4	6.2073E+02	7.9349E+02	1.8542E+01	-7.1920E+01
5	-7.0734E+02	5.9817E+02	4.5782E+02	2.8076E+01
6	-2.5818E+02	2.3696E+02	-7.9613E+02	4.9552E+02
7	-8.7308E+02	1.4480E+02	2.4803E+03	-9.5732E+02
8	1.1657E+03	-1.7082E+03	-2.4985E+03	1.5508E+03
9	4.0898E+01	-2.8806E+02	1.4778E+02	2.1874E+02
10	-4.3232E+00	1.2159E+01	2.1559E+01	4.6039E-01

THE AAUG MATRIX (20 BY 20)

	1	2	3	4	5	6	7	8	9
1	-1.2080E+00	9.4400E-01	-6.0167E-01	0.	3.3864E-01	-1.3362E-01	8.0682E-03	-4.4155E-03	-2.8840E-01
2	-7.0590E+00	-2.0180E+00	-2.1732E+00	0.	6.9773E+00	-5.9954E+00	2.8174E-01	-2.0148E-01	-1.5030E+01
3	-2.6344E-02	0.	-2.5000E-02	-3.3720E-02	0.	0.	0.	0.	0.
4	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	1.0000E+00	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	1.0000E+00	0.
7	1.9491E+01	3.5191E+00	3.5952E-02	0.	-7.6800E+01	-1.1764E+01	-9.6220E-01	6.9965E-01	5.8872E+01
8	-2.6186E+01	-1.9336E+00	1.2791E-01	0.	9.1505E+00	-4.5650E+02	-1.5958E-01	-8.4850E-01	-2.1206E+01
9	-1.0247E+02	-1.7188E+01	-3.0709E+00	2.8828E+00	1.0857E+02	2.1364E+00	-8.2962E+00	2.9527E+00	-5.0047E+02
10	-4.2798E+03	-7.2918E+02	-2.1433E+02	-8.8438E-01	-7.2753E+03	7.8094E+02	-9.1527E+02	-1.0789E+02	-2.6299E+01
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	0.	0.	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	0.	0.	0.	0.	0.	0.	0.
17	0.	0.	0.	0.	0.	0.	0.	0.	0.
18	0.	0.	0.	0.	0.	0.	0.	0.	0.
19	0.	0.	0.	0.	0.	0.	0.	0.	0.
20	0.	0.	0.	0.	0.	0.	0.	0.	0.

Figure A.5.3
Augmented System Matrices - Example 5.1, $\kappa = 10^{-6}$

	10	11	12	13	14	15	16	17	18
1	-4.4000E-03	0.	0.	0.	0.	0.	0.	0.	0.
2	3.5310E-01	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	2.4362E+00	0.	0.	0.	0.	0.	0.	0.	0.
8	2.0252E-01	0.	0.	0.	0.	0.	0.	0.	0.
9	-5.3256E+00	1.0247E+02	1.7188E+01	3.0709E+00	-2.8828E+00	-1.0857E+02	-2.1364E+00	8.2962E+00	-2.9527E+00
10	-1.0025E+03	4.2798E+03	7.2918E+02	2.1433E+02	8.8438E-01	7.2753E+03	-7.8094E+02	9.1527E+02	1.0789E+02
11	0.	-5.5393E+04	-2.4578E+03	-2.4360E+04	-9.1390E+01	-3.2698E+04	2.8569E+04	7.0311E+02	-2.4795E+02
12	0.	-3.1597E+04	-3.1923E+03	-1.2530E+04	-2.5098E+02	4.8632E+01	-3.7656E+04	7.1546E+02	-1.0764E+02
13	0.	-4.0215E+03	-2.4241E+03	5.1320E+02	-3.2572E+02	2.8866E+04	-8.8147E+04	-1.6807E+02	-7.0575E+02
14	0.	3.0566E+03	-4.5176E+02	1.1624E+03	-6.2073E+02	-3.2988E+01	6.2614E+03	7.4597E+02	8.3578E+02
15	0.	-1.3515E+04	-2.6300E+02	-6.7341E+03	7.0734E+02	-1.9425E+04	3.7622E+04	6.8994E+02	6.1934E+02
16	0.	-3.3511E+03	-3.2573E+03	1.0171E+03	2.5818E+02	3.1513E+04	-9.7373E+04	4.6055E+02	-1.0757E+02
17	0.	-1.8469E+04	6.6426E+03	-1.4689E+04	8.7308E+02	-1.0077E+05	2.6885E+05	-1.4182E+02	8.8937E+02
18	0.	-1.0329E+04	-7.7533E+03	3.2773E+03	-1.1657E+03	1.0089E+05	-3.0383E+05	-1.0097E+03	-2.7910E+03
19	0.	-1.4706E+04	-5.4268E+02	-6.2927E+03	-4.0898E+01	-6.3004E+03	6.7993E+02	-1.1183E+02	-4.0014E+02
20	0.	-5.9391E+02	7.4830E+00	-3.0019E+02	4.3232E+00	-8.8451E+02	1.8489E+03	1.5830E+01	1.3645E+01

