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On the Maxwellian Distribution, Symmetric Form, and Entropy Conservation for the Euler Equations

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Summary

The Euler equations of gas dynamics have some very interesting properties in that the flux vector is a homogeneous function of the unknowns and the equations can be cast in symmetric hyperbolic form and satisfy the entropy conservation. Since the Euler equations are the moments of the Boltzmann equation of the kinetic theory of gases when the velocity distribution function is a Maxwellian, it would be interesting to look for the relation between the symmetrizability and the Maxwellian velocity distribution. The present paper precisely shows this relationship. The functions that symmetrize the Euler equations are density and mass flux, which are integrals of the Maxwellian distribution multiplied by unity and molecular velocity, respectively. The field vector and the flux vector in the Euler equations are then the gradients of these functions with respect to new transformed variables $\mathbf{q}$. In terms of these transformed variables, the Euler equations assume the symmetric hyperbolic form.

The entropy conservation is in terms of the H-function, which is a slight modification of the H-function first introduced by Boltzmann in his famous H-theorem. The modified H-function is equal to the negative of specific thermodynamic entropy of the gas. Further, it is an integral of the Legendre transform of the Maxwellian distribution with respect to $\mathbf{q}$, thus establishing its convexity. It is therefore possible to design a numerical method using the convexity of H so that the total H in the computational domain will monotonically decrease with time in conformity with the Boltzmann H-theorem. In view of the H-theorem it is suggested that the development of total H-diminishing (THD) numerical methods may be more profitable than the usual total variation diminishing (TVD) methods for obtaining "wiggle-free" solutions.

Introduction

The Euler equations of gas dynamics can be obtained as the moments of the Boltzmann equation of the kinetic theory of gases (ref. 1) provided the velocity distribution function, which is the basic unknown, is a Maxwellian distribution. The collision term in the Boltzmann equation vanishes when the velocity distribution function is a Maxwellian. The Euler equations therefore are the moments (the moment functions being collisional invariants) of the collisionless Boltzmann equation that are linear hyperbolic equations with a wave velocity independent of space and time. Because the wave velocity does not depend on space and time, the collisionless Boltzmann equation can easily be cast in the strong conservation law form. Taking the moments of this equation therefore yields the Euler equations in the strong conservation law form. This fundamental connection between the linear hyperbolic Boltzmann equation and the Euler equations has been used in reference 2 to construct a new class of upwind methods for the numerical solution of the Euler equations. These methods, called "kinetic numerical methods," are based on the principle that moments of every upwind method for the Boltzmann equation yield an upwind method for the Euler equations. Central to the moment-method strategy is the Maxwellian distribution.

The Euler equations can be cast in two forms: the strong conservation law form and the symmetric hyperbolic form. The strong conservation law form, which is a consequence of the conservation of mass, momentum, and energy, is the basis for constructing fully conservative methods. However, the wave nature of the Euler equations is not apparent when the equations are cast in this form. The symmetric hyperbolic form studied by Harten (ref. 3) makes the hyperbolicity of the Euler equations very transparent. This effect is due to the symmetry of the Jacobian matrices occurring in the symmetric hyperbolic form. The symmetrizability is a very important property of the Euler equations. The excellent report by Harten (ref. 3) shows the connection between the symmetrizability, entropy function, and Roe linearization. According to the Godunov theorem (ref. 3), symmetrizability implies the existence of the entropy function which, in turn, according to the Harten-Lax theorem (ref. 3), implies that the equations admit Roe linearization. Hence, the symmetrizability of the Euler equations allows the equations to be locally linearized so as to preserve hyperbolicity and conservation. Symmetrizability is thus a very important property of the Euler equations both from theoretical as well as numerical points of view. In fact, Abarbanel and Gottlieb (ref. 4) have used the symmetrizability property to analyze rigorously the splitting algorithms for the Navier-Stokes equations. Further, it is possible to construct upwind methods (ref. 5) based on Roe linearization, for example, that lead to the improved structure of iteration matrices, i.e., diagonal dominance.

With the symmetrizability and the entropy condition being very important properties of the Euler equations, it would be very interesting to seek their basis in the fact that the Euler equations are the moments of the Boltzmann equation. The motivation of the present paper lies precisely in this observation. The present paper shows that the Maxwellian distribution plays a very important role in the theory of the Euler equations. The integrals of the products of the Maxwellian distribution and first two collisional invariants (corresponding to mass and momentum conservation) are the functions that accomplish symmetrization. The entropy condition is related to the Boltzmann H-theorem. This relation is to be expected especially when it is noted that according to the analysis of the kinetic numerical method of reference 2, the collision phase decreases H, but the convection phase conserves H.

It is believed that the results of the present paper will be valuable in constructing a new class of upwind methods using the moment-method strategy for obtaining "wiggle-free" solutions. For the Euler equations the moment-method strategy is based on the Maxwellian distribution and the Boltzmann equation.

Symbols

$\mathbf{A}$	Jacobian matrix of flux $\mathbf{g}$ with respect to $\mathbf{w}$
A_{ij}	element i, j of matrix $\mathbf{A}$
$\mathbf{B}$	symmetric matrix in equation (12)
C_1, C_2	constants dependent on γ
e	specific total internal energy per unit mass
F	Maxwellian distribution
$\tilde{F}$	contracted Maxwellian distribution defined by equation (24)
f	arbitrary distribution
$\mathbf{g}$	flux vector in Euler equations
g_i	ith element of vector $\mathbf{g}$
$\mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_z$	vector components of $\mathbf{g}$ along x-, y-, and z-directions, respectively
$\mathbf{g}^+$	flux vector with v-integration over positive half-interval $(0, \infty)$
$\mathbf{g}^-$	flux vector with v-integration over negative half-interval $(-\infty, 0)$
H	Boltzmann H-function
H_v	flux of H in one-dimensional case
H_{v_i}	flux of H in direction of v_i
H_{vx}, H_{vy}, H_{vz}	flux of H along x-, y-, and z-directions, respectively
I	internal energy variable due to nontranslational degrees of freedom
I_o	equilibrium internal energy due to nontranslational degrees of freedom

M_o	integral of F with respect to molecular velocity and I
M_o^+, M_o^-	integrals of F with respect to I and v, the limits for v being half-intervals $(0,\infty)$ and $(-\infty,0)$
$\tilde{M}_o$	any one of M_o, M_o^+, and M_o^-
M_x, M_y, M_z	integrals of v_1F, v_2F, and v_3F, respectively, with respect to v_1, v_2, v_3, and I
M_1	integral of vF with respect to v and I
dM_o, dM_1	differentials of M_o and M_1, respectively
dM_x, dM_y	differentials of M_x and M_y, respectively
$\mathbf{P}$	matrix appearing in symmetric hyperbolic form
p	pressure
Q	quadratic form
$\mathbf{q}$	transformed variables in symmetric hyperbolic form
q_i	element of $\mathbf{q}$
q^n	value of $\mathbf{q}$ at time level n
q', q''	values of $\mathbf{q}$ at neighboring points
R	gas constant per unit mass
S	thermodynamic entropy per unit volume
T	temperature
t	time
u	fluid velocity
v	molecular velocity
v_i	molecular velocity in i-direction
$\mathbf{w}$	field vector in Euler equations
w_i, w_j	elements i and j of field vector $\mathbf{w}$
$\mathbf{w}^+, \mathbf{w}^-$	field vectors with v-integration over half-intervals $(0,\infty)$ and $(-\infty,0)$
dX, dY	differentials in equations (53)
x, y, z	coordinates
β	square of thermal speed $1/(2RT)$
γ	ratio of specific heats
$\boldsymbol{\eta}$	general vector
$\boldsymbol{\eta}^t$	transpose of vector $\boldsymbol{\eta}$
θ	coefficient between 0 and 1
ρ	mass density

$\boldsymbol{\Psi}$	collisional invariant vector
Ψ_i	ith component of vector $\boldsymbol{\Psi}$

Abbreviations:

THD	total H-diminishing
TVD	total variation diminishing

Homogeneity and the Maxwellian Distribution

The one-dimensional unsteady Euler equations of gas dynamics can be written in the vector conservation law form

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{g}}{\partial x} = 0 \tag{1}$$

where

$$\mathbf{w} = \begin{bmatrix} \rho \\ \rho u \\ \rho e \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (\rho e + p)u \end{bmatrix} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \frac{\gamma}{\gamma - 1} pu + \frac{1}{2}\rho u^3 \end{bmatrix} \tag{2}$$

Here, ρ, u, and p are, respectively, the mass density, fluid velocity, and pressure, and e is the specific total internal energy given by

$$\rho e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \tag{3}$$

Equations (1) can be obtained from the Boltzmann equation of the kinetic theory of gases (refs. 1 and 2) and can be written as

$$\left\langle \boldsymbol{\Psi}, \frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} \right\rangle = 0 \tag{4}$$

where $\boldsymbol{\Psi}$ denotes the moment functions 1, v, and $I + (v^2/2)$ corresponding to the collisional invariants, and F is the Maxwellian velocity distribution

$$F = \frac{\rho}{I_o}\sqrt{\frac{\beta}{\pi}} \exp\left[-\beta(v - u)^2 - \frac{I}{I_o} \right] \tag{5}$$

Here, $\beta = 1/(2RT)$, R is the gas constant per unit mass, $I_o = (3 - \gamma)/[4(\gamma - 1)\beta]$ that denotes the internal energy due to the nontranslational degrees of freedom, v is the molecular velocity, I is the molecular internal energy variable, and the inner product is given as

$$\langle \boldsymbol{\Psi}, F \rangle = \int_{-\infty}^{\infty} dv \int_{0}^{\infty} dI \, (\boldsymbol{\Psi} F) \tag{6}$$

The formula for the Maxwellian distribution (eq. (5)) contains two independent variables, the molecular velocity v and the internal energy variable I. The latter variable is required to ensure the existence of additional degrees of freedom that are necessary to satisfy the constraint

$$\left\langle I + \frac{v^2}{2}, F \right\rangle = \frac{\rho}{2(\gamma - 1)\beta} + \frac{1}{2}\rho u^2$$

The parameter γ is assumed to be a constant for the analysis in the present paper. Notice that I_o, which is the average energy in nontranslational degrees of freedom, will be positive if $1 \leq \gamma \leq 3$. For a monatomic or polyatomic perfect gas, $1 \leq \gamma \leq 5/3$ and, hence, I_o will always be positive for such gases.

