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NUMERICAL QUADRATURE METHODS FOR INTEGRALS OF SINGULAR PERIODIC FUNCTIONS AND THEIR APPLICATION TO SINGULAR AND WEAKLY SINGULAR INTEGRAL EQUATIONS*

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ABSTRACT

High accuracy numerical quadrature methods for integrals of singular periodic functions are proposed. These methods are based on the appropriate Euler-Maclaurin expansions of trapezoidal rule approximations and their extrapolations. They are used to obtain accurate quadrature methods for the solution of singular and weakly singular Fredholm integral equations. Such periodic equations are used in the solution of planar elliptic boundary value problems, elasticity, potential theory, conformal mapping, boundary element methods, free surface flows, etc. The use of the quadrature methods is demonstrated with numerical examples.

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1. INTRODUCTION

In this work we shall present some Romberg-type quadrature methods for the numerical solution of singular and weakly singular Fredholm integral equations of the first and second kinds

$$\omega f(t) + \int_a^b K(t,x) f(x) dx = g(t), \quad a \le t \le b \ , \tag{1.1}$$

where $\omega=0$ and $\omega=1$ for first and second kinds respectively, and the corresponding eigenvalue problems

$$f(t) + \lambda \int_a^b K(t,x) f(x) dx = 0, \quad a \le t \le b, \tag{1.2}$$

that arise, for example, from some boundary value problems over two dimensional domains with smooth and rectifiable boundaries. These quadrature methods are ultimately based on appropriate Euler-Maclaurin expansions for the trapezoidal rule, and treat the singularity in the kernel $K(t,x)$ systematically. As such, these methods are simple, easy to implement, and have high order of accuracy.

We assume that $K(t,x)$ is periodic in both t and x with period $T=b-a$, and $g(t)$ is periodic in t with period T. We furthermore assume that $K(t,x)$ is of the form

$$\begin{aligned} K(t,x) = \sum_{k=1}^{M} H_k(t,x) |t-x|^{\alpha_k} (\log|t-x|)^{p_k} \\ + \frac{\hat{H}_1(t,x)}{t-x} + \hat{H}_2(t,x) \ , \end{aligned} \tag{1.3}$$

where α_k are real numbers satisfying $\alpha_k > -1$, and p_k are non-negative integers, $H_k(t,x)$, $1 \le k \le M$, and $\hat{H}_j(t,x)$, $j=1,2$, are differentiable in t and x as many times as needed. In (1.3) it is assumed that $H_k(t,t) \ne 0$ whenever $H_k(t,x) \not\equiv 0$ for some k. It is also assumed that $\hat{H}_1(t,t) \ne 0$ whenever $\hat{H}_1(t,x) \not\equiv 0$, and in this case the integrals in (1.1) and (1.2) are to be taken as Cauchy principal value integrals. When $H_k(t,x) \equiv 0$, $1 \le k \le M$, and $\hat{H}_1(t,x) \not\equiv 0$, the integral equations (1.1) and (1.2) are called singular, and when $H_k(t,x) \not\equiv 0$ for some k, $1 \le k \le M$, and $\hat{H}_1(t,x) \equiv 0$, they are called weakly singular.

It is worth mentioning that the important cases of $K(t,x)$ that arise in applications are $K(t,x) = H_1(t,x)\log\ |t-x| + \hat{H}_2(t,x)$, $K(t,x) = \frac{\hat{H}_1(t,x)}{t-x} + \hat{H}_2(t,x)$, and combinations of the two.

We shall assume that $f(t)$, as $K(t,x)$ and $g(t)$, is periodic in t, with period T, and is as many times differentiable as needed. We note that when (1.1) and (1.2) arise from some boundary value problems in two dimensions, defined over bounded domains with *smooth* boundaries, $K(t,x)$, $g(t)$, and $f(t)$ also, in general, turn out to be smooth.

One of the methods for solving (1.1) and (1.2) numerically is the quadrature method (see Baker (1977, Chap. 4, Section 3)), in which one replaces the integral $\int_a^b K(t,x)f(x)dx$ by a numerical quadrature formula, whose abscissas are x_j, $j=1,\ldots,n$, with $t=x_i$, $i=1,\ldots,n$, then replaces the $f(x_j)$ by their approximation $\tilde{f}_j$, and finally solves the resulting system of linear equation for the $\tilde{f}_j$. Obviously the accuracy of this method depends on the accuracy of the numerical quadrature formula being used, which in turn depends on the analytic properties of both the kernel $K(t,x)$ and the solution $f(t)$ over $[a,b]$. It can be said, in general, that whenever $K(t,x)$ is weakly singular or singular, the solution $f(t)$ will be singular at the end points a and b. The singularity structure of $f(t)$ may be complicated and difficult to determine; see MacCamy (1958) and Graham (1982) for some general results on this problem. When $K(t,x)$, $g(t)$, and $f(t)$ are (periodic) as assumed in the present work, then a and b in (1.1) and (1.2) can be replaced by a' and b' respectively, where $b'-a' = T$. If we now assume that $f(t)$ has singularities at a and b, then it should be singular at a' and b' ad hence at *all* t. As a result we conclude, heuristically, that $f(t)$ cannot have any singularities, and this is the assumption that we have made above.

Let

$$x_j = a + jh, \quad h = (b-a)/n, \quad n \text{ a positive integer.} \tag{1.4}$$

Using the Euler-MacLaurin expansion for smooth integrands, and their extension to integrands having end point singularities (see Navot (1961, 1962)), in the next section we derive Euler-MacLaurin expansions for the integrals $\int_a^b K(t,x)f(x)dx$, with $K(t,x)$ as given in (1.3). We basically derive asymptotic expansions, for $h \to 0$, for the differences

$$\Delta(t,h) = I[t;f] - I_n[t;f], \tag{1.5}$$

where

$$I[t;f] = \int_a^b K(t,x)f(x)dx \tag{1.6}$$

and

$$I_n[t;f] = \sum_{j=1}^{n} w_n(t,x_j)f(x_j), \tag{1.7}$$

such that t is one of the points x_j, and is being held fixed, and $w_n(t,x_j) = hK(t,x_j)$ for $x_j \neq t$, and $w_n(t,t)$ depend on the type of singularity that $K(t,x)$ has for $t = x$. Using the asymptotic expansions for $\Delta(t,h)$, in Section 3 we derive Romberg-type numerical quadrature formulas, thus increasing the accuracy of $I_n[t;f]$ by as many orders of magnitude as we wish. In Section 4 we present quadrature methods that are based on these Bomberg-type formulas. In Section 5 we illustrate the efficiency of our quadrature methods with numerical examples. In Section 6 we review some quadrature methods that have been proposed and bear some relation to the ones proposed in the present work.

Note: The treatments of singular and weakly singular integral equations are not separated from each other throughout this paper. The reader interested in the treatment of singular integral equations could follow it easily by going through Theorems 2.1, 2.4, 2.7a, 3.1, 3.2, and the first part of Section 4, without having to consider the rest of this work.

2. EULER-MACLAURIN EXPANSIONS

The notation described below will be used throughout the remainder of this work.

Let $x_j = a+jh$, $j=0,1,\ldots,n$, $h=(b-a)/n$, where n is a positive integer. Let $t \in (a,b)$ be fixed and $t \in \{x_j \mid 1 \le j \le n-1\}$ for some $n=n_0$. Obviously, there exists an infinite sequence of integers $\{n_k\}_{k=0}^{\infty}$, $n_{k+1} > n_k$, $k=0,1,\ldots$, such that t is one of the x_j whenever $n=n_k$, $k=0,1,\ldots$. With the exception of Theorems 2.1-2.3, the notation $h \to 0$ will be assumed to mean that $n \to \infty$ through the sequence of integers $\{n_k\}_{k=0}^{\infty}$. We shall take $\sum_{j=m_1}^{m_2}{}' \alpha_j$ (or $\sum_{j=m_1}^{m_2}{}^{*} \alpha_j$) to mean that α_{m_2} (or α_{m_1}) is to be multiplied by 1/2, while $\sum_{j=m_1}^{m_2}{}'' \alpha_j$ will be taken to mean that both α_{m_1} and α_{m_2} are to be multiplied by 1/2, in the respective summations.

Theorems 2.1-2.3, which are stated below without proof form the basis of our development throughout the remainder of this work.

