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ABSTRACT

Nonlinear algebraic functional expansions are used to create a form for the unsteady aerodynamic response that is consistent with solutions of the time-dependent Navier-Stokes equations. An enumeration of means of invalidating Fréchet differentiability of the aerodynamic response, one of which is aerodynamic bifurcation, is proposed as a way of classifying steady and unsteady aerodynamic phenomena that are important in flight dynamics applications. Accommodating bifurcation phenomena involving time-dependent equilibrium states within a mathematical model of the aerodynamic response raises an issue of memory effects that becomes more important with each successive bifurcation.

NOMENCLATURE

C_L	lift coefficient
C_ℓ	rolling-moment coefficient
$\mathscr{L}$	Laplace transform
p	surface pressure
s	Laplace transform variable
Re	Reynolds number
t	observation time
U_0	axial velocity at wing center of gravity
u,v	fluid velocity components in x,y directions
v_c	vertical velocity at wing center of gravity
X,Y	space-fixed coordinates (Fig. 1)
x,y	moving coordinates (Fig. 1)
x,r,θ	moving coordinates (Fig. 3)
α	angle of attack (Fig. 1)
ν	kinematic viscosity
ξ	running variable in time
τ	time at origin of step
ϕ	stream function
ψ	roll angle
ω	vorticity

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INTRODUCTION

There is now widespread recognition that the evolution of modern aircraft design makes mandatory an ever closer interaction between workers in the fields of flight dynamics and unsteady aerodynamics. Testimony to this recognition was the recent joint sponsorship by the AGARD Fluid Dynamics and Flight Dynamics Panels of a meeting entitled "Symposium on Unsteady Aerodynamics-Fundamentals and Applications to Aircraft Dynamics" (1). (See also the useful comments and recommendations by the authors of an evaluation report of this symposium (2)). In a contribution to the symposium, Tobak and Chapman (3) continued their effort to provide a consistent formulation and theoretical method for studying nonlinear problems in flight dynamics. They focused attention on the role of bifurcation theory in the mathematical modeling of the aerodynamic contribution to the aircraft's equations of motion. Demonstrating the necessity of accommodating aerodynamic bifurcation phenomena within the mathematical model raised a number of themes that touch on the concerns of workers in both flight dynamics and unsteady aerodynamics. In the interest of widening the ground for common endeavor, an elaboration of three of these themes will be the subject of this report.

First, we further explore the idea of using nonlinear algebraic functional expansions as a more concrete means of representation than the abstract use of functionals themselves. We have already noted that our approach to modeling, involving nonlinear indicial responses and generalized superposition integrals, could be made compatible with one based on nonlinear functional expansions. Here, we show that the form of the expansions is consistent with solutions of the unsteady Navier-Stokes equations and promotes the introduction of transform methods.

Second, we discuss the issue of Fréchet differentiability of the aerodynamic response. We have already noted that aerodynamic bifurcation is but one means of invalidating Fréchet differentiability, and we have proposed that a theory for enumerating these means offers a possible alternative to bifurcation theory. Inasmuch as it would incorporate the latter, it would be a potentially more inclusive way of classifying steady and unsteady aerodynamic phenomena that are important in flight dynamics applications. Here, we begin such a theory with an account of the means of invalidating Fréchet differentiability that we have currently identified.

Finally, we take up a potentially important issue that emerges when we try to implement the accommodation within the mathematical model of bifurcation phenomena having time-dependent equilibrium states. Recognizing, for example, that with a Hopf bifurcation it is necessary to specify phase as well as amplitude of the periodic equilibrium state, one can frame the general question as follows: given that the past motion determines the initial condition for the specification of an indicial response, how much of the past motion must be acknowledged in order to ensure that the equilibrium state of the response is adequately specified? As we shall see, time-dependent equilibrium states are never entirely free of dependence on the past motion, and the dependence becomes increasingly binding with each successive bifurcation.

NONLINEAR ALGEBRAIC FUNCTIONAL EXPANSIONS

Reference (3) contains an analysis showing how a form for the nonlinear aerodynamic response in lift coefficient C_L to an arbitrary variation in angle of attack α could be constructed by means of a suitable summation of lift responses to pulses in α. To set the stage for what follows, the reasoning behind the form of the result obtained in (3) is summarized briefly below.

Figure 1 illustrates the motion under study when it is referred to an X,Y coordinate system that is fixed in space. The wing moves away from the origin with fixed axial velocity U_o (a slight change from the motion considered in (3) where resultant velocity was kept fixed), and at the same time is allowed to translate vertically with vertical velocity at the center of gravity $v_c(\xi)$ being an arbitrary function of time ξ. The angle of attack α is defined to be the angle between the resultant velocity vector and the chord line of the wing, so that $\alpha = \tan^{-1}\{[v_c(\xi)]/U_o\}$. It was argued that the response in lift at time t to a single pulse in α at time ξ_1 ($\xi_1 < t$) would have the form

$$\Delta C_L(t) = \Delta C_L(t - \xi_1,\alpha(\xi_1))_{dir} = \sum_n \hat{a}_n(t - \xi_1)[\alpha(\xi_1)]^n\, \Delta\xi_1 \quad (1)$$

whereas the response in lift at t to a pair of pulses in α at ξ_1 and ξ_2 (with $\xi_1 < \xi_2 < t$) would have the form

$$\Delta C_L(t) = \Delta C_L(t - \xi_1,\alpha(\xi_1))_{dir} + \Delta C_L(t - \xi_2,\alpha(\xi_2))_{dir} + \Delta^2 C_L(t)_{int,2} \quad (2)$$