The unknown field vector $\mathbf{w}$ and the flux vector $\mathbf{g}$ are related to F by the equations

$$\mathbf{w} = \langle \mathbf{\Psi}, F \rangle = \int \begin{bmatrix} 1 \\ v \\ I + \frac{v^2}{2} \end{bmatrix} F \, dv \, dI \tag{7}$$

$$\mathbf{g} = \langle v\mathbf{\Psi}, F \rangle = \int v \begin{bmatrix} 1 \\ v \\ I + \frac{v^2}{2} \end{bmatrix} F \, dv \, dI \tag{8}$$

where only one sign of integration is displayed for brevity. The homogeneity property of the flux vector is

$$\mathbf{Aw} = \mathbf{g} \tag{9}$$

where $\mathbf{A} = \partial \mathbf{g}/\partial \mathbf{w}$. In terms of F we have

$$(\mathbf{Aw})_i = \sum_j A_{ij} w_j = \sum_j \frac{\partial g_i}{\partial w_j} w_j = \int v \Psi_i \left(\sum_j w_j \frac{\partial F}{\partial w_j} \right) dv \, dI$$

Hence, the homogeneity property (eq. (9)) follows if we can show that

$$\sum_j w_j \frac{\partial F}{\partial w_j} = F \tag{10}$$

By using the chain rule of partial differentiation, we get

$$\sum_j w_j \frac{\partial F}{\partial w_j} = \frac{\partial F}{\partial \rho} \left(\sum w_j \frac{\partial \rho}{\partial w_j} \right) + \frac{\partial F}{\partial u} \left(\sum w_j \frac{\partial u}{\partial w_j} \right) + \frac{\partial F}{\partial \beta} \left(\sum w_j \frac{\partial \beta}{\partial w_j} \right) \tag{11}$$

Using

$$\rho = w_1 \qquad u = \frac{w_2}{w_1} \qquad \frac{1}{\beta} = 2(\gamma - 1) \left(\frac{w_3}{w_1} - \frac{w_2^2}{2w_1^2} \right)$$

gives

$$\frac{\partial \rho}{\partial w_1} = 1 \qquad \frac{\partial \rho}{\partial w_2} = 0 \qquad \frac{\partial \rho}{\partial w_3} = 0$$

$$\frac{\partial u}{\partial w_1} = -\frac{u}{\rho} \qquad \frac{\partial u}{\partial w_2} = \frac{1}{\rho} \qquad \frac{\partial u}{\partial w_3} = 0$$

$$\frac{\partial \beta}{\partial w_1} = \frac{\beta}{\rho} - (\gamma - 1)\frac{\beta^2 u^2}{\rho} \qquad \frac{\partial \beta}{\partial w_2} = \frac{2\beta^2(\gamma - 1)u}{\rho} \qquad \frac{\partial \beta}{\partial w_3} = -\frac{2\beta^2(\gamma - 1)}{\rho}$$

From these equations it follows that

$$\sum_j w_j \frac{\partial \rho}{\partial w_j} = \rho \qquad \sum_j w_j \frac{\partial u}{\partial w_j} = 0 \qquad \sum_j w_j \frac{\partial \beta}{\partial w_j} = 0$$

and, hence, equation (11) yields the desired equation

$$\sum_j w_j \frac{\partial F}{\partial w_j} = \rho \frac{\partial F}{\partial \rho} = F$$

It is interesting to note that the homogeneity property (eq. (9)) depends on the defining relations (eqs. (7) and (8)) and on equation (10). Hence, the velocity distribution need not be a Maxwellian for the validity of equation (9); it is enough if equation (10) is true.

Symmetrization

The equations

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{w}}{\partial x} = 0$$

are said to admit a symmetric hyperbolic form if they can be transformed to

$$\mathbf{P}\frac{\partial \mathbf{q}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{q}}{\partial x} = 0 \tag{12}$$

where $\mathbf{P}$ and $\mathbf{B}$ are symmetric matrices and $\mathbf{P}$ is positive definite. We will accomplish the symmetrization of the Euler equations by obtaining the scalar functions M_o and M_1 such that

$$w_1 = \frac{\partial M_o}{\partial q_1} \qquad w_2 = \frac{\partial M_o}{\partial q_2} \qquad w_3 = \frac{\partial M_o}{\partial q_3} \tag{13}$$

$$g_1 = \frac{\partial M_1}{\partial q_1} \qquad g_2 = \frac{\partial M_1}{\partial q_2} \qquad g_3 = \frac{\partial M_1}{\partial q_3} \tag{14}$$

where q_1, q_2, and q_3 are yet to be determined functions of ρ, u, and β. Assuming that M_o and M_1 have been obtained, the time derivatives of $\mathbf{w}$ and the space derivatives of $\mathbf{g}$ are given by

$$\frac{\partial w_i}{\partial t} = \frac{\partial}{\partial t}\frac{\partial M_o}{\partial q_i} = \sum_j \frac{\partial^2 M_o}{\partial q_i\,\partial q_j}\frac{\partial q_j}{\partial t}$$

$$\frac{\partial g_i}{\partial x} = \frac{\partial}{\partial x}\frac{\partial M_1}{\partial q_i} = \sum_j \frac{\partial^2 M_1}{\partial q_i\,\partial q_j}\frac{\partial q_j}{\partial x}$$

Hence, the Euler equations (eq. (1)) transform to

$$\sum_j \frac{\partial^2 M_o}{\partial q_i\,\partial q_j}\frac{\partial q_j}{\partial t} + \sum_j \frac{\partial^2 M_1}{\partial q_i\,\partial q_j}\frac{\partial q_j}{\partial x} = 0 \tag{15}$$

and the required matrices $\mathbf{P}$ and $\mathbf{B}$ are then given by

$$\mathbf{P} = \left[\frac{\partial^2 M_o}{\partial q_i\,\partial q_j}\right] \qquad \mathbf{B} = \left[\frac{\partial^2 M_1}{\partial q_i\,\partial q_j}\right] \tag{16}$$

We will now prove that the desired scalar functions M_o and M_1 accomplishing symmetrization are, respectively,

$$M_o = \int_{-\infty}^{\infty} dv \int_0^{\infty} dI\,(F) = \int F\,dv\,dI \tag{17}$$

$$M_1 = \int_{-\infty}^{\infty} dv \int_0^{\infty} dI\,(vF) = \int vF\,dv\,dI \tag{18}$$

The differentials dM_o and dM_1 are then given by

$$\begin{bmatrix} dM_o \\ dM_1 \end{bmatrix} = \int \begin{bmatrix} 1 \\ v \end{bmatrix} dF\,dv\,dI \tag{19}$$

Using

$$\left.\begin{aligned}\frac{\partial F}{\partial \rho} &= \frac{F}{\rho} \\ \frac{\partial F}{\partial u} &= 2\beta(v-u)F \\ \frac{\partial F}{\partial \beta} &= \left[\frac{3}{2\beta} - (v-u)^2 - \frac{4(\gamma-1)}{3-\gamma}I\right] F\end{aligned}\right\} \tag{20}$$

we obtain

$$\begin{aligned}\frac{dF}{F} &= \frac{d\rho}{\rho} + 2\beta(v-u)\,du + \left[\frac{3}{2\beta} - (v-u)^2 - \frac{4(\gamma-1)}{3-\gamma}I\right] d\beta \\ &= \frac{d\rho}{\rho} - 2\beta u\,du - u^2\,d\beta + \frac{3}{2\beta}d\beta + (2\beta v\,du + 2vu\,d\beta) - \left[\frac{4(\gamma-1)}{3-\gamma}I + v^2\right] d\beta\end{aligned}$$

Using total differentials we obtain

$$\frac{dF}{F} = d\left(\ln\rho + \frac{3}{2}\ln\beta - \beta u^2\right) + v\,d(2\beta u) + \left(I + \frac{v^2}{2}\right) d(-2\beta) + \frac{2(5-3\gamma)}{3-\gamma}\,I\,d\beta \tag{21}$$

where the identity

$$\frac{4(\gamma-1)}{3-\gamma} = 2 - \frac{2(5-3\gamma)}{3-\gamma}$$

has been made use of. Substituting for dF in equation (19) gives

$$\begin{aligned}\begin{bmatrix} dM_o \\ dM_1 \end{bmatrix} = \int \begin{bmatrix} 1 \\ v \end{bmatrix} \Bigg[& d\left(\ln\rho + \frac{3}{2}\ln\beta - \beta u^2\right) + v\,d(2\beta u) \\ & + \left(I + \frac{v^2}{2}\right) d(-2\beta) + \frac{2(5-3\gamma)}{3-\gamma}\,I\,d\beta\Bigg] F\,dv\,dI\end{aligned} \tag{22}$$

For molecules with additional degrees of freedom, i.e., $\gamma \neq 5/3$, more manipulation is required to absorb the $I\,d\beta$ term with other terms of equation (22) in order to express the integrands as sums of products of collisional invariants and perfect differentials. This manipulation is accomplished by using the following equalities:

$$\begin{aligned}\int \frac{2(5-3\gamma)}{3-\gamma}(I\,d\beta)F\,dv\,dI &= \frac{2(5-3\gamma)}{3-\gamma}\int I_o(\tilde{F}\,d\beta)\,dv \\ &= \int \frac{2(5-3\gamma)}{3-\gamma}\frac{3-\gamma}{4(\gamma-1)\beta}(\tilde{F}\,d\beta)\,dv \\ &= \int \frac{5-3\gamma}{2(\gamma-1)}\left(\frac{d\beta}{\beta}F\right)\,dv\,dI\end{aligned} \tag{23}$$

where $\tilde{F}$ is the contracted Maxwellian given by

$$\tilde{F} = \int_0^\infty F\,dI = \rho\sqrt{\frac{\beta}{\pi}}\exp\left[-\beta(v-u)^2\right] \tag{24}$$

By following a similar procedure we can easily prove

$$\int \frac{2(5-3\gamma)}{3-\gamma}\left[vI\,d\beta\,(F)\right]\,dv\,dI = \frac{5-3\gamma}{2(\gamma-1)}\int v\frac{d\beta}{\beta}\,F\,dv\,dI \tag{25}$$

Using equations (23) and (25), equation (22) reduces to

$$\begin{aligned}\begin{bmatrix} dM_o \\ dM_1 \end{bmatrix} &= \int \begin{bmatrix} 1 \\ v \end{bmatrix}\left[d\left(\ln\rho + \frac{3}{2}\ln\beta - \beta u^2\right) + v\,d(2\beta u)\right. \\ &\quad \left. + \left(I + \frac{v^2}{2}\right)\,d(-2\beta) + \frac{5-3\gamma}{2(\gamma-1)}\,d(\ln\beta)\right]F\,dv\,dI \\ &= \int \begin{bmatrix} 1 \\ v \end{bmatrix}\left[d\left(\ln\rho + \frac{\ln\beta}{\gamma-1} - \beta u^2\right) + v\,d(2\beta u)\right. \\ &\quad \left. + \left(I + \frac{v^2}{2}\right)\,d(-2\beta)\right]F\,dv\,dI\end{aligned} \tag{26}$$

The transformed variables q_1, q_2, and q_3 can therefore be defined as

$$q_1 = \ln\rho + \frac{\ln\beta}{\gamma-1} - \beta u^2 \qquad q_2 = 2\beta u \qquad q_3 = -2\beta \tag{27}$$

Equation (26) then assumes the simple form

$$\begin{bmatrix} dM_o \\ dM_1 \end{bmatrix} = \int \begin{bmatrix} 1 \\ v \end{bmatrix}\left[dq_1 + v\,dq_2 + \left(I + \frac{v^2}{2}\right)\,dq_3\right]F\,dv\,dI \tag{28}$$