Theorem 2.1: Let the function $g(x)$ be $2m$ times differentiable on $[a,b]$. Then

$$\begin{aligned} D(h) &= \int_a^b g(x)dx - \sum_{j=0}^{n}{}'' g(x_j) \\ &= \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!}\Big[g^{(2\mu-1)}(a) - g^{(2\mu-1)}(b)\Big]h^{2\mu} + R_{2m}\Big[g;(a,b)\Big], \end{aligned} \tag{2.1}$$

where

$$R_{2m}[g;(a,b)] = h^{2m}\int_a^b \frac{\overline{B}_{2m}[(x-a)/h] - B_{2m}}{(2m)!} g^{(2m)}(x)dx. \tag{2.2}$$

Here B_μ are the Bernoulli numbers, and $\overline{B}_\mu(x)$ is the periodic Bernoullian function of order μ. In addition, since $\overline{B}_\mu(x)$ are bounded on $(-\infty,\infty)$, it follows that

$$|R_{2m}[g;(a,b)]| \le M_{2m}(b-a)h^{2m} \max_{a \le x \le b} |g^{(2m)}(x)|, \tag{2.3}$$

where

$$M_{2m} = \max_{-\infty<x<\infty} |\overline{B}_{2m}(x) - B_{2m}| / (2m)! , \tag{2.4}$$

and, therefore, is independent of h. Consequently, if $g(x)$ is infinitely differentiable on $[a,b]$, then $D(h)$ has an asymptotic expansion of the form

$$D(h) \sim \sum_{\mu=1}^{\infty} \frac{B_{2\mu}}{(2\mu)!}\left[g^{(2\mu-1)}(a) - g^{(2\mu-1)}(b)\right]h^{2\mu} \quad \text{as } h \to 0 . \tag{2.5}$$

For a proof of this result see Steffensen (1950). The expansion in (2.5) is the classical Euler-Maclaurin expansion for trapezoidal rule approximations of integrals of smooth functions. Navot (1961, 1962) has extended the Euler-Maclaurin expansions to trapezoidal rule approximations of integrals of functions having algebraic and/or logarithmic end point singularities. By a different approach that utilizes generalized functions, Lyness and Ninham (1967) have rederived Navot's results. The results stated as Theorems 2.2 and 2.3 below, are special cases of those proved by Navot.

Theorem 2.2: Let $g(x)$ be $2m$ times differentiable on $[a,b]$ and let $G(x) = (x-a)^s g(x)$, $s>-1$. Then

$$\begin{aligned} D(h) &= \int_a^b G(x)dx - h\sum_{j=1}^{n}{}' G(x_j) \\ &= -\sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} G^{(2\mu-1)}(b)h^{2\mu} \\ &\quad - \sum_{\mu=0}^{2m-1} \frac{\zeta(-s-\mu)}{\mu!} g^{(\mu)}(a)h^{\mu+s+1} + \rho_{2m} , \end{aligned} \tag{2.6}$$

where $\zeta(\tau)$ is the Riemann zeta function initially defined for $\text{Re}\,\tau > 1$ by $\zeta(\tau) = \sum_{k=1}^{\infty} k^{-\tau}$, and then continued analytically, and

$$\rho_{2m} = O\left(h^{2m}\right) \quad \text{as } h \to 0 . \tag{2.7}$$

If $g(x)$ is infinitely differentiable on $[a,b]$, then $D(h)$ has an asymptotic expansion of the form

$$D(h) \sim -\sum_{\mu=1}^{\infty} \frac{B_{2\mu}}{(2\mu)!} G^{(2\mu-1)}(b)h^{2\mu} - \sum_{\mu=0}^{\infty} \frac{\zeta(-s-\mu)}{\mu!} g^{(\mu)}(a)h^{\mu+s+1} \quad \text{as } h \to 0. \tag{2.8}$$

Starting from Theorem 2.2, Navot (1962) shows that the extensions of the Euler-Maclaurin expansion to trapezoidal rule approximations of integrals of function of the form $(x-a)^s[\log(x-a)]^p g(x)$, with p being a positive integer and $g(x)$ being sufficiently smooth in $[a,b]$, can be obtained by differentiating both sides of (2.6) p times with respect to s. For $p=1$ the following results are obtained.

Theorem 2.3: Let $g(x)$ be $2m$ times differentiable on $[a,b]$, and let $G(x) = (x-a)^s \log(x-a)g(x)$, $s>-1$. Then

$$\begin{aligned} D(h) &= \int_a^b G(x)dx - h\sum_{j=1}^{n}{}' G(x_j) \\ &= -\sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} G^{(2\mu-1)}(b)h^{2\mu} \\ &\quad - \sum_{\mu=0}^{2m-1} \left[-\zeta'(-s-\mu) + \zeta(-s-\mu)\log h\right] \frac{g^{(\mu)}(a)}{\mu!} h^{\mu+s+1} + \tilde{\rho}_{2m}, \end{aligned} \tag{2.9}$$

where $\zeta'(\tau) = d\zeta(\tau)/d\tau$, and

$$\tilde{\rho}_{2m} = O\left(h^{2m}\right) \quad \text{as } h \to 0. \tag{2.10}$$

If $g(x)$ is infinitely differentiable on $[a,b]$, then $D(h)$ has an asymptotic expansion of the form

$$D(h) \sim -\sum_{\mu=1}^{\infty} \frac{B_{2\mu}}{(2\mu)!} G^{(2\mu-1)}(b)h^{2\mu} - \sum_{\mu=0}^{\infty} \left[-\zeta'(-s-\mu) + \zeta(-s-\mu)\log h\right] \frac{g^{(\mu)}(a)}{\mu!} h^{\mu+s+1}. \tag{2.11}$$

Note that both Theorem 2.2 and Theorem 2.3 are true for *any* $s>-1$, although they were originally stated for $-1<s\le 0$.

We shall now apply Theorems 2.1-2.3 to integrals of the types $\int_a^b \frac{g(x)}{x-t}dx$ and

$\int_a^b |x-t|^s (\log|x-t|)^p g(x)dx$, the former being defined as a Cauchy principal value integral.

Theorem 2.4: Let $g(x)$ be $2m$ times differentiable on $[a,b]$, and let $G(x) = \frac{g(x)}{x-t}$. Then

$$D(h) = \int_a^b G(x)dx - h\sum_{\substack{j=0 \\ x_j \neq t}}^{n}{}'' G(x_j)$$

$$= hg'(t) + \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!}\left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b)\right]h^{2\mu} \tag{2.12}$$

$$+ 0\left(h^{2m}\right) \quad \text{as } h \to 0.$$

Proof: We can assume without loss of generality that $t-a \leq b-t$. Then, since t is one of the x_j, so is $b'=2t-a$. Furthermore, t is the midpoint of the interval $[a,b']$. Now

$$\int_a^b G(x)dx = \int_a^{b'} G(x)dx + \int_{b'}^b G(x)dx \tag{2.13}$$

and

$$\int_a^{b'} G(x)dx = \int_a^{b'} \frac{g(x)-g(t)}{x-t}dx, \tag{2.14}$$

the integral on the right hand side of (2.14) being an ordinary integral, in which the integrand becomes $g'(t)$ when $x=t$. Applying Theorem 2.1 to the right hand side of (2.14), we have

$$D_1(h) = \int_a^{b'} G(x)dx - h \sum_{\substack{x_j \le b' \\ x_j \ne t}}'' \frac{g(x_j) - g(t)}{x_j - t} - h\, g'(t)$$

$$= \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} \left\{ \frac{d^{2\mu-1}}{dx^{2\mu-1}} \left[\frac{g(x)-g(t)}{x-t} \right] \Big|_{x=a} - \frac{d^{2\mu-1}}{dx^{2\mu-1}} \left[\frac{g(x)-g(t)}{x-t} \right] \Big|_{x=b'} \right\} h^{2\mu} \tag{2.15}$$

$$+ 0\left(h^{2m}\right) \quad \text{as } h \to 0.$$

Now

$$\sum_{\substack{x_j \le b' \\ x_j \ne t}}'' \frac{1}{x_j - t} = 0 \tag{2.16}$$

and

$$\frac{d^r}{dx^r}\left[\frac{1}{x-t}\right] = \frac{(-1)^r r!}{(x-t)^{r+1}}, \qquad r = 0, 1, \ldots \ . \tag{2.17}$$

Combining (2.16) and (2.17) in (2.15), we have

$$\begin{aligned} D_1(h) &= \int_a^{b'} G(x)dx - h \sum_{\substack{x_j \le b' \\ x_j \ne t}}'' G(x_j) - h\, g'(t) \\ &= \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} \left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b') \right] h^{2\mu} + 0\left(h^{2m}\right) \quad \text{as } h \to 0. \end{aligned} \tag{2.18}$$

Applying Theorem 2.1 to $\int_{b'}^{b} G(x)dx$, we obtain

$$\begin{aligned} D_2(h) &= \int_{b'}^{b} G(x)dx - h \sum_{x_j \ge b'}'' G(x_j) \\ &= \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} \left[G^{(2\mu-1)}(b') - G^{(2\mu-1)}(b) \right] h^{2\mu} + 0\left(h^{2m}\right) \quad \text{as} \quad h \to 0. \end{aligned} \tag{2.19}$$

Adding (2.18) and (2.19), (2.12) now follows.

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Corollary: The remainder term $0(h^{2m})$ in (2.12) is actually given by

$$\tilde{R}_{2m}[G;(a,b)] = h^{2m}\int_a^b \frac{\overline{B}_{2m}[(x-a)/h]-B_{2m}}{(2m)!}G^{(2m)}(x)dx, \tag{2.20}$$

this integral being interpreted as a Cauchy principal value integral.

