NASA Contractor Report 178193

ICASE REPORT NO. 86-64

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Pravir Dutt

Shlomo Ta'asan

(NASA-CR-178193) A SPLINE-BASED PARAMETER ESTIMATION TECHNIQUE FOR STATIC MODELS OF ELASTIC STRUCTURES Final Report (NASA) 34 p CSCL 72B
N87-14057
Unclas
G3/64 43947

Contract Nos. NAS1-17070, NAS1-18107

October 1986

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

A SPLINE-BASED PARAMETER ESTIMATION TECHNIQUE FOR STATIC MODELS OF ELASTIC STRUCTURES

P. Dutt and S. Ta´asan
Institute for Computer Applications in Science and Engineering

ABSTRACT

We consider the problem of identifying the spatially varying coefficient of elasticity using an observed solution to the forward problem. Under appropriate conditions this problem can be treated as a first order hyperbolic equation in the unknown coefficient. We develop some continuous dependence results for this problem and propose a spline-based technique for approximating the unknown coefficient, based on these results. We establish the convergence of our numerical scheme and obtain error estimates.

This research was supported under the National Aeronautics and Space Administration under NASA Contract Nos. NAS1-17070 and NAS1-18107 while the authors were in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

1. INTRODUCTION

A class of control and identification problems for which the models are based on the equations for elastic structures are those dealing with large space antennas. Mathematical models of these problems are based on the partial differential equation

$$\frac{\partial}{\partial x}\left(e\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(e\,\frac{\partial u}{\partial y}\right) = f \tag{1.1}$$

where $u(x,y)$ is the vertical displacement of the antenna surface, $f(x,y)$ is the distributed loading force per unit area, and $e(x,y)$ is the distributed coefficient of elasticiy of the antenna surface [1].

The identification of e using measured u and f values for the antenna surface is an important inverse problem. A common identification strategy is the "indirect" one in which one minimizes via an iterative process the deviation between a computed forward solution u_e and the observations (see, for example [1]). Alternatively, e can sometimes be identified by a direct approach involving approximate solution of the hyperbolic equation

$$\nabla e \cdot \nabla u + e\Delta u = f, \tag{1.2}$$

for example, by seeking the finite dimensional representation for e which minimizes the residuals of a difference approximation for equation (1.2). This is referred to as the "equation-error" method.

A practical limitation to the direct approach for identifying e is that the coefficients of the hyperbolic problem (1.2) involve derivatives of the measured quantity u. However, when it is feasible it is simpler and cheaper than the indirect approach.

In [2] Richter presented a systematic analysis of the inverse problem

$$\nabla e \bullet \nabla u + e\Delta u = f, \qquad x \ \Omega \ \mathbb{R}^n, \tag{1.3}$$

in which the coefficient e is to be determined on the basis of an observed (f,u) pair. He showed that the hyperbolic problem (1.3) has a unique solution assuming prescribed values along the inflow portion of $\partial\Omega$ for $f \in L^\infty(\Omega)$, provided

$$\inf_{P\in\Omega}[\max\{|\nabla u(P)|^2, \Delta u(P)\}] > 0. \tag{1.4}$$

He also proved that if condition (1.4) holds, then e depends continuously on f in $L^\infty(\Omega)$.

In this paper, we show that if condition (1.4) holds then e depends continuously on f in $L^p(\Omega)$ for all $p \ [1,\infty)$. We then use this continuous dependence result in $L^2(\Omega)$ to propose a spline-based technique for approximating the unknown coefficient e in equation (1.3). We prove that our scheme converges to the actual solution e of (1.3) and obtain error estimates.

In [2] Richter proposed especially favorable "test conditions" for observing a forward solution u to the elliptic problem for (1.3):

$$\inf_{\Omega} f > 0, \ u = 0 \ \text{ on } \ \partial\Omega. \tag{1.5}$$

Under these conditions (1.4) will be satisfied and the hyperbolic problem (1.3) with the resulting (f,u) pair will require no Cauchy data for e

because the characteristics of e will originate at points of degeneracy within Ω, rather than on $\partial\Omega$.

Typically, condition (1.5) holds for antenna problems. Thus, the numerical algorithm we propose in this paper, and which is based on a continuous dependence result in $L^2(\Omega)$, would be particularly suitable for antenna problems. In a companion paper [4], we present a multigrid algorithm for approximating e numerically.

2. CONTINUOUS DEPENDENCE OF THE HYPERBOLIC PROBLEM

Let $B(u)e$ be the operator

$$B(u)e = \nabla \cdot (e \nabla u) = f, \qquad x \; \Omega \subseteq \mathbb{R}^n, \tag{2.1}$$

where e and u are defined in a connected, bounded domain $\Omega \subseteq \mathbb{R}^n$. Throughout this paper we shall assume that $u \in C^2(\bar{\Omega})$ and $e \in C_p^1(\bar{\Omega})$, where $C_p^1(\bar{\Omega})$ denotes the class of piecewise continuously differentiable functions in $\bar{\Omega}$.

We shall denote the boundary of Ω by γ. Let

$$\gamma_1 = \{x \; \gamma : \frac{\partial u}{\partial n} < 0\},$$

where $\frac{\partial u}{\partial n}$ is the outward normal derivative of u along γ. γ_1 is the inflow portion of the boundary γ.

Suppose that u satisfies the condition

(2.2) $$\inf_{P \;\Omega}[\max\{|\nabla u(P|^2, \Delta u(P)\}] = \alpha > 0,$$

where $|\nabla u(P)|^2 = \sum_{i=1,n} (\frac{\partial u}{\partial x_i})^2$. Then $\bar{\Omega}$ can be divided into compact subregions Ω_1 and Ω_2 such that

$$|\nabla u|^2 \geq \alpha \quad \text{in} \quad \Omega_1, \qquad \Delta u \geq \alpha \quad \text{in} \quad \Omega_2,$$

and $\bar{\Omega} = \Omega_1 \; \Omega_2$.