	19	20
1	0.	0.
2	0.	0.
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	0.	0.
8	0.	0.
9	4.9147E+02	5.3256E+00
10	2.6299E+01	9.9247E+02
11	-2.9038E+04	-1.8193E+03
12	-2.4778E+04	-2.1496E+02
13	-1.5826E+04	1.2429E+03
14	2.6950E+03	-9.1372E+00
15	-2.3297E+03	-9.2046E+02
16	-1.6732E+04	1.3910E+03
17	2.9797E+04	-4.5569E+03
18	-5.2367E+04	4.3672E+03
19	-8.7667E+03	-3.7789E+02
20	-7.6806E+01	-5.3014E+01

Figure A.5.3, continued

```
----------------------------------
THE BBAUG      MATRIX ( 20 BY 20 )
----------------------------------

                 1             2
  1   0.           0.
  2   0.           0.
  3   0.           0.
  4   0.           0.
  5   0.           0.
  6   0.           0.
  7   0.           0.
  8   0.           0.
  9   9.0000E+00   0.
 10   0.           1.0000E+01
 11   0.           0.
 12   0.           0.
 13   0.           0.
 14   0.           0.
 15   0.           0.
 16   0.           0.
 17   0.           0.
 18   0.           0.
 19   0.           0.
 20   0.           0.
```

Figure A.5.3, continued

THE CCAUG MATRIX (7 BY 20)

	1	2	3	4	5	6	7	8	9
1	0.	0.	9.4900E+02	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
3	0.	1.0000E+00	0.	0.	0.	0.	0.	0.	0.
4	-1.0000E+00	0.	0.	1.0000E+00	0.	0.	0.	0.	0.
5	0.	0.	0.	1.0000E+00	-1.0000E+00	-1.0000E+00	0.	0.	0.
6	0.	1.0000E+00	0.	0.	0.	0.	-1.0000E+00	-1.0000E+00	0.
7	2.6494E+01	-1.0067E+00	1.3504E+01	0.	4.0795E+01	-8.7153E+01	-1.5530E-01	-8.1019E-02	2.7472E+00

	10	11	12	13	14	15	16	17	18
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	1.9869E+00	0.	0.	0.	0.	0.	0.	0.	0.

	19	20
1	0.	0.
2	0.	0.
3	0.	0.
4	0.	0.
5	0.	0.
6	0.	0.
7	0.	0.