with

$$\Delta^2 C_{L_{int,2}} = \sum_{m,n} \hat{b}_{mn}(t - \xi_2,\xi_2 - \xi_1)[\alpha(\xi_2)]^m \times [\alpha(\xi_1)]^n\, \Delta\xi_1\, \Delta\xi_2 \quad (3)$$

and it was observed that the process of adding pulses could be continued indefinitely in the same way. Going to the limit of a continuous distribution of pulses starting at time $\xi = 0$ led to a summation of multiple integrals having the form

$$C_L(t) = C_{L_{dir}} + C_{L_{int,2}} + C_{L_{int,3}} + \cdots \quad (4)$$

with

$$C_{L_{dir}} = \sum_n \int_0^t \hat{a}_n(t - \xi_1)[\alpha(\xi_1)]^n\, d\xi_1 \quad (5)$$

$$C_{L_{int,2}} = \sum_{m,n} \int_0^t [\alpha(\xi_2)]^m d\xi_2 \int_0^{\xi_2} \hat{b}_{mn}(t - \xi_2,\xi_2 - \xi_1) \times [\alpha(\xi_1)]^n d\xi_1 \quad (6)$$

$$C_{L_{int,3}} = \sum_{m,n,p} \int_0^t [\alpha(\xi_3)]^m d\xi_3 \int_0^{\xi_3} [\alpha(\xi_2)]^n d\xi_2 \times \int_0^{\xi_2} [\alpha(\xi_1)]^p \hat{c}_{mnp}(t - \xi_3,\xi_3 - \xi_2,\xi_2 - \xi_1) d\xi_1 \quad (7)$$

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Written as a nonlinear functional expansion, Eq. (4) represents the lift coefficient at time t in response to an arbitrary variation of angle of attack α over the time interval 0 to t.

The form of Eq. (4) could be put on a firmer basis if it could be shown to be consistent with a solution of the Navier-Stokes equations. To that end, we shall write an analogous form for the stream function $\phi(\vec{x},t)$ governing the unsteady flow past a two-dimensional wing and inquire whether in fact it is consistent with a solution of the time-dependent Navier-Stokes equations. Referring again to Fig. 1, let us transfer coordinates to the moving inertial frame (x,y) in which the fluid velocity components are respectively u and v, and where u aproaches the constant velocity U_o as $|\vec{x}| \to \infty$. We confine the study to two-dimensional flow, which enables the introduction of a stream function $\phi(\vec{x},t)$. Let us further require that the flow be incompressible and that the stream function satisfy the corresponding Navier-Stokes equation, written in the convenient form

$$\omega_t + u\omega_x + v\omega_y - \nu\nabla^2\omega = 0 \qquad (8)$$

with

$$\left.\begin{aligned} \omega &= u_y - v_x = \nabla^2\phi \\ u &= \phi_y \\ v &= -\phi_x \end{aligned}\right\} \qquad (9)$$

Guided by the form of Eq. (4), we assign $\phi(\vec{x},t)$ the form

$$\begin{aligned} \phi(\vec{x},t) = \phi_o(\vec{x}) &+ \sum_n \int_o^t a_n(t-\xi_1,\vec{x})[\alpha(\xi_1)]^n d\xi_1 \\ &+ \sum_{m,n} \int_o^t [\alpha(\xi_2)]^m d\xi_2 \int_o^{\xi_2} [\alpha(\xi_1)]^n \\ &\times b_{mn}(t-\xi_2,\xi_2-\xi_1,\vec{x})d\xi_1 + \cdots \end{aligned} \qquad (10)$$

and ask whether the form is consistent with a possible solution of Eq. (8). After finding the forms of u, v, and ω from Eq. (9) and substituting in Eq. (8), we attempt to match terms of like powers of α. Let us assume that prior to time zero the flow is steady and the angle of attack is zero. Then to $O(\alpha^o)$ we have the initial flow $\phi_o(\vec{x})$ which must satisfy the time-independent Navier-Stokes equation

$$u_o\omega_{o_x} + v_o\omega_{o_y} - \nu\nabla^2\omega_o = 0 \qquad (11)$$

To $O(\alpha^1)$ we have a linearized form of the time-dependent Navier-Stokes equation

$$\frac{\partial}{\partial t}\omega_1 + u_o\omega_{1_x} + u_1\omega_{o_x} + v_o\omega_{1_y} + v_1\omega_{o_y} - \nu\nabla^2\omega_1 = 0 \qquad (12)$$

with

$$\left.\begin{aligned} \omega_1 &= \int_o^t \nabla^2 a_1(t-\xi_1,\vec{x})\alpha(\xi_1)d\xi_1 \\ u_1 &= \int_o^t a_{1_y}(t-\xi_1,\vec{x})\alpha(\xi_1)d\xi_1 \\ v_1 &= -\int_o^t a_{1_x}(t-\xi_1,\vec{x})\alpha(\xi_1)d\xi_1 \end{aligned}\right\} \qquad (13)$$