We introduce some notation we shall need later. Let

$$u_{max} = \max_{P \; \bar{\Omega}} u(P),$$

$$u_{min} = \min_{P \; \bar{\Omega}} u(P),$$

$$[u] = u_{max} - u_{min},$$

$$s = \max_{P \in \gamma} |\frac{\partial u}{\partial n} (P)| \; ,$$

$$\beta = \max_{P \in \Omega_1} \{\frac{-\Delta u(P)}{|\nabla u(P)|^2}\}.$$

We now obtain an a priori bound on the stability of the hyperbolic problem (2.1).

Theorem 1: *Suppose* u *satisfies condition (2.2). Then for any* f *for which* $B(u)e = f$ *has a solution* e *assuming prescribed values along* γ_1 *the solution is unique, and*

$$(2.3) \qquad \|e\|_p \le A_p(u)\|f\|_p + D_p(u)\Big(\int_{\gamma_1} |e|^p ds\Big)^{1/p}$$

for all $p \in [1,\infty)$. *Here*

$$A_p(u) = \frac{1}{\alpha}\Big(\frac{1}{p}\Big)^{1/p}\Big(\frac{p-1}{p}\Big)^{\left(\frac{p-1}{p}\right)} \frac{2}{C_p(u)}, \quad \textit{and}$$

$$D_p(u) = \Big(\frac{2s}{\alpha C_p(u)}\Big)^{1/p}.$$

$C_p(u)$ *is defined as*

$$C_p(u) = \min\Big\{\frac{\beta}{p}, 1 - \frac{1}{p}\Big\} \exp(-p\beta[u]) \quad \textit{if} \quad \beta > 0,$$

$$C_p(u) = \min\Big\{\frac{1}{p}, 1 - \frac{1}{p}\Big\} \exp(-[u]) \quad \textit{if} \quad \beta \le 0.$$

Proof: Let $g(u)$ be a smooth function of u, which we shall specify later. Multiplying equation (2.1) by $g(u)|e|^{p-1}\mathrm{sgn}(e)$ and integrating over Ω, we obtain

$$(2.4) \qquad \int_\Omega g(u)|e|^{p-1}\mathrm{sgn}(e) f \, dx = \int_\Omega g(u)|e|^{p-1}\mathrm{sgn}(e) \, (\nabla\cdot(e\nabla u))dx.$$

Here sgn(e) is the function defined as

$$\mathrm{sgn}(e) = \begin{cases} -1, & e < 0, \\ 0, & e = 0, \\ 1, & e > 0. \end{cases}$$

Integrating the r.h.s. of (2.4) by parts gives

$$\int_\Omega g(u)|e|^{p-1}\mathrm{sgn}(e)(\nabla\cdot(e\nabla u))dx = -\int_\Omega \frac{dg}{du}\,|\nabla u|^2|e|^p dx$$

(2.5)

$$-\int_\Omega (p-1)(g(u)\nabla u)\cdot(|e|^{p-1}\mathrm{sgn}(e)\nabla e)dx + \int_\gamma g(u)|e|^p \frac{\partial u}{\partial n}\,ds.$$

Now

$$(2.6) -\int_\Omega (p-1)(g(u)\nabla u)\cdot(|e|^{p-1}\mathrm{sgn}(e)\nabla e)dx = -\int_\Omega \left(\frac{p-1}{p}\right)(g(u)\nabla u)\cdot\nabla(|e|^p)dx.$$

Integrating the r.h.s. of (2.6) by parts, we obtain

$$-\int_\Omega \left(\frac{p-1}{p}\right)(g(u)\nabla u)\cdot(|e|^{p-1}\mathrm{sgn}(e)\nabla e)dx = \left(\frac{p-1}{p}\right)\int_\Omega \frac{dg}{du}\,|\nabla u|^2|e|^p dx$$

(2.7)

$$+\left(\frac{p-1}{p}\right)\int_\Omega (g(u)\Delta u)|e|^p dx - \left(\frac{p-1}{p}\right)\int_\gamma g(u)\,\frac{\partial u}{\partial n}\,|e|^p ds.$$

Combining (2.5) and (2.7) gives

$$\int_\Omega g(u)|e|^{p-1}\mathrm{sgn}(e)(\nabla\cdot(e\nabla u))dx = \int_\Omega [-\frac{1}{p}\frac{dg}{du}|\nabla u|^2 + (1-\frac{1}{p})g(u)\Delta u]|e|^p dx$$

(2.8)

$$+\frac{1}{p}\int_\gamma (g(u)\frac{\partial u}{\partial n})|e|^p ds.$$

We wish to choose $g(u)$ so that

$$g(u) > 0, \text{ and}$$

$$[-\frac{1}{p}\frac{dg}{du} + (1-\frac{1}{p})g(u)\Delta u] > 0,$$

for all $x \in \overline{\Omega}$.

