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NASA

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Summary

A study of plane-acoustic-wave propagation in small tubes with a cross section in the shape of a flattened oval is described. To begin, theoretical descriptions of a plane wave propagating in a tube with circular cross section and between a pair of infinite parallel plates, including viscous and thermal damping, are expressed in similar form. For a wide range of useful duct sizes, the propagation constant (whose real and imaginary parts are the amplitude attenuation rate and the wave number, respectively) is very nearly the same function of frequency for both cases if the radius of the circular tube is the same as the distance between the parallel plates. This suggests that either a circular-cross-section model or a flat-plate model can be used to calculate wave propagation in flat-oval tubing, or any other shape tubing, if its size is expressed in terms of an equivalent radius, given by $g = 2 \times$ (cross-sectional area)/ (length of perimeter). Measurements of the frequency response of two sections of flat-oval tubing agree with calculations based on this idea. Flat-plate formulas are derived, the use of transmission-line matrices for calculations of plane waves in compound systems of ducts is described, and examples of computer programs written to carry out the calculations are shown.

Introduction

"The problem of the propagation of sound waves in gases contained in cylindrical tubes is a classical one, to which famous names are connected like Helmholtz, Kirchhoff, and Rayleigh. Since then many papers have been written on the subject, often in relation to studies dealing with the dynamic responses of pressure transmission lines." So states H. Tijdeman, referring to 34 previous papers in the first paragraph of one of his reports on the subject (ref. 1). (See also refs. 2 and 3.)

Much less has been written about the propagation of sound waves in ducts of noncircular cross section, however, particularly for propagation in narrow tubes, where viscous and thermal losses at the walls are significant. It was the goal of this study to develop an analytical framework for calculating the pressure-transfer properties of the flat-oval tubing shown in cross section in figure 1, so that its behavior could be predicted when it is used as probe tubing for a pressure

transducer. The hope was that results could be cast in a form similar to the recursion relation of Bergh and Tijdeman (ref. 4), which has been used to analyze pressure probes of circular cross section (ref. 5). This recursion relation is obtained in a slightly more general form in this report (appendix C).

However, in the section Solving Fluid Equations for Noncircular Tubing, it is argued that a Bergh and Tijdeman type recursion relation cannot be found for tubing of noncircular cross section for a viscous, thermally conducting gas. That is to say, the fundamental differential equations describing the behavior of a fluid in a duct cannot be solved in closed form to the same degree of approximation used for the Bergh and Tijdeman analysis in any coordinate system other than circular cylindrical.

The solution for a plane wave propagating between two infinite parallel plates can be found, however, and the section Comparison With Flat-Plate Geometry is concerned with a comparison of the circular-tube and parallel-plate results. Over a useful range of tubing sizes and frequencies, the two solutions are remarkably similar if a generalized radius $g = 2 \times$ (area of cross section)/(length of perimeter of cross section) is properly used to characterize the size of the waveguides. Thus, within the range where the solutions are similar, either should give a close approximation to the acoustical properties of flat-oval tubing, described by its generalized radius. This conclusion applies also to tubing with other cross-sectional shapes as well.

The pressure-transfer properties of short sections of flat-oval tubing were measured, and the results are presented in the section Measurements on Flat-Oval Tubing. The theoretical predictions agree very well with the measurements. Limits to this technique for calculating the pressure-transfer properties of noncircular tubing are suggested in the section Concluding Remarks.

Four appendixes conclude the report. A list of symbols used in the equations is given as appendix A. Appendix B gives details of the solution for plane waves propagating between infinite parallel plates. Appendix C describes the use of transmission-line matrices for calculating the acoustical properties of systems of tubing. (This method is equivalent to but more straightforward and powerful than the Bergh and Tijdeman recursion relation.) Listings of several computer programs written to implement the transmission-line calculations are presented and explained in appendix D, with several numerical examples for reference.
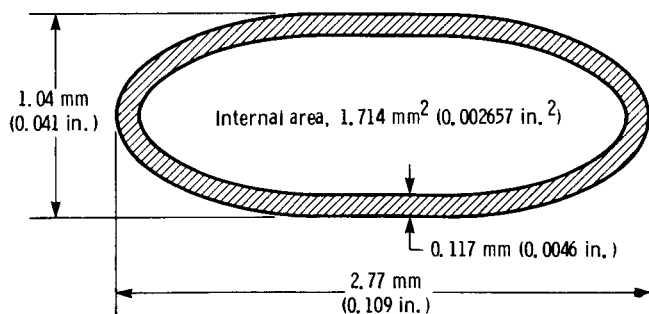
Figure 1.—Cross section of flat-oval tubing investigated.