Equation (28) then implies

$$w_i = \frac{\partial M_o}{\partial q_i} = \int \Psi_i F\,dv\,dI \tag{29}$$

$$g_i = \frac{\partial M_1}{\partial q_i} = \int v\Psi_i F\,dv\,dI \tag{30}$$

Equation (28) is clearly the crucial relation in accomplishing symmetrization for it reveals that the gradients of scalar functions M_o and M_1 are, respectively, equal to $\mathbf{w}$ and $\mathbf{g}$. It is therefore worthwhile to see more transparently the connection between equation (28) and the Maxwellian distribution. In terms of q_1, q_2, and q_3 defined by equation (27), the expression for dF reduces to

$$\begin{aligned}\frac{dF}{F} &= dq_1 + v\,dq_2 + \left(I + \frac{v^2}{2}\right) dq_3 + \frac{2(5-3\gamma)}{3-\gamma} I\,d\beta - \frac{5-3\gamma}{2(\gamma-1)}\frac{d\beta}{\beta} \\ &= dq_1 + v\,dq_2 + \left(I + \frac{v^2}{2}\right) dq_3 + \frac{5-3\gamma}{4(\gamma-1)\beta} dq_3 - \frac{5-3\gamma}{3-\gamma} I\,dq_3 \\ &= dq_1 + v\,dq_2 + \left(I + \frac{v^2}{2}\right) dq_3 + \frac{5-3\gamma}{3-\gamma}(I_o - I)\,dq_3 \end{aligned} \tag{31}$$

The validity of equation (28) is obvious when examining equation (31). Thus,

$$\int_0^\infty (I_o - I)F\,dI = 0 \tag{32}$$

The above analysis makes use of the Maxwellian distribution to establish the validity of equations (29) and (30). Once q_1, q_2, and q_3 are defined by equation (27), the validity of equations (29) and (30) can be directly verified from the definitions of M_o and M_1. Equations (17) and (18) give, respectively,

$$M_o = \rho \qquad M_1 = \rho u$$

From equation (27) we obtain the total differentials

$$\left.\begin{aligned} dq_1 &= \frac{d\rho}{\rho} + \frac{d\beta}{(\gamma-1)\beta} - u^2\,d\beta - 2\beta u\,du \\ dq_2 &= 2u\,d\beta + 2\beta\,du \\ dq_3 &= -2\,d\beta \end{aligned}\right\} \tag{33}$$

The expressions for dq_2 and dq_3 yield

$$du = \frac{dq_2 + u\,dq_3}{2\beta} \tag{34}$$

Substitution of du and $d\beta$ into the expression for dq_1 gives

$$\begin{aligned} dq_1 &= \frac{d\rho}{\rho} + \frac{1}{(\gamma-1)\beta}\left(-\frac{dq_3}{2}\right) + u^2\left(\frac{dq_3}{2}\right) - u(dq_2 + u\,dq_3) \\ &= \frac{d\rho}{\rho} - \left[\frac{u^2}{2} + \frac{1}{2(\gamma-1)\beta}\right] dq_3 - u\,dq_2 \end{aligned}$$

Hence,

$$dM_o = d\rho = \rho\,dq_1 + \rho u\,dq_2 + \left[\frac{\rho u^2}{2} + \frac{\rho}{2(\gamma-1)\beta}\right] dq_3 \tag{35}$$

which implies that $w_i = \partial M_o/\partial q_i$. Similarly, for dM_1 we get

$$\left.\begin{aligned} dM_1 &= \rho\,du + u\,d\rho = \rho\frac{dq_2 + u\,dq_3}{2\beta} + u\left\{\rho\,dq_1 + \rho u\,dq_2 + \left[\frac{\rho}{2(\gamma-1)\beta} + \frac{\rho u^2}{2}\right] dq_3\right\} \\ dM_1 &= \rho u\,dq_1 + \left(\frac{\rho}{2\beta} + \rho u^2\right) dq_2 + \left[\frac{\gamma\rho u}{2(\gamma-1)\beta} + \frac{\rho u^3}{2}\right] dq_3 \\ dM_1 &= g_1\,dq_1 + g_2\,dq_2 + g_3\,dq_3 \end{aligned}\right\} \tag{36}$$

Thus far we have shown that the derivatives of the scalar functions M_o and M_1 with respect to q_i are, respectively, equal to w_i and g_i. Performing differentiation once again yields the elements of the matrices **P** and **B**. The symmetric hyperbolic form for the Euler equations can then be written in the expanded form:

$$\begin{bmatrix} \rho & \rho u & \frac{p}{\gamma-1} + \frac{\rho u^2}{2} \\ \rho u & p + \rho u^2 & \sigma p u + \frac{\rho u^3}{2} \\ \frac{p}{\gamma-1} + \frac{\rho u^2}{2} & \sigma p u + \frac{\rho u^3}{2} & \sigma^2 \frac{p^2}{\rho} + \sigma p u^2 + \frac{\rho u^4}{4} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \ln \rho + \frac{\ln \beta}{\gamma-1} - \beta u^2 \\ 2\beta u \\ -2\beta \end{bmatrix}$$

$$+ \begin{bmatrix} \rho u & p + \rho u^2 & \sigma p u + \frac{\rho u^3}{2} \\ p + \rho u^2 & 3pu + \rho u^3 & \frac{\sigma p^2}{\rho} + \left(\sigma + \frac{3}{2}\right) p u^2 + \frac{\rho u^4}{4} \\ \sigma p u + \frac{\rho u^3}{2} & \frac{\sigma p^2}{\rho} + \left(\sigma + \frac{3}{2}\right) p u^2 + \frac{\rho u^4}{4} & (\sigma^2 + \sigma) \frac{p^2 u}{\rho} + (\sigma + 1)\, p u^3 + \frac{\rho u^5}{4} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \ln \rho + \frac{\ln \beta}{\gamma-1} - \beta u^2 \\ 2\beta u \\ -2\beta \end{bmatrix} = 0 \quad (37)$$

where $\sigma = \gamma/(\gamma - 1)$. It is interesting to note that the scalar functions M_o and M_1 symmetrizing the Euler equations are, respectively, mass density ρ and mass flux ρu. The Euler equations (eq. (1)) are in conservation form (**w**-representation) and transform to symmetric hyperbolic form (**q**-representation) in terms of new variables **q**. The **w**-**q** transformation is given by

$$\mathbf{w} = \frac{\partial M_o}{\partial \mathbf{q}} = \frac{\partial \rho}{\partial \mathbf{q}} \qquad \mathbf{g} = \frac{\partial M_1}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}}(\rho u)$$

The **w**-representation reflects the physical principle of conservation, whereas the **q**-representation clearly reflects the hyperbolicity. The mass density and the mass flux are at the root of the above transformation. This preferential role played by the mass density and mass flux is physically due to the fact that mass is the carrier of the momentum and energy.

Finally, we show that equations (13) and (14) are valid even when the integration with respect to v in equations (17) and (18) is over either of the two half-intervals $(0, \infty)$ and $(-\infty, 0)$. The integrations over half-intervals are required when the Maxwellian distribution is split into two truncated Maxwellians—one corresponding to particles with positive v and the other corresponding to negative v. Such a splitting gives rise to splitting of the flux vector **g** into $\mathbf{g}^+$ and $\mathbf{g}^-$. The split Euler equations (corresponding to the splitting of **g**) can also be cast in the symmetric hyperbolic form using the integrals of the truncated Maxwellian. To demonstrate this property, we offer the following definitions:

$$\begin{bmatrix} w_1^+ \\ w_2^+ \\ w_3^+ \end{bmatrix} = \int_0^\infty dv \int_0^\infty dI \begin{bmatrix} 1 \\ v \\ I + \frac{v^2}{2} \end{bmatrix} F \tag{38}$$

$$\begin{bmatrix} g_1^+ \\ g_2^+ \\ g_3^+ \end{bmatrix} = \int_0^\infty dv \int_0^\infty dI\, v \begin{bmatrix} 1 \\ v \\ I + \frac{v^2}{2} \end{bmatrix} F \tag{39}$$

$$\begin{bmatrix} M_o^+ \\ M_1^+ \end{bmatrix} = \int_0^\infty dv \int_0^\infty dI \begin{bmatrix} 1 \\ v \end{bmatrix} F \tag{40}$$

Proceeding as before we obtain

$$\begin{bmatrix} dM_o^+ \\ dM_1^+ \end{bmatrix} = \int_0^\infty dv \int_0^\infty dI \begin{bmatrix} 1 \\ v \end{bmatrix} F \left[dq_1 + v\, dq_2 + \left(I + \frac{v^2}{2} \right) dq_3 \right.$$
$$\left. + \frac{5 - 3\gamma}{3 - \gamma} (I_o - I)\, dq_3 \right]$$

In view of equation (32), the above equation simplifies to

$$\begin{bmatrix} dM_o^+ \\ dM_1^+ \end{bmatrix} = \int_0^\infty dv \int_0^\infty dI \begin{bmatrix} 1 \\ v \end{bmatrix} F \left[dq_1 + v\, dq_2 + \left(I + \frac{v^2}{2} \right) dq_3 \right] \tag{41}$$

Equation (41) then immediately implies that

$$w_i^+ = \frac{\partial M_o^+}{\partial q_i} \qquad g_i^+ = \frac{\partial M_1^+}{\partial q_i} \tag{42}$$

Thus, when v-integration is over the half-interval, the transformed variables q_1, q_2, and q_3 are the same as before but the scalar functions are M_o^+ and M_1^+ instead of M_o and M_1.

By proceeding on similar lines we can establish the following results. Let

$$\begin{bmatrix} w_1^- \\ w_2^- \\ w_3^- \end{bmatrix} = \int_{-\infty}^0 dv \int_0^\infty dI \begin{bmatrix} 1 \\ v \\ I + \frac{v^2}{2} \end{bmatrix} F \tag{43}$$

$$\begin{bmatrix} g_1^- \\ g_2^- \\ g_3^- \end{bmatrix} = \int_{-\infty}^0 dv \int_0^\infty dI\, v \begin{bmatrix} 1 \\ v \\ I + \frac{v^2}{2} \end{bmatrix} F \tag{44}$$

$$\begin{bmatrix} M_o^- \\ M_1^- \end{bmatrix} = \int_{-\infty}^0 dv \int_0^\infty dI \begin{bmatrix} 1 \\ v \end{bmatrix} F \tag{45}$$

The integration with respect to v in equations (43), (44), and (45) is over the half-interval $(-\infty, 0)$. Then, the following relations hold:

$$w_i^- = \frac{\partial M_o^-}{\partial q_i} \qquad g_i^- = \frac{\partial M_1^-}{\partial q_i} \qquad (i = 1, 2, 3) \tag{46}$$

Here q_1, q_2, and q_3 are the same as before and are defined by equation (27).

