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Group-Kinetic Theory of Turbulence

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GROUP-KINETIC THEORY OF TWO-PHASE TURBULENCE

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Abstract

The two phases are governed by two coupled systems of Navier-Stokes equations. The couplings are nonlinear. These equations describe the microdynamical state of turbulence, and are transformed into a master equation. By scaling, a kinetic hierarchy is generated in the form of groups, representing the spectral evolution, the diffusivity, and the relaxation. The loss of memory in formulating the relaxation yields the closure. The network of sub-distributions that participates in the relaxation is simulated by a self-consistent porous medium, so that the average effect on the diffusivity is to make it approach equilibrium. The kinetic equation of turbulence is derived. The method of moments reverts it to the continuum. The equation of spectral evolution is obtained, and the transport properties (eddy diffusivity and eddy viscosity) are calculated. In inertia turbulence, the Kolmogoroff law $k^{-5/3}$ for the weak coupling and the spectrum k^{-1} for the strong coupling are found. The density fluctuations have similar spectra. The numerical coefficients are determined analytically.

1. Introduction

The motion of small particles in an incompressible turbulent fluid is important for many applications (Soo, 1967), and from the fundamental point of view has attracted the attention of many authors as a statistical problem of many bodies. The earliest treatment of suspension (Tchen, 1947) used the Langevin equation (Langevin, 1908)

$$\frac{d\tilde{\underset{\sim}{v}}(t)}{dt} + \alpha_v \tilde{\underset{\sim}{v}}(t) = \tilde{\underset{\sim}{X}}_v(t) \tag{1}$$

for the motion of a single particle, where $\tilde{\underset{\sim}{v}}(t)$ is the velocity fluctuation, α_v is the coupling coefficient, and $\tilde{\underset{\sim}{X}}_v(t)$ is the driving force. A second Langevin equation may also be written for the Lagrangian representation of the modified fluid.

The driving force is due to the drag that is experienced by the particle from the moving fluid. It is proportional to the fluid velocity $\tilde{\underset{\sim}{u}}(t)$ as $\alpha_v\tilde{\underset{\sim}{u}}(t)$, and may include a Basset memory. The interaction of many particles calls for a system of many coupled Langevin equations (Herczyński and Pieńkowska, 1980; Mazur, 1982; Deutch and Oppenheim, 1971; Ermak and McCammon, 1978; Ramshaw, 1979).

From the Langevin equation with a drag and a memory, Tchen (1947) calculated the Lagrangian correlations of velocities and found the equality

$$\int_0^{\infty} d\tau \left\langle \tilde{\underset{\sim}{v}}(t)\tilde{\underset{\sim}{v}}(t-\tau)\right\rangle = \int_0^{\infty} d\tau \left\langle \tilde{\underset{\sim}{u}}(t)\tilde{\underset{\sim}{u}}(t-\tau)\right\rangle . \tag{2}$$

A fluctuating function

$$\hat{()} = \bar{()} + \tilde{()} \tag{3}$$

can be decomposed into an ensemble average

$$\bar{()} \equiv \langle \ \rangle \tag{4}$$

and a fluctuation $(\tilde{\ })$. The operators

$$\hat{A} = \bar{A} + \tilde{A} \tag{5}$$

may be used for the scaling, where $\hat{A} = 1$ is the unit operator.

By the use of the evolution operators

$$\hat{U}_v(t,t-\tau) \quad \text{and} \quad \hat{U}_u(t,t-\tau) \tag{6}$$

to specify the trajectories of the $\tilde{\underset{\sim}{v}}$-fluctuation and the $\tilde{\underset{\sim}{u}}$-fluctuation, respectively, we can write the Langevin equation (1) and the equality (2) as

$$\frac{d\tilde{\underline{v}}(t)}{dt} + \alpha_v \tilde{\underline{v}}(t) = \tilde{A}\hat{U}_v(t,t-\tau)\tilde{\underline{X}}_v(t-\tau) \quad , \tag{7}$$

and

$$\int_0^\infty d\tau \left\langle \tilde{\underline{v}}(t,\underline{x})\tilde{A}\hat{U}_v(t,t-\tau)\tilde{\underline{v}}(t-\tau)\right\rangle = \int_0^\infty d\tau \left\langle \tilde{\underline{u}}(t,\underline{x})\tilde{A}\hat{U}_v(t,t-\tau)\tilde{\underline{u}}(t-\tau)\right\rangle , \tag{8}$$

The fluctuating operation $\hat{A}$ makes the quantity which follows to be a fluctuation. The force $\dot{\tilde{\underline{X}}}_v(t-\tau)$ in (7) is registered at the time $t-\tau$ and at the position $\tilde{\underline{X}}(t-\tau)$ by an observer riding on the particle that follows the perturbed trajectory of the $\tilde{\underline{v}}$-motion, while at time t, the force is

$$\tilde{\underline{X}}_v(t) = \tilde{\underline{X}}_v(t,\underline{x}) \quad . \tag{9}$$

The integrals

$$\int_0^\infty d\tau \left\langle \tilde{\underline{v}}(t,\underline{x})\tilde{A}\hat{U}_v(t,t-\tau)\tilde{\underline{v}}(t-\tau)\right\rangle \tag{10}$$

and

$$\int_0^\infty d\tau \left\langle \tilde{\underline{u}}(t,\underline{x})\tilde{A}U_u(t,t-\tau)\tilde{\underline{u}}(t-\tau)\right\rangle \tag{11}$$

can be called diffusivities. The correlation functions are calculated from the velocity fluctuations at the two instants along their own trajectories, as specified by the evolution operators (6).

With this distinction between

$$\hat{U}_v(t,t-\tau)\hat{\underset{\sim}{u}}(t-\tau) \quad \text{and} \quad U_u(t,t-\tau)\hat{\underset{\sim}{u}}(t-\tau), \qquad (12)$$

the quantity

$$\int_0^\infty d\tau \left\langle \tilde{\underset{\sim}{u}}(t,\underset{\sim}{x})\tilde{\underset{\sim}{A}}\hat{U}_v(t,t-\tau)\tilde{\underset{\sim}{u}}(t-\tau)\right\rangle, \qquad (13)$$

as formed by the $\tilde{\underset{\sim}{u}}$-fluctuations along the $\tilde{\underset{\sim}{v}}$-trajectory, is not a diffusivity as a property, but is a functional of the diffusivity (11). In this sense, the Langevin equation (7) and the kinematic relation (8) are in fact nonlinear. The nonlinearity arises from the transformation between the evolution operators (6).

The diffusivity in the operator form calls for the need for a Lagrangian-Eulerian transformation (Pismen and Nir, 1978; Reeks and McKee, 1984; Gitterman and Steinberg, 1980; Gouesbet et al., 1984). It soon becomes clear that without the dynamical knowledge of turbulence, the transformations between the Lagrangian and Eulerian representations and between the two evolution operators can not be successful.

The treatment of many interacting particles and the transport theory of the stochastic system belong in a natural way to the many-body statistical mechanics (Bogoliubov, 1962). Yanko (1980) has suggested Bogoliubov's method for treating the suspension of particles. The method is best suited for a system of large numbers of particles that interact through their own potential of interaction, e.g. the Coulomb potential in plasmas, but is not well-developed for the interaction among the particles in a flowing fluid. It would be more pertinent to describe the interaction by a system of many Langevin equations for the many particles embedded in the incompressible flow.

If we are not concerned with the microdynamics of the interactions and the microhydrodynamics of the intrinsic viscosity in our theory of turbulence, we can choose an effective friction constant, such as the Darcy damping for the ensemble of Langevin equations treated as a porous medium. Through the transformation into the Fokker-Planck equation, and the subsequent reduction into the continuum, we will obtain two systems of Navier-Stokes equations for the two phases, with the effective coupling constants as given parameters. These systems describe the microdynamical state of two-phase turbulence.

For the development of a theory of turbulence, the direct hydrodynamic method has shown the difficulties of the closure of hierarchy from the start of the problem (Genchev and Karpuzov, 1980; Elgobashi and Abou-Arab, 1981). The statistical method by Bogoliubov's approach with the closure at the triplet distribution seems unsatisfactory for turbulence. For this reason, we develop a group-kinetic method to derive the kinetic equation for the distribution function in the macro-group. The closure is found by memory loss. We investigate the eddy viscosity and analyze the spectral structure of turbulence.

It is generally believed that a small amount of particles present in the fluid will drastically reduce the drag of the fluid. The laboratory measurement of the turbulent intensity indeed shows a decrease of fluid turbulence with the increasing concentration of particles (Elgobashi and Abou-Arab, 1983). Other measurements of the energy spectrum in the inertia turbulence confirm the Kolmogoroff law $k^{-5/3}$, but show unexpectedly an increased spectral level (Goldschmidt, 1972). To lift this paradox, we investigate the inertia turbulence with strong coupling and weak coupling. In the direction of increasing wave numbers, our results show that the spectral law k^{-1} for strong coupling precedes the spectral tail of power law $k^{-5/3}$ for weak coupling. The power law k^{-1} is reduced by the particles and the power law $k^{-5/3}$ is increased by the particles. The tendency of this shift from reduction to increase is indicated by the lidar measurement in the atmosphere before and after the rain (Gurvich and Pokasov, 1972).

2. Fundamental equations of two-phase flow

We consider two fluids, called the two-phases v and u. Both fluids are governed by the Navier-Stokes equations with coupling, in the form

$$\hat{\rho}_v(\partial_t+\hat{\underset{\sim}{v}}\cdot\underset{\sim}{\nabla})\hat{\underset{\sim}{v}} = -\underset{\sim}{\nabla}\hat{p}_v + \gamma_v\hat{n}_v(\hat{\underset{\sim}{u}} - \hat{\underset{\sim}{v}}) \tag{14}$$

$$\partial_t \hat{\rho}_v + \underset{\sim}{\nabla}\cdot \hat{\rho}_v\hat{\underset{\sim}{v}} = 0 , \tag{15}$$

for the velocity $\hat{\underset{\sim}{v}}$, the pressure $\hat{p}_v$, and the density $\hat{\rho}_v$, and

$$\hat{\rho}_u(\partial_t+ \hat{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla})\hat{\underset{\sim}{u}} = -\underset{\sim}{\nabla}\hat{p}_u - \gamma_u\hat{n}_u(\hat{\underset{\sim}{u}}-\hat{\underset{\sim}{v}}) \tag{16}$$

$$\partial_t \hat{\rho}_u + \underset{\sim}{\nabla}\cdot \hat{\rho}_u \hat{\underset{\sim}{u}} = 0 , \tag{17}$$

for the velocity $\hat{\underset{\sim}{u}}$, the pressure $\hat{p}_u$, and the density $\hat{\rho}_u$. The coupling constants are γ_v,γ_u.

By definition, the mass densities

$$\hat{\rho}_v = m_v\hat{n}_v , \quad \hat{\rho}_u = m_u\hat{n}_u \tag{18}$$

can be written as the product of masses m_v, m_u, with the number densities $\hat{n}_v, \hat{n}_u$. The kinematic viscosities ν_v, ν_u are not written here since they are negligible as compared with the eddy transport properties, but will be included in the energy considerations.

The two fluids are incompressible, but the coupling depends on the density. In investigations of this kind, we allow for the variation of density insofar as it modifies the coupling only. This procedure is called the Boussinesq approximation (Chandrasekhar, 1983). Thus, in eqs.(14)-(17) we replace $\hat{\rho}_v, \hat{\rho}_u$ by their stationary values ρ_v, ρ_u and reduce the hydrodynamic systems into the following:

(i) for the v-phase

$$(\partial_t + \hat{\underset{\sim}{v}} \cdot \underset{\sim}{\nabla} + \hat{\alpha}_v)\hat{\underset{\sim}{v}} = \hat{\underset{\sim}{X}}_v \tag{19}$$

$$\underset{\sim}{\nabla} \cdot \hat{\underset{\sim}{v}} = 0 \tag{20}$$

$$(\partial_t + \hat{\underset{\sim}{v}} \cdot \underset{\sim}{\nabla})\hat{N}_v = \nu_{N,v} \nabla^2 \hat{N}_v \;, \tag{21}$$

with

$$\hat{\underset{\sim}{X}}_v \equiv \hat{\underset{\sim}{E}}_v + \hat{\alpha}_v \hat{\underset{\sim}{u}} \;, \quad \hat{\underset{\sim}{E}}_v \equiv -\frac{1}{\rho_v} \underset{\sim}{\nabla} \hat{p}_v \;, \tag{22}$$

(ii) for the u-phase

$$(\partial_t + \hat{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla} + \hat{\alpha}_v)\hat{\underset{\sim}{u}} = \hat{\underset{\sim}{X}}_u \tag{23}$$

$$\underset{\sim}{\nabla}\cdot\hat{\underset{\sim}{u}} = 0 \tag{24}$$

$$(\partial_t + \hat{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla})\hat{N}_u = \nu_{N,u}\nabla^2\hat{N}_u\ , \tag{25}$$

with

$$\hat{\underset{\sim}{X}}_u \equiv \hat{\underset{\sim}{E}}_u + \alpha_u\hat{\underset{\sim}{v}}\ , \quad \hat{\underset{\sim}{E}}_u \equiv -\frac{1}{\rho_u}\underset{\sim}{\nabla}\hat{p}_v\ . \tag{26}$$

The coupling coefficients

$$\hat{\alpha}_v \equiv \alpha_v\hat{N}_v\ , \quad \hat{\alpha}_u \equiv \alpha_u\hat{N}_u \tag{27}$$

fluctuate by the fluctuating number-densities, and the stationary coupling coefficients are

$$\alpha_v = \gamma_v/m_v\ , \quad \alpha_u = \gamma_u/m_u\ . \tag{28}$$

The normalized densities

$$\hat{N}_v = \hat{n}_v/n_v\ , \qquad \hat{N}_u = \hat{n}_u/n_u \tag{29}$$

are the densities $\hat{n}_v, \hat{n}_u$ normalized by the stationary values n_v, n_u. The stationary quantities are written without the symbols (3).

It is to be remarked that the Navier-Stokes equations, in the form (14)-(17) or (19)-(26), are valid for two gases of number-densities (29) or for two liquids of densities (18). In the special case of the suspension of particles in a liquid, where the coupling is due to the friction of the particles only, we have the simplification

$$\hat{N}_u = \hat{N}_v \quad , \tag{30}$$

without altering the form of the basic equations (19)-(26).

In view of the symmetry, it suffices to treat the first system and omit the subscript v, unless needed for clarity. The results will equally apply to the second system by interchanging $\underset{\sim}{v}$ and $\underset{\sim}{u}$.

3. Group-kinetic method

3.1. Master equation

The inhomogeneous partial differential equation (14) can be transformed into the homogeneous partial differential equation

$$(\partial_t + \hat{L})\hat{f} = 0 \quad , \tag{31}$$

called the master equation for the distribution function $\hat{f}(t,\underset{\sim}{x},\underset{\sim}{v})$, with the differential operator

$$\hat{L} = \underset{\sim}{v}\cdot\underset{\sim}{\nabla} + \hat{\alpha} + \hat{\underset{\sim}{X}}\cdot\underset{\sim}{\partial} \tag{32}$$

and the notations

$$\partial_t \equiv \partial/\partial t \quad , \qquad \underset{\sim}{\partial} \equiv \partial/\partial \underset{\sim}{v} \quad . \tag{33}$$

For establishing the self-consistency between the master equation and the hydrodynamics of the two phases under the Boussinesq approximation, we write

$$\hat{f}(t,\underset{\sim}{x},\underset{\sim}{v}) = \delta\left[\underset{\sim}{v} - \hat{\underset{\sim}{v}}(t,\underset{\sim}{x})\right] \tag{34}$$

and

$$\hat{f}(t,\underset{\sim}{x},\underset{\sim}{v}) = \hat{N}(t,\underset{\sim}{x})\delta\left[\underset{\sim}{v}-\hat{\underset{\sim}{v}}(t,\underset{\sim}{x})\right] \quad , \tag{35}$$

for (19) and (21), respectively.

3.2 Group-scaling

The master equation describes the microdynamical state of turbulence in too many minute details. A smoothing procedure is necessary for a statistical treatment. By the Reynolds decomposition (3), we divide

$$\hat{f} = \bar{f} + \tilde{f} \tag{36}$$

into an ensemble average $\bar{f}$ and a fluctuation $\tilde{f}$. The group-scaling by

$$\tilde{A} = A^{0} + A' \tag{37}$$

$$A' = A^{1} + A'' \tag{38}$$

decomposes the fluctuation

$$\tilde{f} = f^{0} + f' \tag{39}$$

into a macro-group f^0 and a micro-group

$$f' = f^{1} + f'' \quad . \tag{40}$$

The micro-group is re-decomposed into the first-order group f^1 and the group

$$f'' = f^{(2)} + f^{(3)} + \cdots \tag{41}$$

of sub-distributions.