Solving Fluid Equations for Noncircular Tubing

There are four basic equations describing the properties of a viscous gas with a finite thermal conductivity. These four equations are the following:

The force equation, which includes viscosity, is the Navier-Stokes equation (refs. 6 and 7), which may be written

$$\rho'\frac{Du'}{Dt} = -\nabla p' + \mu\left[(\nabla\cdot\nabla)u' + \frac{1}{3}\nabla(\nabla\cdot u')\right] \quad (1)$$

In this equation, which is a vector equation having three components, ρ' is the fluid density, u' is the fluid velocity, p' is the absolute pressure, and μ is the constant shear viscosity coefficient. The Stokes assumption that the bulk (or volume) viscosity is zero has been made, and the differential operator

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u'\cdot\nabla \quad (2)$$

In curvilinear coordinates, the scalar operator $\nabla\cdot\nabla$ can be evaluated by using the identity

$$(\nabla\cdot\nabla)u' = \nabla(\nabla\cdot u') - \nabla\times(\nabla\times u') \quad (3)$$

The equation of continuity of the fluid (or conservation of mass) may be written (ref. 8, p. 8, p. 512)

$$\frac{D\rho'}{Dt} + \rho'\nabla\cdot u' = 0 \quad (4)$$

The thermodynamic properties of the gas are assumed to be described by the ideal gas equation of state

$$p' = \rho'R_0T' \quad (5)$$

and an energy equation describing heat conduction (refs. 9 and 10)

$$\rho'c_p\frac{DT'}{Dt} - \frac{Dp'}{Dt} - \lambda\nabla^2 T' = 0 \quad (6)$$

Here T' is absolute temperature, c_p is the specific heat at constant pressure, and λ is the thermal conductivity of the fluid; a second-order term describing heat transfer due to internal friction has been neglected (see refs. 1 and 4).

When written in cylindrical coordinates, equations (1), (4), (5), and (6) form the starting point for the analysis in the appendixes of references 1 and 4, where it is assumed from symmetry that there is no azimuthal velocity. The solutions for the circular tube proceed by setting

$$\left.\begin{array}{ll} p' = p_a + pe^{j\omega t} & T' = T_a + Te^{j\omega t} \\ \rho' = \rho_a + \rho e^{j\omega t} & u' = ue^{j\omega t} \end{array}\right\} \quad (7)$$

where $j = \sqrt{-1}$, and p_a, ρ_a, and T_a are constants representing the ambient or average values of pressure, density, and temperature. As usual, the position-dependent disturbance amplitudes p, ρ, and T are assumed to be small compared to the ambient values. Similarly, the magnitude of the fluid velocity amplitude u is assumed small compared to c, the free-space adiabatic phase velocity of sound given by $c=\sqrt{\gamma p_a/\rho_a}$, where γ is the ratio of specific heats $\gamma = c_p/c_v$. In addition, it is assumed that (1) the internal tube radius r is small compared to the free-space wavelength ($\omega r/c \ll 1$), (2) the radial velocity component is smaller than the axial velocity by about this same factor, and (3) the flow is laminar throughout the system. (Laminar flow is interpreted to mean that differentiation with respect to the axial coordinate yields a quantity of the same order as does dividing by the free-space wavelength; differentiation with respect to the radial coordinate gives a result of the same order as does dividing by r.) Under these assumptions, closed-form solutions to these equations in cylindrical coordinates are obtained in references 1 and 4.

For purposes of this investigation, it is natural to consider solving the basic equations to the same degree of approximation in other coordinate systems. The difficulty encountered in such attempts may be illustrated by using rectangular coordinates as an example.