Proof: The remainder term $0(h^{2m})$ in (2.15) is

$$R_{2m}^1 = h^{2m}\int_a^{b'} \frac{\overline{B}_{2m}[(x-a)/h]-B_{2m}}{(2m)!}\frac{d^{2m}}{dx^{2m}}\left[\frac{g(x)-g(t)}{x-t}\right]dx. \tag{2.21}$$

Now making the change of variable $x=t+\xi$, we have

$$\begin{aligned}\int_a^{b'} \frac{\overline{B}_{2m}[(x-a)/h]-B_{2m}}{(2m)!}\frac{d^{2m}}{dx^{2m}}\left[\frac{1}{x-t}\right]dx &= \\ \int_{-(t-a)}^{t-a} \frac{\overline{B}_{2m}[\xi/h+(t-a)/h]-B_{2m}}{(2m)!}\frac{d^{2m}}{d\xi^{2m}}\left[\frac{1}{\xi}\right]d\xi.\end{aligned} \tag{2.22}$$

Recall that

$$\overline{B}_k(N+x) = \overline{B}_k(x),\ N \text{ integer},\ k \geq 2, \tag{2.23}$$

and

$$\overline{B}_k(-x) = (-1)^k\overline{B}_k(x), \qquad k \geq 2. \tag{2.24}$$

Since $(t-a)/h$ is an integer, it follows using (2.23) and (2.24) that the integrand of the integral on the right hand side of (2.22) is odd. Consequently, when taken as a Cauchy principal value integral, this integral is zero. The result now follows by using this in (2.21), and adding to R_{2m}^1 the remainder term of the Euler-Maclaurin expansion for the integral $\int_{b'}^{b} G(x)dx$.

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Theorem 2.5: Let $g(x)$ be $2m$ times differentiable on $[a,b]$, and let $G(x) = |x-t|^s g(x)$. Then

$$D(h) = \int_a^b G(x)dx - h \sum_{\substack{j=0 \\ x_j \neq t}}^{n}{}'' G(x_j)$$

$$= \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} \left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b)\right] h^{2\mu} \tag{2.25}$$

$$- 2 \sum_{\mu=0}^{m-1} \frac{\zeta(-s-2\mu)}{(2\mu)!} g^{(2\mu)}(t) h^{2\mu+s+1} + 0\left(h^{2m}\right) \quad \text{as} \quad h \to 0.$$

Proof: Applying Theorem 2.2 to $\int_t^b G(x)dx$, we have

$$D_1(h) = \int_t^b G(x)dx - h \sum_{x_j > t}{}' G(x_j)$$

$$= - \sum_{\mu=0}^{m-1} \frac{B_{2\mu}}{(2\mu)!} G^{(2\mu-1)}(b) h^{2\mu} \tag{2.26}$$

$$- \sum_{\mu=0}^{2m-1} \frac{\zeta(-s-\mu)}{\mu!} g^{(\mu)}(t) h^{\mu+s+1} + 0\left(h^{2m}\right) \quad \text{as} \quad h \to 0.$$

Next applying Theorem 2.2 to $\int_0^{t-a} \xi^s g(t-\xi) d\xi$ $(= \int_a^t G(x)dx)$, we have

$$D_2(h) = \int_a^t G(x)dx - h \sum_{x_j < t}{}^{*} G(x_j)$$

$$= \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} G^{(2\mu-1)}(a) h^{2\mu} \tag{2.27}$$

$$- \sum_{\mu=0}^{2m-1} (-1)^{\mu} \frac{\zeta(-s-\mu)}{\mu!} g^{(\mu)}(t) h^{\mu+s+1} + 0\left(h^{2m}\right) \quad \text{as} \quad h \to 0.$$

Adding (2.26) and (2.27), (2.25) follows.

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Theorem 2.6: Let $G(x) = |x-t|^s \log|x-t| g(x)$ in the statement of Theorem 2.5, everything else being the same. Then

$$D(h) = \int_a^b G(x)dx - h \sum_{\substack{j=0 \\ x_j \neq t}}^{n}{}'' G(x_j)$$

$$= \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!}\left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b)\right]h^{2\mu} \tag{2.28}$$

$$-2\sum_{\mu=0}^{m-1}\left[-\zeta'(-s-2\mu)+\zeta(-s-2\mu)\log h\right] \frac{g^{(2\mu)}(t)}{(2\mu)!}h^{2\mu+s+1} + 0\left(h^{2m}\right) \quad \text{as } h \to 0.$$

Proof: Similar to that of Theorem 2.5.

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Corollary: For $s=0$, (2.28) becomes

$$D(h) = g(t)h \log h + \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!}\left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b)\right]h^{2\mu}$$
$$+ 2\sum_{\mu=0}^{m-1} \frac{\zeta'(-2\mu)}{(2\mu)!} g^{(2\mu)}(t)h^{2\mu+1} + 0\left(h^{2m}\right) \quad \text{as } h \to 0. \tag{2.29}$$

Proof: The proof follows by setting $s=0$ in (2.28) and using the facts that

$$\zeta(0) = -1/2, \quad \zeta(-2\mu) = 0, \quad \mu = 1,2,\dots\ , \tag{2.30}$$

(see Abramowitz and Stegun (1964, p. 807)).

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The results in the following theorem will be the ones on which our quadrature methods will be based.

Theorem 2.7: Assume that the functions $g(x)$ and $\tilde{g}(x)$ are $2m$ times differentiable on $[a,b]$. Assume also that the functions $G(x)$ are periodic with period $T=b-a$, and that they are $2m$ times differentiable on $\tilde{R} = (-\infty,\infty)\setminus\{t+kT\}_{k=-\infty}^{\infty}$. Then

a) if $G(x) = \dfrac{g(x)}{x-t} + \tilde{g}(x)$, and

$$Q_n[G] = h \sum_{\substack{j=1 \\ x_j \neq t}}^{n} G(x_j)\ , \tag{2.31a}$$

then

$$E_n[G] = [\tilde{g}(t)+g'(t)]h + 0\left(h^{2m}\right) \quad \text{as} \quad h \to 0, \tag{2.32a}$$

b) if $G(x) = |x-t|^s g(x) + \tilde{g}(x)$, $s > -1$, and

$$Q_n[G] = h \sum_{\substack{j=1 \\ x_j \neq t}}^{n} G(x_j) + \tilde{g}(t)h - 2\zeta(-s)g(t)h^{s+1}, \tag{2.31b}$$

then

$$E_n[G] = -2\sum_{\mu=1}^{m-1} \frac{\zeta(-s-2\mu)}{(2\mu)!} g^{(2\mu)}(t)h^{2\mu+s+1} + 0\left(h^{2m}\right) \quad \text{as} \quad h \to 0, \tag{2.32b}$$

c) if $G(x) = |x-t|^s \log|x-t|g(x) + \tilde{g}(x)$, $s > -1$, and

$$Q_n[G] = h\sum_{\substack{j=1 \\ x_j \neq t}}^{n} G(x_j) + \tilde{g}(t)h + 2\left[\zeta'(-s) - \zeta(-s)\log h\right]g(t)h^{s+1}, \tag{2.31c}$$

then

$$E_n[G] = 2\sum_{\mu=1}^{m-1}\left[\zeta'(-s-2\mu) - \zeta(-s-2\mu)\log h\right]\frac{g^{(2\mu)}(t)}{(2\mu)!}h^{2\mu+s+1} + 0\left(h^{2m}\right) \quad \text{as} \quad h \to 0. \tag{2.32c}$$

c') When $s=0$ in c), by (2.30) and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, (see Abramowitz and Stegun (1964, p. 807)), (2.31c) and (2.32c) reduce to

$$Q_n[G] = h\sum_{\substack{j=1 \\ x_j \neq t}}^{n} G(x_j) + \tilde{g}(t)h + \log\left(\frac{h}{2\pi}\right)g(t)h, \tag{2.31c'}$$

and

$$E_n[G] = 2\sum_{\mu=1}^{m-1} \frac{\zeta'(-2\mu)}{(2\mu)!} g^{(2\mu)}(t)h^{2\mu+1} + 0\left(h^{2m}\right) \quad \text{as} \quad h \to 0, \tag{2.32c'}$$

where $E_n[G] = \int_a^b G(x)dx - Q_n[G]$ in all cases.

Remarks:

1) As is seen from (2.32b), (2.32c), and (2.32c'), $E_n[G]$ depends only on t, and is *independent* of a and b. This is a consequence of the periodicity of $G(x)$, and is an important property that shall be exploited in the derivation of Romberg-type quadrature formulas in the next section.

2) Until now we assumed that t is one of the points x_j. When t is arbitrary the

periodicity of $G(x)$ can be used to shift the interval $[a,b]$ to $[a',b']$ such that t coincides with one of the x_j in the new interval $[a',b']$. This, combined with the observation of Remark 1, means that a) $E_n[G]$ in all parts of Theorem 2.7 stays the same if the sum $\sum_{\substack{j=1\\ x_j\neq t}}^{n} G(x_j)$ in (2.31a,b,c,c') is replaced by $\sum_{j=1}^{n-1} G(t+jh)$ or by the identical sum $\sum_{\substack{j\neq 0\\ a<t+jh\leq b}} G(t+jh)$, and b) Theorem 2.7 holds for *all* positive integers n. These facts will be repeatedly used in Section 4 without further explanation.

3) In each of the cases of Theorem 2.7, the numerical quadrature formula $Q_n[G]$ is computed by using the function values only. The quadrature formula (2.31a) has an error which is of order h, and it would seem that one would have to know $g'(t)$ with a high accuracy in order to improve this formula. But as we shall see in the next section, the term $[\tilde{g}(t)+g'(t)]h$ that appears in $E_n[G]$ is easily removed by one extrapolation.