The three groups

$$f^{0}\ ,\quad f'\ ,\quad f'' \tag{42}$$

represent the three transport processes of evolution, transport property, and relaxation. The evolution of f^0 occurs in a medium that offers a transport property as shaped by the micro-fluctuation f'. Subsequently, the transport property approaches its equilibrium ($t \to \infty$) by a relaxation from f''. The relaxation is a complicated process obtained by following the perturbed trajectory and by closing the memory chain. The distinction of the transport processes provides a physical demarcation in the groups. Mathematically, the groups are sets of Fourier components, so that the intensities of the field fluctuations

$$\langle \underset{\sim}{X}'^2 \rangle = 2\int_0^k dk' G(k') \,, \langle \underset{\sim}{X}'^2 \rangle = 2\int_k^\infty dk'' G(k'') \,, \langle \underset{\sim}{X}''^2 \rangle = 2\int_{k''}^\infty dk''' G(k''') \tag{43}$$

are portions of the spectral distribution $G(k)$ within the wave number intervals

$$(0,k)\,,\quad (k,\infty)\,,\quad \text{and}\quad (k'',\infty)\,, \tag{44}$$

respectively.

The three groups of increasing orders by the scalings

$$A^{\circ}\ ,\ A'\ \ ,\ A'' \tag{45}$$

do not refer to decreasing magnitudes, but to decreasing coherence. Thus, the decreasing times of correlation, by the inequalities

$$\tau_v^{\circ} > \tau_v' > \tau_v'' \quad , \tag{46}$$

will introduce the quasi-homogeneity and quasi-stationarity at successive group levels to limit the interactions to nearest-neighbours.

3.3 Hierarchy and closure

By A^0 and A^1, we scale the master equation (31) into the following equations for the evolution of groups:

$$(\partial_t + A^{\circ}\hat{L}^{\circ}) f^{\circ} = - L^{\circ}\bar{f} - A^{\circ}L^1 f^1 \tag{47}$$

$$(\partial_t + A^1\hat{L}^1) f^1 = - L^1(\bar{f} + f^{\circ}) - A^1 L^{(2)} f^{(2)}, \ \ldots \tag{48}$$

and the sequence continues in a hierarchy. The sub-distributions form a global network (41) like a lattice.

On the left hand sides of (47) and (48), the advections can be written as

$$A^{0} \hat{L} f^{0} = L_{0} f^{0} - A^{0} L^{1} f^{0} \tag{49}$$

$$A^{1} L^{1} f^{1} = L_{1} f^{1} - A^{1} L^{1} f^{1} \quad . \tag{50}$$

The notations

$$L_{0} \equiv \bar{L} + L^{0}, \quad L_{1} \equiv \bar{L} + L^{0} + L^{1} \tag{51}$$

are used. By the approximation of nearest-neighbour interactions, the last terms

$$- A^{0} L^{1} f^{1} \cong - A^{0} L' f' \equiv \mathcal{C}' \left\{ f^{0} \right\} \tag{52}$$

$$-A^1L^{(2)}f^{(2)} \cong -A^1L''f'' \equiv \mathcal{C}''\left\{f^1\right\}, \quad \cdots \tag{53}$$

in (47) and (48) are called collision and sub-collision, respectively. Since

$$\mathcal{C}'\{\;\} \quad , \quad \mathcal{C}''\{\;\} \tag{54}$$

are operators in $\underset{\sim}{v}$, the memory continues.

Physically, the evolution of f^0 proceeds in a collisional environment where the fluctuations by f' can shape an eddy viscosity. For its approach to equilibrium, this transport property will need a relaxation, where we must find a way to cut the memory and thereby to close the hierarchy.

By an inspection of the chain of collisions in (52) and (53) and of the linkage among the sub-distributions in the network (41), we visualize that the memory is at the same time essential for maintaining a non-Markovian behaviour in the kinetic equation of turbulence, and should ultimately vanish for obtaining the closure and the irreversibility. It can be seen that the memory continues through the v-dependence among the sub-distributions linked in the network (41). In fact, it is the averaged statistical effects that play the role of relaxation and controls the evolution of f^1. Consequently, we can replace the network by a porous medium that offers a friction $\mathcal{C}''$, called the Darcy damping (Brinkman, 1947). This amounts to replacing $\mathcal{C}''\{\,\}$ in the role of operator by $\mathcal{C}''$ as a function independent of $\underset{\sim}{v}$, i.e.

$$\mathcal{C}''\{\,\} \cong \mathcal{C}'' . \tag{55}$$

This function will be determined self-consistently. In so doing, the sub-distributions in the network lose their individuality as v-dependent and thereby are not capable of linking the memory in the homogeneized lattice that turns into a porous medium.

4. Kinetic equation of turbulence

By substituting for the sub-collision (55), we transform (48) into the form

$$(\partial_t + A^1 \hat{L}) f^1 = - L^1(\bar{f}+f^o) + \mathcal{C}'' f^1 \tag{56}$$

or

$$(\partial_t + A^1 \hat{\mathcal{L}}) f^1 = - L^1(\bar{f}+f^o) , \tag{57}$$

with the differential operators

$$\hat{L} = \underset{\sim}{v}\cdot\underset{\sim}{\nabla} + \hat{\alpha} + \hat{\underset{\sim}{X}}\cdot\underset{\sim}{\partial} \quad \text{and} \quad \hat{\mathcal{L}} = \hat{L} - \mathcal{C}'' . \tag{58}$$

The two operators differ by the Darcy damping ζ'' for representing the porous medium at large.

It is to be remarked that the inhomogeneous equation (57) can be integrated by means of a kernal $\hat{\mathcal{U}}(t, t-\tau)$, called the evolution operator, that satisfies the homogeneous equation

$$\left[\frac{\partial}{\partial t_1} + \hat{\mathcal{L}}(t_1)\right]\hat{\mathcal{U}}(t, t_1) = 0 , \tag{59}$$

with the condition $\hat{\mathcal{U}}(t,t) = 1$. The solution is

$$\hat{\mathcal{U}}(t, t_1) = \exp\left[-\int_{t_1}^{t} dt'_1 \ \hat{\mathcal{L}}(t_1)\right] . \tag{60}$$

By means of this operator, we integrate (57) to get the distribution

$$f^1 = -\int_0^t d\tau \ A^1 \hat{\mathcal{U}}(t, t-\tau)\left\{L^1(\bar{f}+f^{\circ})\right\} , \tag{61}$$

and calculate the collision (52) as

$$\mathcal{C}'\left\{f^{\circ}\right\} \cong -\overset{\circ}{A} L^1 f^1$$

$$= \overset{\circ}{A}\int_0^t d\tau \ L^1(t, \underset{\sim}{x}, \underset{\sim}{v}) A^1 \hat{\mathcal{U}}(t, t-\tau)\left\{L^1(\bar{f}+f^{\circ})\right\} . \tag{62}$$

A final substitution into (47) yields the kinetic equation:

$$(\partial_t + L_0) f^0 = - L^0 \bar{f} + \mathcal{C}' \left\{ f^0(t-\tau) \right\} . \qquad (63)$$

This equation differs from the Bogoliubov equation and its generalization to include the memory (Tchen, 1959), and will serve as a basis for deriving the transport properties and the spectral structure without the need of the accompanying pair-distribution function.

It is worth noting the following properties of the collision $\mathcal{C}'\{f^0\}$:

(i) As a collision operator $\mathcal{C}'\{\ \}$ is deterministic and stationary, so that the upper limit of integration in (62) can be replaced by ∞ without altering the value of the integral, by

$$\tau_v^0 > \tau_v' , \qquad (64)$$

from (46).

(ii) The nearest-neighbour interaction approximation between groups is valid by (46), so that the correlation of L^1 in (62) can be replaced by a correlation of L'.

(iii) The approach to equilibrium depends on the loss of memory by relaxation, so that we can write

$$\hat{\mathcal{U}}(t, t-\tau) \cong \bar{\mathcal{U}}(t, t-\tau) . \qquad (65)$$

By these approximations, we transform the collision (62) into

$$\mathcal{C}'\{f^{o}\} = \int_0^{t\to\infty} d\tau \left\langle L'(t,\underset{\sim}{x},\underset{\sim}{v})\bar{\mathcal{U}}(t,t-\tau)L'(t-\tau)\right\rangle\{f^{o}\} , \quad (66)$$

where the scaling gives

$$\overset{o}{A}(\bar{f} + f^{o}) = f^{o} . \quad (67)$$

By the commutability

$$\underset{\sim}{X}'(t,\underset{\sim}{x})\cdot\underset{\sim}{\partial} = \underset{\sim}{\partial}\cdot\underset{\sim}{X}'(t,\underset{\sim}{x}) , \quad (68)$$

we can write the correlation

$$\left\langle L'(t,\underset{\sim}{x},\underset{\sim}{v})\bar{\mathcal{U}}(t,t-\tau)L'(t-\tau)\right\rangle = \underset{\sim}{\partial}\cdot\left\langle \underset{\sim}{X}'(t,\underset{\sim}{x})\bar{\mathcal{U}}(t,t-\tau)\underset{\sim}{X}'(t-\tau)\right\rangle\cdot\underset{\sim}{\partial}$$

$$+ \alpha^2\left\langle N'(t,\underset{\sim}{x})\bar{\mathcal{U}}(t,t-\tau)N'(t-\tau)\right\rangle \quad (69)$$

and the collision operators

$$\mathcal{C}'\{\} = \mathcal{C}'_X\{\} + \mathcal{C}'_N \quad (70)$$

in two components

$$\mathcal{C}'_X\{\} = \underset{\sim}{\partial}\cdot\underset{\approx}{\mathcal{D}}'_X\cdot\underset{\sim}{\partial}\{\} . \quad (71a)$$

and

$$\mathcal{C}'_N = \alpha^2 \mathcal{D}'_N \quad , \tag{71b}$$

with the diffusivities

$$\underline{\underline{\mathcal{D}}}'_X = \int_0^\infty d\tau \left\langle \underline{X}'(t,\underline{x}) \overline{\mathcal{U}}(t,t-\tau) \underline{X}'(t-\tau) \right\rangle \tag{72}$$

$$\mathcal{D}'_N = \int_0^\infty d\tau \left\langle N'(t,\underline{x}) \overline{\mathcal{U}}(t,t-\tau) N'(t-\tau) \right\rangle . \tag{73}$$

It is to be remarked that the auto-correlation of

$$L' = \alpha N' + \underline{X}' \cdot \underline{\partial} \tag{74}$$

will entail the auto-correlation of X' and of N', while the cross-correlations are negligible.

In conclusion, the kinetic equation (63) with the collision (66) will serve as a basis of a transport theory.

5. Hydrodynamic equations of turbulence and cascade transfer

5.1 Kinetic form of cascade

The hydrodynamic equations (19) and (20) describe the microdynamic state of turbulence. They were transformed into a master equation as the starting point of our kinetic theory. By group-scaling and the closure by memory loss, we derived the kinetic equation of turbulence (63). By the moment method, we can convert the kinetic representation back into the continuum to find the hydrodynamic equations of turbulence. as follows:

$$(\partial_t + \underset{\sim}{v}^0 \cdot \underset{\sim}{\nabla} - \nu\nabla^2 + A^0\hat{\alpha})\underset{\sim}{v}^0 = \underset{\sim}{X}^0 + \underset{\sim}{J}^0 \tag{75}$$

and

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v}^0 = 0 \quad . \tag{76}$$

It is to be stipulated that the macro-kinetic equation for f^0 is an irreversible equation, and does not allow to be reverted from f^0 into $\hat{f}$ to generate higher-order groups of moment, as did the reversible master equation for $\hat{f}$.

The driving force is

$$\underset{\sim}{X}^0 = \underset{\sim}{E}^0 + A^0\hat{\alpha}\,\underset{\sim}{u}^0 \quad , \quad \text{with} \quad \underset{\sim}{E}^0 \equiv -\frac{1}{\rho}\underset{\sim}{\nabla}\, p^0 \quad , \tag{77}$$

and the hydrodynamic force due to the collision is

$$\underset{\sim}{J}^0 = \int d\underset{\sim}{v}\, \underset{\sim}{v}\, \mathcal{C}' \left\{ f^0 \right\} . \tag{78}$$

We have added the kinematic viscosity ν.

By (70)-(73), we can write

$$\underset{\sim}{J}^{0} = \underset{\sim}{J}_X^{0} + \underset{\sim}{J}_N^{0} , \tag{79}$$

with

$$\begin{aligned} J_X^{0} &= \int d\underset{\sim}{v}\, \underset{\sim}{v}\, \mathcal{C}_X'\left\{ f^{0}(t-\tau)\right\} \\ &= \int d\underset{\sim}{v}\, \underset{\sim}{v}\, \underset{\sim}{\partial}\cdot \mathcal{D}_X' \cdot \underset{\sim}{\partial}\left\{ f^{0}(t-\tau)\right\} \end{aligned} \tag{80}$$

$$\begin{aligned} J_N^{0} &= \int d\underset{\sim}{v}\, \underset{\sim}{v}\, \mathcal{C}_N'\ f^{0} \\ &= \alpha^2 \mathcal{D}_N'\, \underset{\sim}{v}^{0} . \end{aligned} \tag{81}$$

Upon multiplying by $\underset{\sim}{v}^{0}$ and averaging, we get the energy equation

$$\frac{1}{2}\, \partial_t \left\langle \underset{\sim}{v}^{0\,2}\right\rangle = - T^{0} + W^{0} - \mathcal{E}_{\alpha}^{0} - \mathcal{E}_{\nu}^{0} . \tag{82}$$

We have assumed homogeneity and isotropy, so that

$$\begin{aligned} \left\langle \underset{\sim}{v}^{0} \cdot \underset{\sim}{\nabla}\, p^{0}\right\rangle &= \underset{\sim}{\nabla} \cdot \left\langle \underset{\sim}{v}^{0}\, p^{0}\right\rangle - \left\langle p^{0}\, \underset{\sim}{\nabla}\cdot \underset{\sim}{v}^{0}\right\rangle \\ &= 0 , \end{aligned} \tag{83}$$

The production function

$$W^{0} = \left\langle \underset{\sim}{v}^{0} \cdot A^{0} \, \hat{\alpha}_{v} \, \underset{\sim}{u}^{\prime} \right\rangle \tag{84}$$

and the damping function

$$\varepsilon_{\alpha}^{0} = -\left\langle \hat{\alpha} \, \underset{\sim}{v}^{\prime 2} \right\rangle \tag{85}$$

are originated from the couplings. The dissipation function is

$$\varepsilon_{\nu}^{0} = \nu \left\langle (\nabla v^{\prime})^{2} \right\rangle \quad . \tag{86}$$

Finally the transfer function

$$\begin{aligned} T^{0} &= -\left\langle v^{0} \cdot J^{0} \right\rangle \\ &= - \underset{\sim}{v}^{\prime} \cdot \int d\underset{\sim}{v} \, \underset{\sim}{v} \, \zeta\left\{ f^{0} \right\} . \end{aligned} \tag{87}$$

It is the result of interaction between the macro-dustribution and the micro-field fluctuations. It governs the cascade flow across the spectrum.

The inertia turbulence is governed by

$$T^{0} = \varepsilon \, , \tag{88}$$

where the energy dissipation

$$\mathcal{E} = (\mathcal{E}^{o}_{\nu} + \mathcal{E}^{o}_{\alpha})_{k=\infty} \tag{89}$$

is increased by the drag α.

From the collision of two components (71), it entails the transfer function

$$T^{o} = T^{o}_{N} + T^{o}_{X} \tag{90}$$

of two components

$$\begin{aligned} T_{N} &= - \int d\underset{\sim}{v} \langle \underset{\sim}{v}^{o} \cdot \underset{\sim}{v} \, \mathcal{C}'_{N} \, f^{o} \rangle \\ &= - \alpha^{2} \mathcal{D}'_{N} \langle \underset{\sim}{v}^{o\,2} \rangle \end{aligned} \tag{91}$$

and

$$\begin{aligned} T^{o}_{X} &= - \int d\underset{\sim}{v} \langle \underset{\sim}{v}^{o} \cdot \underset{\sim}{v} \, \mathcal{C}'_{X} \{ f^{o} \} \\ &= - \int d\underset{\sim}{v} \langle \underset{\sim}{v}^{o} \cdot \underset{\sim}{v} \, \underset{\sim}{\partial} \cdot \underset{\approx}{\mathcal{D}}'_{X} \cdot \underset{\sim}{\partial} \{ f^{o} \} \rangle . \end{aligned} \tag{92}$$

5.2 Fluid form of cascade

From the scaling of the fluid equation (19) by A^{0} and A', we get the equation of evolution for the macro-velocity

$$(\partial_t + \underset{\sim}{v}^{\circ} \cdot \underset{\sim}{\nabla})\underset{\sim}{v}^{\circ} = \underset{\sim}{X}^{\circ} - A^{\circ}\hat{\alpha}\,\underset{\sim}{v}^{\circ} + \nu\nabla^2\underset{\sim}{v}^{\circ} + \underset{\sim}{J}^{\circ} \quad . \tag{93}$$

The force of dissipation

$$\underset{\sim}{J}^{\circ} = \underset{\sim}{J}_X^{\circ} + \underset{\sim}{J}_N^{\circ} \tag{94}$$

has the two components

$$J_X^{\circ} = -\underset{\sim}{\nabla} \cdot A^{\circ}\,\underset{\sim}{v}'\,\underset{\sim}{v}' \tag{95}$$

$$J_N^{\circ} = -\alpha\, A^{\circ}\, N'\,\underset{\sim}{v}' \quad , \tag{96}$$

in contrast to the kinetic form given by (80) and (81). These are due to the stresses

$$-A^{\circ}\;\underset{\sim}{v}'\;\underset{\sim}{v}' \tag{97}$$

and

$$-A^{\circ}\;N'\;\underset{\sim}{v}' \tag{98}$$

from the micro-fluctuations. They represent the statistical effects of the micro-fluctuations upon the evolution of the macro-momentum.