We first treat the case $\beta > 0$, where $\beta = \max_{P\in\Omega_1}\{\frac{-\Delta u(P)}{|\nabla u(P)|^2}\}$. Let

$$g(u) = \frac{1}{\alpha}\exp(-p\beta(u-u_{min})).$$

Then

$$\frac{-1}{p}\frac{dg}{du}|\nabla u|^2 + (1-\frac{1}{p})\ g(u)\Delta u = [\beta\frac{|\nabla u|^2}{\alpha} + (1-\frac{1}{p})\frac{\Delta u}{\alpha}]\ \exp(-p\beta(u-u_{min})).$$

If $\Delta u \geq \alpha$ it is easy to see that

$$\frac{-1}{p}\frac{dg}{du}|\nabla u|^2 + (1-\frac{1}{p})\ g(u)\Delta u \geq (1-\frac{1}{p})\exp(-p\beta[u]).$$

Next, suppose $\Delta u < \alpha$. Then $|\nabla u|^2 \geq \alpha$. From the definition of β we have

$$\Delta u \geq - \beta|\nabla u|^2.$$

Hence we obtain

$$\frac{-1}{p}\frac{dg}{du}|\nabla u|^2 + (1 - \frac{1}{p})g(u)\Delta u \geq \frac{\beta}{p}\exp[-p\beta[u]).$$

Clearly

$$\frac{\exp(-p\beta[u])}{\alpha} \leq g(u) \leq \frac{1}{\alpha}.$$

Next, we treat the case $\beta \leq 0$. Then $\Delta u \geq 0$ for all $x \in \Omega$. Let

$$g(u) = \frac{1}{\alpha}\exp(-(u - u_{min})).$$

Then

$$\frac{-1}{p}\frac{dg}{du}|\nabla u|^2 + (1 - \frac{1}{p})g(u)\Delta u = (\frac{|\nabla u|^2}{p\alpha} + (1 - \frac{1}{p})\frac{\Delta u}{\alpha})\exp(-[u]).$$

It is easy to see that in this case

$$-\frac{1}{p}\frac{dg}{du}|\nabla u|^2 + (1 - \frac{1}{p})g(u)\Delta u \geq \min(\frac{1}{p}, 1 - \frac{1}{p})\exp(-[u]).$$

Also

$$\frac{\exp(-[u])}{\alpha} \leq g(u) \leq \frac{1}{\alpha}.$$

Substituting $g(u)$ in (2.8) we obtain the following inequality

$$C_p(u) \int_\Omega |e|^p dx + \frac{1}{p} \int_\gamma \left(g(u) \frac{\partial u}{\partial n}\right)|e|^p ds \tag{2.9}$$
$$\leq \left|\int_\Omega g(u)|e|^{p-1} \operatorname{sgn}(e) f \, dxdy\right|,$$

where

$$C_p(u) = \min\left(\left(1 - \frac{1}{p}\right), \frac{\beta}{p}\right) \exp(-p\beta[u]) \quad \text{if} \quad \beta > 0,$$

$$= \min\left(\frac{1}{p}, 1 - \frac{1}{p}\right) \exp(-[u]) \quad \text{if} \quad \beta \leq 0.$$

Now

$$\left|\int_\Omega g(u)|e|^{p-1} \operatorname{sgn}(e) f \, dx\right| \leq \frac{1}{\alpha} \int_\Omega |e|^{p-1} |f| \, dx.$$

But by Hölder's inequality

$$\int_\Omega |e|^{p-1}|f| dx \leq \left(\int_\Omega |e|^{(p-1)q} dx\right)^{1/q} \|f\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Hence

$$\left|\int_\Omega g(u)|e|^{p-1} \operatorname{sgn}(e) f \, dx\right| \leq \frac{1}{\alpha} \|e\|_p^{p/q} \|f\|_p. \tag{2.10}$$

Applying Young's inequality to the r.h.s. of (2.10), we obtain

$$|\int g(u)\ |e|^{p-1}\mathrm{sgn}(e)f\ dx| \le \frac{C_p(u)}{2}\ \|e\|_p^p$$

(2.11) $$+\left\{\frac{(\frac{2}{C_p(u)q})^{p/q}\ (\frac{1}{\alpha})^p}{p}\right\}\ \|f\|_p^p.$$

Thus

$$\frac{C_p(u)}{2}\ \|e\|_p^p \le \frac{s}{\alpha p}\left[(\int_{\gamma_1}|e|^p ds)^{1/p}\right]^p + \left\{\frac{\left(\frac{2}{C_p(u)q}\right)^{p/q}(\frac{1}{\alpha})^p}{p}\right\}\|f\|_p^p.$$

Since $p \geq 1$, this implies

$$(\frac{C_p(u)}{2})^{1/p}\ \|e\|_p \le (\frac{s}{\alpha p})^{1/p}\ (\int_{\gamma_1}|e|^p ds)^{1/p} + \{\frac{1}{\alpha}\ (\frac{2}{C_p(u)q})^{1/q}(\frac{1}{p})^{1/p}\}\ \|f\|_p.$$

Hence

(2.12) $$\|e\|_p \le (\frac{2s}{\alpha p C_p(u)})^{1/p}(\int_{\gamma_1}|e|^p ds)^{1/p} + (\frac{1}{\alpha}\ (\frac{1}{p})^{1/p}(\frac{1}{q})^{1/q}\ \frac{2}{C_p(u)}\ \|f\|_p.$$

And this gives us the required result.

∎

Remark 1: It is interesting that the continuous dependence estimate (2.3) breaks down for $p = \infty$. Richter [2] proved that the result holds for $p = \infty$. Combining Richter's results with Theorem 1, we conclude that e depends continuously on f in $L^p(\Omega)$ for all $p\in[1,\infty]$, assuming the value of e is prescribed along the inflow boundary.

We would now like to investigate the rather general situation where at least one of the coefficients of the hyperbolic problem is nonvanishing at each point of Ω, i.e., we wish to see whether the continuous dependence results remain valid if we replace condition (2.2) by the condition

$$\inf_{P\in\Omega} [\max \{|\nabla u(P)|^2, \ |\Delta u(P)|\}] > 0. \tag{2.13}$$

First suppose that

$$\inf_{P\in\Omega} [\max \{|\nabla u(P)|^2, -\Delta u(P)\}] > 0. \tag{2.14}$$

Replacing u by $-u$ and f by $-f$ in (2.1) so that it takes the form

$$B(-u)e = -f,$$

it is trivial to see that Theorem 1 remains valid if the following modifications are made:

(i) $\inf_{P\in\Omega} [\max \{|\nabla u(P)|^2, - \Delta u(P)\}] = \alpha > 0.$

(ii) $\sup_{P\in\Omega_1} [\frac{\Delta u(P)}{|\nabla u(P)|^2}] = \beta$

(iii) $\Omega_2 = \{P\in\bar{\Omega}: \ - \Delta u(P) \geq \alpha\}$

(iv) $\gamma_1 = \{P \gamma: \frac{\partial u}{\partial n} > 0\}$, i.e., γ_1 now represents the outflow portion of the boundary γ.