To summarize, the important result of this section is that

$$\left\langle \boldsymbol{\Psi}, \frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} \right\rangle = 0 \qquad \boldsymbol{\Psi} = 1, v, I + \frac{v^2}{2}$$

can be cast in the symmetric hyperbolic form (eq. (15)) irrespective of whether the integration in the inner product with respect to v is over the doubly infinite interval $(-\infty, \infty)$ or the half-intervals $(0, \infty)$ and $(-\infty, 0)$. The transformed variables appearing in the symmetric hyperbolic form are q_1, q_2, and q_3 defined by equation (27), and the scalar functions accomplishing the symmetrization are mass and mass flux.

Positive Definiteness of P

The scalar function M_o is a convex function of q_1, q_2, and q_3 if the matrix

$$\mathbf{P} = \left[\frac{\partial^2 M_o}{\partial q_i\, \partial q_j} \right]$$

is positive definite. The positivity property of $\mathbf{P}$ can be easily proved when the integration with respect to v is over the full space $(-\infty, \infty)$. We first obtain expressions for $\partial^2 M_o / \partial q_i \, \partial q_j$ and then determine whether the determinants corresponding to the leading minors are positive. For this purpose we have

$$d\left(\frac{\partial M_o}{\partial q_1}\right) = d\rho = \rho \, dq_1 + \rho u \, dq_2 + \left[\frac{\rho u^2}{2} + \frac{\rho}{2(\gamma-1)\beta}\right] dq_3 \tag{47}$$

$$\begin{aligned} d\left(\frac{\partial M_o}{\partial q_2}\right) &= d(\rho u) = dM_1 \\ &= \rho u \, dq_1 + \left(\frac{\rho}{2\beta} + \rho u^2\right) dq_2 + \left[\frac{\gamma \rho u}{2(\gamma-1)\beta} + \frac{\rho u^3}{2}\right] dq_3 \end{aligned} \tag{48}$$

$$\begin{aligned} d\left(\frac{\partial M_o}{\partial q_3}\right) &= d\left[\frac{\rho u^2}{2} + \frac{\rho}{2(\gamma-1)\beta}\right] \\ &= \left[\frac{u^2}{2} + \frac{1}{2(\gamma-1)\beta}\right] d\rho - \left[\frac{\rho}{2(\gamma-1)\beta^2}\right] d\beta + \rho u \, du \\ &= \left[\frac{u^2}{2} + \frac{1}{2(\gamma-1)\beta}\right] \left\{\rho \, dq_1 + \rho u \, dq_2 + \left[\frac{\rho u^2}{2} + \frac{\rho}{2(\gamma-1)\beta}\right] dq_3\right\} \\ &\quad + \frac{\rho}{2(\gamma-1)\beta^2} \frac{dq_3}{2} + \rho u \frac{dq_2 + u \, dq_3}{2\beta} \end{aligned}$$

where use has been made of equations (33) to (36). Rearrangement of terms gives

$$\begin{aligned} d\left(\frac{\partial M_o}{\partial q_3}\right) &= \left[\frac{\rho u^2}{2} + \frac{\rho}{2(\gamma-1)\beta}\right] dq_1 + \left(\frac{\gamma}{\gamma-1} \frac{\rho u}{2\beta} + \frac{\rho u^3}{2}\right) dq_2 \\ &\quad + \left[\frac{\gamma}{(\gamma-1)^2} \frac{\rho}{4\beta^2} + \frac{\gamma}{\gamma-1} \frac{\rho u^2}{2\beta} + \frac{\rho u^4}{4}\right] dq_3 \end{aligned} \tag{49}$$

The second derivatives of M_o are then given by

$$\frac{\partial^2 M_o}{\partial q_1^2} = \rho \qquad \frac{\partial^2 M_o}{\partial q_1 \, \partial q_2} = \frac{\partial^2 M_o}{\partial q_2 \, \partial q_1} = \rho u$$

$$\frac{\partial^2 M_o}{\partial q_1 \, \partial q_3} = \frac{\partial^2 M_o}{\partial q_3 \, \partial q_1} = \frac{\rho u^2}{2} + \frac{\rho}{2(\gamma-1)\beta}$$

$$\frac{\partial^2 M_o}{\partial q_2^2} = \frac{\rho}{2\beta} + \rho u^2 \qquad \frac{\partial^2 M_o}{\partial q_2 \, \partial q_3} = \frac{\partial^2 M_o}{\partial q_3 \, \partial q_2} = \frac{\gamma}{\gamma-1} \frac{\rho u}{2\beta} + \frac{\rho u^3}{2}$$

$$\frac{\partial^2 M_o}{\partial q_3^2} = \frac{\gamma}{(\gamma-1)^2} \frac{\rho}{4\beta^2} + \frac{\gamma}{\gamma-1} \frac{\rho u^2}{2\beta} + \frac{\rho u^4}{4}$$

The determinants corresponding to the leading minors are given by

$$\frac{\partial^2 M_o}{\partial q_1^2} = \rho > 0$$

$$\begin{vmatrix} \frac{\partial^2 M_o}{\partial q_1^2} & \frac{\partial^2 M_o}{\partial q_1 \partial q_2} \\ \frac{\partial^2 M_o}{\partial q_2 \partial q_1} & \frac{\partial^2 M_o}{\partial q_2^2} \end{vmatrix} = \rho\left(\rho u^2 + \frac{\rho}{2\beta}\right) - \rho^2 u^2 = \frac{\rho^2}{2\beta} > 0$$

$$\det \mathbf{P} = \begin{vmatrix} \rho & \rho u & \frac{\rho}{2(\gamma-1)\beta} + \frac{\rho u^2}{2} \\ \rho u & \frac{\rho}{2\beta} + \rho u^2 & \frac{\gamma}{\gamma-1}\frac{\rho u}{2\beta} + \frac{\rho u^3}{2} \\ \frac{\rho u^2}{2} + \frac{\rho}{2(\gamma-1)\beta} & \frac{\gamma}{\gamma-1}\frac{\rho u}{2\beta} + \frac{\rho u^3}{2} & \frac{\gamma}{(\gamma-1)^2}\frac{\rho}{4\beta^2} + \frac{\gamma}{\gamma-1}\frac{\rho u^2}{2\beta} + \frac{\rho u^4}{4} \end{vmatrix}$$

A sequence of elementary transformations gives

$$\begin{aligned} \det \mathbf{P} &= \begin{vmatrix} \rho & \rho u & \frac{\rho}{2(\gamma-1)\beta} + \frac{\rho u^2}{2} \\ 0 & \frac{\rho}{2\beta} & \frac{\rho u}{2\beta} \\ \frac{\rho}{2(\gamma-1)\beta} & \frac{\gamma}{\gamma-1}\frac{\rho u}{2\beta} & \frac{\gamma}{(\gamma-1)^2}\frac{\rho}{4\beta^2} + \frac{2\gamma-1}{\gamma-1}\frac{\rho u^2}{4\beta} \end{vmatrix} \\ &= \begin{vmatrix} \rho & \rho u & \frac{\rho}{2(\gamma-1)\beta} + \frac{\rho u^2}{2} \\ 0 & \frac{\rho}{2\beta} & \frac{\rho u}{2\beta} \\ 0 & \frac{\rho u}{2\beta} & \frac{\rho u^2}{2\beta} + \frac{\rho}{4(\gamma-1)\beta^2} \end{vmatrix} \\ &= \rho\left[\frac{\rho^2 u^2}{4\beta^2} + \frac{\rho^2}{8(\gamma-1)\beta^3} - \frac{\rho^2 u^2}{6\beta^2}\right] \\ &= \frac{\rho^3}{8(\gamma-1)\beta^3} > 0 \qquad (\gamma > 1) \end{aligned}$$

Hence, all the leading minors of $\mathbf{P}$ have positive determinants and $\mathbf{P}$ is therefore a positive matrix implying the convexity of M_o. It is interesting to note that the only restriction on γ in the above analysis is that $\gamma > 1$.

The proof of positivity of $\mathbf{P}$ for the cases when the integration is over half-negative and half-positive intervals can be constructed on the above guidelines. The algebra becomes very involved and the main thrust of the present paper, that many results become transparent in view of the Maxwellian distribution, is then defeated. We will now demonstrate the positivity of $\mathbf{P}$ in all the cases using the fact that M_o is an integral of the Maxwellian. To this end, we first write M_o as

$$\tilde{M}_o = \int_a^b dv \int_0^\infty dI\, F \tag{50}$$

where the limit (a, b) could be any one of the intervals $(-\infty, \infty)$, $(0, \infty)$, or $(-\infty, 0)$. Using equation (31) gives

$$\begin{aligned} d\tilde{M}_o &= \int_a^b dv \int_0^\infty dI\,(dF) \\ &= \int_a^b dv \int_0^\infty dI\, F \left[dq_1 + v\, dq_2 + \left(I + \frac{v^2}{2}\right) dq_3 + \frac{5-3\gamma}{3-\gamma}(I_o - I)\, dq_3\right] \end{aligned}$$

In view of equation (32), the expression for $d\tilde{M}_o$ becomes

$$d\tilde{M}_o = \int_a^b dv \int_0^\infty dI\, F \left[dq_1 + v\, dq_2 + \left(I + \frac{v^2}{2}\right) dq_3\right]$$

Taking the differential once more yields

$$d^2\tilde{M}_o = \int_a^b dv \int_0^\infty dI\,(dF)\left[dq_1 + v\,dq_2 + \left(I + \frac{v^2}{2}\right) dq_3\right] \tag{51}$$

Substitution of dF from equation (31) into equation (51) further gives

$$\begin{aligned} d^2\tilde{M}_o = \int_a^b dv \int_0^\infty dI\, &\left[dq_1 + v\,dq_2 + \left(I + \frac{v^2}{2}\right) dq_3\right]\left[dq_1 + v\,dq_2\right. \\ &\left. + \left(I + \frac{v^2}{2}\right) dq_3 + \frac{5-3\gamma}{3-\gamma}(I_o - I)\,dq_3\right] F \end{aligned} \tag{52}$$

Let

$$dX = dq_1 + v\,dq_2 + \left(I + \frac{v^2}{2}\right) dq_3 \tag{53a}$$

$$\begin{aligned} dY &= dq_1 + v\,dq_2 + \left(I + \frac{v^2}{2}\right) dq_3 + \frac{5-3\gamma}{3-\gamma}(I_o - I)\,dq_3 \\ &= dq_1 + v\,dq_2 + \left[\frac{2(\gamma-1)}{3-\gamma} I + \frac{v^2}{2} + \frac{5-3\gamma}{3-\gamma} I_o\right] dq_3 \end{aligned} \tag{53b}$$