4) Let $G(x) = \sum_{k=1}^{M} g_k(x)\log|x-t|^{\alpha_k}(\log|x-t|)^{p_k} + \hat{g}_1(x)/(x-t) + \hat{g}_2(x)$, where α_k and p_k are as described following (1.3), and $g_k(x)$, $1\leq k\leq M$, $\hat{g}_1(x)$, and $\hat{g}_2(x)$ are $2m$ times differentiable on $[a,b]$ and $G(x)$ is periodic with period $T=b-a$ and is $2m$ times differentiable on $\tilde{R}$. Inspection of Theorems 2.4-2.7 reveals that if we form $Q_n[G]$ as the sum of the quadrature formulas for each one of the terms in $G(x)$, then the error $E_n[G]$ does not contain any contribution from $G(x)$ and its derivatives at the end points, and the only contribution to $E_n[G]$ comes from $G(x)$ and its derivatives at $x=t$, as in Theorem 2.7.

3. ROMBERG-TYPE NUMERICAL QUADRATURE FORMULAS

Using the results of Theorem 2.7, we can apply extrapolation techniques to derive Romberg-type numerical quadrature formulas for $\int_a^b G(x)dx$.

The simplest case is that of Theorem 2.7a, and we deal with it first.

Theorem 3.1: Let $G(x)$ and $Q_n[G]$ be as in Theorem 2.7a. Let $h_k = T/k$ and

$$\tilde{Q}_n[G] = 2Q_{2n}[G] - Q_n[G] = h_n \sum_{j=1}^{n} G(a+jh_n-h_n/2), \tag{3.1}$$

i.e., $\tilde{Q}_n[G]$ is a mid-point rule approximation. Then

$$\tilde{E}_n[G] = \int_a^b G(x)dx - \tilde{Q}_n[G] = 0\left(h_n^{2m}\right) \text{ as } h_n \to 0. \tag{3.2}$$

Proof: (3.2) follows directly from Theorem 2.7a, the h term in $E_n[G]$ being eliminated when $\tilde{Q}_n[G]$ is formed.

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As a result of Theorem 3.1 we conclude that if $G(x)$ is infinitely differentiable on $\tilde{R}=(-\infty,\infty)\setminus\{t+kT\}_{k=-\infty}^{\infty}$, then $\tilde{E}_n[G]$ tends to zero more quickly than any inverse power of n, as $n\to\infty$. We can improve this result considerably whenever $G(z)$ is analytic in a strip in the complex z-plane, which, with the exception of the points $t+kT$, $k=0, \pm1,\ldots,$ contains the real line Im $z=y=0$ in its interior.

Theorem 3.2: Let $G(z)$ in the previous theorem be analytic in the strip $|\text{Im } z|<\sigma$, except at the simple poles $t+kT$, $k=0, \pm1, \pm2,\ldots$. Then

$$|\tilde{E}_n[G]| \le 2TM(\sigma') \frac{exp[-2\pi n\sigma'/T]}{1-exp[-2\pi n\sigma'/T]}, \quad \sigma'<\sigma, \tag{3.3}$$

where

$$M(\tau) = \max\left\{\max_{-\infty<x<\infty} |G_e(x+i\tau)|, \max_{-\infty<x<\infty} |G_e(x-i\tau)|\right\}, \tag{3.4}$$

and

$$G_e(\xi) = \frac{1}{2}\left[G(t+\xi) + G(t-\xi)\right]. \tag{3.5}$$

Proof: By the periodicity of $G(x)$, we can write

$$\int_a^b G(x)dx = \int_{t-T/2}^{t+T/2} G(x)dx = \int_{-T/2}^{T/2} G_e(\xi)d\xi. \tag{3.6}$$

$G_e(z)$ is analytic in the strip $|\mathrm{Im}\, z| < \sigma$ and is periodic with period T. After some algebra it can be shown that the $n(odd)$-point trapezoidal rule approximation or the $n(even)$-point mid-point rule approximation to $\int_{-T/2}^{T/2} G_e(\xi)d\xi$ is just $\tilde{Q}_n[G]$. Applying now a theorem due to Davis (1955) (see also Davis and Rabinovitz (1984, pp. 314-316)), (3.3)-(3.5) follow. (Actually Davis' result is stated for the trapezoidal rule. However, inspection of his proof shows it to be valid for the off-set trapezoidal rule. The mid-point rule is an off-set trapezoidal rule.)

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For arbitrary t, by Remark 2 following Theorem 2.7, the approximation $\tilde{Q}_n[G]$ can be replaced by

$$\tilde{Q}_n[G] = h_n \sum_{j=1}^{n} G(t + jh_n - h_n/2), \tag{3.7}$$

and again Theorems 3.1 and 3.2 apply.

$\tilde{Q}_n[G]$ in Theorem 3.1 has been obtained by employing the Richardson extrapolation process once, eliminating the term $0(h)$ in the error $E_n[G]$. Since $\tilde{E}_n[G] = 0(h^{2m})$ with m as large as we wish, there is no need for further extrapolation. For $Q_n[G]$ as given in Theorem 2.7 b,c,c', however, we apply the Richardson extrapolation process (or generalizations of it) repeatedly in order to eliminate successive terms in $E_n[G]$, thus obtaining Romberg-type numerical quadrature formulas with higher degrees of accuracy. (A summary of those extrapolation methods relevant to the present work is given in the appendix.)

Let $G(x)$ and $Q_n[G]$ be as in Theorem 2.7 b,c,c', and define $A = \int_a^b G(x)dx$ and $A(h) = Q_n[G]$ in the notation of the appendix. Select a sequence of integers

$\{n_l\}_{l=0}^{\infty}$, $1 \leq n_0 < n_1 < \cdots$, and set $h_l = T/n_l$, $l=0,1,\ldots$. Obviously $\lim_{l\to\infty} h_l = 0$, as required; see the appendix. Now the Romberg-type quadrature formulas $A_q^{(m)}$, based on $A(h_l)$, $m \leq l \leq m+q$, are of the form

$$A_q^{(m)} = \sum_{k=0}^{q} d_{q,k}^{(m)} A(h_{m+k}), \tag{3.8}$$

where $d_{q,k}^{(m)}$, $0 \leq k \leq q$, are constants determined by the nature of the asymptotic expansion of $A - A(h) = E_n[G]$ as $h \to 0$, as described in the appendix. Some of the details for two special cases are given below:

1) If $G(x)$ is as in theorem 2.7b (or Theorem 2.7c'), then $A(h)$ is of the form given in a) of the appendix with $r=2$, $\gamma_i = s+1+2i$, and $\beta_i = -2\zeta(-s-2i)g^{(2i)}(t)/(2i)!$ (or $r=2$, $\gamma_i = 1+2i$, and $\beta_i = 2\zeta'(-2i)g^{(2i)}(t)/(2i)!$), $i=1,2,\ldots$. Hence for a sequence of the form $h_l = h_0\rho^l$, $l=0,1,\ldots$, the $d_{n,k}^{(j)}$ can be computed from (A.6). For arbitrary h_l, the algorithm given in (A.11) and (A.12) in b) of the appendix is appropriate with $\varphi(h) = h^{s+3}$ and $r=2$ (or $\varphi(h) = h^3$ and $r=2$).

2) If $G(x)$ is as in Theorem 2.7c with $s \neq 0$, then $A(h)$ is of the form given in (A.1) with $e_i(h) = [\zeta'(-s-2i) - \zeta(-s-2i)\log h]h^{2i+s+1}$, and $\beta_i = 2g^{(2i)}(t)/(2i)!$, $i=1,2,\ldots$. The $d_{n,k}^{(j)}$ then can be obtained by solving the equations in (A.5).

Before closing this section we recall that, for *any* positive integer n, $A(h) = Q_n[G]$ in Theorem 2.7 b,c,c' has the form

$$A(h) = h \sum_{\substack{j\neq 0 \\ a < t+jh \leq b}} G(x_j) + C(t,h), \tag{3.9}$$

where

$$C(t,h) = \begin{cases} \tilde{g}(t)h - 2\zeta(-s)g(t)h^{s+1} & \text{for Theorem 2.7b} \\ \tilde{g}(t)h + 2[\zeta'(-s) - \zeta(-s)\log h]g(t)h^{s+1} & \text{for Theorem 2.7c} \\ \tilde{g}(t)h + \log\left(\dfrac{h}{2\pi}\right)g(t)h & \text{for Theorem 2.7c'.} \end{cases} \tag{3.10}$$

4. THE QUADRATURE METHODS FOR INTEGRAL EQUATIONS

In what follows we consider the integral equation (1.1), with $f(t)$ and $g(t)$ being periodic in t with period T, and $K(t,x)$ being periodic both in t and x with period T. Of course, a similar treatmet can be given to the eigenvalue problem in (1.2).