Richter notes in [2] that if Δu is not always positive or negative where $\nabla u = 0$ it may or may not be the case that $B(u)e = f$ has a unique solution for all $f \in L^\infty(\Omega)$, with appropriate initial data for e. For the sake of completeness we cite two examples from his paper illustrating this. In Figures 1 and 2, the curves indicate characteristics of u, with arrows pointed in the direction of increasing u. Both configurations have one maxima and one minima.

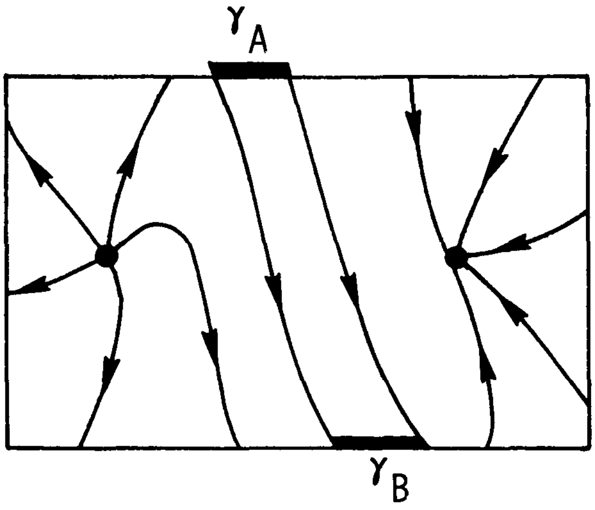

Figure 1

The first depicts a situation where Ω can be separated along a characteristic (any one from γ_A to γ_B) into two subregions Ω_1 and Ω_2 such that

$$(2.15) \qquad |\nabla u| \text{ or } \Delta u > 0 \text{ in } \Omega_1, \qquad |\nabla u| \text{ or } -\Delta u > 0 \text{ in } \Omega_2.$$

Thus the corresponding hyperbolic problem can be solved uniquely for e with initial data specified along γ_A or γ_B. Clearly the continuous dependence result (2.3) remains valid for this situation.

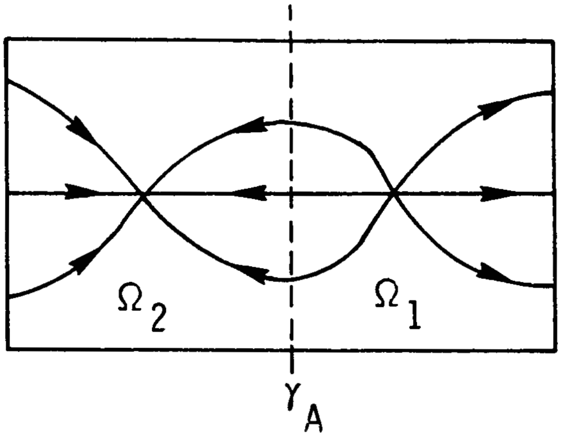

Figure 2

This is not so for the second configuration because of the presence of characteristics going between the maximum and minimum points. This difficulty can be circumvented by cutting the domain across characteristics by a line γ_A into subregions Ω_1 and Ω_2. The resulting e would in general be discontinuous at the interface γ_A. Clearly such a solution depends on the choice of γ_A and hence is not unique. Note that Cauchy data would not be required at such an interface.

3. THE NUMERICAL SCHEME

Our method uses the results of Theorem 2.1 with $p = 2$. We describe our method for the two dimensional parameter estimation problem restricting ourselves to the case where the domain Ω is the unit square, i.e.,

$$\bar{\Omega} = [0,1] \times [0,1].$$

The results described would carry through for the more general situation where the boundary of Ω is piecewise smooth, with some technical modifications.

The two dimensional problem can be formulated as follows:

Let L be a grid of points

$$L = \{(\tilde{x}_i, \tilde{y}_j): \ (i,j) \in \Lambda\},$$

where Λ is a finite index set in $\mathbb{Z}^2$. Given a data set of observations $\{u(\tilde{x}_i, \tilde{y}_j)\}$, $\{f(\tilde{x}_i, \tilde{y}_j)\}$ for u and f at the points $(\tilde{x}_i, \tilde{y}_j) \in L$ determine $e(x)$ that satisfies the equation

$$B(u)e = \frac{\partial}{\partial x}\left(e \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(e \frac{\partial u}{\partial y}\right) = f, \qquad (x,y) \in \bar{\Omega}, \tag{3.1}$$

assuming that the inflow section of the boundary γ_1 is void.

We assume that u, e, and f are smooth and that u satisfies condition (2.2).

Divide the square $[0,1] \times [0,1]$ into a grid of points

$$L_h = \{(x_i, y_j): \ (i,j) \in \Lambda_h\},$$

where

$$\Lambda_h = \{(i,j): \ 1 \leq i \leq N+1, \qquad 1 \leq j \leq N+1\}.$$

We assume that $L_h \subseteq L$.

The grid is depicted in Figure 3. Consider now the case where the inflow portion of the boundary $\gamma_1 \neq \phi$. Let

$$\gamma_1' = \{P \in \gamma: \frac{\partial u}{\partial n}(P) < \delta\}$$

for any $\delta > 0$. We assume that we are provided Cauchy data for e at a set of points

$$B = \{(\tilde{x}_i, \tilde{y}_j): (i,j) \in \Delta \subseteq \Lambda\}.$$

Here $(i,j) \in \Delta$ whenever $(\tilde{x}_i, \tilde{y}_j) \in \gamma_1'$, but Δ may contain other elements besides. We define Δ precisely below.