Choose the x-axis to lie along the center axis of a long tube with rectangular cross section, with the y- and z-axes perpendicular to the duct walls. Denote the x-component of the harmonic amplitude of the fluid velocity by u, the y-component by v, and the z-component by w. Writing the basic equations (1), (4), (5), and (6) in rectangular coordinates, substituting equations (7), and retaining only the most significant terms in each equation (as in the circular tube case) yield the following six equations:

$$jwu = -\frac{1}{\rho_a}\frac{\partial p}{\partial x} + \frac{\mu}{\rho_a}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (8)$$

$$0 = -\frac{\partial p}{\partial y} \qquad (9a)$$

$$0 = -\frac{\partial p}{\partial z} \qquad (9b)$$

$$jw\rho = -\rho_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad (10)$$

$$\rho = \frac{\gamma}{c^2}\left(p - \rho_a R_0 T\right) \qquad (11)$$

$$jw\rho_a c_p T = \lambda\left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + jwp \qquad (12)$$

These are similar to equations (6) through (10) in the appendix of reference 4 or equations (B1) through (B5) of reference 1, except that they contain two independent coordinates or components perpendicular to the tube axis instead of one.

To this approximation as in the circular case, equations (9a) and (9b), which are the two transverse components of the Navier-Stokes equation, directly imply that the pressure waves are plane waves. The planes of constant pressure coincide with planes of constant x. However, the rectangular case differs from the circular case because it has less symmetry.

The rectangular case has two independent transverse components of fluid velocity, denoted by v and w, whereas by symmetry the circular case has only one transverse velocity, the radial velocity. In order to solve the problem, then, we need one more equation in the rectangular case than in the circular geometry to determine an additional unknown. This additional equation is the third component of the Navier-Stokes equation given in equation (9), but unfortunately it provides no information about v or w. The two transverse velocities appear only together in equation (10) with no additional relation which can be used to separate them. In addition, if one tries to solve equation (12) for temperature fluctuations, one finds that with both y- and z-derivatives present the separation of variables procedure is not successful.

Thus, the equations do not appear solvable to this approximation in rectangular coordinates. Furthermore, we would expect the same difficulty in any other coordinate system lacking circular symmetry and thus having two independent transverse velocity components, if we seek a plane wave in pressure. In spite of the Bessel functions which appear, the circular geometry is fundamentally the simplest case to consider and is the only shape duct for which a closed solution is possible.

Comparison with Flat-Plate Geometry

As mentioned by Lord Rayleigh, another effectively two-dimensional case which can be solved, in addition to the circular duct, is a plane wave propagating between a pair of infinite parallel plates (ref. 11). A solution for that case is derived in appendix B, and the results may be summarized as follows: If a plane pressure wave propagates in the x-direction between a pair of isothermal, rigid, infinite planes located at $y = \pm h/2$, the x-dependent amplitude of the pressure disturbance at frequency w is

$$p(x) = Ae^{\varphi x} + Be^{-\varphi x} \qquad (13)$$

Here, A and B are complex constants determined by boundary conditions, and the complex propagation constant φ is

$$\varphi = \frac{w}{c}\left[F(\alpha)\right]^{-1/2}\sqrt{\frac{\gamma}{n}} \qquad (14)$$

In this expression,

$$n \equiv \left[1 + \frac{\gamma-1}{\gamma}F\left(\alpha\sqrt{Pr}\right)\right]^{-1} \qquad (15)$$

$$F(\zeta) \equiv \left(\frac{2}{\zeta}\right)\tan\left(\frac{\zeta}{2}\right) - 1 \qquad (16)$$

$$\alpha \equiv j^{3/2}h\sqrt{\frac{\rho_a w}{\mu}} \qquad (17)$$

and Pr is the Prandtl number. (See appendix B.) At any particular x, the average amplitude of the longitudinal fluid velocity is

$$\bar{u}(x) = \frac{\varphi}{jw\rho_a}F(\alpha)\left(Ae^{\varphi x} - Be^{-\varphi x}\right) \qquad (18)$$

Interestingly enough these expressions are of precisely the same form as the corresponding quantities in the circular duct solution, as found in equations (29), (30), and (42) in the appendix of Bergh and Tijdeman (ref. 4). The only difference is that for the infinite-flat-plate geometry the function F has replaced the Bessel function ratio J_2/J_0 appearing in the circular-tube case. The arguments of the respective functions are the same if one chooses the spacing h between the plates equal to the radius r of the circular tube.