Figure A.5.3, concluded

SPECTRAL DECOMPOSITION OF AAUG

	REAL PART	IMAGINARY PART	FREQUENCY (HERTZ)	DAMPING RATIO	TIME CONST. (SEC/RAD)
1	-9.99859E+02	0.	0.	1.00000000	1.00014E-03
2	-5.00453E+02	0.	0.	1.00000000	1.99819E-03
3	-5.35097E-01	2.13497E+01	3.39898E+00	.02505558	1.86882E+00
5	-1.80654E+00	8.60820E+00	1.39988E+00	.20538894	5.53543E-01
7	-1.49301E+00	2.36555E+00	4.45204E-01	.53373164	6.69789E-01
9	-7.51900E-03	6.78634E-02	1.08669E-02	.11012221	1.32996E+02
11	-1.10555E+05	0.	0.	1.00000000	9.04524E-06
12	-7.32565E+04	0.	0.	1.00000000	1.36507E-05
13	-1.60613E+03	0.	0.	1.00000000	6.22614E-04
14	-1.70565E+03	0.	0.	1.00000000	5.86287E-04
15	-9.69527E+01	0.	0.	1.00000000	1.03143E-02
16	-2.24419E+00	2.03507E+00	4.82160E-01	.74077843	4.45595E-01
18	-6.20948E+00	0.	0.	1.00000000	1.61044E-01
19	-9.97416E+00	0.	0.	1.00000000	1.00259E-01
20	-1.56634E+00	0.	0.	1.00000000	6.38432E-01

Figure A.5.4
Spectral Decomposition - Example 5.1, $\kappa = 10^{-6}$

```
---------------------------------
THE EE        MATRIX ( 20 BY 20 )
---------------------------------

                        1                       2                        3                        4                        5
 1 ( 1.20E-04, 0.        ) (-4.13E-04, 0.        ) ( 5.55E-04, 1.13E-03) ( 5.55E-04,-1.13E-03) (-3.64E-03, 3.84E-03)
 2 (-3.01E-03, 0.        ) (-2.38E-02, 0.        ) (-1.64E-02, 1.88E-02) (-1.64E-02,-1.88E-02) (-9.49E-03,-8.59E-03)
 3 ( 3.25E-09, 0.        ) (-1.86E-08, 0.        ) (-2.52E-06, 2.16E-06) (-2.52E-06,-2.16E-06) (-1.88E-05,-1.01E-05)
 4 ( 3.01E-06, 0.        ) ( 4.76E-05, 0.        ) ( 8.97E-04, 7.43E-04) ( 8.97E-04,-7.43E-04) (-7.34E-04, 1.26E-03)
 5 ( 4.78E-05, 0.        ) (-1.87E-04, 0.        ) ( 1.17E-03,-2.90E-03) ( 1.17E-03, 2.90E-03) (-1.06E-01,-4.88E-02)
 6 (-3.39E-07, 0.        ) ( 6.69E-05, 0.        ) ( 5.51E-02,-1.56E-02) ( 5.51E-02, 1.56E-02) (-4.26E-04,-1.49E-05)
 7 (-4.78E-02, 0.        ) ( 9.37E-02, 0.        ) ( 6.13E-02, 2.66E-02) ( 6.13E-02,-2.66E-02) ( 6.12E-01,-8.26E-01)
 8 ( 3.39E-04, 0.        ) (-3.35E-02, 0.        ) ( 3.04E-01, 1.18E+00) ( 3.04E-01,-1.18E+00) ( 8.98E-04,-3.64E-03)
 9 ( 1.65E-01, 0.        ) (-7.90E-01, 0.        ) ( 2.52E-03, 6.59E-03) ( 2.52E-03,-6.59E-03) (-3.50E-02,-8.52E-03)
10 ( 1.56E+01, 0.        ) (-8.13E-02, 0.        ) (-4.83E-02,-1.61E-01) (-4.83E-02, 1.61E-01) ( 2.45E-01, 1.10E+00)
11 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