Upon multiplying (93) by $\underset{\sim}{v}^{0}$ and averaging, we get the energy equation in the same form as (82), with the transfer function also in two components now in the fluid form:

$$T_X^0 = \langle \underset{\sim}{v}^0 \cdot \underset{\sim}{\nabla} \cdot A^0 \underset{\sim}{v}' \underset{\sim}{v}' \rangle \tag{99}$$

$$T_N^0 = \alpha \langle \underset{\sim}{v}^0 \cdot \underset{\sim}{\nabla} \cdot A^0 N' \underset{\sim}{v}' \rangle , \tag{100}$$

in contrast to the kinetic form given by (91) and (92).

5.3 Transport of the gradient type and transport of the non-gradient type

The flux (97)

$$- A^0 v'_j v'_i = K'_{js} \nabla_s u_i^0 \tag{101}$$

for the transport of v'_i by the velocity fluctuation v'_j is proportional to the velocity gradient $\nabla_s u_i^0$. The transport coefficient K'_{js} is called eddy viscosity. This transport relation was used by Heisenberg (1948) to represent the force of dissipation

$$\underset{\sim}{J}_X^0 = K' \nabla^2 \underset{\sim}{u}^0 \tag{102}$$

in the notation (95) with groups. It served to determine the transfer function

$$T_X^0 = K' \langle (\underset{\sim}{\nabla} \underset{\sim}{u}^0)^2 \rangle . \tag{103}$$

The same relation for this transport of the gradient type is called the Onsager relation in the thermodynamics of irreversible processes. These phenomenological arguments can not lead to the determination of the transport coefficient, for which a kinetic approach is necessary. This is true for the eddy viscosity as well as for the molecular viscosity.

By comparing (102) with (80), or (103) with (92), we find the relationship in the form

$$\mathcal{C}'_{X} = K' \nabla^2 \tag{104a}$$

and

$$\mathcal{C}''_{X} = K'' \nabla^2 \quad , \tag{104b}$$

between the collision coefficient and the transport coefficient, when they are independent of v, i.e. after the memory cutoff. Therefore, this relationship identifies the collisions in the calculation of the relaxation, and enables the self-consistent determination of the eddy viscosity.

On the other hand, the transport

$$\overset{o}{A} \;\; \overline{N' \, \underset{\sim}{v}'} = \alpha \, \mathcal{D}'_{N} \, \underset{\sim}{v}^{o} \quad , \tag{105}$$

unlike $-A^0 \underset{\sim}{v}'N'$, is not of the gradient type, as seen by comparing (96) with (81).

6. Perturbations of the trajectory

The Lagrangian correlation

$$\left\langle \underset{\sim}{X}'(t,\underset{\sim}{x})\, \bar{\mathcal{U}}(t,t-\tau)\, \underset{\sim}{X}'(t-\tau) \right\rangle = \left\langle \underset{\sim}{X}'(t,\underset{\sim}{x})\, \underset{\sim}{X}'\left[t-\tau,\ \hat{\underset{\sim}{x}}(t-\tau)\right] \right\rangle \qquad (106)$$

is defined as the correlation between two fluctuations at two instants $(t, t-\tau)$ along the trajectory. The variable position

$$\hat{\underset{\sim}{x}}(t-\tau) = \underset{\sim}{x} - \underset{\sim}{v}\tau - \hat{\underset{\sim}{\ell}}(\tau) \qquad (107)$$

consists of an initial position

$$\overline{\hat{\underset{\sim}{x}}(t)} = \underset{\sim}{x} \;, \qquad (108)$$

a free flight $\underset{\sim}{v}\tau$, and a perturbed path

$$\hat{\underset{\sim}{\ell}}(\tau) = \bar{\underset{\sim}{\ell}}(\tau) + \tilde{\underset{\sim}{\ell}}(\tau) \;. \qquad (109)$$

By Fourier decomposition, we have the fields

$$\underset{\sim}{X}'(t,\underset{\sim}{x}) = \iint d\omega'\ d\underset{\sim}{k}'\ \underset{\sim}{X}'(\omega',\underset{\sim}{k}')\ e^{-i(\omega' t-\underset{\sim}{k}'\cdot\underset{\sim}{x})} \tag{110}$$

$$\underset{\sim}{X}'\left[t-\tau,\underset{\sim}{x}(t-\tau)\right] = \iint d\omega''\ d\underset{\sim}{k}''\ \underset{\sim}{X}'(\omega'',\underset{\sim}{k}'')e^{-i\omega''(t-\tau)}e^{i\underset{\sim}{k}''\cdot\underset{\sim}{x}(t-\tau)}, \tag{111}$$

and the correlation

$$\left\langle \underset{\sim}{X}'(t,\underset{\sim}{x})\underset{\sim}{X}'\left[t-\tau,\hat{\underset{\sim}{x}}(t-\tau)\right]\right\rangle = \iint d\omega'\ d\underset{\sim}{k}' \iint d\omega''\ d\underset{\sim}{k}'' \left\langle \underset{\sim}{X}'(\omega',\underset{\sim}{k}')\underset{\sim}{X}'(\omega'',\underset{\sim}{k}'')\right\rangle$$

$$e^{-i\left[(\omega'+\omega'')t - (\underset{\sim}{k}'+\underset{\sim}{k}'')\cdot\underset{\sim}{x}\right]}h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}). \tag{112}$$

The orbit function is

$$h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}) = e^{-i\underset{\sim}{k}''\cdot\underset{\sim}{v}\tau}\left\langle e^{-i\underset{\sim}{k}''\cdot\hat{\underset{\sim}{\ell}}(\tau)}\right\rangle, \tag{113}$$

by (111) and (107).

If the Fourier decomposition is made in an interval of time 2 T and an interval of length 2 L in three dimensions within which the field is stationary and homogeneous, we can average (112) in time and space, integrate it with respect to τ, to obtain the diffusivity

$$\underset{\approx}{\mathcal{D}}'_X = \int_0^\infty d\tau \iint d\omega''\ d\underset{\sim}{k}''\ \chi_{TL}\left\langle \underset{\sim}{X}'(\omega'',\underset{\sim}{k}'')\underset{\sim}{X}'(-\omega'',-\underset{\sim}{k}'')\right\rangle\ h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}), \tag{114}$$

as defined by (72). The factor of Fourier truncation is

$$\chi_{TL} = \frac{\pi}{T}\left(\frac{\pi}{L}\right)^3 . \tag{115}$$

With a free flight in a frictionless medium, the diffusivity

$$\underset{\approx}{\mathcal{D}}'_X = \pi \iint d\omega'' \, d\underset{\sim}{k}'' \, \chi_{TL} \left\langle \underset{\sim}{X}'(\omega'', \underset{\sim}{k}'') \underset{\sim}{X}'(-\omega'', \underset{\sim}{k}'') \right\rangle \delta(\omega'' - \underset{\sim}{k}'' \cdot \underset{\sim}{v}) \tag{116}$$

has a resonance, as is common in linear instability. In weak turbulence, the trajectory of $\underset{\sim}{X}'$ is perturbed by a smooth streaming from $\underset{\sim}{\ell}^0$-fluctuations.

We deal with a strong turbulence by neglecting ω'' in (114) and reduce it into the form:

$$\underset{\approx}{\mathcal{D}}'_X = \int_0^{t\to\infty} d\tau \int d\underset{\sim}{k}'' \, \chi \left\langle \underset{\sim}{X}'(\underset{\sim}{k}'') \underset{\sim}{X}'(-\underset{\sim}{k}'') \right\rangle h(\tau, \underset{\sim}{k}'', \underset{\sim}{v}) \quad , \tag{117}$$

where $\chi \equiv (\pi/L)^3$ is the factor of Fourier truncation within the interval of length $2L$ in three dimensions.

The orbit function (113) has a component

$$\left\langle e^{-i\underset{\sim}{k}'' \cdot \underset{\sim}{\tilde{\ell}}(\tau)} \right\rangle = \int d\underset{\sim}{\ell} \; e^{-i\underset{\sim}{k}'' \cdot \underset{\sim}{\ell}} p(\tau, \underset{\sim}{\ell}) \tag{118}$$

from the path fluctuation $\underset{\sim}{\tilde{\ell}}(\tau)$, and can be calculated by means of the probability of transition $p(\tau, \underset{\sim}{\ell})$. A theory of transition can be developed on the basis of a master equation by a method similar to the one presented earlier. As a result, we will equally have a Darcy damping ζ'' to represent the friction by the porous medium:

$$\left\langle e^{-i\underset{\sim}{k}''\cdot\underset{\sim}{\ell}''(\tau)}\right\rangle = e^{\zeta''\tau} \equiv h_D \quad . \tag{119}$$

Analogously, we have

$$e^{-i\underset{\sim}{k}''\cdot\underset{\sim}{\ell}'(\tau)} = e^{\zeta'\tau} \equiv h_\ell \quad , \tag{120}$$

with $\zeta^1 \cong \zeta'$, by the approximation of the nearest-neighbour interaction. Finally, the coupling constant α enters as an external parameter, giving

$$h_\alpha \equiv e^{-\alpha\tau} \quad . \tag{121}$$

By collecting (119)-(121), we find the orbit function

$$h = h_\alpha \; h_v \; h_D \; h_\ell \quad , \tag{122}$$

with the following components:

$$h_v(\tau, \underset{\sim}{k}'', \underset{\sim}{v}) = e^{-i\underset{\sim}{k}''\cdot\underset{\sim}{v}\tau} \tag{123}$$

$$h_D(\tau,k'') = e^{\mathcal{C}''\tau} \tag{124}$$

$$h_{\ell}(\tau,k'') = e^{\mathcal{C}'\tau} \quad . \tag{125}$$

By (104), the last two orbit components can be written in the form:

$$h_D(\tau,k'') \cong \exp(-k''^2 K''\tau) \tag{126}$$

$$h_{\ell}(\tau,k'') \cong \exp(-k''^2 K'_{\ell}\tau) \quad . \tag{127}$$

Note that $\bar{\underset{\sim}{\ell}} = 0$, and that $\underset{\sim}{\ell}^0(\tau)$ is neglected, because the evolution operator

$$A'\hat{\mathcal{U}}(t,t-\tau) \tag{128}$$

in (61) selects the micro-group by A'. It may play a role of streaming.

The only component that represents the path-perturbation is $h_{\ell}(\tau,k'')$. It performs the physical role of relaxation for the eddy viscosity K' as a transport property to approach the equilibrium at a self-consistent rate $k''^2K'_{\ell}$.

7. Cascade transfer by eddy viscosity

By (92) and (71), the transfer function T^0_X can be written in the kinetic form, as follows:

$$\begin{aligned} T_X &= -\int d\underset{\sim}{v}\ \underset{\sim}{v}\cdot\left\langle \underset{\sim}{v}^{o}(t,\underset{\sim}{x})\ \mathcal{C}'_X\{f^{o}\}\right\rangle \\ &= -\int d\underset{\sim}{v}\ \underset{\sim}{v}\cdot\left\langle \underset{\sim}{v}^{o}(t,\underset{\sim}{x})\ \underset{\sim}{\partial}\cdot\underset{\approx}{\mathcal{D}}'_X\{\cdot\underset{\sim}{\partial}\ f^{o}(t-\tau)\}\right\rangle \\ &= -\int d\underset{\sim}{v}\ \underset{\sim}{v}\ \underset{\sim}{\partial}\cdot\underset{\approx}{\mathcal{D}}'_X\{\cdot\underset{\sim}{\partial}\ \underset{\sim}{\varphi}^{o}(\tau,\underset{\sim}{v})\} \\ &= -\mathcal{D}'_X\left\{\int d\underset{\sim}{v}\ \underset{\sim}{v}\ \partial^2\ \varphi^{o}(\tau,\underset{\sim}{v})\right\}, \end{aligned} \tag{129}$$

with

$$\underset{\sim}{\varphi}^{o}(\tau,\underset{\sim}{v}) = \left\langle \underset{\sim}{v}^{o}(t,\underset{\sim}{x}) f^{o}(t-\tau)\right\rangle \tag{130}$$

and

$$\mathcal{D}'_X = \mathrm{tr}\ \underset{\approx}{\mathcal{D}}'_X . \tag{131}$$

In Fourier form, we write

$$\underset{\sim}{\varphi}^{o}(\tau,v) = \int d\underset{\sim}{k}'\,\chi\left\langle v^{o}(\underset{\sim}{k}')\ f^{o}(-\underset{\sim}{k}',\underset{\sim}{v})\right\rangle h(\tau,\underset{\sim}{k}',\underset{\sim}{v}) \tag{132}$$

and

$$\partial^2 \underset{\sim}{\varphi}^0(\tau,\underset{\sim}{v}) = \int d\underset{\sim}{k}' \left\langle v^0(k') \; f^0 \; (-\underset{\sim}{k}',\underset{\sim}{v})\right\rangle \partial^2 h(\tau,\underset{\sim}{k}',\underset{\sim}{v}) . \tag{133}$$

The differentiation applies to the fast varying function $h(\tau,\underset{\sim}{k}',\underset{\sim}{v})$ and not to the slowly varying function $f^0(-\underset{\sim}{k}',\underset{\sim}{v})$ as an approximation. In the same manner, we write

$$\underset{\sim}{\mathcal{D}}'_X = \int_0^\infty d\tau \int d\underset{\sim}{k}'' \chi \left\langle \underset{\sim}{X}'(\underset{\sim}{k}'') \underset{\sim}{X}'(-\underset{\sim}{k}'')\right\rangle h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}=0) \quad , \tag{134}$$

from (117) and (128).

By collecting (133) and (134), we transform (129) into

$$T^0_X = -\int_0^\infty d\tau \int d\underset{\sim}{k}'' \chi \left\langle \underset{\sim}{X}'(\underset{\sim}{k}'') \underset{\sim}{X}'(-\underset{\sim}{k}'')\right\rangle h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}=0)$$

$$\times \int d\underset{\sim}{v} \; \underset{\sim}{v} \int d\underset{\sim}{k}' \chi \left\langle \underset{\sim}{v}^0(\underset{\sim}{k}') \; f^0 \; (-\underset{\sim}{k}'\underset{\sim}{v})\right\rangle \partial^2 h(\tau,\underset{\sim}{k}',\underset{\sim}{v}) \quad . \tag{135}$$

Here we have two orbit components

$$h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}=0)\partial^2 h(\tau,\underset{\sim}{k}',\underset{\sim}{v}) = -k'^2 \tau^2 h(\tau,\underset{\sim}{k}'',v=0) \; h(\tau,\underset{\sim}{k}',\underset{\sim}{v}) \quad , \tag{136}$$

The second component can also drop its $\underset{\sim}{v}$-dependence after its differentiation by ∂^2, so that

$$h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}=0) \; h(\tau,\underset{\sim}{k}',\underset{\sim}{v}) \simeq h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}=0) \; h(\tau,\underset{\sim}{k}',\underset{\sim}{v}=0) . \tag{137}$$

In the competition for the dampings, the second component is ineffective, and can be approximated by

$$h(\underset{\sim}{\tau},\underset{\sim}{k}',v=0) \cong 1 \quad . \tag{138}$$

By the use of (136)-(138), we transform (135) into

$$T_X^0 = \mathrm{tr}\int d\underset{\sim}{k}''\,\chi\left\langle \underset{\sim}{X}'(\underset{\sim}{k}'')\underset{\sim}{X}'(-\underset{\sim}{k}'')\right\rangle\int_0^\infty d\tau\,\tau^2\,h(\tau,\underset{\sim}{k}'',v=0)$$

$$\times\int d\underset{\sim}{k}'\;k'^2\chi\left\langle \underset{\sim}{v}^0(\underset{\sim}{k}')\cdot\underset{\sim}{v}^0(-\underset{\sim}{k}')\right\rangle$$

$$= K'\,R^0 \quad . \tag{139}$$

By separating the integrals into two parts, we recognize the vorticity function as

$$R^0 = \int d\underset{\sim}{k}'\;k'^2\,\chi\left\langle \underset{\sim}{v}^0(\underset{\sim}{k}')\cdot\underset{\sim}{v}^0(-\underset{\sim}{k}')\right\rangle \quad , \tag{140}$$

and identify the eddy viscosity as

$$K' = \mathrm{tr}\int d\underset{\sim}{k}''\,\chi\left\langle \underset{\sim}{X}'(\underset{\sim}{k}'')\underset{\sim}{X}'(-\underset{\sim}{k}'')\right\rangle\int_0^\infty d\tau\,\tau^2\,h(\tau,\underset{\sim}{k}'',v=0) \quad , \tag{141}$$

or as

$$K' = \mathcal{D}_X'\{\tau^2\} \quad . \tag{142}$$

Note that the diffusivity as a transport property

$$\mathcal{D}'_{X} = \int d\underset{\sim}{k}'' \chi \langle \underset{\sim}{X}'(\underset{\sim}{k}'')\underset{\sim}{X}'(-\underset{\sim}{k}'')\rangle \int_0^\infty d\tau \; h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}=0), \qquad (143)$$

by (117), can be used as operating on τ^2.