We see that with ω_1, u_1, v_1 all having the form of convolution integrals, we are able to introduce Laplace transforms to seek a solution of Eq. (12). Defining the Laplace transform as

$$\mathcal{L}_t^s[f(t)] \equiv \int_o^\infty e^{-st} f(t)dt \qquad (14)$$

(where the superscript s and subscript t will be useful later to identify the transform variable and transformed variable in question) and letting

$$\left.\begin{aligned} \mathcal{L}_t^s[\alpha(t)^n] &= \mathcal{A}(s;n) \\ \mathcal{L}_t^s[a_1(t,\vec{x})] &= A_1(s;\vec{x}) \end{aligned}\right\} \qquad (15)$$

we get in Eq. (12)

$$\begin{aligned} \mathcal{A}(s;1)\{s\nabla^2 A_1 + u_o\nabla^2 A_{1_x} + v_o\nabla^2 A_{1_y} + \omega_{o_x} A_{1_y} \\ - \omega_{o_y} A_{1_x} - \nu\nabla^4 A_1\} = 0 \end{aligned} \qquad (16)$$

where, as required by Eq. (13), we have imposed the initial condition $\omega_1(0,\vec{x}) = 0$. For arbitrary $\alpha(t)$, the bracketed term must be zero, which provides a form for the solution of $a_1(t,\vec{x})$.

To $O(\alpha^2)$ we have

$$\begin{aligned} \frac{\partial}{\partial t}\omega_2 + u_o\omega_{2_x} + u_1\omega_{1_x} + u_2\omega_{o_x} + v_o\omega_{2_y} + v_1\omega_{1_y} \\ + v_2\omega_{o_y} - \nu\nabla^2\omega_2 = 0 \end{aligned} \qquad (17)$$

with

$$\begin{aligned} \phi_2 = \int_o^t a_2(t-\xi_1,\vec{x})\alpha^2(\xi_1)d\xi_1 + \int_o^t \alpha(\xi_2)d\xi_2 \\ \times \int_o^{\xi_2} \alpha(\xi_1)b_{11}(t-\xi_2,\xi_2-\xi_1,\vec{x})d\xi_1 \end{aligned} \qquad (18)$$

and

$$\left.\begin{aligned} \omega_2 &= \nabla^2\phi_2 \\ u_2 &= \phi_{2_y} \\ v_2 &= -\phi_{2_x} \end{aligned}\right\} \tag{19}$$

In addition to terms similar in form to those in Eq. (12), Eq. (17) contains terms consisting of double integrals and products of single integrals. Provided these can be shown to be transformable, Eq. (17) also will yield to Laplace transformation. The double-integral terms arise from the second contribution to ϕ_2 (Eq. (18)). Its Laplace transform takes the form

$$\mathscr{L}_t^s[\phi_{22}] = \int_0^\infty e^{-st}\, dt \int_0^t \alpha(\xi_2)d\xi_2 \int_0^{\xi_2} \alpha(\xi_1) \times b_{11}(t - \xi_2, \xi_2 - \xi_1, \vec{x})d\xi_1 \tag{20}$$

After reversals of the order of integration and the substitution $t - \xi_2 = u$,

$$\mathscr{L}_t^s[\phi_{22}] = \int_0^\infty \alpha(\xi_2)e^{-s\xi_2}\, d\xi_2 \int_0^{\xi_2} \alpha(\xi_1)d\xi_1 \int_0^\infty e^{-su} \times b_{11}(u, \xi_2 - \xi_1, \vec{x})du \tag{21}$$

Letting

$$\mathscr{L}_u^s[b_{11}(u, \xi_2 - \xi_1, \vec{x})] = B_{11}(s, \xi_2 - \xi_1; \vec{x}) \tag{22}$$

we have

$$\mathscr{L}_t^s[\phi_{22}] = \int_0^\infty \alpha(\xi_2)e^{-s\xi_2}\, d\xi_2 \int_0^{\xi_2} \alpha(\xi_1) \times B_{11}(s, \xi_2 - \xi_1; \vec{x})d\xi_1 \tag{23}$$

Recognizing that the inner integral is in the form of a convolution, we let

$$\mathscr{L}_w^{s_1}[B_{11}(s, w; \vec{x})] \equiv \int_0^\infty e^{-s_1 w} B_{11}(s, w; \vec{x})dw = \hat{B}_{11}(s, s_1; \vec{x}) \tag{24}$$

so that

$$\int_0^{\xi_2} \alpha(\xi_1)B_{11}(s, \xi_2 - \xi_1; \vec{x})d\xi_1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathscr{A}(s_1;1)\hat{B}_{11}(s, s_1; \vec{x})e^{s_1\xi_2}\, ds_1 \tag{25}$$

whereupon Eq. (23) becomes

$$\mathscr{L}_t^s[\phi_{22}] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathscr{A}(s - s_1;1)\mathscr{A}(s_1;1)\hat{B}_{11}(s, s_1; \vec{x})ds_1 \tag{26}$$

Products of integrals arise from terms such as, e.g., $u_1\omega_{1_x}$ in Eq. (17), the Laplace transform of which has the form

$$\mathscr{L}_t^s[u_1\omega_{1_x}] = \int_0^\infty e^{-st}\, dt\left(\int_0^t a_{1_y}(t - \xi_1, \vec{x})\alpha(\xi_1)d\xi_1\right) \times \left(\int_0^t \nabla^2 a_{1_x}(t - \xi_2, \vec{x})\alpha(\xi_2)d\xi_2\right) \tag{27}$$

Terms such as this can be handled by the rule (4)

$$\mathscr{L}_t^s[f_1(t)\cdot f_2(t)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_1(s_1)F_2(s - s_1)ds_1 \tag{28}$$