Equation (52) then reduces to

$$d^2\tilde{M}_o = \int_a^b dv \int_0^\infty dI\, dX\, dY\, F \tag{54}$$

Noting that

$$dY = dX + \frac{5-3\gamma}{3-\gamma}(I_o - I)\,dq_3$$

$$dX = dY - \frac{5-3\gamma}{3-\gamma}(I_o - I)\,dq_3$$

equation (54) yields

$$d^2\tilde{M}_o = \int_a^b dv \int_0^\infty dI\,(dX)^2 F + \int_a^b dv \int_0^\infty dI\,\left[\frac{5-3\gamma}{3-\gamma}(I_o - I)\,dq_3\,dX\right] F$$

and, alternatively,

$$d^2\tilde{M}_o = \int_a^b dv \int_0^\infty dI\,(dY)^2 F - \int_a^b dv \int_0^\infty dI\,\left[\frac{5-3\gamma}{3-\gamma}(I - I_o)\,dq_3\,dY\right] F$$

Substituting for dX and dY from equations (53a) and (53b), respectively, in the second term of the above formulas for $d^2\tilde{M}_o$ and using the easily provable relations gives

$$\int_0^\infty (I - I_o) F\,dI = 0 \qquad \int_0^\infty I(I - I_o) F\,dI = I_o^2 \tilde{F} = \int_0^\infty I_o^2 F\,dI$$

The two equations for $d^2\tilde{M}_o$ then simplify to

$$d^2\tilde{M}_o = \int_a^b dv \int_0^\infty dI\,(dX)^2 F - \int_a^b dv \int_0^\infty dI\,\left[\frac{5-3\gamma}{3-\gamma} I_o^2\,(dq_3)^2\right] F \tag{55a}$$

$$d^2\tilde{M}_o = \int_a^b dv \int_0^\infty dI\,(dY)^2\,F + \int_a^b dv \int_0^\infty dI\,\left[\frac{2(\gamma-1)(5-3\gamma)}{(3-\gamma)^2} I_o^2\,(dq_3)^2\right] F \tag{55b}$$

The quadratic form Q corresponding to the matrix $\mathbf{P}$ where

$$Q = \boldsymbol{\eta}^t \left(\frac{\partial^2 M_o}{\partial q_i\,\partial q_j}\right) \boldsymbol{\eta}$$

can be obtained from equations (55) by replacing dq_i by η_i. We then obtain

$$\begin{aligned} Q &= \int_a^b dv \int_0^\infty dI \left[\eta_1 + v\eta_2 + \left(I + \frac{v^2}{2}\right)\eta_3\right]^2 F \\ &\quad - \int_a^b dv \int_0^\infty dI \left[\frac{5-3\gamma}{3-\gamma}(I_o\eta_3)^2\right] F \\ &= \int_a^b dv \int_0^\infty dI \left\{\eta_1 + v\eta_2 + \left[\frac{2(\gamma-1)}{3-\gamma}I + \frac{v^2}{2} + \frac{5-3\gamma}{3-\gamma}I_o\right]\eta_3\right\}^2 F \\ &\quad + \int_a^b dv \int_0^\infty dI \left[\frac{2(\gamma-1)(5-3\gamma)}{(3-\gamma)^2}(I_o\eta_3)^2\right] F \end{aligned}$$

It is obvious that $Q > 0$ as long as $1 < \gamma < 3$. Hence, $\mathbf{P}$ is a positive matrix. Two expressions for Q were derived to show that irrespective of whether $\gamma \geq 5/3$ or $\gamma \leq 5/3$, we obtain a positive expression for Q. If $5 - 3\gamma < 0$, the positivity of $\mathbf{P}$ follows from the first of the above two formulas for Q, and if $5 - 3\gamma > 0$, then the positivity of $\mathbf{P}$ follows from the second formula.

The above proof regarding the convexity of M_o (or equivalently positive definiteness of $\mathbf{P}$) rests upon the equation $dF = F\,dY$, which in turn is a consequence of the Maxwellian distribution. Once again the connection between the convexity of the function symmetrizing the Euler equations and the Maxwellian distribution is obvious.

Entropy Function and the H-Theorem

The Boltzmann H-theorem has been described in the kinetic theory of gases as the bridge connecting the thermodynamics and the statistical mechanics of particles. Briefly stated, the theorem says that the H-function (ref. 1) defined by

$$H = \int (f \ln f)\,dv_1\,dv_2\,dv_3$$

monotonically decreases with time as a homogeneous gas in statistical nonequilibrium evolves to equilibrium. In case of spatial inhomogeneity, the theorem states that

$$\frac{\partial H}{\partial t} + \sum_i \frac{\partial H_{v_i}}{\partial x_i} \leq 0 \tag{56}$$

where H_{vi} is the flux defined by

$$H_{vi} = \int v_i (f \ln f)\,dv_1\,dv_2\,dv_3$$

It is therefore natural to define an entropy function for the Euler equations using the above definition of H with the arbitrary distribution f replaced by the Maxwellian. We then obtain H by

$$H \equiv H(F) = \int\int (F \ln F)\,dv\,dI \tag{57a}$$

and H-flux by

$$H_v = \int\int v(F \ln F)\,dv\,dI \tag{57b}$$

When F is a Maxwellian, $\ln F$ is given by

$$\ln F = \ln \rho + \frac{3}{2}\ln \beta + \ln\left[\frac{4(\gamma-1)}{\sqrt{\pi}(3-\gamma)}\right] - \beta(v-u)^2 - \frac{4(\gamma-1)}{3-\gamma}\beta I$$

which can be equivalently written as

$$\ln F - \frac{2(5-3\gamma)}{3-\gamma}\beta I = \left\{\ln \rho + \frac{3}{2}\ln \beta + \ln\left[\frac{4(\gamma-1)}{\sqrt{\pi}(3-\gamma)}\right] - \beta u^2\right\} + (2\beta u)v + (-2\beta)\left(I + \frac{v^2}{2}\right) \tag{58}$$

The right-hand side of equation (58) is a linear combination of collisional invariants and, hence, substituting equation (4) into (58) gives

$$\int\int\left[\ln F - \frac{2(5-3\gamma)}{3-\gamma}\beta I\right]\left(\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x}\right) dv\, dI = 0 \tag{59}$$

We will show that equation (59) is the basis of the entropy conservation. Considering first that $\gamma = 5/3$, equation (59) simplifies to

$$\int\int \ln F\left(\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x}\right) dv\, dI = 0 \tag{60}$$

Hence, in view of equation (60),

$$\left[\int\int\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)(F\ln F)\right] dv\, dI = \int\int (1+\ln F)\left(\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x}\right) dv\, dI = 0$$

In terms of H and H_v, the above equation becomes

$$\frac{\partial H}{\partial t} + \frac{\partial H_v}{\partial x} = 0 \tag{61}$$

which is the entropy conservation for a smooth solution. When $\gamma \neq 5/3$, the gas has additional degrees of freedom apart from the translational ones. The $\ln F$ appearing in the definition of H is then no longer a linear combination of collisional invariants and, consequently, equation (60) will not be valid. Therefore, a slight modification in the definitions of H and H_v is required. To obtain these new expressions for H and H_v we can proceed in the following manner. Equation (59) can be written as

$$\int\int\left[1 + \ln F - \frac{2(5-3\gamma)}{3-\gamma}\beta I\right]\left(\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial t}\right) dv\, dI = 0 \tag{62}$$

Using $d(F\ln F) = (1+\ln F)\,dF$ gives

$$\int\int dv\, dI\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)(F\ln F) - \frac{2(5-3\gamma)}{3-\gamma}\int\int \beta I\left(\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x}\right) dv\, dI = 0 \tag{63}$$

The "trick" lies in transforming the second term in equation (63) as an integral of a perfect differential. We note that

$$\begin{aligned}\int\int \beta I\left(\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x}\right) dv\, dI &= \int\int\left[\frac{\partial}{\partial t}(\beta I F) + v\frac{\partial}{\partial x}(\beta I F)\right] dv\, dI \\ &\quad - \int\int IF\left(\frac{\partial \beta}{\partial t} + v\frac{\partial \beta}{\partial x}\right) dv\, dI \\ &= \frac{3-\gamma}{4(\gamma-1)}\left(\frac{\partial \tilde{w}_1}{\partial t} + \frac{\partial \tilde{g}_1}{\partial x}\right) - I_o\left(\tilde{w}_1\frac{\partial \beta}{\partial t} + \tilde{g}_1\frac{\partial \beta}{\partial x}\right)\end{aligned} \tag{64}$$

where $\tilde{w}_1$ is equal to ρ, w_1^+, or w_1^-, and $\tilde{g}_1$ is equal to g_1, g_1^+, or g_1^-, depending on the interval of v integration. Further manipulation gives

$$\begin{aligned}
&\int\!\!\int \beta I \left(\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x}\right) dv\, dI \\
&\quad = \frac{3-\gamma}{4(\gamma-1)} \left\{(1+\ln\beta)\left(\frac{\partial \tilde{w}_1}{\partial t} + \frac{\partial \tilde{g}_1}{\partial x}\right) - \left[\frac{\partial}{\partial t}(\tilde{w}_1 \ln\beta) + \frac{\partial}{\partial x}(\tilde{g}_1 \ln\beta)\right]\right\}
\end{aligned} \tag{65}$$

When the integration is over the full interval,

$$\frac{\partial w_1}{\partial t} + \frac{\partial g_1}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

When the integration is over the half-interval, split Euler equations

$$\frac{\partial \mathbf{w}^\pm}{\partial t} + \frac{\partial \mathbf{g}^\pm}{\partial x} = 0 \tag{66}$$

will have to be imposed. Keeping this fact in mind, equation (65) gives

$$\begin{aligned}
\int\!\!\int \beta I \left(\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x}\right) dv\, dI &= -\frac{3-\gamma}{4(\gamma-1)} \left[\frac{\partial}{\partial t}(\tilde{w}_1 \ln\beta) + \frac{\partial}{\partial x}(\tilde{g}_1 \ln\beta)\right] \\
&= -\frac{3-\gamma}{4(\gamma-1)} \int\!\!\int \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)(F \ln\beta)\, dv\, dI
\end{aligned} \tag{67}$$

Combining equation (67) with equation (63) gives

$$\int\!\!\int \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left[\ln F + \frac{5-3\gamma}{2(\gamma-1)}\ln\beta\right] F\, dv\, dI = 0 \tag{68}$$

The H-function and H-flux can hence be defined, respectively, as

$$H = \int\!\!\int dv\, dI \left[F\ln F + \frac{5-3\gamma}{2(\gamma-1)} F \ln\beta + C_1 F\right] \tag{69}$$

$$H_v = \int\!\!\int dv\, dI\, v \left[F\ln F + \frac{5-3\gamma}{2(\gamma-1)} F \ln\beta + C_2 F\right] \tag{70}$$

where C_1 and C_2 are constants independent of v, I, ρ, β, and u. In view of the definitions given in equations (69) and (70), equation (68) can be cast in the compressed form

$$\frac{\partial H}{\partial t} + \frac{\partial H_v}{\partial x} = 0$$

which is the entropy conservation. The thermodynamic entropy per unit mass for a perfect gas is

$$S = -R\left(\ln\rho + \frac{\ln\beta}{\gamma-1} + \text{Constant}\right)$$

The appearance of a constant in the above formula is due to the fact that the entropy is indeterminate within a constant. The H-function defined by equation (69) after integration with respect to v and I becomes

$$H = \rho\left\{\ln\rho + \frac{\ln\beta}{\gamma-1} + C_1 - \frac{3}{2} + \ln\left[\frac{4(\gamma-1)}{\sqrt{\pi}(3-\gamma)}\right]\right\}$$

which is therefore negative of thermodynamic entropy per unit volume. Since the H-function is a measure of the information content of a distribution, it will be negative of entropy. The modified H-function is thus a physically meaningful quantity.