8. Eddy viscosity

We have derived the formulae for the transfer function T^0_X, the eddy viscosity K', and the orbit function $h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}=0)$, in (139), (141), and (122), respectively. We show that the spectral evolution is characterized by the eddy viscosity K'. This transport property approaches its equilibrium by a relaxation process. The analysis of the perturbations of the trajectory results in the orbit function $h(\tau,\underset{\sim}{k}'',\underset{\sim}{v}=0)$.

From (141) we can distinguish two eddy viscosities:

$$\begin{aligned} K' &= \mathrm{tr} \int d\underset{\sim}{k}'' \chi \langle \underset{\sim}{X}'(\underset{\sim}{k}'')\underset{\sim}{X}'(-\underset{\sim}{k}'')\rangle \mathcal{G}(\underset{\sim}{k}'') \\ &= \frac{2}{3}\int_k^\infty dk''\; G(k'')\, \mathcal{G}(\underset{\sim}{k}'') \end{aligned} \qquad (144)$$

$$\begin{aligned} K'_{\ell} &= \mathrm{tr} \int d\underset{\sim}{k}'' \chi \langle \underset{\sim}{X}'(\underset{\sim}{k}'')\underset{\sim}{X}'(-\underset{\sim}{k}'')\rangle \mathcal{G}_{\ell}(k'') \\ &= \frac{2}{3}\int_k^\infty dk''\; G(k'')\, \mathcal{G}_{\ell}(k'') \quad . \end{aligned} \qquad (145)$$

They control the cascade transfer and determine the relaxation, respectively. They are energized by the field fluctuations of spectrum G(k"), and are characterized by the two time moments:

$$\mathcal{G}(k") = \int_0^{\infty} d\tau \; \tau^2 \; h(\tau, \underset{\sim}{k}", \underset{\sim}{v}=0) \tag{146}$$

$$\mathcal{G}_{\ell}(k") = \int_0^{\infty} d\tau \; \tau^2 \; h(\tau, k") \quad . \tag{147}$$

The latter is the moment of h_{ℓ} in its role of relaxation, in the form

$$\mathcal{G}_{\ell}(k") = \int_0^{\infty} d\tau \; \tau^2 \; \exp(-k"^2 K'_{\ell} \;)$$

$$= 2\, k"^2 \, K'_{\ell} \quad , \tag{148}$$

so that, by a substitution into (145), yields

$$K'_{\ell} = \frac{4}{3}\int_k^{\infty} dk" \; G(k") \; (k"^2 K'_{\ell} \;)^{-3} \; . \tag{149}$$

The eddy viscosity K'_{ℓ}, as appearing in both sides of (149), is equally stationary and homogeneous since it is a function of k. By grouping them together, we simplify (149) into the form

$$K'_{\ell}(k") = \left[\frac{4}{3}\int_{k"}^{\infty} dk"' \; k"'^{-6} G(k"') \right]^{1/4} \tag{150}$$

For the calculation of K', we distinguish between the weak coupling and the strong coupling in the following.

8.1 Weak coupling $\alpha \ll k''^2 K'_\ell$

In weak coupling we neglect α in (122) and (146) to get

$$\mathcal{G}(k'') = \int_0^\infty d\tau\, \tau^2 \exp\left\{-k''^2\left[K''(k'') + K'_\ell(k'')\right]\right\}$$

$$= 2\left\{k''^2\left[K''(k'') + K'_\ell(k'')\right]\right\}^{-3}$$

$$\cong 2\left[2k''^2 K'_\ell(k'')\right]^{-3}. \qquad (151)$$

We note that by its higher order in the group, K"(k") is qualified to play the same role of relaxation as does $K'_\ell(k'')$. Use of (126) and (127) has been made.

It is not difficult to demonstrate the relation

$$K' = \frac{1}{2} K'_\ell \qquad (152)$$

by differentiating the two eddy viscosities (144) and (150) and by comparing them.

8.2 Strong coupling $k''^2 K'_\ell \ll \alpha$

The calculation of G(k"), as defined by (146), is more involved. The function has two time scales

$$\alpha^{-1} \quad \text{and} \quad \left[k''^2 (K' + K'') \right]^{-1} \tag{153}$$

to distribute. As an optimization for fast approach to equilibrium, we write

$$\mathcal{G}(k'') = \int_0^\infty d\tau \, e^{-\alpha\tau} \int_0^{\tau\to\infty} d\tau' \, \tau' \exp\left\{-k''^2 \left[K''(k'') + K'_\ell (k'')\right] \tau'\right\} , \tag{154}$$

or

$$\mathcal{G}(k'') = \frac{1}{\alpha} \left\{ k''^2 \left[K''(k'') + K'_\ell (k'') \right] \right\}^{-2}$$

$$= \frac{1}{\alpha} \left[2k''^2 K'_\ell (k'') \right]^{-2} , \tag{155}$$

when K" joins with K'_ℓ for the relaxation process by its qualification of belonging to a high-order group. The upper limit τ of integration in (154) can be put to ∞ in view of the condition

$$\alpha \gg k''^2 K'_\ell \quad . \tag{156}$$

Upon substituting (155) into (144), we find

$$K' = \frac{2}{3\alpha}\int_k^{\infty} dk''\, k''^{-4}\, G(k'')\left[2K'_\ell\,(k'')\right]^{-2} \quad . \tag{157}$$

The proportionality α^{-1} fulfills the Einstein condition in Brownian motion.

9. Equation of state relating pressure and velocity fluctuations

The transfer functions T^0_N and T^0_X as obtained in (91) and (92) depend on N' and $\underset{\sim}{X}'$, respectively. For T^0_X, a relation between $\underset{\sim}{X}'$ and $\underset{\sim}{v}'$ is needed. It is a kinematic relation without involving dynamical processes and is called the equation of state. Since T^0_X describes a momentum transfer at the exclusion of the density fluctuations, we can use the hydrodynamic equation (19) with constant coupling, i.e. $\hat{\alpha} \cong \alpha$, as our basis of derivation. By taking the divergence, we find the Poisson equation

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\tilde{E}} = \tilde{r} \quad , \tag{158}$$

with the source

$$\begin{aligned} \tilde{r} &= \underset{\sim}{\nabla}\underset{\sim}{\nabla} : \tilde{A}\,\underset{\sim}{\hat{v}}\,\underset{\sim}{\hat{v}} \\ &= \underset{\sim}{\nabla}\underset{\sim}{\nabla} : \left(\tilde{v}\,\tilde{v} - \langle \tilde{v}\,\tilde{v} \rangle\right) . \end{aligned} \tag{159}$$

By Fourier decomposition, we transform the equation (158) for the $\tilde{E}$-field into the following equation for the intensities, as

$$\left\langle \tilde{E}(k)\cdot\tilde{E}(-k)\right\rangle = k^{-2}\left\langle \tilde{r}(k)\,\tilde{r}(-k)\right\rangle . \tag{160}$$

It is evident by (159) that the right-hand side will have quadruple moments. These will be decoupled into products of quadratic ones by the assumption of quasi-normality. This is legitimate, since the equation of state deals with a stationary, homogeneous, and isotropic state of fluctuations, and not with a dynamic process. We find the

pressure-velocity relation in the form:

$$\left\langle r_v^{\circ\,2}\right\rangle = \frac{2}{9}\, R_v^{\circ\,2} \quad . \tag{161}$$

The details of the derivation which uses the Fourier method are omitted here.

For the sake of brevity, it will be convenient to introduce the notation

$$R_w^{\circ} = \int dk'\; k'^2 \chi \left\langle w^{\circ}(k')\cdot w^{\circ}(-k')\right\rangle \quad , \tag{162}$$

with $\hat{w} = (\hat{v},\hat{u},\hat{E},\hat{X})$. We find

$$R_E^{\circ} = \left\langle r_v^{\circ\,2}\right\rangle \tag{163}$$

and

$$R_X^{\circ} = R_E^{\circ} + \alpha_v^2 R_u^{\circ} \tag{164}$$

from (160) and (22), respectively.

By substituting (161) into (163) and subsequently into (164), we transform it into

$$R_X^{\circ} = \frac{2}{9} (R_V^{\circ})^2 + \alpha_v^2 R_u^{\circ} \quad . \tag{165}$$

By differentiating with respect to k, we obtain the following relation between the spectra:

$$G(k) = \frac{4}{9} R_V^{\circ} F_V(k) + \alpha_v^2 F_u(k) \quad . \tag{166}$$

This relation will serve in transforming the transport coefficients ϑ_X' and K' that were written in terms of $\underset{\sim}{X}'$ into those in terms of velocities.

The functions G(k), $F_v(k)$ and $F_u(k)$ are spectral densities, giving the spectral intensities:

$$\int d\underset{\sim}{k}'' \chi \left\langle \underset{\sim}{X}'(\underset{\sim}{k}'') \cdot \underset{\sim}{X}'(-\underset{\sim}{k}'') \right\rangle = 2\int_k^{\infty} dk'' \; G(k'') \tag{167}$$

$$\int d\underset{\sim}{k}' \chi \left\langle \underset{\sim}{v}^{\circ}(\underset{\sim}{k}') \cdot \underset{\sim}{v}^{\circ}(-\underset{\sim}{k}') \right\rangle = 2\int_0^{k} dk' \; F_V(k') \tag{168}$$

$$\int dk' \chi \left\langle \underset{\sim}{u}^{\circ}(\underset{\sim}{k}') \cdot \underset{\sim}{u}^{\circ}(-\underset{\sim}{k}') \right\rangle = 2\int_0^{k} dk' \; F_u(k') \quad . \tag{169}$$

10. Spectral structure of velocity fluctuations

10.1 Inertia turbulence with weak coupling

The inertia turbulence is governed by the cascade flow at the constant rate of dissipation. If the coupling is negligible, i.e.

$$\alpha \ll k^2 K' \;, \; (R^{o})^{1/2} \;, \tag{170}$$

the two phases are separated.

The spectral flow is governed by

$$T_X^{o} \equiv K' \, R^{o} = \mathcal{E} \;, \tag{171}$$

and is independent of α. The small-scale transfer is chosen, because we are treating the range near the dissipation end of the spectrum.

The expression (150) for the eddy viscosity K'_{ℓ} in terms of the spectrum G(k) of the field spectrum can be converted in terms of the velocity spectrum $F_v(k)$ by the equation of state (166) which is

$$G(k) \cong \frac{4}{9} R_V^{o} (k) \; F_V(k) \tag{172}$$

in weak coupling. We find

$$K'_{\ell}(k'') = \left[\frac{16}{27}\int_{k''}^{\infty} dk''' \, k'''^{-6} \, R_v^0 (k''') \, F_v(k''')\right]^{1/4} . \qquad (173)$$

This leads to K' by the relation (152).

By the use of the formula for K', the spectral balance (171) yields the following results:

$$F_v(k) = 1.633 \ \varepsilon_v^{2/3} \, k^{-5/3} \qquad (174)$$

$$K'_v = 0.408 \ \varepsilon_v^{1/3} \, k^{-4/3} \quad . \qquad (175)$$

and, by an interchange of indices v, u,

$$F_u(k) = 1.633 \ \varepsilon_u^{2/3} \, k^{-5/3} \qquad (176)$$

$$K'_u = 0.408 \ \varepsilon_u^{1/3} \, k^{-4/3} \quad . \qquad (177)$$

The numerical coefficients are predicted analytically, and are in good agreement with the best experimental values.

The -5/3 law is identical to the Kolmogoroff law, except the rates of dissipation ε_v, ε_u are augmented by the drag coefficients, by (89). Such an increase has been found in experiments, where the coupling coefficients are small to satisfy the condition (179), but not so small as to be negligible in (89) (Goldschmidt, 1972).

10.2 Inertia turbulence with strong coupling

In strong coupling, i.e.

$$\alpha \quad \gg \quad k^2 K' \quad , \quad (R^{\circ})^{1/2} \quad , \tag{178}$$

the spectral flow is governed by equation (171), rewritten as

$$T^{\circ}_{X,v} = K'_v R^{\circ}_v \tag{179}$$

for the $\underset{\sim}{v}$-phase.

The vorticity function is

$$R^{\circ}_v = 2\int_0^k dk' \, k'^2 \, F_v(k') \quad . \tag{180}$$

The eddy viscosity K'_v is given by (157) in terms of the spectrum $G_v(k)$. It can be converted in terms of the spectrum $F_u(k)$ by means of the equation of state (166), which becomes

$$G_v(k) \cong \alpha^2 \, F_u(k) \tag{181}$$

in strong coupling. We get

$$K'_v = \frac{2\alpha_v}{3}\int_k^{\infty} dk'' \, k''^{-4} \, F_u(k'') \, \left[2K'_{\ell,v}(k'') \right]^{-2} \quad , \tag{182}$$

with

$$K'_{\ell,v} = \left[\frac{16}{27} \int_k^\infty dk''\, k''^{-6}\, R_v\,(k'')\; F_v(k'') \right]^{1/4} \tag{183}$$

by (173).

The system (179), (180), (182) and (183) governs the v-cascade. A parallel system governs the û-cascade, and is obtained by interchanging v̂ and û. The two systems are coupled.

By omitting the details of calculation, we obtain the following results:

$$F_v(k) = A_\alpha u_\varepsilon^{\,2}\, k^{-1} \tag{184}$$

$$F_u(k) = A_\alpha v_*^{\,2}\, k^{-1} \tag{185}$$

$$K'_v = A_\alpha^{\;-1} \alpha_v (v_\varepsilon / u_\varepsilon)^2\, k^{-2} \tag{186}$$

$$K'_u = A_\alpha^{\;-1} \alpha_u (u_\varepsilon / v_\varepsilon)^2\, k^{-2} \tag{187}$$

with the friction velocities

$$v_\varepsilon \equiv (\varepsilon_v/\alpha_v)^{1/2}, \quad u_\varepsilon = (\varepsilon_u/\alpha_u)^{1/2} . \tag{188}$$

The numerical coefficient is predicted as:

$$A_\alpha = 8/\sqrt{3} \quad . \tag{189}$$

The solutions can be obtained by a method of verification using a power law spectral distribution.

11. Spectral structure of density fluctuations

The kinetic equation (63) is valid for density fluctuations, by definition (35). By taking the zeroth moment, we find the equation of evolution

$$\left(\partial_t + \underset{\sim}{v}^{\circ}\cdot\underset{\sim}{\nabla}\right)N^{\circ} = -\underset{\sim}{v}^{\circ}\cdot\underset{\sim}{\nabla}\bar{N} + \nu_N \nabla^2 N^{\circ} + \int d\underset{\sim}{v}\, \mathcal{C}' \left\{ f^{\circ}(t-\tau) \right\} \quad . \tag{190}$$

Upon multiplying it by N° and averaging, we derive the equation of spectral flow

$$\frac{1}{2}\partial_t \left\langle N^{\circ 2} \right\rangle = -\left\langle N^{\circ} \underset{\sim}{v}^{\circ} \right\rangle \cdot \underset{\sim}{\nabla}\bar{N} - \varepsilon_N^{\circ} - \Theta^{\circ} \quad . \tag{191}$$

This is governed by the production function

$$-\left\langle N^{\circ} \underset{\sim}{v}^{\circ} \right\rangle \cdot \nabla \bar{N} \quad , \tag{192}$$

the dissipation function

$$\varepsilon_N^{\circ} = \nu_N \left\langle (\nabla N^{\circ})^2 \right\rangle \tag{193}$$

by molecular diffusivity ν_N, and the transfer function

$$\Theta^{0} = - \int d\underset{\sim}{v} \quad N^{0} \mathcal{G}' \left\{ f^{0}(t-\tau) \right\} \quad . \tag{194}$$

The transfer function

$$\Theta^{0} = \Theta^{0}_{X} + \Theta^{0}_{N} \tag{195}$$

has again two components

$$\Theta^{0}_{X} = - \int d\underset{\sim}{v} \left\langle N^{0} \mathcal{G}'_{X} \{ f^{0}(t-\tau) \} \right\rangle \tag{196}$$

$$\Theta^{0}_{N} = - \int d\underset{\sim}{v} \left\langle N^{0} \mathcal{G}'_{N} \{ f^{0}(t-\tau) \} \right\rangle \quad . \tag{197}$$

By the same procedures as were used for deriving the velocity transfer, we find the components

$$\Theta^{0}_{X} = K' \left\langle (\nabla N^{0})^{2} \right\rangle \tag{198}$$

$$\Theta^{0}_{N} = - \kappa^{2} \mathcal{D}'_{N} \left\langle N'^{2} \right\rangle \quad , \tag{199}$$

for the direct cascade and for the reverse cascade, respectively.