Using Eq. (28) and noting that each of the product terms is itself a convolution, we get for the transform of $u_1\omega_{1_x}$

$$\mathscr{L}_t^s[u_1\omega_{1_x}] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathscr{A}(s_1,1)\mathscr{A}(s - s_1,1)A_{1_y}(s_1;\vec{x}) \times \nabla^2 A_{1_x}(s - s_1;\vec{x})ds_1 \tag{29}$$

So Eq. (17) yields to Laplace transformation. We get

$$\begin{aligned} &\mathscr{A}(s;2)\{s\nabla^2 A_2 + u_o\nabla^2 A_{2_x} + v_o\nabla^2 A_{2_y} + \omega_{o_x}A_{2_y} - \omega_{o_y}A_{2_x} \\ &- \nu\nabla^4 A_2\} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathscr{A}(s_1;1)\mathscr{A}(s - s_1;1)\{s\nabla^2\hat{B}_{11}(s, s_1;\vec{x}) \\ &+ u_o\nabla^2\hat{B}_{11_x} + v_o\nabla^2\hat{B}_{11_y} + \omega_{o_x}\hat{B}_{11_y} - \omega_{o_y}\hat{B}_{11_x} - \nu\nabla^4\hat{B}_{11} \\ &+ A_{1_y}(s_1)\nabla^2 A_{1_x}(s - s_1) - A_{1_x}(s_1)\nabla^2 A_{1_y}(s - s_1)\}ds_1 = 0 \end{aligned} \tag{30}$$

Again, with $\alpha(t)$ arbitrary, each of the bracketed terms must be independently zero, which provides forms for the solution of $a_2(t,\vec{x})$ and $b_{11}(t_1,t_2,\vec{x})$. We see that $a_2(t,\vec{x})$, and in fact each subsequent $a_i(t,\vec{x})$, is governed by the same linear form as that of $a_1(t,\vec{x})$. Each will have the same set of eigensolutions, but their combinations will differ. The homogeneous operator on b_{11} also has the same form (as will that on each subsequent b_{ij}), but b_{11} depends additionally on $a_1(t,\vec{x})$.

It is clear that the procedure can be continued to higher powers of α. At each stage, the multiple integrals can be transformed to functions of a single transform variable and forms will be available for the solution of the various transformed coefficients. Thus, although the problem of satisfying boundary conditions remains to be addressed, it appears that the form of Eq. (10) will be consistent with a possible solution of the Navier-Stokes equations. It is in effect a power-series expansion in α around $\alpha = 0$, and should be valid for at least a small range of α in the vicinity of $\alpha = 0$. The ways in which it can fail brings up the subject of Fréchet differentiability.

FRÉCHET DIFFERENTIABILITY

To study the issue of Fréchet differentiability, let us consider the general problem of forming the indicial response imagining that it can be obtained from solutions of the Navier-Stokes equations. We need to consider two motions (cf. (3) and Fig. 2). In the first, the body undergoes the motion under study $\alpha(\xi)$ from time zero to time $\xi = \tau$. Subsequent to τ, α is held constant at $\alpha(\tau)$. The motion history is prescribed as $\alpha_1(\xi)$:

$$\left.\begin{aligned} \alpha_1(\xi) &= \alpha(\xi) \ ; \quad 0 < \xi < \tau \\ &= \alpha(\tau) \ ; \quad \xi \geq \tau \end{aligned}\right\} \tag{31}$$

We ask for the response in, say, surface pressure $p(\vec{x}_s,t)$, which can be written in functional notation as

$$p(\vec{x}_s,t) = p[\alpha_1(\xi);\vec{x}_s] = p[\alpha(\xi);t,\tau,\vec{x}_s] \tag{32}$$

Let us assume that a solution for $p(\vec{x}_s,t)$ can be found from the appropriate two- or three-dimensional time-dependent Navier-Stokes equations. In the second motion, the body undergoes the same angle-of-attack history $\alpha(\xi)$ up to time τ. Subsequent to τ, the angle of attack is again held constant, but is given an incremental step change $\Delta\alpha$ over its previous value of $\alpha(\tau)$. Following (3), we represent the second motion as

$$\alpha_2(\xi) = \alpha_1(\xi) + \epsilon\eta \tag{33}$$

with

$$\left.\begin{aligned} \epsilon &= \Delta\alpha \\ \eta &= 0 \ ; \quad 0 < \xi < \tau \\ &= 1 \ ; \quad \tau \leq \xi \end{aligned}\right\} \tag{34}$$

The surface-pressure response to this motion is

$$p(\vec{x}_s,t) = p[\alpha_1(\xi) + \epsilon\eta;\vec{x}_s] \tag{35}$$

Again we assume that a solution can be found. The indicial response in surface pressure is the Fréchet derivative, which is defined as

$$\begin{aligned} \lim_{\Delta\alpha\to 0} \frac{\Delta p(\vec{x}_s,t)}{\Delta\alpha} &= \lim_{\epsilon\to 0}\left(\frac{p[\alpha_1(\xi)+\epsilon\eta;\vec{x}_s] - p[\alpha_1(\xi);\vec{x}_s]}{\epsilon}\right) \\ &= \frac{d}{d\epsilon}\, p[\alpha_1(\xi)+\epsilon\eta;\vec{x}_s]_{\epsilon=0} \\ &= p'[\alpha_1(\xi);x_s]\eta \equiv p_\alpha[\alpha(\xi);t,\tau,\vec{x}_s] \end{aligned} \tag{36}$$

Let us assume that the derivative exists for every α that occurs in a motion over some time interval. We obtain the surface pressure response to the angle-of-attack variation by forming a generalized superposition integral:

$$p(\vec{x}_s,t) = p(t,\alpha(0);\vec{x}_s) + \int_0^t p_\alpha[\alpha(\xi);t,\tau,\vec{x}_s]\,\frac{d\alpha}{d\tau}\,d\tau \tag{37}$$

Validity of the approach thus rests on the Fréchet differentiability of the surface-pressure response over the interval $0 < \tau < t$.