12 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
13 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
14 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
15 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
16 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
17 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
18 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
19 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
20 ( 0.       , 0.       ) ( 0.       , 0.       ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )

                        6                       7                        8                        9                       10
 1 (-3.64E-03,-3.84E-03) (-1.91E-01,-2.11E-01) (-1.91E-01, 2.11E-01) (-7.84E-02,-9.24E-02) (-7.84E-02, 9.24E-02)
 2 (-9.49E-03, 8.59E-03) ( 5.94E-01,-3.97E-01) ( 5.94E-01, 3.97E-01) ( 2.38E-02, 3.31E-02) ( 2.38E-02,-3.31E-02)
 3 (-1.88E-05, 1.01E-05) ( 3.25E-04,-5.65E-03) ( 3.25E-04, 5.65E-03) ( 1.74E-01, 2.35E-01) ( 1.74E-01,-2.35E-01)
 4 (-7.34E-04,-1.26E-03) (-2.33E-01,-1.04E-01) (-2.33E-01, 1.04E-01) ( 4.43E-01,-3.99E-01) ( 4.43E-01, 3.99E-01)
 5 (-1.06E-01, 4.88E-02) (-2.28E-04,-2.40E-03) (-2.28E-04, 2.40E-03) ( 1.44E-04,-4.13E-03) ( 1.44E-04, 4.13E-03)
 6 (-4.26E-04, 1.49E-05) ( 7.94E-03, 1.25E-02) ( 7.94E-03,-1.25E-02) ( 3.94E-03, 4.86E-03) ( 3.94E-03,-4.86E-03)
 7 ( 6.12E-01, 8.26E-01) ( 6.02E-03, 3.05E-03) ( 6.02E-03,-3.05E-03) ( 2.79E-04, 4.09E-05) ( 2.79E-04,-4.09E-05)
 8 ( 8.98E-04, 3.64E-03) (-4.15E-02, 1.05E-04) (-4.15E-02,-1.05E-04) (-3.60E-04, 2.30E-04) (-3.60E-04,-2.30E-04)
 9 (-3.50E-02, 8.52E-03) ( 1.30E-02, 4.29E-02) ( 1.30E-02,-4.29E-02) ( 1.38E-02, 9.40E-03) ( 1.38E-02,-9.40E-03)
10 ( 2.45E-01,-1.10E+00) ( 3.93E-01, 1.22E+00) ( 3.93E-01,-1.22E+00) ( 2.81E-01, 3.54E-01) ( 2.81E-01,-3.54E-01)
11 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
12 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
13 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
14 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
15 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
16 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
17 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
18 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
19 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
20 ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      ) ( 0.       , 0.      )
```