4.1 The Singular Case

Let $K(t,x) = \hat{H}_1(t,x)/(t-x) + \hat{H}_2(t,x)$. For a given integer N, let $h \equiv h_{2N} = (b-a)/(2N)$, and $x_j = a+jh$, $j=1,\ldots,2N$. Then setting $t=x_i$ for some i, and approximating the integral $\int_a^b K(x_i,x)f(x)dx$ by the rule $\tilde{Q}_N$ in (3.1), we write down the following set of equations for the $2N$ unknowns $\tilde{f}_j$ (the approximations to the corresponding $f(x_j)$):

$$\omega\tilde{f}_i + 2h\sum_{j=1}^{2N} \varepsilon_{ij}K(x_i,x_j)\tilde{f}_j = g(x_i), \quad i=1,2,\ldots,2N, \tag{4.1}$$

where

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } |i-j| \text{ odd} \\ 0 & \text{if } |i-j| \text{ even.} \end{cases} \tag{4.2}$$

4.2 The Weakly Singular Case

We mentioned in the previous section that when $G(x)$ is a known function any set of integers $\{n_l\}_{l=0}^{\infty}$, $1\le n_0<n_1<\cdots$, can be chosen for computing the approximation $A_q^{(m)}$ to the integrals $\int_a^b G(x)dx$ in Theorem 2.7b,c,c'. If, however, we want to use the Romberg-type formula $A_q^{(m)}$ for solving integral equations by quadrature methods, the n_l cannot be arbitrary. In fact, we should choose the n_l (hence the $h_l=T/n_l$), such that the sets of abscissas that enter the computation of $A(h_{m+k}) = Q_{n_{m+k}}[G]$, $0\le k\le q-1$, where $G(x) = K(t,x)f(x)$, are all subsets of the set of abscissas that enter the computation of $A(h_{m+q})$. This is achieved by picking the n_l, $m\le l\le m+q-1$, as divisors of n_{m+q}. With this choice of the n_l let

$x_j = jh_{m+q} = a + jT/n_{m+q}$, $1 \le j \le n_{m+q}$. With the help of (3.9) it can be verified that, for $t = x_i$,

$$Q_{n_{m+k}}[G] = h_{m+k} \sum_{\substack{j=1 \\ j \ne i}}^{n_{m+q}} \varepsilon_{ij}^{m,q,k} G(x_j) + C(x_i, h_{m+k}), \tag{4.3}$$

where

$$\varepsilon_{ij}^{m,q,k} = \begin{cases} 1 & \text{if } |i-j| \text{ is divisible by } n_{m+q}/n_{m+k} \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Thus (3.8) becomes, for $t = x_i$,

$$A_q^{(m)} = \sum_{\substack{j=1 \\ j \ne i}}^{n_{m+q}} \left[\sum_{k=0}^{q} \varepsilon_{ij}^{m,q,k} d_{q,k}^{(m)} h_{m+k} \right] G(x_j) + \sum_{k=0}^{q} d_{q,k}^{(m)} C(x_i, h_{m+k}) . \tag{4.5}$$

Now for $j \ne i$, $G(x_j) = K(x_i, x_j) f(x_j)$. By Theorem 2.7b,c,c' we note that when b) $K(t,x) = H_1(t,x)|t-x|^s + \hat{H}_2(t,x)$, $g(x) = H_1(t,x) f(x)$ and $\tilde{g}(x) = \hat{H}_2(t,x) f(x)$ in (2.31b). Thus $C(t,h) = \hat{C}(t,h) f(t)$, where

$$\hat{C}(t,h) = h\left[\hat{H}_2(t,t) - 2\zeta(-s) H_1(t,t) h^s\right] . \tag{4.6b}$$

c) $K(t,x) = H_1(t,x)|t-x|^s \log|t-x| + \hat{H}_2(t,x)$, $g(x)$ and $\tilde{g}(x)$ in (2.31c) are as in b) above. Thus $C(t,h) = \hat{C}(t,h) f(t)$, where

$$\hat{C}(t,h) = h\left\{\hat{H}_2(t,t) + 2[\zeta'(-s) - \zeta(-s)\log h] H_1(t,t) h^s\right\}. \tag{4.6c}$$

c') $K(t,x) = H_1(t,x)\log|t-x| + \hat{H}_2(t,x)$, $g(x)$ and $\tilde{g}(x)$ in (2.31c') are as in b) and c) above. Thus $C(t,h) = \hat{C}(t,h) f(t)$, where

$$\hat{C}(t,h) = h\left[\hat{H}_2(t,t) + \log\left[\frac{h}{2\pi}\right] H_1(t,t)\right] . \tag{4.6c'}$$

Combining the above in (4.5), approximating the integral $\int_a^b K(t,x) f(x) dx$ in (1.1) by $A_q^{(m)}$, and replacing the $f(x_j)$ by the corresponding approximations $\tilde{f}_j$, $1 \le j \le n_{m+q}$, we obtain the appropriate quadrature methods for (1.1), which are defined by the systems of linear equations

$$\omega \tilde{f}_i + \sum_{j=1}^{n_{m+q}} \tilde{K}_{ij} \tilde{f}_j = g(x_i), \quad 1 \le i \le n_{m+q}, \tag{4.7}$$

where

$$\tilde{K}_{ij} = \left[\sum_{k=0}^{q} \varepsilon_{ij}^{m,q,k} d_{q,k}^{(m)} h_{m+k} \right] K(x_i, x_j), \quad j \ne i , \tag{4.8}$$

and

$$\tilde{K}_{ii} = \sum_{k=0}^{q} d_{q,k}^{(m)} h_{m+k} \hat{\mathcal{C}}(x_i, h_{m+k}), \tag{4.9}$$

with $\hat{\mathcal{C}}(t,h)$ as defined in (4.6b,c,c').

It is not the purpose of this work to give precise error bounds or convergence results for $\Lambda = \max_j |f(x_j) - \tilde{f}_j|$. However, in general, we would expect Λ to be of the order of magnitude of the error in the numerical quadrature formula used in approximating the integral $\int_a^b K(t,x) f(x) dx$. Thus, for the singular case, if $K(t,z)$ is meromorphic in the strip $|Im\ z| < \sigma$ with its only poles at $t + kT$, $k = 0, \pm 1, \pm 2, \ldots$, and $g(z)$ is analytic in the same strip, we would expect $f(z)$ to be analytic in this strip too. Therefore, by Theorem 3.2, $\int_a^b G(x) dx - \tilde{Q}_n[G] = 0\left(e^{-2\pi N\sigma/T}\right)$ as $N \to \infty$, thus we would expect $\Lambda = 0\left(e^{-2\pi N\sigma/T}\right)$ as $N \to \infty$ too. For the weakly singular case, by the appendix, $\int_a^b G(x) dx - A_q^{(m)} = 0\left(e_{q+1}(h_m)\right)$ as $m \to \infty$, in general. Thus, we would expect $\Lambda = 0\left(h_m^{s+1+2q} (\log h_m)^p\right)$ as $m \to \infty$, where $s = \max_{1 \le k \le M} \alpha_k$ and $p = \max_{1 \le k \le M} p_k$, (cf. (1.3)).

Finally, we could use the approximations $\tilde{f}_i$ to $f(x_i)$, $i = 1, \ldots, \hat{N}$, where $\hat{N} = 2N$, for singular equations and $\hat{N} = n_{m+q}$ for weakly singular equations, to construct a trigonometric interpolation polynomial $P_{\hat{N}}(x)$ in $\cos(2\pi kx/T)$, $\sin(2\pi kx/T)$, $k = 0, 1, \ldots$, satisfying $P_{\hat{N}}(x_i) = \tilde{f}_i$, $i = 1, \ldots, \hat{N}$, thus obtaining an approximation to $f(x)$ for all x in $[a, b]$.

5. NUMERICAL EXAMPLES

5.1 The Singular Case

Example 5.1.1 (See Mikhlin (1964, pp. 122-124))

$$af(t) + \frac{b}{2\pi}\int_0^{2\pi} \cot\left(\frac{x-t}{2}\right) f(x)dx = u(t). \tag{5.1}$$

When $a \neq 0$ and $a^2+b^2 \neq 0$ (a and b may be complex) a unique solution exists and is given by

$$\begin{aligned} f(t) = {} & \frac{a}{a^2+b^2}u(t) - \frac{b}{2\pi(a^2+b^2)}\int_0^{2\pi} u(x)\cot\left(\frac{x-t}{2}\right) \\ & + \frac{b^2}{2\pi a(a^2+b^2)}\int_0^{2\pi} u(x)dx. \end{aligned} \tag{5.2}$$

We first observe that the kernel function $K(t,x) = \frac{b}{2\pi}\cot\left(\frac{x-t}{2}\right)$ is periodic with period 2π in both x and t. Also for fixed t, $K(t,z)$ is meromorphic in the whole z-plane with simple poles at $t+2\pi k$, $k=0,\pm1,\pm2,\ldots$, thus being of the form described in Section 4.1. Next, we observe that if $u(t)$ is periodic with period 2π, then so is $f(t)$. Also if $u(z)$ is analytic in a strip $|\text{Im } z| < \sigma$, then so is $f(z)$. The last two assertions can be verified with the help of (5.2).

In our numerical experiments we chose $u(t) = (D+\cos t)^{-1}$, $D>1$, so that both $u(z)$ and $f(z)$ are analytic in the strip $|\text{Im } z| < \sigma = \log(D + \sqrt{D^2-1})$. With this choice of $u(t)$, (5.2) becomes

$$f(t) = \frac{1}{a^2+b^2}\left\{\frac{1}{D+\cos t}\left(a - \frac{b \sin t}{\sqrt{D^2-1}}\right) + \frac{b^2}{a\sqrt{D^2-1}}\right\}.$$

Denote $2N$, the number of abscissas in (4.1), by $\hat{N}$. Then $h = 2\pi/\hat{N}$ and $x_j = jh$, $1 \le j \le \hat{N}$. Denote $\tilde{f}_{\hat{N},j} \equiv \tilde{f}_j$, $1 \le j \le \hat{N}$, and let $\Lambda_{\hat{N}} = \max_{1\le j\le \hat{N}} |f(x_j) - \tilde{f}_{\hat{N},j}|$.