As we have noted (3), a theory for enumerating the means of invalidating Fréchet differentiability of the aerodynamic response offers the possibility of a more inclusive way than bifurcation theory is of classifying steady and unsteady aerodynamic phenomena that are important in flight-dynamics applications. At present we have identified three distinct means.

Aerodynamic Bifurcation

We define aerodynamic bifurcation as the replacement of an unstable equilibrium flow by a new stable equilibrium flow at a critical value of a parameter. The onset of instability of the equilibrium state is a linear problem, capturable, at least in principle, through analysis of a linearized form of the time-dependent Navier-Stokes equations. For two-dimensional flow, the governing equation for the perturbation stream function is the same as that of Eq. (12), with the exception that the basic state (u_o,v_o,ω_o in Eq. (12)) must be the equilibrium state that is existent at the angle of attack $\alpha(\tau)$ in question. For this reason, analysis based on the solution form Eq. (10) will not reach far enough to identify bifurcation phenomena without further generalization.

An example that illustrates aerodynamic bifurcation in a context relevant to current interests is the slender body of revolution at high angle of attack. We shall describe this problem heuristically by adapting the impulsive flow analogy, both to demonstrate the continued usefulness of the analogy as a conceptual device and to make a point that will be important later. The analogy has been used extensively to help explain the presence of the steady symmetric vortices that are found on the leeward of slender wings and bodies at high angles of attack. To review briefly, we refer to Fig. 3 where now the moving axis system (x,r,θ) is fixed to the body nose and the body is sinking with uniform velocity v_c. Fixing attention to a vertical plane fixed in space, e.g., the plane $X = 0$, and ignoring the growth of the body, one observes the flow about a cylinder that appears to have started from rest and is sinking at a uniform rate v_c. On the assumption that the flow in the vertical plane is essentially two-dimensional, one is able to relate the steady growth of the vortices in the body-axis system with distance x along the body to their time-dependent growth behind the impulsively started cylinder, as viewed in the $X = 0$ plane. The latter problem has been studied extensively both experimentally (5) and via numerical computation (6). The first appearance of separation on the leeward ray, followed by the growth of the primary vortices, then the secondary vortices, and so forth, has been thoroughly documented. We also know that beyond a critical Reynolds number ($Re \approx 50$ based on v_c and on body diameter), the equilibrium flow becomes time-dependent, leading to the periodic shedding of vortices in the wake. This phenomenon also has been studied extensively. The work of Nishioka and Sato (7) is particularly noteworthy, inasmuch as it demonstrates clearly that the onset of periodicity occurs at a critical Reynolds number and constitutes, in effect, a Hopf bifurcation. We expect, then, that at a critical Reynolds number a perturbation velocity within a fixed plane will have behavior that can be expressed initially as

$$v = g(r,\theta)e^{i\omega t} \tag{38}$$

where if v is radial velocity, then $g(r,\theta)$ is antisymmetric with respect to $\theta = 0$. Now, we have assumed that the axial flow induced by the cylinder as it moves through the plane is more or less independent of X. As we shall note again later, a nominally two-dimensional flow will permit a wave-form perturbation solution in the direction of the third coordinate. A more general form of the perturbation velocity v is then

$$v = g(r,\theta)e^{i\omega t}\, e^{i\lambda X} \tag{39}$$

Transforming this solution to the body axis system with $X = x - U_o t$, we get

$$v = g(r,\theta)e^{i\lambda[x-U_o t+(\omega/\lambda)t]} \tag{40}$$

and we see that a stationary perturbation solution is possible in the body-axis system when the wave speed ω/λ equals the axial speed U_o. Experimental results indicate that a stationary asymmetric flow is the preferred solution over a considerable range of angle of attack. We conjecture that it is easy for an equality between U_o and the wave speed ω/λ to be achieved for the following reason. Owing to the presence of the vortices, the axial velocity profile has a considerable overshoot beyond the free-stream value U_o. The wave speed ω/λ must lie between the extremes of the axial velocities available and hence can easily match U_o. On the other hand, as $\alpha \to 90°$, $\lambda \to 0$, a match becomes impossible, and the perturbation form reverts to Eq. (38) in either axis system.

The analysis leads to the following conclusions. (1) Having demonstrated its ability to fit observations, the impulsive flow analogy remains a useful conceptual device for understanding complex three-dimensional flows. (2) Accepting (1) encourages the belief that instabilities in complex three-dimensional flows may be essentially local phenomena which will yield to localized analyses. (3) The form of bifurcation, and hence its classification, may depend on the coordinate system in which it is observed. Fréchet differentiability on the other hand, remains invariant across coordinate transformations.

Loss of Analytic Dependence on a Parameter

The following refers principally to time-invariant equilibrium flows. The variation with a parameter (e.g., roll angle ψ) of the aerodynamic response may develop a fold at a critical value of the parameter ψ_c, so that the slope of the response becomes infinite there. A jump in the response necessarily ensues to another branch of the folded curve with an infinitesimal increase in ψ beyond ψ_c, and hysteresis follows on the return route.