Not only are the overall expressions of the same form, but, for equal arguments of the type used in these equations, the magnitude and phase of the function F are actually quite similar to the corresponding quantities for the Bessel function ratio.

3

If both are evaluated for an argument written as $\zeta = j^{3/2}s$, where s is real (this is the particular form of α), then

$$\lim_{s \to 0} \frac{J_2(\zeta)}{J_0(\zeta)} = -\frac{js^2}{8} \tag{19}$$

$$\lim_{s \to 0} F(\zeta) = -\frac{js^2}{12} \tag{20}$$

$$\lim_{s \to \infty} \frac{J_2(\zeta)}{J_0(\zeta)} = \lim_{s \to \infty} F(\zeta) = -1 + \frac{2j}{\zeta} \tag{21}$$

For large arguments, both have the limiting value -1, so that for very wide conduits, very high frequencies, high ambient pressure, or vanishing viscosity and thermal conductivity the propagation constant becomes $j\omega/c$. Between the extreme limits, the magnitudes and phases of the two quantities are plotted in figures 2 and 3. There it can be seen that the two quantities are not identical, but are surprisingly similar. To indicate the frequency range in which the important variation of these two functions occurs in the system of interest, figures 2 and 3 show the values of s appropriate for 100 and 329 Hz if $s = h\sqrt{\rho_a \omega/\mu}$ for air at atmospheric pressure and 300 °C and $h = 0.061$ cm.

Quantities that are calculated by using these two similar functions are also very much alike. Graphs of the real and imaginary parts of the propagation constant φ as a function of frequency are shown in figures 4 through 6, for several
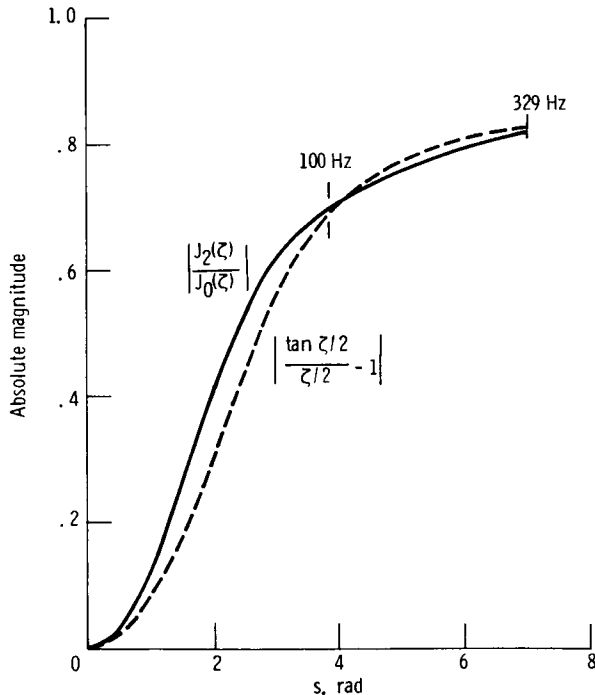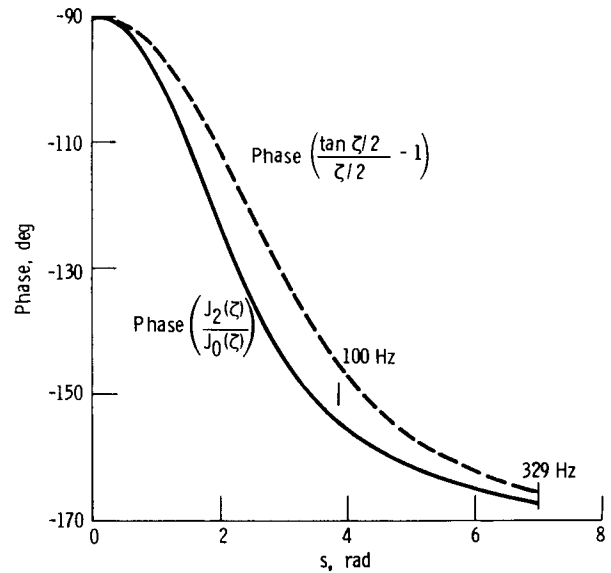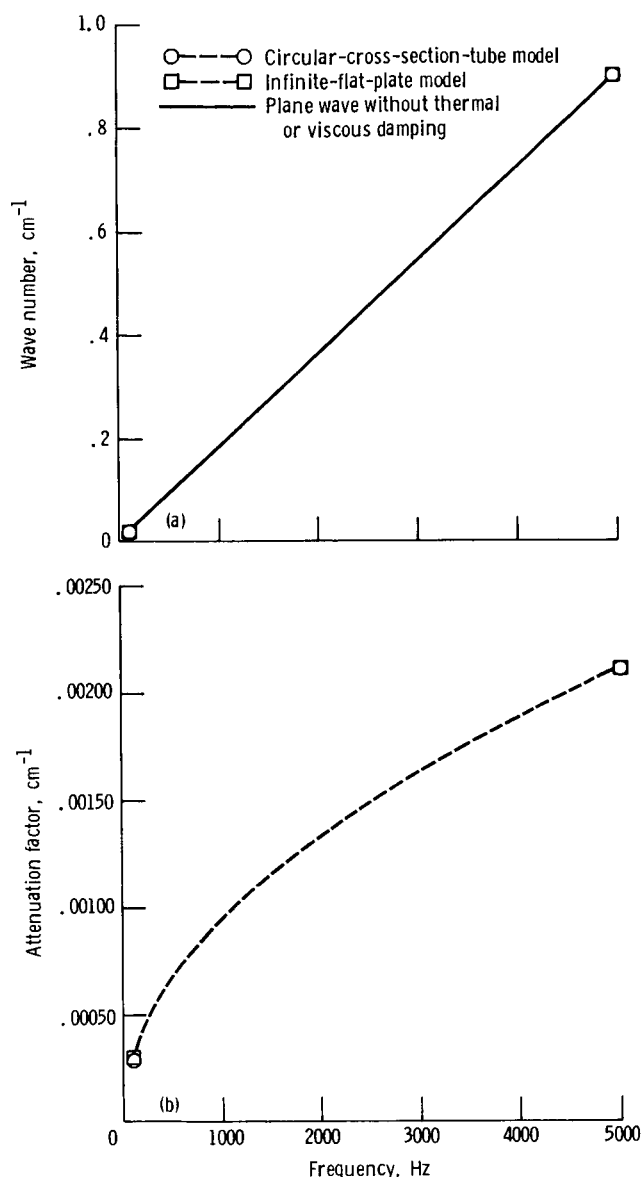