Figure A.5.4, continued

	11	12	13	14	15
1	(-2.50E-09, 0.)	(1.57E-08, 0.)	(-1.03E-05, 0.)	(5.43E-07, 0.)	(5.27E-03, 0.)
2	(1.60E-07, 0.)	(1.13E-07, 0.)	(2.98E-03, 0.)	(-9.73E-03, 0.)	(1.67E+00, 0.)
3	(-5.97E-16, 0.)	(5.64E-15, 0.)	(-2.09E-10, 0.)	(1.21E-10, 0.)	(-4.55E-06, 0.)
4	(-1.45E-12, 0.)	(-1.54E-12, 0.)	(-1.86E-06, 0.)	(5.70E-06, 0.)	(-1.72E-02, 0.)
5	(-1.16E-11, 0.)	(6.91E-11, 0.)	(-6.72E-06, 0.)	(1.41E-05, 0.)	(3.15E-02, 0.)
6	(-7.12E-13, 0.)	(-7.05E-12, 0.)	(-1.54E-06, 0.)	(5.23E-06, 0.)	(-2.04E-02, 0.)
7	(1.28E-06, 0.)	(-5.06E-06, 0.)	(1.08E-02, 0.)	(-2.40E-02, 0.)	(-3.06E+00, 0.)
8	(7.87E-08, 0.)	(5.16E-07, 0.)	(2.47E-03, 0.)	(-8.93E-03, 0.)	(1.98E+00, 0.)
9	(-1.17E-04, 0.)	(2.63E-03, 0.)	(9.66E-02, 0.)	(-4.52E-01, 0.)	(8.45E+00, 0.)
10	(-5.52E-02, 0.)	(8.86E-02, 0.)	(-9.45E+00, 0.)	(2.77E+01, 0.)	(-8.59E+01, 0.)
11	(2.61E-01, 0.)	(-8.90E-01, 0.)	(-2.99E-02, 0.)	(-1.76E-01, 0.)	(-1.66E+00, 0.)
12	(-3.65E-01, 0.)	(-4.67E-01, 0.)	(-7.80E-02, 0.)	(-5.24E-01, 0.)	(-3.12E+00, 0.)
13	(-8.41E-01, 0.)	(4.42E-03, 0.)	(-4.11E-01, 0.)	(1.53E+00, 0.)	(5.68E+00, 0.)
14	(5.75E-02, 0.)	(4.56E-02, 0.)	(-1.10E-01, 0.)	(-1.79E+00, 0.)	(-1.22E+01, 0.)
15	(3.58E-01, 0.)	(-2.44E-01, 0.)	(7.81E-01, 0.)	(-2.58E+00, 0.)	(-8.69E+00, 0.)
16	(-9.26E-01, 0.)	(1.89E-02, 0.)	(2.89E-01, 0.)	(-9.92E-01, 0.)	(-3.50E+00, 0.)
17	(2.56E+00, 0.)	(-4.95E-01, 0.)	(7.63E-01, 0.)	(-1.53E+00, 0.)	(-1.20E+01, 0.)
18	(-2.90E+00, 0.)	(7.09E-02, 0.)	(-1.54E+00, 0.)	(6.25E+00, 0.)	(1.03E+01, 0.)
19	(3.72E-03, 0.)	(-2.31E-01, 0.)	(-1.57E-01, 0.)	(9.33E-01, 0.)	(4.67E+00, 0.)
20	(1.75E-02, 0.)	(-1.07E-02, 0.)	(8.59E-03, 0.)	(-4.23E-02, 0.)	(-2.08E-01, 0.)