Because of the appearance of an additional term involving $F \ln \beta$ in equation (69), the convexity of H is not obvious. We will now show that H is a convex function of q_1, q_2, and q_3. The proof rests upon showing that H is the Legendre transform of M_o; that is,

$$H = q_1 \frac{\partial M_o}{\partial q_1} + q_2 \frac{\partial M_o}{\partial q_2} + q_3 \frac{\partial M_o}{\partial q_3} - M_o \tag{71}$$

The function H will then be convex if M_o is convex. Since M_o has been shown to be convex, it is enough if H defined by equation (69) satisfies equation (71). For this purpose we observe that

$$\frac{\partial F}{\partial q_1} = F \qquad \frac{\partial F}{\partial q_2} = vF \qquad \frac{\partial F}{\partial q_3} = \left[\frac{2(\gamma-1)}{3-\gamma} I + \frac{v^2}{2} + \frac{5-3\gamma}{4(\gamma-1)\beta}\right] F \tag{72}$$

These equations directly follow from equation (31). Further, equation (71) can be written in the form

$$H = \int \int \left(q_1 \frac{\partial F}{\partial q_1} + q_2 \frac{\partial F}{\partial q_2} + q_3 \frac{\partial F}{\partial q_3} - F \right) dv\, dI \tag{73}$$

which says that the integrand of H defined by equation (73) is also a Legendre transform of F. Using equation (72) gives

$$q_1 \frac{\partial F}{\partial q_1} + q_2 \frac{\partial F}{\partial q_2} + q_3 \frac{\partial F}{\partial q_3} = F \left[\ln \rho + \frac{\ln \beta}{\gamma - 1} - \beta u^2 + 2\beta u v - \beta v^2 - \frac{4(\gamma-1)}{3-\gamma} \beta I - \frac{5-3\gamma}{2(\gamma-1)} \right] \tag{74}$$

Equation (58) for $\ln F$ can be slightly rewritten as

$$\ln F + \frac{5-3\gamma}{2(\gamma-1)} \ln \beta = \ln \rho + \frac{\ln \beta}{\gamma-1} + \ln \left[\frac{4(\gamma-1)}{\sqrt{\pi}(3-\gamma)} \right] - \beta (v-u)^2 - \frac{4(\gamma-1)}{3-\gamma} \beta I$$

Equation (74) then simplifies to

$$q_1 \frac{\partial F}{\partial q_1} + q_2 \frac{\partial F}{\partial q_2} + q_3 \frac{\partial F}{\partial q_3} = F \left[\ln F + \frac{5-3\gamma}{2(\gamma-1)} \ln \beta \right] - F \left\{ \ln \left[\frac{4(\gamma-1)}{\sqrt{\pi}(3-\gamma)} \right] + \frac{5-3\gamma}{2(\gamma-1)} \right\}$$

Hence, the Legendre transform of F is given by

$$\sum q_i \frac{\partial F}{\partial q_i} - F = F \left(\ln F + \frac{5-3\gamma}{2(\gamma-1)} \ln \beta - \left\{ \ln \left[\frac{4(\gamma-1)}{\sqrt{\pi}(3-\gamma)} \right] + \frac{3-\gamma}{2(\gamma-1)} \right\} \right) \tag{75}$$

Equation (75) is purely a consequence of the Maxwellian distribution and the definitions of q_1, q_2, and q_3. A comparison of equations (69) and (70) with equation (75) shows that the H-function and H-flux defined by equations (69) and (70), respectively, can be equivalently written as

$$H = \int \int dv\, dI \left(\sum q_i \frac{\partial F}{\partial q_i} - F \right) \tag{76}$$

$$H_v = \int \int dv\, dI\, v \left(\sum q_i \frac{\partial F}{\partial q_i} - F \right) \tag{77}$$

if we choose

$$C_1 = -\ln \left[\frac{4(\gamma-1)}{\sqrt{\pi}(3-\gamma)} \right] - \frac{3-\gamma}{2(\gamma-1)}$$

The above analysis reveals a very interesting property of the H-function. First, the H-function is based on the Boltzmann H-function and is negative of the density of thermodynamic entropy. Second, because of the appearance of $\ln F$ in its integrand, the H-function is a Legendre transform of another convex function. The convexity of H is a consequence of this fact. Further, because of the presence of the $F \ln F$ term in the H-function, the H-function satisfies the entropy conservation. Thus, the Maxwellian distribution is at the root of the convexity of H and the satisfaction of the entropy conservation.

Finally, we return to the important point mentioned at the beginning of this section, namely, the connection between the H-theorem and the entropy condition (eq. (56)). The convexity of the H-function implies that

$$H\left(q^{n+1}\right) \leq H\left(q^{n}\right)$$

if the numerical solution q^{n+1} at the $n+1$ time level is given by

$$q^{n+1} = (1-\theta)(q')^{n} + \theta(q'')^{n} \tag{78}$$

Here q' and q'' could be the values of $\mathbf{q}$ at neighboring grid points. Thus, if a numerical method for the solution of the Euler equations is of the type used in equation (78), then the H-function will decrease with increasing n for that method.

Extension to Multidimensional Case

The extension of the results given in the previous section to the multidimensional case is fairly straightforward. We will now briefly outline various significant steps involved in extending the above analysis to two-dimensional Euler equations:

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{g}_x}{\partial x} + \frac{\partial \mathbf{g}_y}{\partial y} = 0 \tag{79}$$

where

$$\mathbf{w} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho e \end{bmatrix} \qquad \mathbf{g}_x = \begin{bmatrix} \rho u_1 \\ p + \rho u_1^2 \\ \rho u_1 u_2 \\ (\rho e + p) u_1 \end{bmatrix} \qquad \mathbf{g}_y = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 \\ p + \rho u_2^2 \\ (\rho e + p) u_2 \end{bmatrix} \tag{80}$$

As before, equation (79) can be written in the moment form

$$\left\langle \mathbf{\Psi}, \frac{\partial F}{\partial t} + v_1 \frac{\partial F}{\partial x} + v_2 \frac{\partial F}{\partial y} \right\rangle = 0 \tag{81}$$

where the moment function $\mathbf{\Psi}$ is equal to 1, v_1, v_2, and $I + \left[\left(v_1^2 + v_2^2\right)/2\right]$, and F is the Maxwellian. Thus,

$$F = \frac{\rho}{I_o} \frac{\beta}{\pi} \exp\left[-\beta(v_1 - u_1)^2 - \beta(v_2 - u_2)^2 - \frac{I}{I_o}\right] \tag{82}$$

$$I_o = \frac{2-\gamma}{2(\gamma - 1)\beta} \tag{83}$$

where I_o is positive for $1 \leq \gamma \leq 2$. Notice that

$$\mathbf{w} = \langle \mathbf{\Psi}, F \rangle \qquad \mathbf{g}_x = \langle v_1 \mathbf{\Psi}, F \rangle \qquad \mathbf{g}_y = \langle v_2 \mathbf{\Psi}, F \rangle \tag{84}$$

Equation (84) states that the unknown vector $\mathbf{w}$ and the flux vectors $\mathbf{g}_x$ and $\mathbf{g}_y$ in equation (79) are the moments of the Maxwellian F. We will now show that

$$w_i = \frac{\partial M_o}{\partial q_i} \qquad (g_x)_i = \frac{\partial M_x}{\partial q_i} \qquad (g_y)_i = \frac{\partial M_y}{\partial q_i} \qquad (i = 1,2,3,4) \tag{85}$$

where

$$\left.\begin{aligned} q_1 &= \ln\rho + \frac{\ln\beta}{\gamma - 1} - \beta u^2 \\ q_2 &= 2u_1\beta \\ q_3 &= 2u_2\beta \\ q_4 &= -2\beta \end{aligned}\right\} \qquad \left(u^2 = u_1^2 + u_2^2\right) \tag{86}$$

and

$$\left.\begin{aligned} M_o &= \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \int_0^{\infty} dI\, F \\ M_x &= \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \int_0^{\infty} dI\, (v_1 F) \\ M_y &= \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \int_0^{\infty} dI\, (v_2 F) \end{aligned}\right\} \tag{87}$$

The differentials are given by

$$\begin{bmatrix} dM_o \\ dM_x \\ dM_y \end{bmatrix} = \int dv_1\, dv_2\, dI \begin{bmatrix} 1 \\ v_1 \\ v_2 \end{bmatrix} dF \tag{88}$$

where, for the sake of notational brevity, only one integration symbol is shown and the limits are also not explicitly displayed. Using

$$dF = \frac{\partial F}{\partial \rho} d\rho + \frac{\partial F}{\partial u_1} du_1 + \frac{\partial F}{\partial u_2} du_2 + \frac{\partial F}{\partial \beta} d\beta$$

$$\frac{\partial F}{\partial \rho} = \frac{F}{\rho} \qquad \frac{\partial F}{\partial u_1} = 2\beta(v_1 - u_1)F \qquad \frac{\partial F}{\partial u_2} = 2\beta(v_2 - u_2)F$$

$$\frac{\partial F}{\partial \beta} = \left[\frac{2}{\beta} - (v_1 - u_1)^2 - (v_2 - u_2)^2 - \frac{2(\gamma - 1)}{2 - \gamma} I\right] F$$

yields

$$\begin{aligned} \frac{dF}{F} &= \frac{d\rho}{\rho} + 2\beta(v_1 - u_1)\, du_1 + 2\beta(v_2 - u_2)\, du_2 + \left[\frac{2}{\beta} - (v_1 - u_1)^2 \right. \\ &\qquad \left. - (v_2 - u_2)^2 - \frac{2(\gamma - 1)}{2 - \gamma} I\right] d\beta \\ &= \left(\frac{d\rho}{\rho} - 2\beta u_1\, du_1 - u_1^2\, d\beta - 2\beta u_2\, du_2 - u_2^2\, d\beta + \frac{2\, d\beta}{\beta}\right) \\ &\qquad + (2\beta v_1\, du_1 + 2v_1 u_1\, d\beta) + (2\beta v_2\, du_2 + 2v_2 u_2\, d\beta) \\ &\qquad - 2\left(I + \frac{v^2}{2}\right) d\beta + \frac{2(3 - 2\gamma)}{2 - \gamma} I\, d\beta \\ &= d(\ln\rho + 2\ln\beta - \beta u^2) + v_1\, d(2\beta u_1) + v_2\, d(2\beta u_2) \\ &\qquad + \left(I + \frac{v^2}{2}\right) d(-2\beta) + \frac{2(3 - 2\gamma)}{2 - \gamma} I\, d\beta \end{aligned}$$