Then, by what has been said in the paragraph following (4.9), we would expect to have $\Lambda_{\hat{N}} = 0(e^{-\sigma\hat{N}/2})$ as $\hat{N} \to \infty$. This is born out by the numerical results, where, for each $\hat{N}$, σ can be estimated by the formula

$$\sigma \approx \frac{2}{\hat{N}_{\max}-\hat{N}} \log \frac{\Lambda_{\hat{N}}}{\Lambda_{\hat{N}_{\max}}} \equiv \sigma_{\hat{N},\hat{N}_{\max}} \ , \tag{5.3}$$

where $\hat{N}_{\max}$ is the maximum of the $\hat{N}$'s used in the computation. This estimate, of course, is based on the $\Lambda_{\hat{N}}$, which in turn are based on $f(x)$. An estimate based solely on the computed values $\tilde{f}_{\hat{N},j}$ can be obtained from the formula

$$\sigma \approx \frac{2}{J} \log \left| \frac{\tilde{f}_{\hat{N}+2J,\hat{N}+2J} - \tilde{f}_{\hat{N}+J,\hat{N}+J}}{\tilde{f}_{\hat{N}+J,\hat{N}+J} - \tilde{f}_{\hat{N},\hat{N}}} \right| \equiv \sigma_{\hat{N}} \ . \tag{5.4}$$

This follows from the following expected behavior of $\tilde{f}_{\hat{N},\hat{N}}$:

$$\tilde{f}_{\hat{N},\hat{N}} \sim f(x_{\hat{N}}) + C(x_{\hat{N}}) e^{-\sigma \hat{N}/2} \quad \text{as } \hat{N} \to \infty \ . \tag{5.5}$$

Here, of course, $\tilde{f}_{\hat{N},\hat{N}}$ and $\hat{x}_N = 2\pi$ can be replaced by $\tilde{f}_{\hat{N},j(\hat{N})}$ and $\bar{x}$ respectively, where $x_{j(\hat{N})} = \bar{x}$ is the same for all $\hat{N}$ used in the computations (in (5.4) they are $\hat{N}$, $\hat{N}+J$, and $\hat{N}+2J$).

Table 5.1.1 gives the results obtained for $\Lambda_{\hat{N}}$, $\sigma_{\hat{N},\hat{N}_{\max}}$, and $\sigma_{\hat{N}}$, $\hat{N} = 4(4)44$, $\hat{N}_{\max} = 44$, with $\alpha = b = 1$. Note that $\sigma_{44,44}$, σ_4 and σ_{12} are not defined. Note also that, for $D=2$, $\sigma_{\hat{N},\hat{N}_{\max}}$ and $\sigma_{\hat{N}}$ deteriorate for $\hat{N}$ large. This is due to the fact that there is a loss of significance in the arguments of log in (5.3) and (5.4), which is caused by the high accuracy of the $\tilde{f}_{\hat{N},j}$.

$\hat{N}$	D=1.1			D=2		
	$\Lambda_{\hat{N}}$	$\sigma_{\hat{N},\hat{N}_{\max}}$	$\sigma_{\hat{N}}$	$\Lambda_{\hat{N}}$	$\sigma_{\hat{N},\hat{N}_{\max}}$	$\sigma_{\hat{N}}$
4	2.03×10^{0}	0.426		6.10×10^{-2}	1.3136	
8	1.12×10^{0}	0.4407		4.60×10^{-3}	1.3157	
12	4.93×10^{-1}	0.0445	0.67	3.37×10^{-4}	1.3170	1.354
16	2.01×10^{-1}	0.4442	0.52	2.41×10^{-5}	1.31677	1.3195
20	7.98×10^{-2}	0.4411	0.474	1.73×10^{-6}	1.31683	1.3171
24	3.33×10^{-2}	0.4420	0.456	1.25×10^{-7}	1.3171	1.316971
28	1.39×10^{-2}	0.44339	0.4485	8.94×10^{-9}	1.3169586	1.3169588
32	5.73×10^{-3}	0.44303	0.4456	6.42×10^{-10}	1.316996	1.3169580
36	2.33×10^{-3}	0.4400	0.4444	4.62×10^{-11}	1.3173	1.3169587
40	9.72×10^{-4}	0.4420	0.44391	3.31×10^{-12}	1.3172	1.316948
44	4.01×10^{-4}		0.44371	2.38×10^{-13}		1.3176

Table 5.1.1: Results for $\Lambda_{\hat{N}}, \sigma_{\hat{N},\hat{N}_{\max}}$, and $\sigma_{\hat{N}}$, $\hat{N}=4(4)44$, $\hat{N}_{\max}=44$, for Example 5.1.1. Exact values of σ are 0.44357 for D=1.1 and 1.3169579 for D=2.

5.2 The Weakly Singular Case

In both of the examples below the kernel function is of the form described in c') of Section 4.2, namely $K(t,x) = H_1(t,x)\log|t-x| + \hat{H}_2(t,x)$. Therefore, the approximations $\tilde{f}_j$ to $f(x_j)$, the solution to the integral equation, satisfy (4.7)-(4.9) with (4.4) and (4.6c'). Here the $d_{q,k}^{(m)}$, $0 \le k \le q$, in (4.8) and (4.9) can be determined from (A.5) with $e_i(h) = h^{2i+1}$, $i=1,2,\ldots$. If we let $n_l = 2^l$, thus $h_l = T/2^l$, $l=0,1,2,\ldots$, as we do in the examples below, then the $d_{q,k}^{(m)}$ are independent of m, and can be computed recursively from (A.6) with $\sigma_i = 2^{-2i-1}$, $i=1,2,\ldots$. Thus we find $d_{0,0}^{(m)} = 1$, $d_{1,0}^{(m)} = -1/7$, $d_{1,1}^{(m)} = 8/7$, $d_{2,0}^{(m)} = 1/217$, $d_{2,1}^{(m)} = -40/217$, $d_{2,2}^{(m)} = 256/217$, etc. From what has been said in paragraph following (4.9) and from (A.7), we would expect to have $\Lambda_q^{(m)} = 0(\sigma_{q+1}^m) = 0(2^{-(2q+3)m})$ as $m \to \infty$, where $\Lambda_q^{(m)} \equiv \max_{n_m \le j \le n_{m+q}} |f(x_j) - \tilde{f}_j|$. This is indeed observed numerically.

Example 5.2.1 (See Christiansen (1971))

$$\int_0^{2\pi} \log\left(2a \sin \frac{|t-x|}{2}\right) f(x)dx = -\frac{\pi}{2}\cos 2t.$$

Provided $a \ne 1$, the unique solution to this equation is $f(t) = \cos 2t$. Otherwise, the solution is $f(t) = \cos 2t + c$, c being an arbitrary constant. We observe that the kernel $K(t,x) = \log\left(2a \sin \frac{|t-x|}{2}\right)$ is periodic with period 2π in both t and x, and is of the form described in c') of Section 4.2, namely, $K(t,x) = H_1(t,x)\log|t-x| + \hat{H}_2(t,x)$, with $H_1(t,t) = 1$ and $\hat{H}_2(t,t) = \log a$.

Table 5.2.1 gives some of the results obtained with $a = \sqrt{e}$ for $\Lambda_q^{(m)}$, $N = n_{m+q} = 2^{m+q}$, $m+q = 3(1)7$. Note that, for a given row in this table, N, the number of abscissas in (4.7)-(4.9), is he same for every member of this row. Thus the first column is the result of no extrapolation, the second, of one extrapolation, the third, of two extrapolations, etc. As can be seen for a given number of abscissas $N=2^r$, the best results are obtained roughly for $q \approx r/2$.

$m+q$	q							
	0	1	2	3	4	5	6	7
3	3.8×10^{-2}	$\underline{9.9\times10^{-3}}$	4.0×10^{-2}	4.9×10^{-2}				
4	4.7×10^{-3}	2.3×10^{-4}	$\underline{7.4\times10^{-5}}$	3.7×10^{-4}	4.7×10^{-4}			
5	5.9×10^{-4}	6.9×10^{-6}	4.3×10^{-7}	$\underline{1.4\times10^{-7}}$	8.8×10^{-7}	1.1×10^{-6}		
6	7.3×10^{-5}	2.1×10^{-7}	3.2×10^{-9}	2.1×10^{-10}	$\underline{6.9\times10^{-11}}$	5.0×10^{-10}	6.3×10^{-10}	
7	9.2×10^{-6}	6.6×10^{-9}	2.5×10^{-11}	5.0×10^{-13}	1.5×10^{-13}	$\underline{1.2\times10^{-13}}$	1.8×10^{-13}	1.8×10^{-13}

Table 5.2.1: Results for $\Lambda_q^{(m)}$ for Example 5.2.1.
Here $N=2^{m+q}$ is the number of abscissas in the quadrature method and q is the number of extrapolations in the corresponding numerical quadrature formula. The best result for fixed N (or $m+q$) is underlined.