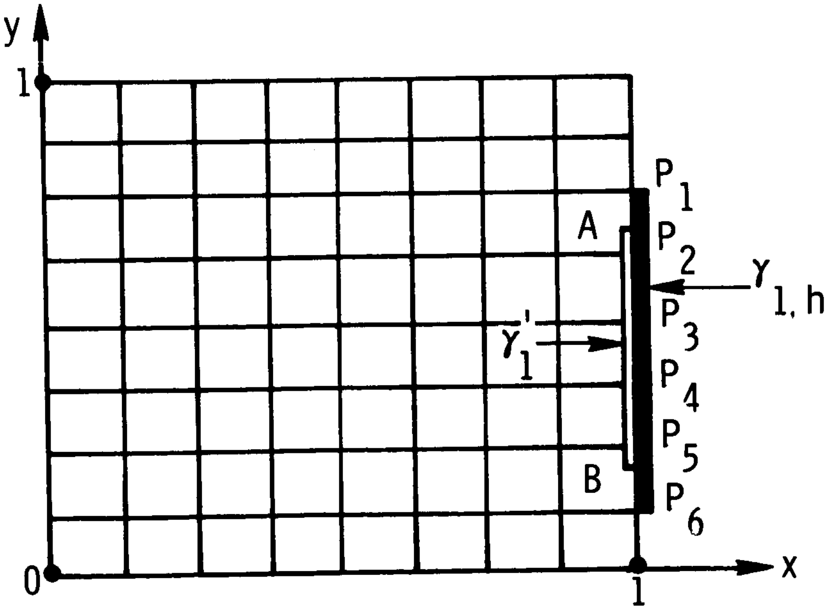

Figure 3

Consider the situation in Figure 3 where γ_1' is the line segment AB of the boundary $x = 1$. Here A and B do not belong to the grid of points L_h. We extend AB to the smallest possible line segment $\gamma_{1,h}$, $P_1 P_6$ in this case, such that

(i) $\gamma_1' \subseteq \gamma_{1,h}$.

(ii) the end points of $\gamma_{1,h}$ belong to the grid L_h.

Then

$$B = \{(\tilde{x}_i, \tilde{y}_j): \ (i,j)\in\Delta \subseteq \Lambda\}$$

where

$$\Delta = \bigcup_h \{(i,j): \ (\tilde{x}_i,\tilde{y}_j)\in\gamma_{1,h}\}.$$

Here the union is taken over all h such that $L_h \subseteq L$.

Thus we are provided with Cauchy data for e

$$e(\tilde{x}_i,\tilde{y}_j) = w(\tilde{x}_i,\tilde{y}_j), \ (i,j)\in\Delta.$$

We assume the function w is smooth.

Let

$$B_h = \{(x_i,y_j): (i,j)\in\Delta_h\},$$

where

$$\Delta_h = \{(i,j): \ (x_i,y_j)\in\gamma_{1,h}\}.$$

In the situation depicted in Figure 3, B_h consists of the points P_1 through P_6.

From the data set of observations $\{f_{ij}\}$ for f where

$$f_{ij} = f(x_i,y_j),$$

we construct the piecewise bilinear interpolant $f_h(x,y)$ of $f(x,y)$, i.e.,

$$f_h(x_i,y_j) = f_{ij}$$

and f_h is piecewise bilinear in $\overline{\Omega}$.

We approximate the unknown function $e(x,y)$ by a piecewise bilinear function $e_h(x,y)$. In case $\gamma_{1,h}$ is not void there would be M grid points belonging to B_h. Then $e_h(x,y)$ is the bilinear spline function that assumes the values

$$e_h(x_i,y_j) = w(x_i,y_j) \quad \text{for} \quad (i,j)\in\Delta_h$$

and whose values at the other mesh points $\{(x_i,y_j)\colon\ (i,j)\in\Lambda_h\setminus\Delta_h\}$ have to be determined.

Let $\mathcal{S}_h$ denote the space of piecewise bilinear functions $v_h(x,y)$ where

$$v_h(x_i,y_j) = v_{ij} \quad \text{for } (i,j)\in\Lambda_h\setminus\Delta_h, \quad \text{and}$$

$$v_h(x_i,y_j) = w(x_i,y_j) \quad \text{for} \quad (i,j)\in\Delta_h.$$

Let $\underline{E}_h$ and $\underline{V}_h$ denote the vectors

$$\underline{E}_h = \{(e_{ij})\}^T_{(i,j)\in\Lambda_h\setminus\Delta_h},$$

$$\underline{V}_h = \{(v_{ij})\}^T_{(i,j)\in\Lambda_h\setminus\Delta_h}.$$

Clearly there is a one to one correspondence between the linear spline functions $e_h(x,y)$, $v_h(x,y)$, and the vectors $\underline{E}_h$ and $\underline{V}_h$. $\underline{E}_h$ and $\underline{V}_h$ belong to a vector space of dimension $K = (N+1)^2 - M$.