Using the identity

$$\frac{1}{\gamma - 1} = 2 + \frac{3 - 2\gamma}{\gamma - 1}$$

simplifies the above equation for dF to

$$\frac{dF}{F} = dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(I + \frac{v^2}{2}\right) dq_4 - \frac{3-2\gamma}{\gamma-1}\frac{d\beta}{\beta} + \frac{2(3-2\gamma)}{2-\gamma} I\,d\beta$$
$$= dq_1 + v_1\,dq_2 - v_2\,dq_3 + \left(I + \frac{v^2}{2}\right) dq_4 + \frac{3-2\gamma}{2-\gamma}(I_o - I)\,dq_4 \tag{89}$$

Substituting dF from equation (89) into equation (88) gives

$$\begin{bmatrix} dM_o \\ dM_x \\ dM_y \end{bmatrix} = \int dv_1\,dv_2\,dI \begin{bmatrix} 1 \\ v_1 \\ v_2 \end{bmatrix} \left[dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(I + \frac{v^2}{2}\right) dq_4 + \frac{3-2\gamma}{2-\gamma}(I_o - I)\,dq_4\right] F \tag{90}$$

Again, from observing that

$$\int (I_o - I) F\,dI = 0 \tag{91}$$

equation (90) simplifies to

$$\begin{bmatrix} dM_o \\ dM_x \\ dM_y \end{bmatrix} = \int dv_1\,dv_2\,dI \begin{bmatrix} 1 \\ v_1 \\ v_2 \end{bmatrix} \left[dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(I + \frac{v^2}{2}\right) dq_4\right] F \tag{92}$$

which implies the validity of equations (85). In terms of the functions M_o, M_x, and M_y, the Euler equations (eq. (79)) assume the symmetric hyperbolic form

$$\sum_i \sum_j \left(\frac{\partial^2 M_o}{\partial q_i\,\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial^2 M_x}{\partial q_i\,\partial q_j}\frac{\partial q_j}{\partial x} + \frac{\partial^2 M_y}{\partial q_i\,\partial q_j}\frac{\partial q_j}{\partial y} \right) = 0 \tag{93}$$

The proof about the positive definiteness of the matrix $\mathbf{P}$ where

$$\mathbf{P} = \left[\frac{\partial^2 M_o}{\partial q_i\,\partial q_j}\right] \tag{94}$$

proceeds along exactly the same lines as before. We first observe that

$$d^2 M_o = \int dv_1\,dv_2\,dI \left[dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(I + \frac{v^2}{2}\right) dq_4\right] dF \tag{95}$$

Substituting dF from equation (89) into equation (95) gives

$$d^2 M_o = \int dv_1\,dv_2\,dI \left[dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(I + \frac{v^2}{2}\right) dq_4\right] \Bigg[dq_1 + v_1\,dq_2$$
$$+ v_2\,dq_3 + \left(I + \frac{v^2}{2}\right) dq_4 + \frac{3-2\gamma}{2-\gamma}(I_o - I)\,dq_4\Bigg] F \tag{96}$$

Using $\int dI\,(I_o - I)F = 0$, equation (96) reduces to two forms:

$$d^2M_o = \int dv_1\,dv_2\,dI\,\left[dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(I + \frac{v^2}{2}\right)\,dq_4\right]^2 F$$
$$+ \int dv_1\,dv_2\,dI\,\left[\frac{3-2\gamma}{2-\gamma}(dq_4)^2\,(II_o - I^2)\right] F$$

and

$$d^2M_o = \int dv_1\,dv_2\,dI\,\left[dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(\frac{\gamma-1}{2-\gamma}I + \frac{v^2}{2} + \frac{3-\gamma}{2-\gamma}I_o\right)\,dq_4\right]^2 F$$
$$+ \int dv_1\,dv_2\,dI\,\left[\frac{3-\gamma}{2-\gamma}(I^2 - II_o)\,(dq_4)^2\right] F$$

With the use of the easily provable result

$$\int I^2 F\,dI\,dv = \int 2I_o^2 F\,dI\,dv$$

we obtain

$$d^2M_o = \int dv_1\,dv_2\,dI\,\left[dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(I + \frac{v^2}{2}\right)\,dq_4\right]^2 F$$
$$- \frac{3-2\gamma}{2-\gamma}\int dv_1\,dv_2\,dI\,\left[I_o(dq_4)^2\right] F \tag{97}$$

$$d^2M_o = \int dv_1\,dv_2\,dI\,\left[dq_1 + v_1\,dq_2 + v_2\,dq_3 + \left(\frac{\gamma-1}{2-\gamma}I + \frac{v^2}{2} + \frac{3-2\gamma}{2-\gamma}I_o\right)\,dq_4\right]^2 F$$
$$+ \frac{3-2\gamma}{2-\gamma}\int dv_1\,dv_2\,dI\,\left[I_o(dq_4)^2\right] F \tag{98}$$

If $\gamma \geq 1.5$, then the positive definiteness of **P** follows from equation (97); and if $\gamma \leq 1.5$, then the positive property of **P** follows from equation (98).

We now come to the derivation of the H-function for the two-dimensional case. As before, the H-function is defined as the Legendre transform

$$H(F) = H = \int \left(\sum q_i \frac{\partial F}{\partial q_i} - F\right) dv_1\,dv_2\,dI \tag{99}$$

and the fluxes are defined by

$$\left.\begin{aligned} H_{vx} &= \int v_1 \left(\sum q_i \frac{\partial F}{\partial q_i} - F\right) dv_1\,dv_2\,dI \\ H_{vy} &= \int v_2 \left(\sum q_i \frac{\partial F}{\partial q_i} - F\right) dv_1\,dv_2\,dI \end{aligned}\right\} \tag{100}$$

From equation (89) it follows that

$$\frac{1}{F}\frac{\partial F}{\partial q_1} = 1 \qquad \frac{1}{F}\frac{\partial F}{\partial q_2} = v_1 \qquad \frac{1}{F}\frac{\partial F}{\partial q_3} = v_2$$

$$\frac{1}{F}\frac{\partial F}{\partial q_4} = \frac{\gamma-1}{2-\gamma}I + \frac{v^2}{2} + \frac{3-2\gamma}{2-\gamma}I_o$$

Hence,

$$
\begin{aligned}
\frac{1}{F}\sum q_i \frac{\partial F}{\partial q_i} &= \ln\rho + \frac{\ln\beta}{\gamma-1} - \beta u^2 + 2\beta v_1 u_1 + 2\beta v_2 u_2 \\
&\quad - 2\beta\left(\frac{\gamma-1}{2-\gamma} I + \frac{v^2}{2} + \frac{3-2\gamma}{2-\gamma} I_o\right) \\
&= \ln\rho + \frac{\ln\beta}{\gamma-1} - \beta(v_1-u_1)^2 - \beta(v_2-u_2)^2 - \frac{\gamma-1}{2-\gamma} 2\beta I - \frac{3-2\gamma}{\gamma-1}
\end{aligned}
$$

The Legendre transform of F is given by

$$
\begin{aligned}
\sum q_i \frac{\partial F}{\partial q_i} - F &= F\left[\ln\rho + \frac{\ln\beta}{\gamma-1} - \beta(v_1-u_1)^2 - \beta(v_2-u_2)^2 - \frac{\gamma-1}{2-\gamma} 2\beta I - \frac{2-\gamma}{\gamma-1}\right] \\
&= F\left[\ln\rho + \frac{\ln\beta}{\gamma-1} - \beta(v_1-u_1)^2 - \beta(v_2-u_2)^2 - \frac{I}{I_o} - \frac{2-\gamma}{\gamma-1}\right] \qquad (101)
\end{aligned}
$$

In order to express the right-hand side of equation (101) in terms of $\ln F$, we make use of

$$
\ln F = \ln\rho + 2\ln\beta + \ln\left[\frac{2(\gamma-1)}{(2-\gamma)\pi}\right] - \beta(v_1-u_1)^2 - \beta(v_2-u_2)^2 - \frac{I}{I_o} \qquad (102)
$$

Combining equations (101) and (102) gives

$$
\begin{aligned}
\sum q_i \frac{\partial F}{\partial q_i} - F &= F\left\{\ln F + \frac{3-2\gamma}{\gamma-1}\ln\beta - \frac{2-\gamma}{\gamma-1} - \ln\left[\frac{2(\gamma-1)}{(2-\gamma)\pi}\right]\right\} \\
&= F\left[\ln F + \frac{3-2\gamma}{\gamma-1}\ln\beta - C\right] \qquad (103)
\end{aligned}
$$

where

$$
C = \frac{2-\gamma}{\gamma-1} + \ln\left[\frac{2(\gamma-1)}{(2-\gamma)\pi}\right]
$$

The definitions in equations (99) and (100) for H and H-fluxes, respectively, therefore reduce to

$$
\begin{bmatrix} H \\ H_{vx} \\ H_{vy} \end{bmatrix} = \int \begin{bmatrix} 1 \\ v_1 \\ v_2 \end{bmatrix} F\left(\ln F + \frac{3-2\gamma}{\gamma-1}\ln\beta - C\right) dv_1\, dv_2\, dI \qquad (104)
$$

Using equation (104) now shows that

$$
\frac{\partial H}{\partial t} + \frac{\partial H_{vx}}{\partial x} + \frac{\partial H_{vy}}{\partial y} = 0 \qquad (105)
$$

For this purpose we observe that

$$
\begin{aligned}
\frac{\partial H}{\partial t} + \frac{\partial H_{vx}}{\partial x} + \frac{\partial H_{vy}}{\partial y} &= \int (1 - C + \ln F)\left(\frac{\partial F}{\partial t} + v_1\frac{\partial F}{\partial x} + v_2\frac{\partial F}{\partial y}\right) dv_1\, dv_2\, dI \\
&\quad + \frac{3-2\gamma}{\gamma-1}\int\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\right)(F\ln\beta)\, dv_1\, dv_2\, dI \qquad (106)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int \beta I &\left(\frac{\partial F}{\partial t} + v_1 \frac{\partial F}{\partial x} + v_2 \frac{\partial F}{\partial y} \right) dv_1\, dv_2\, dI \\
&= \beta \left[\frac{\partial}{\partial t}(\rho I_o) + \frac{\partial}{\partial x}(\rho u_1 I_o) + \frac{\partial}{\partial y}(\rho u_2 I_o) \right] \\
&= \frac{2-\gamma}{2(\gamma-1)} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_1) + \frac{\partial}{\partial y}(\rho u_2) - \frac{\rho}{\beta}\frac{\partial \beta}{\partial t} - \frac{\rho u_1}{\beta}\frac{\partial \beta}{\partial x} - \frac{\rho u_2}{\beta}\frac{\partial \beta}{\partial y} \right] \\
&= -\frac{2-\gamma}{2(\gamma-1)} \left(\rho \frac{\partial}{\partial t} \ln \beta + \rho u_1 \frac{\partial}{\partial x} \ln \beta + \rho u_2 \frac{\partial}{\partial y} \ln \beta \right) \\
&= -\frac{2-\gamma}{2(\gamma-1)} \left[\frac{\partial}{\partial t}(\rho \ln \beta) + \frac{\partial}{\partial x}(\rho u_1 \ln \beta) + \frac{\partial}{\partial y}(\rho u_2 \ln \beta) \right] \\
&= -\frac{2-\gamma}{2(\gamma-1)} \int \left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right) (F \ln \beta)\, dv_1\, dv_2\, dI \qquad (107)
\end{aligned}
$$