Folds need not be the result of bifurcation; no new branches of equilibrium flow solutions need appear. On the other hand, folds may be the indirect result of bifurcation. An example based on the preceding study is the slender delta wing. Just as for the slender body of revolution, bifurcation to an asymmetric flow pattern occurs at a critical value of α, beyond which a finite value of the rolling-moment coefficient C_ℓ may exist at zero roll angle. As shown in Fig. 4, both a positive and a negative value of C_ℓ is possible, depending on the sense of the asymmetry. The solution $C_\ell = 0$, corresponding to the unstable symmetric flow, also exists at $\psi = 0$, but is itself unstable. So folds must occur to accommodate the three solution points.

Disconnected Bifurcation

By disconnected bifurcation we mean the existence of an isolated branch of equilibrium solutions that is not connected to other solution branches at bifurcation points. The sources of this form of invalidation are still obscure although evidence abounds of its existence in experimental studies. It is clear that the state of the flow at the origin of the indicial response $\xi = \tau$ must be involved. Disturbances there must be large enough and of an appropriate form so as to divert the response toward a new attractor, that is, toward an alternative branch of equilibrium solutions.

As we have already seen, a source to which nominally two-dimensional flows should be particularly susceptible is the presence of three-dimensional disturbances in cellular patterns in the direction of the third coordinate. As candidates for study in this light we cite the following two possibilities. The first one concerns flow at the stagnation line of a cylindrical body. The presence of a regular pattern of disturbances along, e.g., the leading edge of a straight wing has been known for a long time. Attempts to explain the pattern's presence on the basis of stability theory have been neither consistent nor convincing. In a comprehensive and useful review of the subject, Morkovin (8) concludes that there is still no viable explanation of the phenomenon. The second possibility could be axisymmetric vortex breakdown.[3] A solution here on the basis

[3]Joint work with D. Weihs, Technion, Israel.

of a disconnected bifurcation would represent an extension to viscous flows of Brooke Benjamin's notion (9) of "conjugate flows" that he introduced as an aid to the explanation of inviscid axisymmetric vortex breakdown. We believe that an extension of our use of nonlinear functional expansions that accounts for cellular perturbations in the third coordinate direction will provide a focus on the sources that may be available to sustain such perturbations in the initial flow.

MEMORY EFFECTS

We have postulated six major subdivisions in the form of the aerodynamic response by means of a set of sketches (3), and these are reproduced in Fig. 5. Of particular interest here are Figs. 5(d) and (e), depicting time-dependent equilibrium states. Hopf bifurcation is indicated in Fig. 5(d), wherein a formerly stable time-invariant equilibrium state is replaced by a time-varying periodic equilibrium state. The equilibrium state resulting from a Hopf bifurcation is often succeeded by bifurcation to a quasi-periodic equilibrium state with further increase of the relevant parameter (here, angle of attack). The quasi-periodic state may be succeeded by an aperiodic (chaotic) equilibrium state.

We indicated (3) how these bifurcations could be accommodated within our mathematical model. In the implementation, however, an important additional issue first arises with a Hopf bifurcation when we recognize that it is necessary to specify phase as well as amplitude in order to completely determine the periodic equilibrium state. The general problem can be posed as follows: given that the past motion determines the initial condition for the specification of an indicial response, how much of the past motion must be acknowledged in order to ensure that the equilibrium state of the response is adequately specified? We see that there will be a definite sequence, in which at least one piece of additional information about the past motion must be supplied with each successive bifurcation. (The sequence is illustrated in Fig. 6.) Let us note, incidentally, that with a linear system, no information about the past motion is required in the specification of an indicial response. This is, in effect, the most general definition of a linear system. For a nonlinear system in which the equilibrium state is time-invariant, the level of the past motion at the origin of the step is all that is required to specify the equilibrium state (Fig. 6(a)). For the time-periodic equilibrium state that replaces the time-invariant state with a Hopf bifurcation, the amplitude and frequency of the periodic state are again determined by the level of the past motion, but an additional piece of information, the rate of the past motion at the origin of the step must be supplied to fix the phase of the equilibrium state (Fig. 6(b)). If the next bifurcation results in a quasi-periodic equilibrium state, another derivative of the motion at the origin of the step will be required to specify the additional phase (Fig. 6(c)). It is clear that a chaotic equilibrium state will require information about the complete past motion to ensure a complete specification. This is how "sensitive dependence on the initial conditions," the signature of a chaotic state, will manifest itself in the modeling of the indicial response. We recognize the novel appearance of a turbulence modeling question: Is it possible to model the past motion so as to suppress the chaotic part of an equilibrium state? The necessity of coping with this problem in the context of flight dynamics studies demonstrates once again the convergence of interests that is occurring among workers in heretofore widely separate fields.

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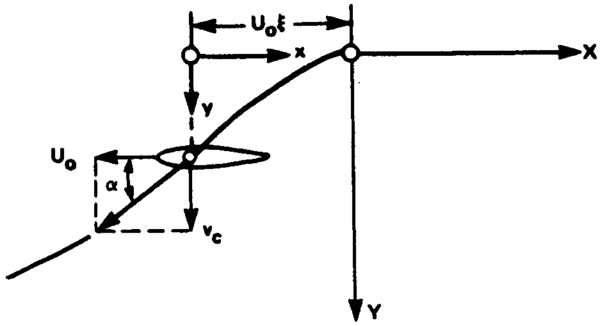

Fig. 1 Wing maneuver referred to space-fixed (X,Y) and moving (x,y) coordinates.