The inertia turbulence for the direct cascade is governed by the spectral flow

$$\Theta_X^0 \equiv K' \left\langle (\nabla N^0)^2 \right\rangle = \varepsilon_N , \tag{200}$$

where K' has been derived in (152) and (183) for the weak coupling and the strong coupling, respectively, and $\varepsilon_N = \varepsilon_N^0(k=\infty)$.

We find the following results for the spectrum $F_{N,v}(k)$ of $\tilde{N}_v$-fluctuations as follows:

$$F_{N,v}(k) = 1.634\ (\varepsilon_{N,v}/\varepsilon_v{}^{1/3})\ k^{-5/3},\ \text{zero coupling} \tag{201}$$

$$F_{N,v}(k) = \frac{8}{\sqrt{3}}\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^2 N_{\varepsilon,v}^2\ k^{-1},\quad \text{strong coupling}\ . \tag{202}$$

The parameter is

$$N_{\varepsilon,v} = (\varepsilon_{N,v}/\alpha_v)^{1/2} \quad . \tag{203}$$

By an interchange of the subscripts v, u, we find the spectrum $F_{N,u}(k)$ of $\tilde{N}_u$-fluctuations as follows:

$$F_{N,u}(k) = 1.634\ (\varepsilon_{N,u}/\varepsilon_u{}^{1/3})k^{-5/3},\quad \text{zero coupling} \tag{204}$$

$$F_{N,u}(k) = \frac{8}{\sqrt{3}}(v_\varepsilon/u_\varepsilon)^2\ N_{\varepsilon,u}^2\ k^{-1},\ \text{strong coupling}, \tag{205}$$

with

$$N_{\varepsilon,u} = (\varepsilon_{N,u}/\alpha_u)^{1/2} \tag{206}$$

12. Summary and discussion

12.1 Criteria of chaos

A nonlinear dissipative system can be represented by the equation of evolution in the form (real or complex), as follows:

$$(\partial_t + \hat{M})\hat{\Psi}(t,\underset{\sim}{x}) = \hat{X}(t,\underset{\sim}{x}) \quad , \tag{207}$$

with

$$\hat{M} = M_\ell + \hat{M}_{n\ell} \tag{208}$$

$$\hat{X} = \hat{X}_{self} + \hat{X}_{ext} \quad , \tag{209}$$

where $\hat{\Psi}$ is an unknown random function (scalar or vector), M_ℓ is a non-random linear differential operator, and $\hat{M}_{n\ell}$ is a differential operator that causes nonlinearity by involving $\hat{\Psi}$ or another random function, linear or nonlinear. The driving force consists of the component $\hat{X}_{self}$ that is reducible to Ψ by a self-consistency condition, and the component $\hat{X}_{ext}$ that is externally given and is random. The hydrodynamic equations of two phases, the nonlinear propagation of light and of solitons, and the dynamics of neurons fall into this category.

The hydrodynamic equation is inhomogeneous and there is an advantage in transforming it into the homogeneous equation

$$(\partial_t + \hat{L})\hat{f}(t,\underset{\sim}{x},\psi) = 0 \tag{210}$$

of higher dimensions, called the master equation. The differential operator is

$$\hat{L} = \hat{M} + \hat{X}\,\partial \;, \tag{211}$$

with $\partial = \partial/\partial\psi$. The random differential operator $\hat{M}$ in (207) is transformed into a non-random differential operator M in (210). The field $\hat{X}$ that was a driving force in (207) becomes the only random convection in (210).

By writing

$$f(t,\underset{\sim}{x},\psi) = \delta\left[\psi - \Psi(t,\underset{\sim}{x})\right] \tag{212}$$

and by taking the moment of (210), we recover the hydrodynamic equation (207) and show the self-consistency of the master equation.

The hydrodynamic equation and the master equation can serve to describe the microdynamical state of turbulence. From the theory of universality of chaos, it will be interesting to know whether such a system can develop chaos and yield a broad spectrum. The theory of chaos is not yet able to predict the criteria. The hydrodynamical closure by the degeneration of high orders into lower orders can not find physical support. With the lack of a small parameter in turbulence, the kinetic method by Bogoliubov can not justify its closure. In our group-scaling method, we scale the master equation into a hierarchy of equations for the macro-group, micro-group, and sub-microgroup, representing the spectral evolution, the transport property, and the relaxation. The last process is represented by a network of random higher-order distributions. By the successive collisions, the distributions in the network lose their memory, and as an ensemble can be simulated by a porous medium which then closes the hierarchy by a self-consistent condition. The relaxation thus formulated makes the transport property approach its equilibrium. Hence, the kinetic equation is obtained. By the return to the continuum, we calculate the transport coefficients, and determine the spectral structure.

We find a direct cascade and a reverse cascade. If one or both cascades can find their spectral balance from the equation of spectral flow, the chaos can develop into a broad spectrum.

12.2 Dusty gas

For a dilute suspension, where the particles are not sufficiently dense to produce a pressure, such that

$$\hat{\alpha}_v \hat{u} \gg \hat{E}_v \quad , \tag{213}$$

we simplify the system (19)-(21) for the v-phase into

$$(\partial_t + \hat{\underset{\sim}{v}} \cdot \underset{\sim}{\nabla} + \hat{\alpha}_v) \hat{\underset{\sim}{v}} = \hat{\alpha}_v \hat{\underset{\sim}{u}} \tag{214}$$

$$(\partial_t + \hat{\underset{\sim}{v}} \cdot \underset{\sim}{\nabla}) \hat{N}_v = \nu_{N,v} \nabla^2 \hat{N}_v \quad , \tag{215}$$

with

$$\hat{\alpha}_v = \alpha_v \hat{N}_v \quad , \tag{216}$$

while we keep the system for the u-phase in the original form (23)-(26). These two new systems consitute the 'dusty gas' model by Saffman (1962).

The spectral results for the dusty gas can be derived by a degeneration of our general formulation, under the condition (213).

When we further neglect the coupling in (214), we obtain the Burgers equation

$$(\partial_t + \hat{\underset{\sim}{v}} \cdot \underset{\sim}{\nabla} - \nu_v \nabla^2) \hat{\underset{\sim}{v}} = 0 \tag{217}$$

and the diffusion equation

$$(\partial_t + \hat{\underset{\sim}{v}}\cdot\underset{\sim}{\nabla} - \nu_{N,v}\nabla^2)\hat{N} = 0 \quad . \tag{218}$$

The treatment of turbulence on the basis of the homogeneous Burgers equation can be made by following the procedures related to the master equation.

12.3 Reduction of turbulence by the particles

The power law

$$K'_u = A_\gamma^{-1}|\underset{\sim}{\nabla}\bar{\underset{\sim}{u}}|\ k^{-2} \tag{219}$$

for the eddy viscosity has been suggested by Prandtl for turbulent boundary layers with a mean velocity gradient $|\underset{\sim}{\nabla}\bar{\underset{\sim}{u}}|$. With this eddy viscosity and the equation of spectral balance in the form

$$K'_u R_u^0 = \varepsilon_u \ , \tag{220}$$

we find the spectrum

$$F_u(k) = A_\gamma\ u_\gamma^2\ k^{-1} \quad , \tag{221}$$

where u_γ is

$$u_\gamma = (\varepsilon_u / |\nabla \bar{u}|)^{1/2} \quad , \tag{222}$$

and A_γ is a numerical coefficient. The power law (209) confirms the earlier prediction by Tchen (1953, 1954). This law has been observed in the atmospheric boundary layers (Tchen et al., 1985) and in the tropopause (otten et al., 1982).

It is interesting to note the analogy between (185) and (221) that can suggest that a shear flow with the combined effects of $|\nabla \bar{u}|$ and α will have a reduced spectrum. This reduction can be seen from the lidar measurements in the atmosphere before and after the rain (Gurvich and Pokasov, 1972).

12.4 Equality of transport coefficients

We choose a strong coupling where the governing parameters are α_v, α_u. By the use of Prandtl's formula, we write

$$K'_{v|v} = \alpha_v k^{-2} \tag{223}$$

and obtain

$$F_v(k) = A_\alpha \alpha_v^2 k^{-3} \tag{224}$$

and similarly

$$F_u(k) = A_\alpha \, \alpha_u^2 \, k^{-3} \quad . \tag{225}$$

The numerical coefficients are not determined in our dimensional reasonings. These spectral distributions occur in a range where the transfer balances the production and where α_v, α_u are the only dominant parameters.

We rewrite the formula (144) for the eddy viscosity in the form

$$\begin{aligned} K'_{v|v} &= \frac{2}{3}\int_k^\infty dk'' \; G_v(k'') \, \mathcal{G}_v(k'') \\ &= \frac{2}{3}\,\alpha_v^2 \int_k^\infty dk'' \; F_v(k'') \, \mathcal{G}_v(k''), \end{aligned} \tag{226}$$

and similarly

$$K'_{u|v} = \frac{2}{3}\,\alpha_u^2 \int_k^\infty dk'' \; F_v(k'') \, \mathcal{G}_v(k'') \quad . \tag{227}$$

Use of (166) is made for strong coupling. The subscript $(\ )_{u|v}$ denotes that the transfer coefficient is endowed by the $\underset{\sim}{u}'$-fluctuations along the $\hat{\underset{\sim}{v}}$-trajectory, and $(\ )_{v|v}$ denotes that the transport coefficient is endowed by the $\underset{\sim}{v}'$-fluctuations along the $\hat{\underset{\sim}{v}}$-trajectory. Upon substituting for (224) and (225), we obtain

$$K'_{v|v} = K'_{u|v} \quad , \tag{228}$$

confirming the equality (8) from the Langevin method.

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GROUP-KINETIC THEORY OF DISPERSIVE SOLITON TURBULENCE

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As the fluid analogue, the nonlinear Schrödinger equation has a driving force in the form of emission of solitons by velocity fluctuations, and is used to describe the micro-dynamical state of turbulence. In order for the emission together with the modulation to participate in the transport processes (spectral evolution, eddy collision, and relaxation), the non-homogeneous Schrödinger equation is transformed into a homogeneous master equation. By group-scaling, the master equation is decomposed into a system of transport equations, replacing the Bogoliubov system of equations of many-particle distributions. It is in the relaxation that the memory is lost when the ensemble of higher-order distributions is simulated by an effective porous medium. The closure is thus found. The kinetic equation is derived and is transformed into the equation of spectral flow. The emission deposits the energy into the spectrum for accumulation toward small wave numbers by a reverse cascade with a negative eddy viscosity. The modulation develops a direct cascade and the modulation collision provides a fluctuation-dissipation for the drain. The spec-

tral laws for the soliton energy are k^{-1}, k^{-3}, and k^{-6} in the inertia, coupling, and dispersion subranges, respectively. The spectral flow determines the statistical criteria for a nonlinear stochastic system to become chaotic. It is found that the homogeneous Schrödinger equation cannot develop strong turbulence.

I. INTRODUCTION

Many nonlinear phenomena in the continuous media exhibit fast density fluctuations superposed with slow ones, and the fast components have envelopes that fluctuate slowly. Such a modeling is called the soliton formalism and is usually represented by a homogeneous partial differential equation of the parabolic type, called the homogeneous Schrödinger equation.[1] This equation is considered more appealing than the Navier-Stokes equation and has been widely used in plasmas, optics, and fluids, including atmospheric waves, gravity waves and sea surface waves. Unfortunately, the numerical computations of the homogeneous Schrödinger equation have not found chaos and deemed it not suitable for turbulence. From the hydrodynamical standpoint, the Navier-Stokes equations for density and velocity can be transformed into a wave equation for the propagation of density with a coupling to velocity. Such a coupling is known as the production of noises by turbulence or the emission of solitons by velocity fluctuations, and must be represented in the Schrödinger equation to make it self-consistent with the Navier-Stokes equations. The fluid analogue of solitons, i.e. the derivation of the

Schrödinger equation from the Navier-Stokes equations, has been treated by several authors.[2-4] It is found that for the Schrödinger equation to develop turbulence, the emission must be present. Hence, it is not justifiable to conclude that solitons cannot coalesce or break by collision on the basis of the Schrödinger equation truncated to its homogeneous form. The dispersion in the differential form ∇^2 is important, without which the partial differential equation of Schrödinger would be reduced into an ordinary differential equation in the class of Duffing oscillator.

As the scope of treatment, we consider the non-homogeneous Schrödinger equation to describe the microdynamic state of soliton turbulence, and transform it into a homogeneous master equation (Section II). By a scaling into groups and by the loss of memory in the relaxation, we find the closure, and derive the kinetic equation of turbulence (Section III-V). By taking the moment we obtain the equation of spectral flow. By the emission, the energy is deposited into the spectrum, to be accumulated toward small wavenumbers by reverse cascade and transferred toward large wavenumbers by direct cascade. A transport theory determines the eddy viscosity (Sections VI, VII). The equation of spectral evolution is solved (Sections VIII, IX). The group-kinetic consideration of spectral flow determines the criteria for a nonlinear dynamical system to develop chaos and turbulence (Section X).

II. KINETIC DESCRIPTION OF THE MICRODYNAMICAL STATE OF SOLITON TURBULENCE

The nonlinear Schrödinger equation

$$(i\,\partial_t + \nu_n \nabla^2 - \tfrac{1}{2}\omega_n \tilde{A}\tilde{N})\tilde{\underset{\sim}{E}} = \tilde{\underset{\sim}{X}} \tag{1}$$

for the soliton field $\tilde{\underset{\sim}{E}}(t,\underset{\sim}{x})$ under the driving force $\tilde{\underset{\sim}{X}}(t,\underset{\sim}{x})$ has a nonlinear modulation by the density fluctuation

$$\tilde{N} = -\alpha\,\tilde{A}\,|\tilde{\underset{\sim}{E}}|^2 \tag{2}$$

and a dispersion by $\nu_n \nabla^2$, where ν_n, ω_n and α are real constants, having the dimensions of viscosity, frequency and the inverse square of acceleration, respectively. The driving force is real, to be specified later.

The random field

$$\hat{\underset{\sim}{E}} = \bar{\underset{\sim}{E}} + \tilde{\underset{\sim}{E}} \tag{3}$$

can be decomposed into an ensemble average $\bar{\underset{\sim}{E}} \equiv \langle \underset{\sim}{E} \rangle$ and a fluctuation $\tilde{\underset{\sim}{E}}$. The operators

$$\hat{A} = \bar{A} + \tilde{A} \tag{4}$$

can be used, where $\hat{A} = 1$ is a unit operator, and

$$\tilde{A} = 1 - \bar{A} \tag{5}$$

is the deviation from the average.

In the following, we assume homogeneous and isotropic turbulence in three dimensions, without mean field, i.e.

$$\bar{\underset{\sim}{E}} = 0. \tag{6}$$

By the distribution function

$$\hat{f}(t,\underset{\sim}{x},\underset{\sim}{E}) = \delta[\underset{\sim}{E} - \hat{\underset{\sim}{E}}(t,\underset{\sim}{x})], \tag{7}$$

the Schrödinger equation in the physical space $t,\underset{\sim}{x}$ is transformed into the master equation

$$(\partial_t + \hat{L})\,\hat{f} = 0 \tag{8}$$

in the phase space $t, \underset{\sim}{x}, \underset{\sim}{E}$. Here $\underset{\sim}{E}$ is an independent variable, as distinct from the function $\underset{\sim}{\hat{E}}(t,\underset{\sim}{x})$ and

$$\hat{L} = L_\nu + \tilde{L}_N + \tilde{L}_X \tag{9a}$$

$$L_\nu = -i\ \nu_n\ \nabla^2 \tag{9b}$$

$$\tilde{L}_N = i\ \tfrac{1}{2}\omega_n\ \tilde{N} \tag{9c}$$

$$\tilde{L}_X = -i\ \underset{\sim}{\tilde{X}} \cdot \underset{\sim}{\partial} \tag{9d}$$

are called differential operators, with

$$\partial_t = \partial/\partial t, \quad \underset{\sim}{\partial} = \partial/\partial \underset{\sim}{E}\ . \tag{10}$$

It is not difficult to verify that the moment of (8) reproduces (1). The kinetic description (8) has the definite advantage over the fluid description (1) by being homogeneous. It is also the only logical way of developing a transport theory of spectrum-dependent eddy viscosity.