In deriving equation (107) repeated use of the mass balance equation is made. Combining equations (106) and (107) gives

$$
\begin{aligned}
\frac{\partial H}{\partial t} + \frac{\partial H_{vx}}{\partial x} + \frac{\partial H_{vy}}{\partial y} &= \int (1 - C + \ln F) \left(\frac{\partial F}{\partial t} + v_1 \frac{\partial F}{\partial x} + v_2 \frac{\partial F}{\partial y} \right) dv_1\, dv_2\, dI \\
&\quad - \frac{2(3-2\gamma)}{2-\gamma} \int \beta I \left(\frac{\partial F}{\partial t} + v_1 \frac{\partial F}{\partial x} + v_2 \frac{\partial F}{\partial y} \right) dv_1\, dv_2\, dI \\
&= \int \left(1 - C + \ln F - \frac{6-4\gamma}{2-\gamma} \beta I \right) \left(\frac{\partial F}{\partial t} + v_1 \frac{\partial F}{\partial x} + v_2 \frac{\partial F}{\partial y} \right) dv_1\, dv_2\, dI \qquad (108)
\end{aligned}
$$

Now, equation (102) gives

$$
\begin{aligned}
1 + \ln F - C &- \frac{6-4\gamma}{2-\gamma} \beta I \\
&= 1 - C + \ln \rho + 2 \ln \beta + \ln \left[\frac{2(\gamma-1)}{(2-\gamma)\pi} \right] - \beta (v_1 - u_1)^2 - \beta (v_2 - u_2)^2 \\
&\quad - \left[\frac{2(\gamma-1)}{2-\gamma} + \frac{6-4\gamma}{2-\gamma} \right] \beta I \\
&= 1 - C + \ln \rho + 2 \ln \beta + \ln \left[\frac{2(\gamma-1)}{(2-\gamma)\pi} \right] - \beta (v_1 - u_1)^2 - \beta (v_2 - u_2)^2 - 2\beta I \\
&= \left\{ 1 + \ln \rho + 2 \ln \beta - \frac{2-\gamma}{\gamma-1} - \beta u^2 \right\} - 2\beta v_1 u_1 - 2\beta u_2 v_2 - 2\beta \left(I + \frac{v^2}{2} \right)
\end{aligned}
$$

which is equal to the linear combination of collisional invariants 1, v_1, v_2, and $I + (v^2/2)$. Hence, the right-hand side of equation (108) vanishes, thus proving the entropy conservation of equation (105). Once again the validity of the entropy conservation has been shown to rest upon the appearance of $\ln F$ in equation (104). The integrand of equation (104) can be expressed as a linear combination of 1, v_1, v_2, and $I + (v^2/2)$.

From the above analysis the results for the three-dimensional Euler equations

$$
\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{g}_x}{\partial x} + \frac{\partial \mathbf{g}_y}{\partial y} + \frac{\partial \mathbf{g}_z}{\partial z} = 0 \qquad (109)
$$

where

$$\mathbf{w} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho u_4 \\ \rho e \end{bmatrix} \qquad \mathbf{g}_x = \begin{bmatrix} \rho u_1 \\ p + \rho u_1^2 \\ \rho u_1 u_2 \\ \rho u_1 u_3 \\ (\rho e + p) u_1 \end{bmatrix} \qquad \mathbf{g}_y = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 \\ p + \rho u_2^2 \\ \rho u_2 u_3 \\ (\rho e + p) u_2 \end{bmatrix}$$

$$\mathbf{g}_z = \begin{bmatrix} \rho u_3 \\ \rho u_1 u_3 \\ \rho u_3 u_3 \\ p + \rho u_3^2 \\ (\rho e + p) u_3 \end{bmatrix} \qquad \rho e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2$$

can be immediately written as

$$\begin{bmatrix} M_o \\ M_x \\ M_y \\ M_z \end{bmatrix} = \int \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} F \, dv_1 \, dv_2 \, dv_3 \, dI \tag{110}$$

$$F = \frac{\rho}{I_o} \left(\frac{\beta}{\pi}\right)^{3/2} \exp\left[-\beta(v-u)^2 - \frac{I}{I_o}\right] \tag{111}$$

$$I_o = \frac{5 - 3\gamma}{4(\gamma - 1)\beta} \tag{112}$$

$$q_1 = \ln \rho + \frac{\ln \beta}{\gamma - 1} - \beta u^2 \qquad q_2 = 2u_1\beta \qquad q_3 = 2u_2\beta \qquad q_4 = 2u_3\beta \qquad q_5 = -2\beta \tag{113}$$

Then,

$$w_i = \frac{\partial M_o}{\partial q_i} \qquad (g_x)_i = \frac{\partial M_x}{\partial q_i} \qquad (g_y)_i = \frac{\partial M_y}{\partial q_i} \qquad (g_z)_i = \frac{\partial M_z}{\partial q_i} \qquad (i = 1, 2, \ldots 5) \tag{114}$$

The H-function and H-fluxes are defined by

$$\begin{bmatrix} H \\ H_{vx} \\ H_{vy} \\ H_{vz} \end{bmatrix} = \int \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \left(\sum_i q_i \frac{\partial F}{\partial q_i} - F \right) dv_1 \, dv_2 \, dv_3 \, dI \tag{115}$$

and they satisfy the entropy conservation

$$\frac{\partial H}{\partial t} + \frac{\partial H_{vx}}{\partial x} + \frac{\partial H_{vy}}{\partial y} + \frac{\partial H_{vz}}{\partial z} = 0 \tag{116}$$

The convexity of H is a consequence of equation (115) and the convexity of M_o.

Review and Discussion

The Euler equations of gas dynamics are the moments of the Boltzmann equation when the distribution function is a Maxwellian. Further, the collision term in the Boltzmann equation vanishes for the Maxwellian distribution implying that the Euler equations are the moments of the collisionless Boltzmann equation

$$\frac{\partial F}{\partial t} + v_1 \frac{\partial F}{\partial x} + v_2 \frac{\partial F}{\partial y} + v_3 \frac{\partial F}{\partial z} = 0$$

This is a first-order hyperbolic partial differential equation and can be cast in the strong conservation law form by simply taking v_1, v_2, and v_3 inside the differentiation symbols. The entire information about the Euler equations is compressed in the single equation for the scalar F. For example, any upwind method for the collisionless Boltzmann equation becomes an upwind method for the Euler equations by taking Ψ moments of the Boltzmann equation. This moment-method strategy has been fully used in reference 2 to construct a new class of upwind methods to obtain the numerical solution of the Euler equations.

The present paper uses the above connection between the Euler and the Boltzmann equations even more and shows that the homogeneity of the flux vector, symmetrizability, and the existence and construction of the entropy function are all consequences of the Maxwellian distribution. It may be noted that the Euler equations can be cast in two forms: the strong conservation law form and the symmetric hyperbolic form. The strong conservation law form is obtained when Ψ moments of the collisionless Boltzmann equation are taken, and it reflects the physical principle of conservation. The symmetric hyperbolic form, which reflects the hyperbolic nature of the Euler equations, is obtained by transforming the field vector $\mathbf{w}$ to the vector $\mathbf{q}$. At the root of the $\mathbf{w}$-$\mathbf{q}$ transformation is the equation

$$\frac{dF}{F} = dq_1 + v_1\,dq_2 + v_2\,dq_3 + v_3\,dq_4 + \left(I + \frac{v^2}{2}\right) dq_5 + \frac{7-5\gamma}{5-3\gamma}(I_o - I)\,dq_5$$

When $\gamma = 5/3$, the gas has only translational degrees of freedom and then I and I_o drop out in equation (111) for the Maxwellian distribution F. In such a case the above equation assumes the simple form

$$\frac{dF}{F} = dq_1 + v_1\,dq_2 + v_2\,dq_3 + v_3\,dq_4 + \frac{v^2}{2}\,dq_5$$

The positive definiteness of matrix $\mathbf{P}$ (or, equivalently, the convexity of scalar function M_o) can be proved from the above equation. It is very interesting to observe that the same equation is used to establish the convexity of the H-function. It is an interesting property of the Maxwellian distribution that the integrand in the definition of the H-function containing $\ln F$ can be obtained as the Legendre transform of F. One of the important results of the present paper is the demonstration of the convexity of the H-function based on the Legendre transform and the positive definiteness of the matrix $\mathbf{P}$. Therefore, the Maxwellian distribution F and the transformed variables q_i play a fundamental role in the theory of Euler equations. Just as the field vector $\mathbf{w}$ naturally arises when the Euler equations are cast in the strong conservation law form, the variables $\mathbf{q}$ naturally arise when these equations are transformed to the symmetric hyperbolic form.

The H-function, which is a slight modification of the Boltzmann H-function, is the negative of specific entropy of thermodynamics. Because of the convexity of the H-function, it is possible to design a method satisfying the entropy condition, that is, a method for which total H in the computational domain decreases with time. A decrease in total H therefore corresponds to the existence of entropy-producing mechanisms in the numerical method. It should therefore be possible to obtain nonoscillatory solutions to the Euler equations by controlling the decrease in H. This fact, together with the observation that a decrease in H is physically meaningful, in that it avoids expansion shocks and is easily extended to multidimensional flows, gives credence to the view that it would be more profitable to construct total H-diminishing (THD) methods instead of the well-known total variation diminishing (TVD) methods.

Finally, the ability of the moment-method strategy to tackle a system of equations by taking moments of a single scalar equation can be used in various ways. First, for a system of conservation equations it is enough to develop a TVD-lgke criterion for a single scalar equation having a physical meaningful connection with that system. Second, in this framework upwind methods for the three-dimensional flows can be constructed without the usual splitting of a three-dimensional problem into three one-dimensional problems.

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16. Abstract
The Euler equations of gas dynamics have some very interesting properties in that the flux vector is a homogeneous function of the unknowns and the equations can be cast in symmetric hyperbolic form and satisfy the entropy conservation. The Euler equations are the moments of the Boltzmann equation of the kinetic theory of gases when the velocity distribution function is a Maxwellian. The present paper shows the relationship between the symmetrizability and the Maxwellian velocity distribution. The entropy conservation is in terms of the H-function, which is a slight modification of the H-function first introduced by Boltzmann in his famous H-theorem. In view of the H-theorem, it is suggested that the development of total H-diminishing (THD) numerical methods may be more profitable than the usual total variation diminishing (TVD) methods for obtaining "wiggle-free" solutions.

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