Finally, we approximate $u(x,y)$ by its cubic B-spline interpolant $u_h(x,y)$. The cubic B-spline $\phi(x)$ in one dimension is the function sketched below [3]:

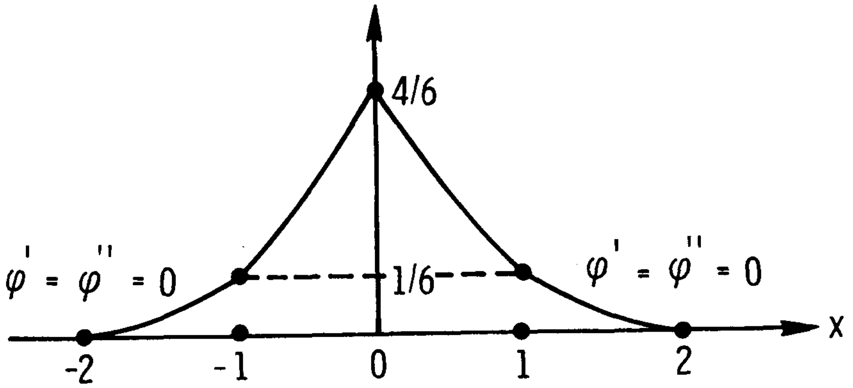

Figure 4

Let $\phi_j^h(x) = \phi(\frac{x}{h} - j)$. Then

$$u_h(x,y) = \sum_i \sum_j q_{ij}\ \phi_i^h(x)\phi_j^h(y)$$

where $\{q_{ij}\}$ are obtained by solving a linear system of equations.

The piecewise bilinear approximation for the unknown function $e(x,y)$ that we choose is the function $e_h(x,y)$ that minimizes

$$\int_0^1 \int_0^1 |B(u_h)\ v_h - f_h|^2 dxdy \tag{3.2}$$

over all $v_h \epsilon S_h$.

Substituting the explicit form of the functions $u_h(x,y)$, $f_h(x,y)$, and $v_h(x,y)$ in (3.2), we obtain

$$\int_0^1 \int_0^1 |B(u_h)v_h - f_h|^2 dxdy = \underline{V}_h^T C_h \underline{V}_h - 2\underline{V}_h^T \underline{G}_h + \lambda_h \tag{3.3}$$

where

C_h is a symmetric matrix of dimension K by K depending only on $u_h(x,y)$;

G_h is a vector of dimension K depending on $u_h(x,y)$, $f_h(x,y)$, and $w(x,y)$;

$\underline{V}_h$ is a vector of dimension K that corresponds to the linear spline function $v_h(x,y)$;

and λ_h is a scalar depending on $f_h(x,y)$ and $w(x,y)$.

We shall prove in Lemma 2 that C_h is positive definite for all $h \leq h_0$, where h_0 is a positive constant.

The piecewise bilinear function $e_h(x,y)$ that minimizes (3.2) corresponds to the vector $\underline{E}_h = \{(e_{ij})\}_{(i,j)\in\Lambda_h\setminus\Delta_h}$ that minimizes

$$\underline{V}_h^T C_h \underline{V}_h - 2\underline{V}_h^T \underline{G}_h \tag{3.4}$$

over all vectors $\underline{V}_h \in \mathbb{R}^K$. Thus $\underline{E}_h$ is obtained by solving the linear system of equations

$$C_h \underline{E}_h = \underline{G}_h. \tag{3.5}$$

Since C_h is positive definite, for h small enough, there exists a unique solution $\underline{E}_h$ of (3.5), and hence the minimization problem (3.2) has a unique solution.

Let $e_h(x,y)$ be the linear spline function that corresponds to the vector $\underline{E}_h$. We claim that $e_h(x,y)$ converges to the true solution $e(x,y)$ in the L^2 norm at a linear rate of convergence. To prove this result we need first to prove a lemma.

Lemma 1: Let $u(x,y)$ be a smooth function that satisfies condition (2.2) in the unit square $\bar{\Omega}$. Let $u_h(x,y)$ be the cubic B-spline interpolant of $u(x,y)$. Then for $h \leq h_0$, where h_0 is a positive constant, and all functions $v(x,y)$ that are piecewise continuously differentiable in $\bar{\Omega}$, the following inequality holds:

(3.6) $$\|v\|_2 \leq 2A_2(u)\|B(u_h)v\|_2 + 2D_2(u)(\int_{\gamma_{1,h}} |v(x,y)|^2 ds)^{1/2},$$

where $A_2(u)$ and $D_2(u)$ are defined in (2.3).

Proof: Since $u_h(x,y)$ is a cubic B-spline interpolant of $u(x,y)$, $u_h(x,y)$ is twice continuously differentiable.

By a standard result in approximation theory, the following inequalities

(3.7a) $$|u - u_h| \leq K_1 h^4,$$

(3.7b) $$|\nabla u - \nabla u_h| \leq K_2 h^3, \quad \text{and}$$

(3.7c) $$|\Delta u - \Delta u_h| \leq K_3 h^2$$

hold, for all $(x,y) \epsilon \overline{\Omega}$. Here the constants K_1 through K_3 depend on higher derivatives of $u(x,y)$.

Since $u(x,y)$ satisfies condition (2.2), there exists subregions Ω_1 and Ω_2 of $\overline{\Omega}$ such that $|\nabla u|^2 \geq \alpha$ in Ω_1, $\Delta u \geq \alpha$ in Ω_2. For any $\epsilon > 0$ we can choose h_0 so small that

(3.8) $$|\nabla u_h|^2 \geq \alpha - \epsilon \quad \text{in } \Omega_1, \qquad \Delta u_h \geq \alpha - \epsilon \quad \text{in } \Omega_2$$

for all $h \leq h_0$.