The spectrum $F_E(k)$ gives the field intensity

$$2\int_0^{k=\infty} dk'\ F_E(k') = \langle |\underset{\sim}{\tilde{E}}|^2 \rangle\ . \tag{11}$$

It is defined by the Fourier transformation

$$F_E(k) = 2\pi\, k^2\, \frac{1}{(2\pi)^3} \int d\underset{\sim}{\ell}\ \ e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\ell}} \langle| \underset{\sim}{\tilde{E}}^*(t,\underset{\sim}{x}) \cdot \underset{\sim}{\tilde{E}}(\underset{\sim}{x}-\underset{\sim}{\ell}) |\rangle \tag{12}$$

of the correlation function

$$\begin{aligned} \langle| \underset{\sim}{\tilde{E}}^*(t,\underset{\sim}{x}_1)\cdot \underset{\sim}{\tilde{E}}(t,\underset{\sim}{x}_2) |\rangle &= \iint d\underset{\sim}{E}_1 d\underset{\sim}{E}_2\ \ \underset{\sim}{E}_1\cdot\underset{\sim}{E}_2\ \langle| \tilde{f}^*(t,\underset{\sim}{x}_1,\underset{\sim}{E}_1)\ \tilde{f}(t,\underset{\sim}{x}_2,\underset{\sim}{E}_2) |\rangle \\ &= \iint d\underset{\sim}{E}_1 d\underset{\sim}{E}_2\ \ \underset{\sim}{E}_1\cdot\underset{\sim}{E}_2\ \bar{f}_{12}(t,\underset{\sim}{x}_1,\underset{\sim}{E}_1;t,\underset{\sim}{x}_2,\underset{\sim}{E}_2) \end{aligned} \tag{13}$$

through the intermediary of the pair-distribution function

$$\bar{f}_{12}(t,\underset{\sim}{x}_1,\underset{\sim}{E}_1;\ t,\underset{\sim}{x}_2,\underset{\sim}{E}_2) = \langle| \tilde{f}^*(t,\underset{\sim}{x}_1,\underset{\sim}{E}_1)\ \tilde{f}(t,\underset{\sim}{x}_2,\underset{\sim}{E}_2) |\rangle . \tag{14}$$

Thus the determination of this spectrum becomes a problem of the many-body statistical mechanics, involving a hierarchy of distribution functions.[5]

Alternatively, by considering a finite upper limit of integration k in (11), we get the spectral intensity

$$2\int_0^k dk' F_E(k') = \langle|\mathbf{E}^0|^2\rangle. \qquad (15)$$

from which we find the spectrum $F_E(k)$ by a differentiation with respect to k. This method deals with the singlet-distribution function, or the moment

$$\mathbf{E}^0(t,\mathbf{x}) = \int d\mathbf{E}\,\mathbf{E}\, f^0(t,\mathbf{x}_1,\mathbf{E}), \qquad (16)$$

or the energy

$$\langle|\mathbf{E}^0|^2\rangle = \int d\mathbf{E}\,\mathbf{E}\cdot\langle\mathbf{E}^{0*}(t,\mathbf{x})\, f^0(t,\mathbf{x},\mathbf{E})\rangle, \qquad (17)$$

without involving the pair-distribution as was with the Bogoliubov method in (13) and (14).[5]

The macro-group $\mathbf{E}^0$ is a component of the decomposition

$$\tilde{\mathbf{E}} = \mathbf{E}^0 + \mathbf{E}'. \qquad (18a)$$

The micro-group

$$\mathbf{E}' = \mathbf{E}^1 + \mathbf{E}'' \qquad (18b)$$

is re-decomposed into a first-order group $\mathbf{E}^1$ and a subgroup

$$\mathbf{E}'' = \mathbf{E}^{(2)} + \mathbf{E}^{(3)} + \ldots \qquad (18c)$$

which comprises all the remaining higher-order groups. Likewise, we decompose the distribution into:

$$\hat{f} = \bar{f} + \tilde{f} \qquad (19a)$$

$$\tilde{f} = f^0 + f' \qquad (19b)$$

$$f' = f^1 + f'' \qquad (19c)$$

$$f'' = f^{(2)} + f^{(3)} + \ldots . \qquad (19d)$$

The operators

$$A^{0}\ ,\ A',\ A'' \qquad (20)$$

can be used for scaling into the macro-, micro-, and sub-groups, respectively.

By the nonlinearity of the master equation, f^{0} will be coupled to f', and subsequently to f'', in a sequence of equations of transport. Hence the present group-kinetic theory deals with the many groups

$$f^{0},\ f',\ f'' \qquad (21)$$

of the singlet-distribution function, while the Bogoliubov theory deals with the distribution functions of many points:[5]

$$\bar{f}(t,\underset{\sim}{x}_1,\underset{\sim}{E}) \qquad (22a)$$

$$\bar{f}_{12}(t,\underset{\sim}{x}_1,\underset{\sim}{E}_1;t,\underset{\sim}{x}_2,\underset{\sim}{E}_2) = \langle \tilde{f}(t,\underset{\sim}{x}_1,\underset{\sim}{E}_1)\tilde{f}(t,\underset{\sim}{x}_2,\underset{\sim}{E}_2)\rangle \qquad (22b)$$

$$\vdots$$

The three groups (21) represent the three transport processes of spectral evolution, eddy viscosity, and relaxation. The closure is not based on the decreasing magnitude of the groups but on their decreasing coherence with correlation times

$$\tau^{0}_{E} > \tau'_{E} > \tau''_{E}, \qquad (23)$$

such that the memory-loss in the relaxation makes the eddy coefficient (e.g. eddy diffusivity) approach its equilibrium and become adiabatic. This adiabaticity is not present in the high-order closure of distributions or of moments.

III. DERIVATION OF THE KINETIC EQUATION OF SOLITION TURBULENCE

By applying the scaling operators $\bar{A}$, A^{0}, A^{1}, $A^{(2)}$, we transform the master equations into the following equations of groups:

$$(\partial_t + \bar{A}\,\hat{L})\bar{f} = -\bar{A}\,L^{0} f^{0} \tag{24a}$$

$$(\partial_t + A^{0}\hat{L})f^{0} = -L^{0}\bar{f} - A^{0}L^{1} f^{1} \tag{24b}$$

$$(\partial_t + A^{1}\hat{L})f^{1} = -L^{1}(\bar{f} + f^{0}) - A^{1}L^{(2)}f^{(2)} . \tag{24c}$$

An operator (e.g. scaling operator, differential operator, collision operator, and evolution operator) applies to all functions which follow.

In the right hand side, we find the collisions from the fluctuations of the higher-order nearest-neighbor group:

$$-\bar{A}\,L^{0} f^{0} \cong -\bar{A}\,\tilde{L}\,\tilde{f} = \mathcal{C}\{\bar{f}\} \tag{25a}$$

$$-A^{0}L^{1}f^{1} \cong -A^{0}L'f' = \mathcal{C}'\{f^{0}\} \tag{25b}$$

$$-A^{1}L^{(2)}f^{(2)} \cong -A^{1}L''f'' = \mathcal{C}''f^{1}, \tag{25c}$$

as related to the adiabatic (non-fluctuating) collision coefficients

$$\mathcal{C}, \quad \mathcal{C}', \quad \mathcal{C}'' \tag{26}$$

to be determined by a transport theory in Section VI. This transforms (24a) and (24b) into the kinetic equations

$$(\partial_t + \bar{A}\hat{L})\bar{f} = -\mathcal{C}\{\bar{f}\} \tag{27}$$

$$(\partial_t + A^{0}\hat{L})f^{0} = -L^{0}\bar{f} + \mathcal{C}'\{f^{0}\} \tag{28}$$

for the probability function $\bar{f}$ and the macro-distribution f^{0}, respectively. In the following we shall deal with the kinetic equation (28) only. The governing collision coefficient and its relaxation are determined by the system of equations

$$(\partial_t + A^{1}\hat{L})f^{1} = -L^{1}(\bar{f} + f^{0}) + \mathcal{C}''f^{1} \tag{29a}$$

and

$$(\partial_t + A''\hat{L})f'' = - L''(\bar{f} + f^0 + f^1). \tag{29b}$$

The function within the brackets $\{\}$ is operated by the collision coefficient in the role of an operator for the propagation of memory.

IV. EQUATION OF SOLITON TURBULENCE

By taking the moment of the kinetic equation (28), we derive the following equation of soliton turbulence:

$$(\partial_t - i\,\nu_n \nabla^2 + i\tfrac{1}{2}\omega_n A^0 N^0)\underset{\sim}{E}^0 = -i\underset{\sim}{X}^0 + \underset{\sim}{J}^0, \tag{30}$$

with the stress

$$\underset{\sim}{J}^0 \equiv \int d\underset{\sim}{E}\,\underset{\sim}{E}\quad \mathcal{C}'\{f^0\}\ . \tag{31}$$

A subsequent multiplication by $\underset{\sim}{E}^{0*}$ and an average give the equation of spectral evolution

$$\tfrac{1}{2}\partial_t \langle|\underset{\sim}{E}^0|^2\rangle = W^0 - T^0 \tag{32}$$

in homogeneous and isotropic turbulence, where

$$W^0 = -\tfrac{1}{2}[i\langle\underset{\sim}{X}^0\cdot\underset{\sim}{E}^{0*}\rangle - i\langle\underset{\sim}{X}^0\cdot\underset{\sim}{E}^0\rangle] \tag{33}$$

is the coupling function, and

$$\begin{aligned} T^0 &= -\langle|\underset{\sim}{E}^{0*}\cdot\underset{\sim}{J}^0\rangle \\ &= -\int d\underset{\sim}{E}\,\underset{\sim}{E}\cdot\mathcal{C}'\{\langle|\underset{\sim}{E}^{0*}(t,\underset{\sim}{x})f^0(t-\tau)|\rangle\} \end{aligned} \tag{34}$$

is the transfer function. The Lagrangian correlation

$$\langle|\underset{\sim}{E}^{0*}(t,\underset{\sim}{x})f^0(t-\tau)|\rangle \tag{35}$$

contains a memory. Since $\mathcal{C}'$ is derived from

$$L' = L'_N + L'_X \tag{36}$$

with

$$L'_N = i\tfrac{1}{2}\omega_n N',\ L'_X = i\underset{\sim}{X}'\cdot\underset{\sim}{\partial}, \tag{37}$$

by (9), the collision coefficient can be written into the two components

$$\mathcal{C}'\{\} = \mathcal{C}'_N\{\} + \mathcal{C}'_X\{\} \quad , \tag{38}$$

as due to L'_N and L'_X, respectively. Similarly, we have

$$\mathcal{C}'' = \mathcal{C}''_N + \mathcal{C}''_X \quad . \tag{39}$$

From (31) and (34), it entails

$$\underset{\sim}{J}^{0} = \underset{\sim}{J}^{0}_N + \underset{\sim}{J}^{0}_X \tag{40}$$

$$T^{0} = T^{0}_N + T^{0}_X, \tag{41}$$

with

$$\underset{\sim}{J}^{0}_N = \int d\underset{\sim}{E}\ \underset{\sim}{E}\ \mathcal{C}'_N\{f^0\} \tag{42a}$$

$$\underset{\sim}{J}^{0}_X = \int d\underset{\sim}{E}\ \underset{\sim}{E}\ \mathcal{C}'_X\{f^0\} \quad . \tag{42b}$$

V. LOSS OF MEMORY IN RELAXATION FOR CLOSURE

The equation of evolution (29a) for f^1 is controlled by the collision

$$\mathcal{C}''\{f^1\} \quad , \tag{43}$$

and is transformed by moment into an equation of evolution for $\underset{\sim}{E}^1$ with the controlling stress

$$\underset{\sim}{J}^1 = \int d\underset{\sim}{E}\ \underset{\sim}{E}\ \mathcal{C}''(t,\underset{\sim}{x})\{f^1\} \tag{44}$$

in analogy with (31). An effective collision would be

$$\mathcal{C}''_{eff}(t,\underset{\sim}{x})\ f^1, \tag{45}$$

giving

$$\begin{aligned}\underset{\sim}{J}^1 &\cong \mathcal{C}''_{eff}(t,\underset{\sim}{x}) \int d\underset{\sim}{E}\ \underset{\sim}{E}\ f^1 \\ &= \mathcal{C}''_{eff}(t,\underset{\sim}{x})\ \underset{\sim}{E}^1(t,\underset{\sim}{x}) \ .\end{aligned} \tag{46}$$

The collision coefficient that operates on f^1 in (44) keeps a memory by its dependence on $\underset{\sim}{E}$. If we wish to lose the memory in the relaxation, it suffices to suppress the $\underset{\sim}{E}$-dependence, corresponding to the fluid simulation

$$\mathcal{C}''(t,\underset{\sim}{x},\underset{\sim}{E})\{\ \} \cong \mathcal{C}''(t,\underset{\sim}{x},\underset{\sim}{E}=0)$$
$$\equiv \mathcal{C}''_N(t,\underset{\sim}{x}) \quad , \tag{47}$$

since

$$L''(t,\underset{\sim}{x},\underset{\sim}{E} = 0) = L''_N(t,\underset{\sim}{x}) \tag{48}$$

by definitions (9a) - (9d). Hence we approximate (44) by

$$\underset{\sim}{J}^1 \cong \underset{\sim}{J}^1_N$$
$$= \mathcal{C}''_N(t,\underset{\sim}{x})\ \underset{\sim}{E}^1(t,\underset{\sim}{x}) \ . \tag{49}$$

Upon comparing (46) with (49), we identify

$$\mathcal{C}''_{eff}(t,\underset{\sim}{x}) \cong \mathcal{C}''_N(t,\underset{\sim}{x}). \tag{50}$$

A fluid simulation can be conceived if the diverse distributions in the subgroup (19d) undergo a large number of subcollisions, so that the subgroup as an ensemble behaves like a porous medium without individuality from the separate $\underset{\sim}{E}$-dependence. The porous medium offers a Darcy coefficient $\mathcal{C}''_N$.

With this approximation, we reduce (29a) and (29b) into

$$(\partial_t + A^1\hat{\mathcal{L}})f^1 = -L^1(\bar{f} + f^0) \tag{51}$$

and

$$(\partial_t + A''\hat{L}_N)f'' = -L''(\bar{f} + f^0 + f^1) \ . \tag{52}$$

The differential operators

$$\hat{\mathcal{L}} = \hat{L} - \mathcal{C}''_N \quad , \tag{53}$$

$\hat{L}_N$ and $\hat{L}$ govern the evolution operators $\mathcal{U}, U_N, U$, as follows:

$$(\partial_t + A^1\hat{\mathcal{L}})\,\mathcal{U}(t,t_1) = 0 \tag{54a}$$

$$(\partial_t + A''L_N)\,U_N(t,t_1) = 0 \tag{54b}$$

$$(\partial_t + A^0\hat{L})\,U(t,t_1) = 0 \ . \tag{54c}$$

We conclude that the sequence of transport equations (28), (51) and (52) for the distribution functions

$$f^{\circ},\ f^{1},\ f'' \tag{55}$$

forms a closed system.

VI. COLLISION COEFFICIENTS

A. Darcy damping $\mathcal{C}''_N$ and collision coefficient $\mathcal{C}'_N$.

The equation (52) for relaxation can be integrated into

$$f'' = -A''\int_0^t d\tau\ U_N(t,t-\tau)\quad L''_N(t-\tau)\ (\bar{f} + f^{\circ} + f^1)\ , \tag{56}$$

by means of the evolution operators (54b). Upon multiplying by L''_N and scaling by A^1, we find the flux in the form

$$-\ A^1 L''_N f'' = A^1\int_0^t d\tau\ L''_N(t,\underset{\sim}{x})\ A''\ U_N(t,t-\tau) L''_N(t-\tau)\ (f+f+f^1), \tag{57a}$$

or in the adiabatic form

$$\begin{aligned} -A^1 L''_N f'' &= \int_0^{t\to\infty} d\tau \langle L''_N(t,\underset{\sim}{x})\ A''\ \mathcal{U}_N(t,t-\tau)\ L''_N(t-\tau)\rangle\ f^1 \\ &= \mathcal{C}''_N\ f^1 \end{aligned} \tag{57b}$$

Hence we derive the adiabatic coefficients

$$\begin{aligned} \mathcal{C}''_N &\equiv \int_0^{\infty} d\tau\ \langle L''_N\ (t,\underset{\sim}{x})\ A''\ \ U_N(t,t-\tau)\ L''_N\ (t-\tau)\rangle \\ &= -\ (\tfrac{1}{2}\ \ \omega_n)^2\ D''_N \end{aligned} \tag{58}$$

and

$$D''_N\ = \int_0^{\infty} d\tau\ \langle N''(t,\underset{\sim}{x})\ A''\ \ U_N(t,t-\tau)\ U_N(t-\tau)\rangle. \tag{59}$$

B. Collision coefficient $\mathcal{C}'_N$

The procedure of calculating $\mathcal{C}'_N$ from (51) by restricting to L^1_N is analogous to that for $\mathcal{C}''_N$. We obtain:

$$\mathcal{C}'_N = \int_0^{t\to\infty} d\tau \, \langle L'_N(t,\underset{\sim}{x}) \, A' \, \mathcal{U}_N(t,t-\tau) \quad L'_N(t-\tau)\rangle$$
$$= -(\tfrac{1}{2}\omega_n)^2 \, D'_N \tag{60}$$
$$D'_N = \int_0^{\infty} d\tau \, \langle N'(t,\underset{\sim}{x}) \, A' \, \mathcal{U}_N(t,t-\tau) \, N'(t-\tau)\rangle. \tag{61}$$

C. Collision coefficient $\mathcal{C}'_X$.

The integration of (51) by restricting to L^1_X gives
$$f^1 = -\int_0^t d\tau \, A^1 \, \mathcal{U}(t,t-\tau) \, L^1_X \, (\bar{f} + f^0) \tag{62}$$
Upon multiplying by L^1_X and scaling, we get
$$-A^0 L^1_X f^1 = A^0\int_0^t d\tau \, L^1_X(t) \, A^1 \mathcal{U}(t,t-\tau) \, L^1_X \, (t-\tau)\{\bar{f} + f^0\}_{t-\tau}$$
$$\cong A^0\int_0^{t\to\infty} d\tau \Big\langle L'_X(t) \, A' \, \mathcal{U}(t,t-\tau) \, L'_X(t-\tau)\Big\rangle \{\bar{f} + f^0\}_{t-\tau}$$
$$= \mathcal{C}'_X\{f^0(t-\tau)\} . \tag{63}$$
The adiabatic coefficients are found as follows:
$$\mathcal{C}'_X\{\,\} = \int_0^{\infty} d\tau \, \langle L'_X(t) \quad A'\mathcal{U}(t,t-\tau) \quad L'_X(t-\tau)\rangle\{\,\}$$
$$= -\underset{\sim}{\partial} \cdot \underset{\approx}{D}'_X \cdot \underset{\sim}{\partial}\{\,\} \tag{64}$$
$$\underset{\approx}{D}'_X = \int_0^{\infty} d\tau \, \langle \underset{\sim}{X}'(t,\underset{\sim}{x}) \, A'\mathcal{U}(t,t-\tau) \quad \underset{\sim}{X}'(t-\tau)\rangle \quad . \tag{65}$$
The coefficients
$$\mathcal{C}'_X(t,\underset{\sim}{x},\underset{\sim}{E}) \quad \text{and} \quad \underset{\approx}{D}'_X(t,\underset{\sim}{x},\underset{\sim}{E}) \tag{66}$$
are operators and generate memory.