Example 5.2.2 (see Henrici (1979, pp. 492-494))

Let $\Gamma: z=z(\tau)$, $0\le\tau\le\beta$, be the boundary curve of a Jordan region D containing the point $z=0$, and let $f(z)$ be the function mapping D conformally onto the unit disk $|\omega|<1$ in such a manner that $f(0)=0$ and $f'(0)>0$. To determine $f(z)$ it is sufficient to know its values on the boundary of D. Because $|f(z)|=1$, it suffices to know $\vartheta(\tau) = \arg(f(z(\tau)))$, $0\le\tau\le\beta$. Let $\xi(\tau) = \vartheta'(\tau)$. Then, provided the capacity of Γ is different than 1, $\xi(\tau)$ is the unique solution of

$$\int_0^\beta \log|z(\sigma)-z(\tau)|\,\xi(\tau)d\tau = 2\pi\log|z(\sigma)|, \quad 0\le\sigma\le\beta, \tag{5.6}$$

which is known as Symm's equation (see Symm (1966)). The capacity of Γ is different than 1, in particular, if Γ is entirely within or entirely without the unit circle.

Obviously, the kernel $K(\sigma,\tau) = \log|z(\sigma)-z(\tau)|$ is periodic in both σ and τ with period β, and is of the form $K(\sigma,\tau) = H_1(\sigma,\tau)\log|\sigma-\tau| + \hat{H}_2(\sigma,\tau)$ with $H_1(\sigma,\sigma) = 1$ and $\hat{H}_2(\sigma,\sigma) = \log|z'(\sigma)|$. Also, $\log|z(\tau)|$ and $\xi(\tau)$ are periodic with period β.

Let $\beta=2\pi$ and let D be the elliptic domain whose boundary curve is Γ: $z(\tau) = c(e^{i\tau}+\varepsilon e^{-i\tau})$, $0\le\tau\le2\pi$, with $0<\varepsilon<1$, and $c>0$ chosen so that Γ is entirely without the unit circle. (Actually the capacity of Γ in this case is c, so that it is sufficient to choose $c\ne1$.) The semi-axes of D are $c(1+\varepsilon)$ and $c(1-\varepsilon)$. The solution for $\xi(\tau)$ can be expressed as

$$\xi(\tau) = 1+4 \sum_{k=1}^{\infty} (-1)^k \frac{\varepsilon^k}{1+\varepsilon^{2k}} \cos (2k\tau).$$

Note that both $\log|z(\tau)|$ and $\xi(\tau)$ are analytic functions of τ.

Observe that $\xi(\tau)$ for this example is symmetric with respect to both the Re z and the Im z axes. This can be utilized to reduce the dimensions of the matrix by 4, thus reducing the storage and computing time considerably.

Tables 5.2.2a and 5.2.2b give some of the results obtained with c=50 and ε=0.1 and ε=0.5 respectively, for $\Lambda_q^{(m)}$, $N = n_{m+q} = 2^{m+q}$, $m+q \le 7$. As in Table 5.2.1, in these tables too, for a given row, N, the number of abscissas in (4.7)-(4.9), is the same for every member of this row.

$m+q$	q			
	0	1	2	3
2	$\underline{1.6\times10^{-1}}$			
3	2.9×10^{-2}	$\underline{2.7\times10^{-2}}$		
4	4.0×10^{-3}	$\underline{8.1\times10^{-4}}$	4.5×10^{-3}	
5	5.0×10^{-4}	$\underline{2.7\times10^{-5}}$	6.1×10^{-5}	
6	6.3×10^{-5}	7.1×10^{-7}	$\underline{1.0\times10^{-7}}$	4.8×10^{-7}
7	7.8×10^{-6}	2.2×10^{-8}	6.7×10^{-10}	$\underline{1.5\times10^{-10}}$

Table 5.2.2a: Results for $\Lambda_q^{(m)}$ for Example 5.2.2 with ε=0.1 and c=50. Here $N=2^{m+q}$ is the number of abscissas in the quadrature method and q is the number of the extrapolations in the corresponding numerical quadrature formula. The best result for fixed N (or $m+q$) is underlined.

$m+q$	q		
	0	1	2
5	$\underline{5.7\times10^{-2}}$	1.9×10^{-1}	
6	$\underline{1.6\times10^{-3}}$	1.5×10^{-3}	
7	9.4×10^{-4}	$\underline{3.2\times10^{-5}}$	3.6×10^{-5}

Table 5.2.2b: Results for $\Lambda_q^{(m)}$ for Example 5.2.2 with ε=0.5 and c=50. Here $N=2^{m+q}$ is the number of abscissas in the quadrature method and q is the number of extrapolations in the corresponding numerical quadrature formula. The best result for fixed N (or $m+q$) is underlined.

For small values of ε the ellipse is close to a circle. Therefore, $\xi(\tau)$ does not change rapidly with σ, and this explains the high accuracy obtained for the approximations to $\xi(\tau)$ even with a small number of abscissas when ε=0.1. For large values of ε, however, the ellipse is elongated, and this leads to rapid changes in $\xi(\tau)$ in the vicinity of $\tau = \pi/2$ and $\tau = 3\pi/2$, i.e., where $\xi(\tau)$ is maximal. This then explains the slow convergence of the approximations for ε=0.5. Furthermore, extrapolation becomes effective in this case starting with a relatively large N.

To improve the performance of the quadrature method above for the cases in which $\xi(\tau)$ has rapid changes we can make a change of variable of integration $\tau = \tau(\psi)$ so that $\xi(\tau(\psi))$ changes slowly as a function of ψ. This can be achieved by picking $\tau(\psi)$ such that $d\tau/d\psi$ becomes small where $d\xi/d\tau$ is large. This is equivalent to having more abscissas in places where $\xi(\tau)$ changes rapidly. Needless to say, the transformation $\tau = \tau(\psi)$ should be such that $d\tau/d\psi$ is a periodic function of ψ.

For the example under consideration we can choose

$$\frac{d\psi}{d\tau} = \frac{\eta\sqrt{1+\eta^2}}{\eta^2+\cos^2\tau}\,, \quad \eta \text{ a positive constant,}$$

so that

$$\psi(\tau) = \tan^{-1}\left(\frac{\sqrt{1+\eta^2}}{\eta}\tan\tau\right), \quad 0\leq\tau\leq\pi/2$$

or

$$\tau(\psi) = \tan^{-1}\left(\frac{\eta}{\sqrt{1+\eta^2}}\tan\psi\right), \quad 0\leq\psi\leq\pi/2\,.$$

$\tau(\psi)$ is now extended so that

$$\tau(\psi) = \pi - \tau(\pi-\psi), \quad \pi/2\leq\psi\leq\pi\,,$$
$$\tau(\psi) = \pi + \tau(\psi-\pi), \quad \tau\leq\psi\leq 2\pi\,.$$

We used this transformation with different values of η. In Table 5.2.3 we give some of the results with c=50 and ε=0.5 obtained for the errors at $\tau = \pi/2$, the point at which the error is maximum, and at τ=0. The number of abscissas in all cases is N=32, and q=0, i.e., no extrapolation is employed. Nevertheless, the improvement in the results is remarkable.

η	error for $\tau=0$	error for $\tau=\pi/2$
0.2	4.0×10^{-3}	8.1×10^{-4}
0.5	1.2×10^{-4}	2.3×10^{-3}
100	2.4×10^{-4}	5.7×10^{-2}

Table 5.2.3: Results for the error at $\tau=0$ and $\tau=\pi/2$ in the approximations to $\xi(\tau)$ in Example 5.2.2, with $c=50$ and $\varepsilon=0.5$, and the change of variable $\tau=\tau(\psi)$ as described in the text. The numerical quadrature formula used has $N=32$ abscissas and does not employ extrapolation. $\xi(0)=0.014671...$ and $\xi(\pi/2)=4.5324....$

6. CONCLUDING REMARKS

In this section we shall briefly discuss some known quadrature methods that are related to those proposed in the present work.

A lot of attention has been paid to Cauchy principal value integrals and singular integral equations. We do not intend to survey all the methods developed for these, but we shall restrict our attention to those that are periodic. When $K(t,x) = \frac{1}{2\pi}\cot\left(\frac{t-x}{2}\right)$, $0\le t,x\le 2\pi$, (the Hilbert kernel),

$$\tilde{K}_{ij} = 2h\varepsilon_{ij}K(x_i,x_j) = \frac{1}{N}\cot\left(\frac{(i-j)\pi}{2N}\right)\varepsilon_{ij},\ \ i,j=1,\dots,2N,$$

(cf. (4.1)). The matrix $\tilde{K} = (\tilde{K}_{ij})$ is called Wittich's matrix (see Gaier (1964, p. 76)) and its properties are well known. Gutknecht (1981) has used Wittich's matrix to discretize Theodorsen's integral equation for conformal mapping and has analyzed various nonlinear iterative techniques for solving the resulting equations. We note that Theodorsen's equation has the Hilbert kernel as its kernel.

The Hilbert kernel arises as part of the Cauchy kernel on closed curves and Atkinson (1972b) has proposed and analyzed product-type integration formulas for the Cauchy transform

$$\int_\Gamma \frac{\varphi(\zeta)}{\zeta - z}, \quad \Gamma \text{ closed}, \quad z \in \Gamma,$$

which are different than those proposed in the present work.