Recall that

$$\beta = \max_{P \epsilon \Omega_1} \{- \frac{\Delta u(P)}{|\nabla u(P)^2}\},$$

$$[u] = \max_{P\in\overline{\Omega}} \{u(P)\} - \min_{P\in\overline{\Omega}} \{u(P)\}, \text{ and}$$

$$s = \max_{P\in\gamma} |\frac{\partial u}{\partial n}(P)|.$$

We can now choose h_0 so small that (3.8) and

$$(3.9) \qquad \begin{cases} \max_{P\in\Omega_1} \{-\dfrac{\Delta u_h(P)}{|\nabla u_h(P)|^2}\} \geq \beta - \epsilon, \\ [u_h] \geq [u] - \epsilon, \quad \text{and} \\ \max_{P\in\gamma} |\frac{\partial u}{\partial n}(P)| \geq s - \epsilon. \end{cases}$$

hold simultaneously for all $h \leq h_0$. Now

$$\gamma_1' = \{P\in\gamma: \ \frac{\partial u}{\partial n} < \delta\}$$

for some $\delta > 0$. Let $\gamma_2 = \gamma_1 \backslash \gamma_1'$. Then

$$\gamma_2 = \{P\in\gamma: \ \frac{\partial u}{\partial n} \geq \delta\}.$$

If we choose h_0 smaller, if necessary, we can obtain

$$\frac{\partial u}{\partial n} h \ (P) \geq \frac{\delta}{2}, \quad \text{for all} \quad P\in\gamma_2,$$

whenever $h \leq h_0$. Clearly the inflow section of the boundary for $u_h \subseteq \gamma_1' \subseteq \gamma_{1,h}$.

If we choose ϵ small enough (and h_0 correspondingly small), we can always arrange

$$(3.10)\qquad A_2(u_h) \leq 2A_2(u), \text{ and}$$

$$D_2(u_h) \leq 2D_2(u)$$

to hold for all $h \leq h_0$. We then obtain a cruder version of estimate (2.3)

$$(3.6)\qquad \|v\|_2 \leq 2A_2(u)\|B(u_h)v\|_2 + 2D_2(u)\Big(\int_{\gamma_{1,h}} |v(x,y)|^2 ds\Big)^{1/2}$$

replacing $\overline{\gamma}_1 = \{P\in\gamma\colon \frac{\partial u}{\partial n} h\ (P) < 0\}$ by $\gamma_{1,h}$ in the second term on the r.h.s. in (2.3).

■

We shall use Lemma 1 to prove that the matrix C_h defined in (3.3) is positive definite for all $h \leq h_0$, where h_0 is a positive constant.

Lemma 2: *Let* C_h *be the matrix defined in (3.3). Then* C_h *is positive definite for all* $h \leq h_0$, *where* h_0 *is a positive constant*.

Proof: Since the matrix C_h depends only on $u_h(x,y)$, we may, without loss of generality, choose $w(x,y) \equiv 0$. Then by (2.3)

$$\int_0^1 \int_0^1 |B(u_h)v_h|^2 dxdy = V_h^T C_h V_h.$$

By Lemma 1

$$\|v_h\|_2^2 \le 4(A_2(u))^2 \, \|B(u_h)v_h\|^2 \tag{3.11}$$

since $v_h(x,y) \equiv 0$, for all $(x,y) \in \gamma_{1,h}$.

By elementary means one can show that

$$\|v_h\|_2^2 \ge \frac{h^2}{24} V_h^T V_h . \tag{3.12}$$

Combining (3.11) and (3.12) we obtain

$$V_h^T C_h V_h \ge \frac{h^2}{96A_2(u)^2} V_h^T V_h . \tag{3.13}$$

The lemma is proved.

■

We can now prove the main result of this section.

Theorem 2: Consider the parameter estimation problem

$$B(u)e = \frac{\partial}{\partial x}\left(e \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(e \frac{\partial u}{\partial y}\right) = f \tag{3.1}$$

in the unit square $\bar{\Omega}$, and assume that we are provided with Cauchy data for e on $\gamma_{1,h}$,

$$e(x,y)\big|_{(x,y)\in\gamma_{1,h}} = w(x,y).$$

Here u, e, w, and f are smooth functions, and we assume that u satisfies condition (2.2). Let $e_h(x,y)$ be the piecewise bilinear function that minimizes

$$\int_0^1\int_0^1 |B(u_h)v_h - f_h|^2 dxdy \tag{3.2}$$

over all piecewise bilinear functions $v_h(x,y)\in S_h$. Then $e_h(x,y)$ converges to the true solution $e(x,y)$ in the L^2 norm at a linear rate of convergence.

<u>Proof</u>: Let $e_h^a(x,y)$ be the piecewise bilinear function $\in S_h$ that interpolates $e(x,y)$ at the mesh points $\{(x_i, y_j): (i,j)\in\Lambda_h\setminus\Delta_h\}$. Then $e_h^a(x,y)$ interpolates $e(x,y)$ at all the mesh points $\{(x_i,y_j): (i,j)\in\Lambda_h\}$.

Since $e_h(x,y)$ minimizes

$$\int_0^1\int_0^1 |B(u_h)v_h - f_h|^2 dxdy \tag{3.2}$$

over all piecewise bilinear functions $v_h(x,y)\in S_h$, the inequality

$$\left(\int_0^1\int_0^1 |B(u_h)e_h - f_h|^2 dxdy\right)^{1/2} \le \left(\int_0^1\int_0^1 |B(u_h)e_h^a - f_h|^2 dxdy\right)^{1/2} \tag{3.14}$$

holds.

By a standard result in approximation theory, we have the inequalities

(3.15a) $$\|e - e_h^a\|_2 \leq K_4 h^2, \text{ and}$$

(3.15b) $$\|\nabla e - \nabla e_h^a\|_2 \leq K_5 h.$$

Here K_4 and K_5 are positive constants depending on higher derivatives of e.

Further, since f_h is the piecewise bilinear interpolant of f we have

(3.16) $$\|f - f_h\|_2 \leq K_6 h^2,$$

where K_6 is a positive constant depending on higher derivatives of f.