VII. TRANSPORT THEORY

A. Direct cascade T^0_N

From (34) and (42a), the direct cascade is written as
$$T^0_N = -\int d\underset{\sim}{E} \; E_i \; \mathcal{C}'_N \quad \langle | E_i{}^*(t,\underset{\sim}{x}) f^0(t-\tau)|\rangle$$
$$\cong \quad -\mathcal{C}'_N \, \langle|\underset{\sim}{E}^0|^2\rangle. \tag{67}$$

The collision coefficient

$$\mathcal{C}'_N < 0$$

is negative by (60), and is not an operator since it is independent of $\underset{\sim}{E}$. The direct cascade performs a transfer toward smaller scales.

B. Reverse cascade T^0_X

From (34) and (42b), we calculate the reverse cascade in the form

$$T^0_X = -\int d\underset{\sim}{E}\; E_i\; \mathcal{C}'_X \left\{ \langle | E_i^{0*}(t, x) f^0(t-\tau) | \rangle \right\}, \tag{68}$$

or

$$T^0_X = \int d\underset{\sim}{E}\; E_i\; \underset{\sim}{\partial} \cdot \underset{\approx}{D}'_X \cdot \underset{\sim}{\partial} \left\{ \varphi^0_i(\tau) \right\} \quad . \tag{69}$$

It contains two Lagrangian correlations: one is the auto-correlation

$$\langle \underset{\sim}{X}'(t,\underset{\sim}{x}) A' \mathcal{U}(t,t-\tau) \underset{\sim}{X}'(t-\tau) | \rangle \tag{70a}$$

of the micro-fluctuations $\underset{\sim}{X}'$ by definition (65), and the other is the cross-correlation

$$\varphi_i(\tau) \quad = \langle | E_i^{0*}(t,\underset{\sim}{x}) A^0 \mathcal{U}(t,t-\tau) f^0(t-\tau) | \rangle \tag{70b}$$

between $\underset{\sim}{E}^{0*}(t,\underset{\sim}{x})$ and $f^0(t-\tau)$.

By Fourier transformation, the correlation (70a) and the diffusivity (65) can be written in the form

$$\langle \underset{\sim}{X}'(t,\underset{\sim}{x})\; A' \mathcal{U}(t,t-\tau) \underset{\sim}{X}'(t-\tau) \rangle \cong \langle \underset{\sim}{X}'(t,\underset{\sim}{x})\; \bar{\mathcal{U}}(t,t-\tau) \underset{\sim}{X}'(t-\tau) \rangle$$

$$= \int d\underset{\sim}{k}''\;\; \chi \langle \underset{\sim}{X}'(\underset{\sim}{k}'')\;\; \underset{\sim}{X}'(-\underset{\sim}{k}'') \rangle\;\; h(\tau,\underset{\sim}{k}'',\underset{\sim}{E}) \tag{71}$$

$$\underset{\approx}{D}'_X = \int d\underset{\sim}{k}''\; \chi \langle \underset{\sim}{X}'(\underset{\sim}{k}'') \underset{\sim}{X}'(-\underset{\sim}{k}'') \rangle \int_0^\infty d\tau \quad h(\tau,\underset{\sim}{k}'',\underset{\sim}{E}), \tag{72}$$

by using the orbit function

$$h(\tau,\underset{\sim}{k}'',\underset{\sim}{E}) \;=\; \bar{\mathcal{U}}_k(t,t-\tau) \tag{73}$$

that is equivalent to the average evolution operator in the

Fourier form. The Fourier transformation is made with respect to position in a length interval 2L in three dimensions within which the function $\underset{\sim}{X}'$ is quasi-homogeneous, resulting in a factor of truncation

$$\chi \equiv (\pi/L)^3.$$

The transformation with respect to time is not necessary, because the temporal fluctuations are slow as compared with the spatial fluctuations in strong turbulence. The transformation with respect to both time and position would be necessary in weak turbulence to establish a resonance.

C. Subdynamics

Note that the Lagrangian function can be written in the form

$$\tilde{\mathcal{U}}(t,t-\tau)\underset{\sim}{X}'(t-\tau) = \underset{\sim}{X}'[t-\tau, \hat{\underset{\sim}{x}}(t-\tau)], \tag{74}$$

where $\hat{\underset{\sim}{x}}(t-\tau)$ satisfies the subdynamics of the detailed trajectory, as governed by

$$\ddot{\hat{\underset{\sim}{x}}}(t_1) = \hat{\underset{\sim}{E}}(t_1) \quad , \qquad \dot{\hat{\underset{\sim}{E}}}(t_1) = -i\hat{\underset{\sim}{X}}(t_1) \quad , \tag{75}$$

with the initial conditions at time t

$$\hat{\underset{\sim}{x}}(t) = \underset{\sim}{x}, \; \dot{\underset{\sim}{x}}(t)=0, \; \hat{\underset{\sim}{E}}(t) = \underset{\sim}{E}, \tag{76}$$

and with $t_1 = t-\tau$. By analysing the trajectory, we find that the orbit function

$$h(\tau,\underset{\sim}{k}'',\underset{\sim}{E}) = h_E(\tau,\underset{\sim}{k}'',\underset{\sim}{E})h_\nu(\tau,\underset{\sim}{k}'')h_D(\tau,\underset{\sim}{k}'')h_\lambda(\tau,\underset{\sim}{k}'') \tag{77}$$

has the following components

$$h_E(\tau, \underset{\sim}{k}'', \underset{\sim}{E}) = \exp(-\tfrac{1}{2}i\underset{\sim}{k}''\cdot \underset{\sim}{E}\,\tau^2) \tag{78a}$$

$$h_\nu(\tau, k'') = \exp\, ik''^2 \nu_n \tau \tag{78b}$$

$$h_D(\tau, k'') = \exp\, \mathcal{C}_N'' \tau \tag{78c}$$

$$h_\ell(\tau, k'') \equiv \langle \exp\, i\underset{\sim}{k}''.\, \tilde{\underset{\sim}{\ell}}(\tau)\rangle = \exp \mathcal{C}_N \tau . \tag{78d}$$

They refer to the streaming by $\underset{\sim}{E}$ for the memory propagation, the dispersion by ν_n, the Darcy damping $\mathcal{C}_N''$, and the Landau damping by path perturbations $\tilde{\underset{\sim}{\ell}}(\tau)$. The average in (78d) can be calculated by means of a probability of transition $p(\tau, \underset{\sim}{\ell})$ for the random path $\underset{\sim}{\ell}$ travelled during a time interval τ. The probability function satisfies an equation of transition with a damping rate $\mathcal{C}_N$.

By definition (65) or (72), the diffusivity is energized by $\langle \underset{\sim}{X}'\underset{\sim}{X}'\rangle$ for a duration of correlation that is determined by the evolution operator in the detailed fluctuating form or its averaged approximate form, through which the memory can propagate and be stopped for the closure and for the adiabaticity of the diffusivity. With this regard, we have represented the transmission of memory through $\underset{\sim}{E}$ by the orbit component h_E. Other factors (the fluctuating $\mathcal{U}$, the $\underset{\sim}{E}$-dependent probability of transition) may cause the memory to continue further, but are suppressed for closure.

D. Reverse cascade by negative viscosity

In analogy with the auto-correlation (79a), we write the cross-correlation (70b) in the form:

$$\varphi_i^o(\tau) = \int dk'\ \chi \langle | E_i^o\ (\underset{\sim}{k}')f^o(-\underset{\sim}{k}', \underset{\sim}{E})|\rangle\, h(\tau, \underset{\sim}{k}', \underset{\sim}{E}). \tag{80}$$

The orbit function $h(\tau, k', E)$ has the same structure as (77).

We collect the results (72) and (80) and transform (69) into

$$T_X^0 = \int_0^\infty d\tau \int d\underset{\sim}{E}\; E_i \int d\underset{\sim}{k}'' \;\; \chi \left\langle X_r'(\underset{\sim}{k}'')X_s'(-\underset{\sim}{k}'')\right\rangle \int d\underset{\sim}{k}' H_{rsi}^0(\tau,\underset{\sim}{k}'',\underset{\sim}{k}',\underset{\sim}{E}), \quad (81)$$

with

$$H_{rsi}^0(\tau,\underset{\sim}{k}',\underset{\sim}{k}'',\underset{\sim}{E}) \equiv \partial_r \;[h(\tau,\underset{\sim}{k}'',\underset{\sim}{E})\partial_s \chi\langle|E_i^0\;(k')\;f^0(-\underset{\sim}{k}',\underset{\sim}{E})|\rangle\, h(\tau,\underset{\sim}{k}',\underset{\sim}{E})]. \quad (82)$$

The two orbit functions

$$h(\tau,\underset{\sim}{k}'',\underset{\sim}{E}) \text{ and } h(\tau,\underset{\sim}{k}',\underset{\sim}{E})$$

govern the small and large scales, respectively. They compete for the role of memory propagation by $\underset{\sim}{E}$. We attribute this role to the large-scale orbit function , by writing

$$\underset{\sim}{\partial}\underset{\sim}{\partial}\, h(\tau,\underset{\sim}{k}',\underset{\sim}{E}) = -\tfrac{1}{4}\;\underset{\sim}{k}'\underset{\sim}{k}'\,\tau^4 \quad h(\tau,\underset{\sim}{k}',\underset{\sim}{E})$$
$$\cong -\tfrac{1}{4}\;k'k'\,\tau^4 \quad (83a)$$

$$h(\tau,\underset{\sim}{k}'',\underset{\sim}{E}) \;\cong\; h(\tau,\underset{\sim}{k}'',\underset{\sim}{E} = 0). \quad (83b)$$

When the memory propagation is terminated, other roles (dispersion and dampings) are taken over by $h(\tau,\;\underset{\sim}{k}'',\;\underset{\sim}{E} = 0)$ in view of the condition $k'<<k''$. These considerations simplify (82) into

$$H_{rsi}^0(\tau,\underset{\sim}{k}',\underset{\sim}{k}'',\underset{\sim}{E}) = -h(\tau,\underset{\sim}{k}'',\underset{\sim}{E}=0)\tfrac{1}{4}k_r'k_s'\,\tau^4\;\chi\langle|E_i^0(\underset{\sim}{k}')\;f^0(-\underset{\sim}{k}',\underset{\sim}{E})|\rangle, \quad (84)$$

and subsequently (81) into

$$\begin{aligned} T_X^0 &= -\int d\underset{\sim}{k}''\;\chi\langle X_r'(\underset{\sim}{k}'')\quad X_s'(-\underset{\sim}{k}'')\rangle \\ &\times \int d\underset{\sim}{k}' k_r'\;k_s'\;\tfrac{1}{4}\int_0^\infty d\tau\,\tau^4\;\; h(\tau,\underset{\sim}{k}'',\underset{\sim}{E}=0) \\ &\times \int d\underset{\sim}{E}\quad E_i\,\chi\langle|E_i^0\,(\underset{\sim}{k}')\;f^0(-\underset{\sim}{k}',\underset{\sim}{E})|\rangle. \end{aligned} \quad (85)$$

After integration with respect to E and rearrangement, we obtain

$$\begin{aligned} T_X^0 &= -\text{trace}\int d\underset{\sim}{k}''\;\chi\langle \underset{\sim}{X}'(\underset{\sim}{k}'')\underset{\sim}{X}'(-\underset{\sim}{k}'')\rangle[\,\tau_X(k'')]^5\;R_E^0 \\ &= -\;K'R_E^0\;. \end{aligned}$$

Here the vorticity function is

$$R_E^0 = \int dk'k'^2\;\;\chi\langle|\underset{\sim}{E}^0(\underset{\sim}{k}')|^2\rangle$$

$$= 2\int_0^k dk'k'^2 F_E(k'), \qquad (87)$$

the relaxation time is $\tau_X(k'')$, such that

$$[\tau_X(k'')]^5 = \text{real}\,\tfrac{1}{4}\int_0^\infty d\tau\ \tau^4\ h(\tau, \underset{\sim}{k}'', \underset{\sim}{E}=0), \qquad (88)$$

and the eddy viscosity is

$$K' = \text{trace}\int d\underset{\sim}{k}''\ \chi < \underset{\sim}{X}'(\underset{\sim}{k}'')\ \underset{\sim}{X}'(-\underset{\sim}{k}'')>[\tau_X(k'')]^5$$

$$= \frac{2}{3}\int_k^\infty dk''\quad F_X(k'')\ [\tau_X(k'')]^5. \qquad (89)$$

The spectral functions are $F_E(k)$ and $F_X(k)$, such that

$$<|\underset{\sim}{E}^0|^2> = 2\int_0^k dk' F_E(k') \qquad (90)$$

$$<\underset{\sim}{X}^{0\,2}> = 2\int_0^k dk'' F_X\ (k''). \qquad (91)$$

The transfer at a negative transport coefficient -K' indicates a reverse cascade.

The relaxation time $\tau_X(k'')$, as defined by (88), is calculated by (78c) and (78d), as follows:

$$[\tau_X(k'')]^5 = \tfrac{1}{4}\int_0^\infty d\tau\ \tau^4\ \exp(-\mathcal{E}_N'' - \mathcal{E}_N)$$

$$= \quad 6(-\mathcal{E}_N'' - \mathcal{E}_N)^{-5}. \qquad (92)$$

Upon substituting (92) into (89), we obtain the eddy viscosity

$$K' = 4\int_k^\infty dk''\ F_X(k'')\ (-\mathcal{E}_N'' - \mathcal{E}_N)^{-5}. \qquad (93)$$

E. Coupling function

The coupling function, as defined by (33), can be interpreted as the flux of transport of $\underset{\sim}{E}^0$ by $\underset{\sim}{X}^0$ - fluctuation. Upon integrating (30), we write

$$\underset{\sim}{E}^0 = -i\int_0^t d\tau\ \ A^0\,\mathcal{U}(t,t-\tau)\ \underset{\sim}{X}^0(t-\tau). \qquad (94)$$

Upon multiplying by $i\underset{\sim}{X}^0$ and averaging, we determine the coupling function

$$W^{o} = 3D_X^{o} \; ,$$

$$= \int d\underset{\sim}{k}'' \, \chi \, \langle \underset{\sim}{X}^{o}(\underset{\sim}{k}') \cdot \underset{\sim}{X}^{o}(-\underset{\sim}{k}') \rangle \int_0^{\infty} d\tau \; h(\tau, \underset{\sim}{k}', \underset{\sim}{E} = 0), \tag{95}$$

with

$$D_X^{o} = \text{trace} \; \underset{\approx}{D}_X^{o} \; , \tag{96}$$

by (72). The evolution operator $\mathcal{U}$ absorbs the damping effect of $\underset{\sim}{J}_N^{o}$ from $\underset{\sim}{J}^{o}$ in (30).