	16	17	18	19	20
1	(-3.34E-01, 1.35E-02)	(-3.34E-01,-1.35E-02)	(-2.53E-01, 0.)	(3.59E-03, 0.)	(2.66E-01, 0.)
2	(3.43E-01,-7.12E-01)	(3.43E-01, 7.12E-01)	(1.43E+00, 0.)	(-3.77E-02, 0.)	(-2.00E-01, 0.)
3	(-4.97E-03,-2.91E-03)	(-4.97E-03, 2.91E-03)	(-2.34E-03, 0.)	(2.23E-05, 0.)	(7.33E-03, 0.)
4	(-2.42E-01, 9.81E-02)	(-2.42E-01,-9.81E-02)	(-2.31E-01, 0.)	(3.78E-03, 0.)	(1.28E-01, 0.)
5	(2.83E-02,-4.79E-01)	(2.83E-02, 4.79E-01)	(4.68E-01, 0.)	(-7.11E-03, 0.)	(1.43E+00, 0.)
6	(1.38E-02, 8.02E-03)	(1.38E-02,-8.02E-03)	(-1.38E-02, 0.)	(7.03E-04, 0.)	(-2.15E-02, 0.)
7	(9.12E-01, 1.13E+00)	(9.12E-01,-1.13E+00)	(-2.91E+00, 0.)	(7.09E-02, 0.)	(-2.24E+00, 0.)
8	(-4.73E-02, 1.01E-02)	(-4.73E-02,-1.01E-02)	(8.59E-02, 0.)	(-7.01E-03, 0.)	(3.37E-02, 0.)
9	(8.43E-02,-3.80E-01)	(8.43E-02, 3.80E-01)	(7.51E-01, 0.)	(-2.20E-02, 0.)	(9.80E-01, 0.)
10	(-3.14E-01,-4.80E+00)	(-3.14E-01, 4.80E+00)	(2.74E+00, 0.)	(7.68E-02, 0.)	(2.00E+01, 0.)
11	(5.32E-01,-1.18E+00)	(5.32E-01, 1.18E+00)	(-4.05E-01, 0.)	(-7.10E-02, 0.)	(2.91E+00, 0.)
12	(-9.66E-01, 8.72E-01)	(-9.66E-01,-8.72E-01)	(6.76E-01, 0.)	(-1.64E-02, 0.)	(5.31E+00, 0.)
13	(-1.21E+00, 2.29E+00)	(-1.21E+00,-2.29E+00)	(-7.22E-01, 0.)	(1.05E-01, 0.)	(-7.87E+00, 0.)
14	(-8.08E-03, 9.67E-02)	(-8.08E-03,-9.67E-02)	(5.48E-01, 0.)	(-1.32E-02, 0.)	(5.78E-01, 0.)
15	(2.38E-03, 7.01E-02)	(2.38E-03,-7.01E-02)	(5.24E-01, 0.)	(-2.29E-02, 0.)	(6.74E-01, 0.)
16	(-1.08E-02, 2.37E-02)	(-1.08E-02,-2.37E-02)	(2.52E-02, 0.)	(9.37E-03, 0.)	(-9.44E-02, 0.)
17	(-2.12E+00,-5.45E+00)	(-2.12E+00, 5.45E+00)	(1.25E+00, 0.)	(-6.64E-01, 0.)	(1.15E+01, 0.)
18	(1.15E+00, 6.32E+00)	(1.15E+00,-6.32E+00)	(-5.77E-01, 0.)	(6.48E-01, 0.)	(-6.16E+00, 0.)
19	(1.00E-02, 1.25E-02)	(1.00E-02,-1.25E-02)	(7.91E-01, 0.)	(-5.99E-03, 0.)	(8.37E-02, 0.)
20	(1.29E-02, 1.80E-02)	(1.29E-02,-1.80E-02)	(-2.39E-02, 0.)	(1.09E+00, 0.)	(-4.55E-02, 0.)

Figure A.5.4, concluded

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15. Supplementary Notes

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16. Abstract

The use of eigenspace assignment techniques to synthesize flight control systems for flexible aircraft is explored. These eigenspace assignment techniques are used to achieve a specified desired eigenspace, chosen to yield desirable system impulse residue magnitudes for selected system responses. Two eigenspace assignment techniques are investigated. The first is a technique for directly determining constant measurement feedback gains that will yield a closed-loop system eigenspace "close" to a desired eigenspace. The second technique is a method for selecting quadratic weighting matrices in a linear quadratic control synthesis that will asymptotically yield the closed-loop achievable eigenspace. Finally, the possibility of using either of these techniques with state estimation is explored.

Application of the methods to synthesize integrated flight-control and structural-mode-control laws for a large flexible aircraft is demonstrated and results discussed. Eigenspace selection criteria based upon the design goals are discussed, and for the study case it would appear that a desirable eigenspace can be obtained. In addition, the importance of state-space selection is noted along with problems with reduced-order measurement feedback. Since the full-state control laws may be implemented with dynamic compensation (state estimation), the use of reduced-order measurement feedback is less desirable. This is especially true since no change in the transient response from the pilot's input results if state estimation is used appropriately. The potential is also noted for high actuator bandwidth requirements if the linear quadratic synthesis approach is utilized. Even with the actuator pole location selected, a problem with unmodeled modes is noted due to high bandwidth.

Some suggestions for future research include investigating how to choose an eigenspace that will achieve certain desired dynamics and stability robustness, determining how the choice of measurements effects synthesis results, and exploring how the phase relationships between desired eigenvector elements effects the synthesis results.

17. Key Words (Suggested by Author(s))

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