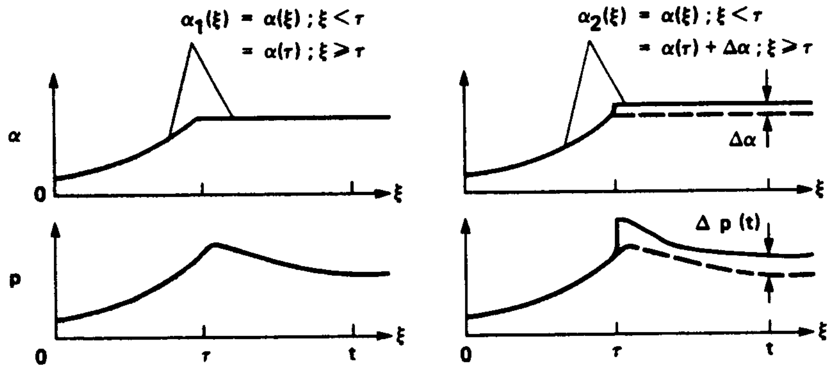

Fig. 2 Formation of indicial response in surface pressure.

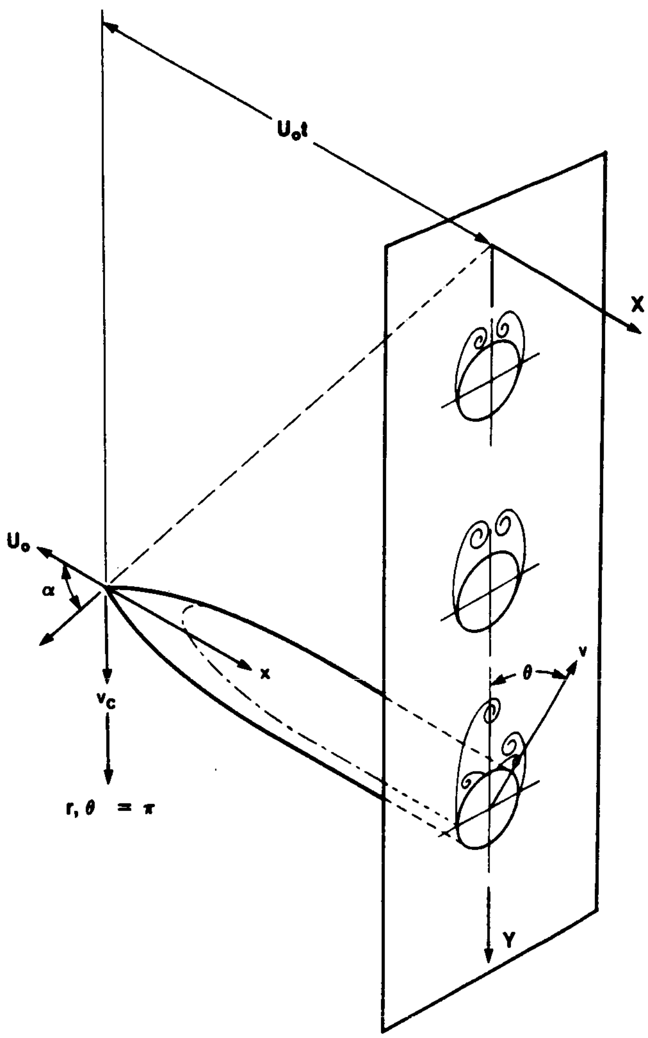

Fig. 3 Body and coordinates for impulsive flow analogy.

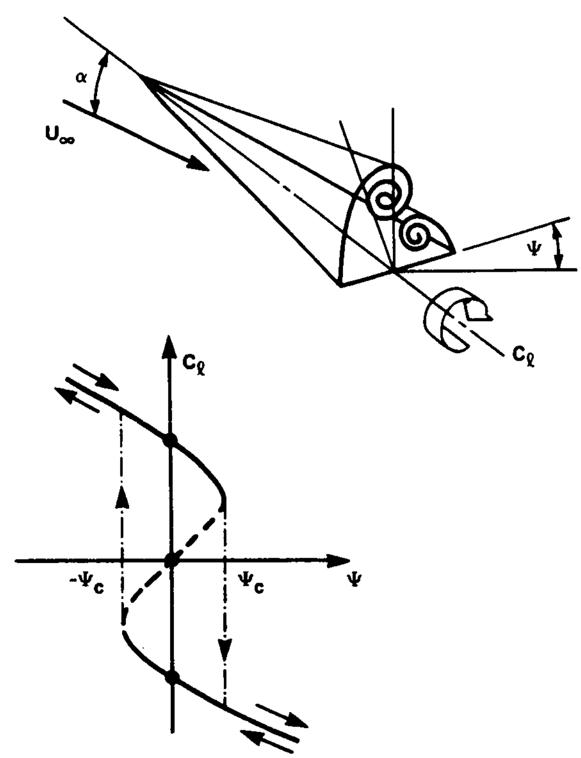

Fig. 4 Folds in solution curve for rolling-moment coefficient C_ℓ.