As for the weakly singular integral equations, the case that has received the widest attention is that of logarithmic singularity. Periodic integral equations with logarithmically singular kernels arise naturally, for example, in conformal mapping (Symm's equation) and two-dimensional potential theory. Two of the quadrature methods that have been considered for such equations are based on the so called *modified quadrature method* (see Kantorovich and Krylov (1964, p. 102), and Baker (1977, Chap. 5, Sect. 4)). In this method we begin by writing (1.1) in the form

$$\omega f(t) + \int_a^b K(t,x)[f(x) - f(t)]dx + f(t)\int_a^b K(t,x)dx = g(t). \tag{6.1}$$

Employing the numerical quadrature formula $\sum_{j=1}^{n} w_j F(x_j)$ to approximate the integral $\int_a^b F(x)dx$, and using the fact that $\lim_{x \to t} K(t,x)[f(x) - f(t)] = 0$, we replace the integral equation above by the linear equations

$$\left[\omega + \int_a^b K(x_i,x)dx\right]\tilde{f}_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j K(x_i,x_j)[\tilde{f}_j - \tilde{f}_i] = g(x_i), \quad i = 1,\ldots,n, \tag{6.2}$$

where $\tilde{f}_i$ are approximations to the $f(x_i)$. Kussmaul and Werner (1968) have applied this method with equidistant abscissas $x_{i+1} - x_i = h$, $i = 1,\ldots,n-1$, to periodic integral equations with logarithmically singular kernels and have shown that the error is $0(h^3)$ as $h \to 0$, assuming $\int_a^b K(t_i,x)dx$ has been computed exactly. For kernels of the type $K(p,q) = \log \rho(p,q)$, where $\rho(p,q) = \sqrt{(x(p) - x(q))^2 + (y(p) - y(q))^2}$, and $(x(p), y(p))$, $a \le p \le b$, is the parametric representation of a simple closed curve in the $x-y$ plane, Christiansen (1971) has used the modified quadrature method in (5.1) with $\int_a^b K(t_i,q)dq$ essentially replaced by a numerical quadrature approximation. The numerical results indicate that this method too has an error of $O(h^3)$ as $h \to 0$. For both methods above we can show, using Theorem 2.7c', that the errors in the numerical quadrature formulas are $O(h^3)$ as $h \to 0$, although this proof for Christiansen's method becomes very complicated.

For weakly singular Fredholm integral equations of the second kind methods based on product integration have also been developed and analyzed by Atkinson (1967, 1972a).

Finally, the approach of this work can easily be extended to coupled Fredholm integral equations for several unknown functions in which several types of singularities occur simultaneously.

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APPENDIX

In this appendix we summarize some of the important aspects of generalizations of the classical Richardson extrapolation process.

Let the function $A(h)$, where $h>0$ is a continuous or discrete variable, have the asymptotic expansion

$$A(h) \sim A + \sum_{i=1}^{\infty} \beta_i e_i(h) \quad \text{as } h \to 0, \tag{A.1}$$

with $e_{i+1}(h) = o(e_i(h))$ as $h \to 0$, $i=1,2,\ldots$. $A(h)$ and the $e_i(h)$ are known, but A and the β_i are not. We are interested in approximating A, which, in general, is $\lim_{h\to 0} A(h)$ when this limit exists.

Select a sequence of h's, namely $h_0 > h_1 > \ldots$, such that $\lim_{l\to\infty} h_l = 0$, and define $A_n^{(j)}$ (the approximation to A) and the $\bar{\beta}_i$ to be the solution of the system of linear equations

$$A(h_l) = A_n^{(j)} + \sum_{i=1}^{n} \bar{\beta}_i e_i(h_l), \quad j \le l \le j+n. \tag{A.2}$$

For some classes of functions $A(h)$ and $e_i(h)$, $i=1,2,\ldots$, and some choices of h_l, $l=0,1,\ldots$, the following can be shown:

1) For fixed n,

$$A - A_n^{(j)} = 0\left[e_{n+1}(h_j)\right] \quad \text{as } j \to \infty \quad (i.e., h_j \to 0), \tag{A.3}$$

2) For fixed j, $A_n^{(j)} \to A$ as $n \to \infty$, and the convergence in this case is better than in the previous case.

For the present work, it is important to note that $A_n^{(j)}$ can be expressed as

$$A_n^{(j)} = \sum_{k=0}^{n} d_{n,k}^{(j)} A(h_{j+k}), \tag{A.4}$$

where the coefficients $d_{n,k}^{(j)}$ can be obtained directly by solving the linear system of equations

$$\sum_{k=0}^{n} d_{n,k}^{(j)} = 1$$
$$\sum_{k=0}^{n} e_i(h_{j+k})d_{n,k}^{(j)} = 0, \quad 1\le i\le n. \tag{A.5}$$

The problem of computing the $A_n^{(j)}$ recursively has been attacked by several authors. Schneider (1975), Havie (1979), and Brezinski (1980) have devised an algorithm that has been denoted the E-algorithm. Recently, a more efficient algorithm has been derived by Ford and Sidi (1984).

Special Cases

a) $e_i(h) = h^{\gamma_i}$, $0<\gamma_1<\gamma_2<\ldots$, $\lim\limits_{i\to\infty}\gamma_i = \infty$.

For the choice $h_l = h_0\rho^l$, $l=0,1,\ldots$, $0<\rho<1$, the $d_{n,k}^{(j)}$ can be computed from the recursion relation

$$d_{n,k}^{(j)} = \frac{\sigma_n d_{n-1,k}^{(j)} - d_{n-1,k-1}^{(j+1)}}{\sigma_n - 1}, \quad 0\le k\le n, \tag{A.6}$$

with $d_{n,-1}^{(j)} = 0 = d_{n,n+1}^{(j)}$, $n=0,1,\ldots$, $d_{0,0}^{(j)}=1$, $j=0,1,\ldots$, and $\sigma_i=\rho^{\gamma_i}$, $i=1,2,\ldots$. Obviously, the $d_{n,k}^{(j)}$ are *independent* of j. This development, in slightly different notation, is due to Bulirsch and Stoer (1964), who also give a recursive algorithm for $A_n^{(j)}$ and a thorough convergence analysis. First,

$$A-A_n^{(j)} = 0(\sigma_{n+1}^j) \quad \text{as } j\to\infty. \tag{A.7}$$

Second, if there exist constants $\tilde{\beta}_k$, $k=1,2,\ldots$, for which

$$|A(h)-A-\sum_{i=1}^{N-1}\beta_i h^{\gamma_i}| \le \tilde{\beta}_N h^N, \quad h\le h_0, \tag{A.8}$$

and, for some fixed $R>0$,

$$\tilde{\beta}_k = 0(k!R^k) \quad \text{as } k\to\infty, \tag{A.9}$$

then, for the special case $\gamma_i = \gamma_0+ir$, for some $\gamma_0>0$ and $r>0$,

$$A-A_n^{(j)} = 0(\omega^{n^2}) \quad \text{as } n\to\infty, \tag{A.10}$$

where $\omega=\rho^{r/2}+\varepsilon<1$, any $\varepsilon>0$. For a similar result pertaining to the case of arbitrary γ_i see Bulirsch and Stoer (1964).

b) $e_i(h) = \varphi(h)h^{ir-r}$, $\quad r>0$.

For arbitrary h_l, the $A_n^{(j)}$ can be computed by using the recursive W-algorithm of Sidi (1982). As a consequence of this algorithm we have

$$d_{n,k}^{(j)} = \frac{\delta_{n,k}^{(j)}}{\sum_{i=0}^{n}\delta_{n,i}^{(j)}}, \qquad 0\le k\le n, \tag{A.11}$$

where $\delta_{n,k}^{(j)}$ can be obtained from the recursion relation

$$\delta_{n,k}^{(j)} = \frac{\delta_{n-1,k-1}^{(j+1)}-\delta_{n-1,k}^{(j)}}{h_{j+n}^{r}-h_{j}^{r}}, \quad 0\le k\le n, \tag{A.12}$$

with $\delta_{n,-1}^{(j)} = 0 = \delta_{n,n+1}^{(j)}$, $n=0,1,\ldots$, and $\delta_{0,0}^{(j)} = 1/\varphi(h_j)$, $j=0,1,\ldots$. We note that this is also a special case of the generalized Richardson extrapolation process of Sidi (1979), a very efficient recursive algorithm for which has recently been given by Ford and Sidi (1984).

We also note that the algorithms given for the two special cases above are more efficient than the algorithms for the general extrapolation algorithms for obtaining $A_n^{(j)}$ defined by (A.2), since they take advantage of the special forms of the problem.

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16. Abstract

High accuracy numerical quadrature methods for integrals of singular periodic functions are proposed. These methods are based on the appropriate Euler-Maclaurin expansions of trapezoidal rule approximations and their extrapolations. They are used to obtain accurate quadrature methods for the solution of singular and weakly singular Fredholm integral equations. Such periodic equations are used in the solution of planar elliptic boundary value problems, elasticity, potential theory, conformal mapping, boundary element methods, free surface flows, etc. The use of the quadrature methods is demonstrated with numerical examples.

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