Using these and the earlier results (3.7) from approximation theory, it is easy to show that

(3.17) $$\int_0^1 \int_0^1 |B(u_h)e_h^a - f_h|^2 dxdy \leq 0(h).$$

By the triangle inequality

(3.18) $$\|B(u_h)(e_h^a - e_h)\|_2 \leq \|B(u_h)e_h^a - f_h\|_2 + \|B(u_h)e_h - f_h\|_2.$$

From (3.14) and (3.17) we conclude that

(3.19) $$\|B(u_h)(e_h^a - e_h)\|_2 \leq 0(h).$$

Since both $e_h^a(x,y)$ and $e_h(x,y)$ interpolate $e(x,y)$ at the points $\{(x_i,y_j)\colon (i,j)\in\Delta_h\}$, it is easy to see that

$$\int_{\gamma_{1,h}} |e_h^a - e_h|^2 ds = 0. \tag{3.20}$$

From Lemma 1 we have that

$$\|v\|_2 \leq 2A_2(u)\|B(u_h)v\|_2 + 2D_2(u)[\int_{\gamma_{1,h}} |v|^2 ds]^{1/2} \tag{3.6}$$

for all functions v that are piecewise continuously differentiable in $\overline{\Omega}$.

Using (3.19) and (3.20) we conclude that

$$\|e_h - e_h^a\|_2 \leq 2A_2(u)\|B(u_h)(e_h^a - e_h)\|_2 \leq O(h). \tag{3.21}$$

By (3.15a)

$$\|e_h^a - e\|_2 \leq O(h^2).$$

Using the triangle inequality once more we obtain the result

$$\|e - e_h\|_2 \leq O(h). \tag{3.22}$$

■

Remark: If we were to interpolate $e(x,y)$ by piecewise Hermite cubics or by a B-cubic spline, our method would converge at a quadratic rate of convergence. It is easy to modify the method to obtain higher orders of convergence.

Finally we indicate how the numerical method we have proposed can be adapted in case u does not satisfy condition (2.2) but the more general condition

$$\inf_{P\in\Omega} [\max\{|\nabla u(P)|^2, |\Delta u(P)|\}] = \alpha > 0. \tag{2.13}$$

Consider the situation depicted in Figure 2 where u has both a maximum and minimum. Let us suppose that f is such that a unique solution e to (2.1) exists. We do not need to specify any Cauchy data for e in this situation. As before, we cut the domain Ω across characteristics into two subdomains Ω_1 and Ω_2 such that u satisfies condition (2.2) in Ω_1 and condition (2.14) in Ω_2.

Let

$$a(x,y) = e(x,y) \quad \text{for} \quad (x,y)\in\Omega_1, \text{ and}$$

$$b(x,y) = e(x,y) \quad \text{for} \quad (x,y)\in\Omega_2.$$

We can find a solution a of 2.1 in the domain Ω_1 without any Cauchy data. The same holds for b in the domain Ω_2. Clearly if a solution e to problem (2.1) exists in Ω then e must be continuous along γ_A.

Hence we must have

$$a(P) = b(P) \quad \text{for all} \quad P \in \gamma_A .$$

Let a_h and b_h denote piecewise bilinear approximations for a and b respectively. Then a_h and b_h are the piecewise bilinear functions defined in Ω_1 and Ω_2 respectively which minimize

$$\int_{\Omega_1}\int |B(u_h)v_h - f_h|^2 dxdy + \int_{\Omega_1}\int |B(u_h)t_h - f_h|^2 dxdy$$

$$+ \theta \int_{\gamma_A} |t_h - v_h|^2 ds$$

over all piecewise bilinear functions v_h and t_h defined in Ω_1 and Ω_2 respectively. Here θ is a weighting parameter.

In case a smooth solution e to (2.1) exists, it is easy to show that the function e_h defined as

$$\begin{cases} e_h(x,y) = a_h(x,y) & \text{for} \quad (x,y) \in \Omega_1, \\ e_h(x,y) = b_h(x,y) & \text{for} \quad (x,y) \in \Omega_2 \end{cases}$$

converges to the true solution $e(x,y)$ in the L^2 norm at a linear rate of convergence.

In a companion paper [4], we present a multigrid algorithm for parameter estimation problems. In this algorithm we seek a finite dimensional representation for e which minimizes the ℓ^2 norm of the residuals of a second order difference approximation of (2.1). Extensive numerical experiments indicate that the method converges to the true solution in the ℓ^2 norm at a quadratic rate of convergence.

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Standard Bibliographic Page

1. Report No. NASA CR-178193 ICASE Report No. 86-64	2. Government Accession No.	3. Recipient's Catalog No.

4. Title and Subtitle

A SPLINE-BASED PARAMETER ESTIMATION TECHNIQUE
FOR STATIC MODELS OF ELASTIC STRUCTURES

5. Report Date
October 1986

6. Performing Organization Code

7. Author(s)

Pravir Dutt and Shlomo Ta'asan

8. Performing Organization Report No.
86-64

9. Performing Organization Name and Address
Institute for Computer Applications in Science
and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23665-5225

10. Work Unit No.

11. Contract or Grant No.
NAS1-17070, NAS1-18107

12. Sponsoring Agency Name and Address

National Aeronautics and Space Administration
Washington, D.C. 20546

13. Type of Report and Period Covered
Contractor Report

14. Sponsoring Agency Code
505-90-21-01

15. Supplementary Notes

Langley Technical Monitor: Math Comp.
J. C. South

Final Report

16. Abstract

We consider the problem of identifying the spatially varying coefficient of elasticity using an observed solutiont to the forward problem. Under appropriate conditions this problem can be treated as a first order hyperbolic equation in the unknown coefficient. We develop some continuous dependence results for this problem and propose a spline-based technique for approximating the unknown coefficient, based on these results. We establish the convergence of our numerical scheme and obtain error estimates.

17. Key Words (Suggested by Authors(s))

parameter estimation, finite elements
equation error

18. Distribution Statement

59 - Mathematical and Computer
Sciences (General)
64 - Numerical Analysis

Unclassified - unlimited

19. Security Classif.(of this report)	20. Security Classif.(of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	33	A03

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1986