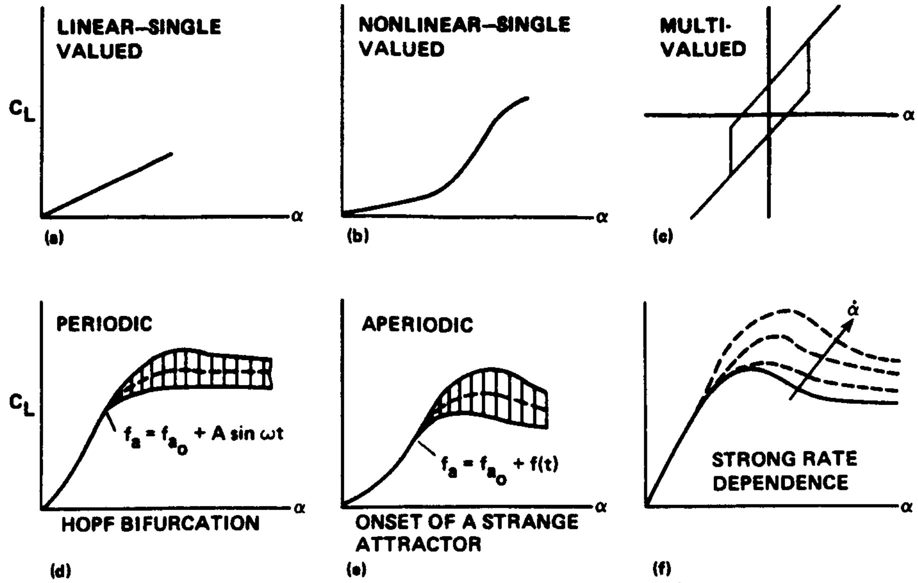

Fig. 5 Major subdivision of aerodynamic forces.

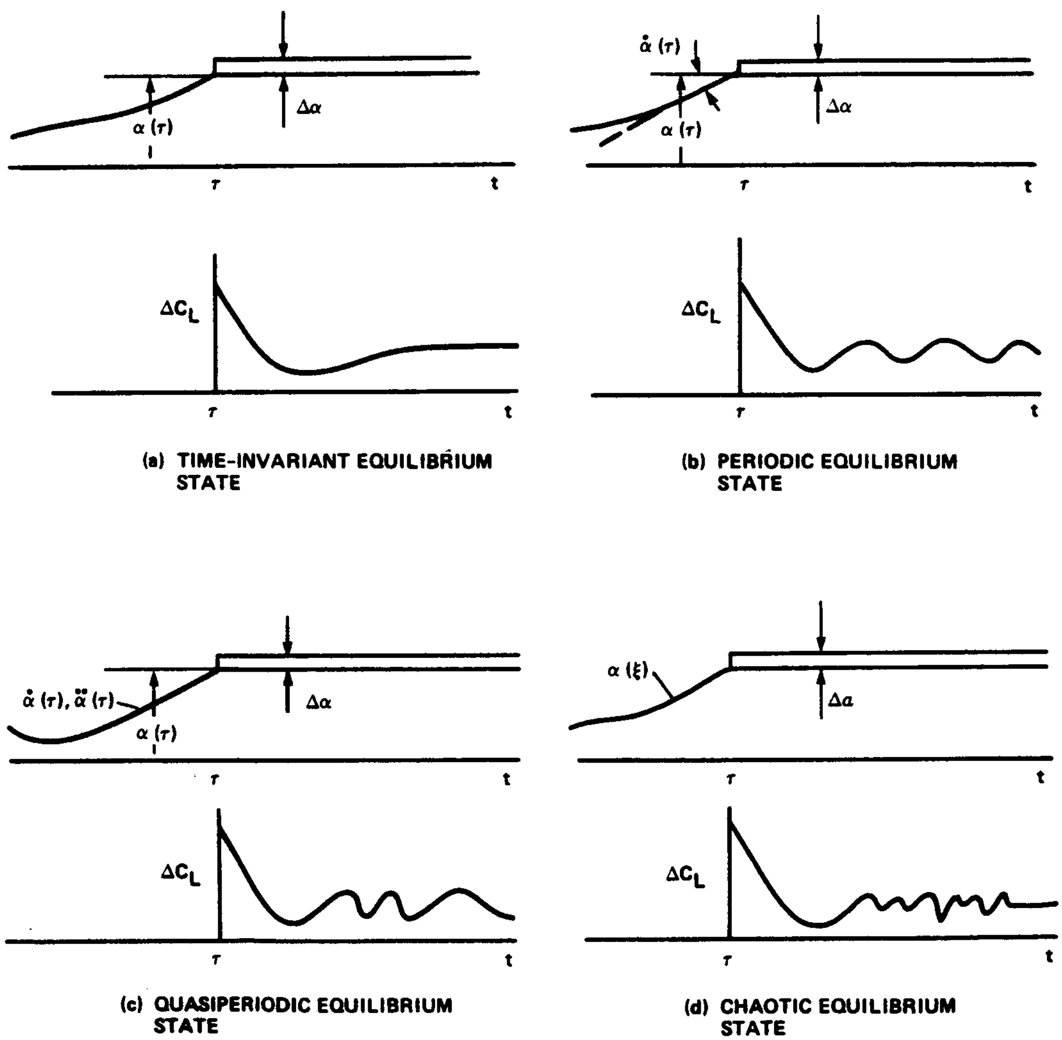

Fig. 6 Memory effects.

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16. Abstract

Nonlinear algebraic functional expansions are used to create a form for the unsteady aerodynamic response that is consistent with solutions of the time-dependent Navier-Stokes equations. An enumeration of means of invalidating Fréchet differentiability of the aerodynamic response, one of which is aerodynamic bifurcation, is proposed as a way of classifying steady and unsteady aerodynamic phenomena that are important in flight dynamics applications. Accommodating bifurcation phenomena involving time-dependent equilibrium states within a mathematical model of the aerodynamic response raises an issue of memory effects that becomes more important with each successive bifurcation.

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