The orbit function that belongs to the trajectory of $\underset{\sim}{E}^{o}$ is

$$h(\tau, \underset{\sim}{k}', \underset{\sim}{E} = 0) = \exp\,(ik'^2\nu_n + \mathcal{E}_N' + \mathcal{E}_N)\tau \tag{97}$$

by (77). A integration and a substitution into (95) yield

$$W^{o} = 2\int_0^{k} dk' \; F_X(k') \; (-\mathcal{E}_N - \mathcal{E}_N')^{-1} \; \psi_W(k'/k_\nu). \quad (98) \tag{98}$$

The function

$$\psi_W = \text{Real}\; [1 - i(k'/k_\nu)^2]^{-1}$$

$$= [1 + (k'/k_\nu')^4]^{-1} \tag{99}$$

regulates the dispersion at the cutoff wavenumber

$$k_\nu = [(-\mathcal{E}_N'' - \mathcal{E}_N)/\nu_n]^{\frac{1}{2}}. \tag{100}$$

VIII. MECHANISMS OF SPECTRAL EVOLUTION

The evolution of the spectrum is governed by the transfer function T_X^{o} for reverse cascade, the coupling function W^{o}, and the transfer function T_N^{o} for direct cascade. Two types of nonlinearity enter: the modulation by $\tilde{N}$ and the emission by $\underset{\sim}{\tilde{X}}$. The spectral flow, as described by (32), involves cascade coupling, source and drain.

A. Modulation and emission

It is to be recalled that the soliton formalism was derived by a fluid analogue from the Navier-Stokes equation for compressible fluid, by considering a density fluctuation of two scales. The slow density wave is related to modulation in the detailed form (2), or in the form of intensities

$$\langle N'^2 \rangle = \alpha^2 [\langle |\underset{\sim}{E}'|^2 \rangle]^2. \tag{101}$$

This relation is called the equation of state. The fast density wave obeys an equation of wave propagation with the emission caused by velocity fluctuations. By an envelope transformation, the wave equation is transformed into the parabolic equation (1), called the Schrödinger equation for driven solitons, with the driving force $\tilde{\underset{\sim}{X}}$, such that [6,7]

$$\underset{\sim}{\nabla} \cdot \tilde{\underset{\sim}{X}} \sim \tilde{A} \; \underset{\sim}{\nabla}\underset{\sim}{\nabla} : \tilde{\underset{\sim}{u}}\tilde{\underset{\sim}{u}} \tag{102}$$

The propagation of finite sound wave and the excitation of acoustic turbulence are based upon the sawtooth shock wave formation having the time τ_s as an invariant. This time is obtained by the effective slope in the propagation of a fluctuating velocity in the $\underset{\sim}{x}$- space. With this invariant as a parameter and by dimensional considerations, it is seen the velocity fluctuations follow the spectral law

$$F_u(k) \sim k^{-3} \tag{103}$$

It follows, from (102), the spectral intensity

$$\langle \underset{\sim}{X}'^2 \rangle = \tau_s^{-6} k^{-2} \tag{104}$$

for the driving force. For the determination of τ_s, see Ref. 6.

B. Instability as a source

At the larger-scale end of the spectrum, the reverse cascade accumulates the energy from the micro-fluctuation into the macro-fluctuations, to be ultimately balanced by the instability of soliton waves as a source in the form

$$\varepsilon_E = \frac{1}{2}\,\partial_t \langle|\tilde{\underset{\sim}{E}}|^2\rangle, \tag{105}$$

so that the governing equation is

$$\begin{aligned} \varepsilon_E &= -\,\overset{o}{T}_X \\ &= K'\overset{o}{R}_E \text{ in inertial subrange,} \end{aligned} \tag{106}$$

from (86).

At present, we shall not examine the spectral composition of the source; however, it would be required for analyzing the wave-turbulence interaction in the non-universal range of the spectrum.

C. Fluctuation-dissipation as a drain

A nonlinear random system can be dissipative. The dissipation can be prescribed by an external parameter, such as the molecular viscosity. A "fluctuation-dissipation" can be induced by density fluctuations and serves as a drain in the spectral flow. Our soliton formalism finds a drain from the fluctuation-dissipation.

The transfer function $\overset{o}{T}_N$ would play the role of a direct cascade in classical fluid turbulence, transferring energy from $\underset{\sim}{E}^{\circ}$ into $\mathcal{E}'_N$ and finally down to the molecular dissipation. With the absence of molecular dissipation in soliton turbulence, the transfer function becomes a drain itself, by writing

$$\mathcal{E}'_N \cong \mathcal{E}_N, \tag{107}$$

to balance the larger-scale coupling function $\overset{\circ}{W}$. The governing equation for this coupling subrange is

$$\overset{o}{W} - \overset{o}{T}_N = 0, \tag{108}$$

from (32). The approximation of the scaled collision coefficient into an unscaled drain simplifies (67) into

$$T_N^0 \cong -\mathcal{C}_N \langle |\underset{\sim}{E}^0|^2 \rangle \qquad (109)$$

and (98) into

$$W^0 = (-2\mathcal{C}_N)^{-1}\ 2\int_0^k dk'\, F_X(k')\ \Psi_W(k'/k_\nu\ ). \qquad (110)$$

Here Ψ_W is defined by (99), with

$$k_\nu \equiv [(-2\mathcal{C}_N)/\nu_n]^{\frac{1}{2}}, \qquad (111)$$

from (100).

It is to be remarked that the structure of the collision coefficient $\mathcal{C}_N$ can be calculated from (60) and (61) for the determination of the probability function $\overline{f}(t,\underset{\sim}{x},\underset{\sim}{E})$ by (27). Since now $\mathcal{C}_N$ serves as a parameter, its explicit structure is not needed.

D. Spectral flow

By the use of (107), we reduce the equation of evolution (32) into

$$\begin{aligned} \varepsilon_E &= -T_X^0 + W^0 - T_N^0 \\ &= 2(-2\mathcal{C}_N)^5 \langle \underset{\sim}{X}'^2 \rangle\ 2\int_0^k dk'\, k'^2\ F_E(k') \\ &\quad + (2\mathcal{C}_N)^{-1}\ 2\int_0^k dk'\ F_X(k')\ \Psi_W(k'/k_\nu) \\ &\quad + \mathcal{C}_N\ \langle |\underset{\sim}{E}^0|^2 \rangle, \text{ with } \mathcal{C}_N < 0. \end{aligned} \qquad (112)$$

This integral equation will be solved in Section IX.

IX. SPECTRAL STRUCTURE

A. Inertia subrange

The inertia subrange is governed by the constant reverse cascade as described by the following integral equation:

$$\varepsilon_E = 2(-2\mathcal{T}_N)^{-5} \langle \underset{\sim}{\chi}'^2 \rangle \, 2\int_0^k dk' k'^2 \, F_E(k'), \tag{113}$$

from (112). The solution for $F_E(k)$ is found upon dividing by $\langle \underset{\sim}{\chi}'^2 \rangle$ and differentiating with respect to k, in the form:

$$F_E(k) = \tfrac{1}{2}(-2\mathcal{T}_N \tau_s)^5 \tau_s \varepsilon_E k^{-1}. \tag{114}$$

Use of (104) has been made. It follows from (101) and (114) the density spectrum:

$$F_N(k) = \alpha^2 \langle |\underset{\sim}{E}'|^2 \rangle (-2\mathcal{T}_N \tau_s)^5 \tau_s \varepsilon_E \, k^{-1}. \tag{115}$$

We can introduce the notations

$$\omega_E = (-2\mathcal{T}_N \tau_s^3)^{-\frac{1}{2}} \tag{116}$$

$$k_\varepsilon = [(-2\mathcal{T}_N \tau_s)^7 \tau_s^5 \, \varepsilon_E]^{-\frac{1}{2}}, \tag{117}$$

to scale the magnitude of the driving force and the source ε_E. In these notations, (114) and (115) become

$$F_E(k) = \tfrac{1}{2}\omega_E^4 \, k_\varepsilon^{-2} k^{-1} \tag{118}$$

$$F_N(k) = \alpha^2 \langle |\underset{\sim}{E}'|^2 \rangle \, \omega_E^4 \, k_\varepsilon^{-2} k^{-1}. \tag{119}$$

An integration with respect to k yields the intensity of field fluctuations

$$\langle |\underset{\sim}{E}'|^2 \rangle = \tfrac{1}{2} \omega_E^4 k_\varepsilon^{-2} \, \ell n(k/k_\varepsilon), \tag{120}$$

and $\langle N'^2)$ is given by (101).

B. Coupling subrange $(k/k_\nu \ll 1)$

The coupling subrange follows the inertia subrange at larger wavenumbers. From (112), we write the spectral flow in the form:

$$(-2\mathcal{T}_N)^{-1} 2\int_0^k dk' \, F_X(k') \, \psi_W(k'/k_\nu) + \mathcal{T}_N \, 2\int_0^k dk' \, F_E(k') = 0, \tag{121}$$

The approximation

$$\psi_W(k/k_\nu) \cong 1, \text{ for } k/k_\nu \ll 1 \tag{122}$$

will be made. Upon differentiating with respect to k and by the use of (104), we solve for $F_E(k)$ in the form

$$F_E(k) = 4\omega_E^4\, k^{-3} \quad , \tag{123a}$$

or

$$\langle|\underset{\sim}{E}'|^2\rangle = 4\,\omega_E^4\, k^{-2} \quad , \tag{123b}$$

where ω_E is given by (116).

From (101) and (123b), we derive the density spectrum

$$\langle N'^2\rangle = (4\alpha\, \omega_E^4)^2\, k^{-4} \tag{124a}$$

and the spectral intensity

$$F_N(k) = (8\alpha\omega_E^4)^2\, k^{-5} \tag{124b}$$

C. Dispersion subrange ($k/k_\nu >> 1$)

The function for the dispersive cutoff is given by (99) in the approximate form:

$$\psi_W(k'/k_\nu) \cong (k'/k_\nu)^4, \text{ for } k'/k_\nu >> 1 \ . \tag{125}$$

Upon substituting for (125), we solve (121) to find the spectra

$$F_E(k) = 2\,(k_\nu\omega_E)^4\, k^{-7} \tag{126a}$$

$$F_N(k) = \frac{4}{3}\,[\alpha\,(k_\nu\,\omega_E)^4]^2\, k^{-13} \tag{126b}$$

and the intensities

$$\left\langle|\underset{\sim}{E}'|^2\right\rangle = \frac{1}{3}\,(k_\nu\omega_E)^4\, k^{-6} \tag{127a}$$

$$\left\langle N'^2\right\rangle = \left[\frac{1}{3}\alpha\,(k_\nu\omega_E)^4\right]^2 \quad k^{-12} \ . \tag{127b}$$

All the spectral results indicate that the spectra are increased by the driving force through the parameter ω_E and by the instability as a source through the length parameter k_{ϵ}^{-1}.

X. DISCUSSION ON THE CRITERIA OF CHAOS

A nonlinear dynamical stochastic system can be represented by the equation

$$(\partial_t + L_{\nu} + \tilde{A}\hat{L}_N)\tilde{E}(t,\underset{\sim}{x}) = \tilde{Y}(t,\underset{\sim}{x}). \tag{128}$$

Here $\tilde{E}(t,\underset{\sim}{x})$ is the unknown random function without average, i.e. (6). The advection consists of a non-random linear part L_{ν} and a random part L_N that causes nonlinearity by being related to $\tilde{E}(t,\underset{\sim}{x})$ or being another random function. The driving force $\tilde{Y}(t,\underset{\sim}{x})$ may be of an external origin or be self-consistent with $\tilde{E}(t,\underset{\sim}{x})$. All the functions may be real or imaginary, scalar or vector, on one or more dimensions. Many stochastic problems in hydrodynamics and optics can be described by this stochastic system. For solitons, the nonlinear Schrödinger equation (1) is identical to (128), by writing

$$\tilde{Y} = -i\tilde{X} \tag{129}$$

and by using (9b) and (9c) as differential operators.

The Duffing oscillator

$$m\ddot{x} + (\sigma - \beta x^2)x = Y(t), \tag{130}$$

with constant coefficients m, σ, and β, is reducible to the cubic equation

$$(i\partial_t - \gamma - \tfrac{1}{2} b\tilde{A}\tilde{N})\tilde{E} = \tilde{X} \tag{131}$$

by an envelope transformation, where $\tilde{N}$ is given by (2), and γ, b are constant coefficients. The Korteweg-de Vries equation

$$(\partial_t + u\frac{\partial}{\partial x} + \lambda\frac{\partial^3}{\partial x^3} + \nu)u = 0 \tag{132}$$

can be considered as an alternative to the Schrödinger equation for long waves. Here λ and γ are constants.

A large amount of numerical effort has been devoted to calculating the soliton models, and found that the simple ones that are analytically integrable cannot develop chaos. This has lead to the belief that an integrable nonlinear equation cannot develop chaos, as exemplified by the Schrödinger equation, the Korteweg-de Vries equation, and the Burgers equation in their restricted form. This conclusion seems to be direct, and calls for the need of a causal relation between chaos and existence of solution, or for a statistical reason why certain dynamical properties are necessary to produce cascades and maintain a broad spectrum.

The statistical theory as developed here may clarify to some extent this problem. As the starting point, we consider the dynamical equation (128) to describe the microdynamic state of turbulence. The equation has the difficulty of being non-homogeneous, with the consequence that the driving force will cause a hierarchy of high-order moments and is hindered to participate in the transport processes as does the nonlinear operator. To lift this hindrance, we use a kinetic method by raising (128) to a high dimension, and transform it into the master equation (8). The nonlinear operator then becomes

$$\hat{L} = L_{\nu} + \hat{L}_N + \tilde{L}_Y, \tag{133}$$

with

$$\tilde{L}_Y \equiv \underset{\sim}{\tilde{Y}} \cdot \underset{\sim}{\partial} . \tag{134}$$

The group-kinetic method derives the kinetic equation of turbulence and establishes the spectral flow in the form (112). The governing transport functions are the coupling functions W^{0}, the transfer function T^{0}_{X} for the reverse cascade, the transfer function T^{0}_{N} for the direct cascade, the source ε_{E} and the sink ε_{ν}. The notation (129) for the solitons is followed. The solitons gain energy from the pumping $\underset{\sim}{X}^{0}$ into $\underset{\sim}{E}^{0}$ by an amount W^{0}. The energy deposited is accumulated toward the small wavenumber end of the spectrum by the reverse cascade T^{0}_{X} and is transferred toward high wavenumbers across the spectrum by the direct cascade T^{0}_{N}. At this end of the spectrum, the transfer function

$$T^{0}_{N} \equiv - \mathcal{C}'_{N}|\underset{\sim}{E}^{0}|^{2} \cong - \mathcal{C}_{N}\langle|\underset{\sim}{E}^{0}|^{2}\rangle \tag{135}$$

serves as a drain in such a way that a collision coefficient is found and plays the role of a "dissipation" induced by turbulence, called "fluctuation-dissipation." The energy balance between the reverse cascade and the instability on the one hand, and between the coupling and the drain on the other hand, even when the sink ε_{ν} is absent, determines the two subranges and provides a broad spectrum of turbulence.

If the soliton dynamics is described by the homogeneous Schrödinger equation, i.e. $\underset{\sim}{\tilde{X}} = 0$, the spectral flow is governed by the direct cascade

$$\partial_{t}\langle|\underset{\sim}{E}^{0}|^{2}\rangle = - T^{0}_{N} \quad , \text{ with } \quad T^{0}_{N} > 0 \quad . \tag{136}$$

The cascade can not find a dissipation to complete the transfer, except in a decaying system, i.e.

$$\partial_t \langle |\underset{\sim}{E}'|^2 \rangle > 0.$$

The statistical considerations above help to determine the criteria of chaos.

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16. ABSTRACT

The two phases are governed by two coupled systems of Navier-Stokes equations. The couplings are nonlinear. These equations describe the microdynamical state of turbulence, and are transformed into a master equation. By scaling, a kinetic hierarchy is generated in the form of groups, representing the spectral evolution, the diffusivity and the relaxation. The loss of memory in formulating the relaxation yields the closure. The network of sub-distributions that participates in the relaxation is simulated by a self-consistent porous medium, so that the average effect on the diffusivity is to make it approach equilibrium. The kinetic equation of turbulence is derived. The method of moments reverts it to the continuum. The equation of spectral evolution is obtained and the transport properties are calculated. In inertia turbulence, the Kolmogoroff law for weak coupling and the spectrum for the strong coupling are found.

As the fluid analogue, the nonlinear Schrodinger equation has a driving force in the form of emission of solitons by velocity fluctuations, and is used to describe the microdynamical state of turbulence. In order for the emission together with the modulation to participate in the transport processes, the non-homogeneous Schrodinger equation is transformed into a homogeneous master equation. By group-scaling, the master equation is decomposed into a system of transport equations, replacing the Bozoliubov system of equations of many-particle distributions. It is in the relaxation that the memory is lost when the ensemble of higher-order distributions is simulated by an effective porous medium. The closure is thus found. The kinetic equation is derived and transformed into the equation of spectral flow.

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