Figure 3.—Comparison of phases of $J_2(\zeta)/J_0(\zeta)$ and $[(\tan \zeta/2)/(\zeta/2)] - 1$ as function of real parameter s, where $\zeta = j^{3/2}s$.

values of $r = h$, given in centimeters. The imaginary part of φ is the wave number, inversely proportional to the phase velocity of the wave in the conduit. It is the primary quantity determining the resonant frequencies of standing waves in a tube. The real part of φ is the factor giving the amplitude attenuation rate with distance. In figures 4 through 6, the solid straight line is the wave number for an ideal, inviscid gas, the dashed lines ending in a circle are computed by using the circular-tube expressions, and the dashed lines ending in a square show the infinite-flat-plate values. The most noticeable difference between the circular-tube and flat-plate propagation constants is that for the smallest tubing the flat-plate geometry has a larger attenuation factor, or greater viscous and thermal damping, than the circular case. But the smallest case shown has such large damping that tubing of that size would not be useful for pressure-transmission purposes. The intermediate pair of graphs turns out to show the propagation constant appropriate for the flat-oval tubing of interest, and there the flat-plate damping is only slightly greater than the circular-tube damping.

For the particular case of infinite flat plates, figures 4 through 6 illustrate the assertion that theoretical expressions derived for waves propagating in a circular tube may be adapted to describe the properties of waves in ducts of arbitrary cross-sectional shape. This may be done by simply replacing r, the radius of the circular tube, by a generalized quantity calculated as

$$g = 2 \times (\text{cross-sectional area of duct})/(\text{length of perimeter of cross section}) \tag{22}$$

wherever r appears in the equations (refs. 12 through 14). For a circular tube, $g = r$, while in the limit that the width of a
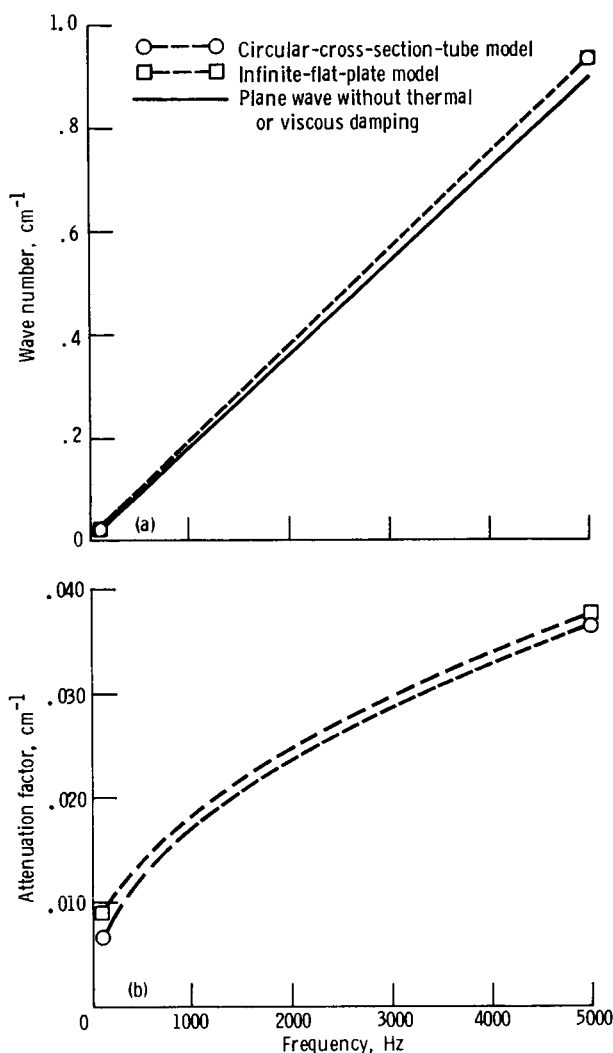

Figure 2.—Comparison of magnitudes of $J_2(\zeta)/J_0(\zeta)$ and $[(\tan \zeta/2)/(\zeta/2)] - 1$ as function of real parameter s, where $\zeta = j^{3/2}s$.

(a) Wave number for circular-cross-section tube with radius of 1.0 cm, wave between flat plates separated by 1.0 cm, and plane wave without thermal or viscous damping.

(b) Attenuation factor for circular-cross-section tube with radius of 1.0 cm, and wave between flat plates separated by 1.0 cm.

Figure 4.—Wave number and attenuation factor, imaginary and real parts, respectively, of propagation constant of equations (13) and (14) as functions of frequency.

(a) Wave number for circular-cross-section tube with radius of 0.060 cm, wave between flat plates separated by 0.060 cm, and plane wave without thermal or viscous damping.

(b) Attenuation factor for circular-cross-section tube with radius of 0.060 cm, and wave between flat plates separated by 0.060 cm.

Figure 5.—Wave number and attenuation factor, imaginary and real parts, respectively, of propagation constant of equations (13) and (14) as functions of frequency.

rectangular duct becomes infinite, $g = h$. The assertion seems true at least in that $p(x)$ and $\overline{u}(x)$ are very nearly the same function of g, frequency, and gas properties for waves propagating both in a circular duct and between infinite parallel plates over a wide range of sizes. It would be reasonable to expect $p(x)$ and $\overline{u}(x)$ to depend in the same way on the same parameters in flat-oval tubing or tubing of other cross-sectional shapes as well.

Shown in figure 7 are rough sketches of circular, flat-oval, and infinite-plate waveguides drawn with the same value of g. The pressure and average-velocity waves in these three systems, then, are described by almost the same expressions. The properties of systems made up of tubing with any cross-sectional shape should be calculable with a recursion relation like that of Bergh and Tijdeman (ref. 4) or equivalent transmission-line formulas (see appendix C), where in most places r is replaced by the appropriate g.

However, care must be exercised, for not all such replacements would be correct. The exceptions occur in expressions for the characteristic impedance of a tube found
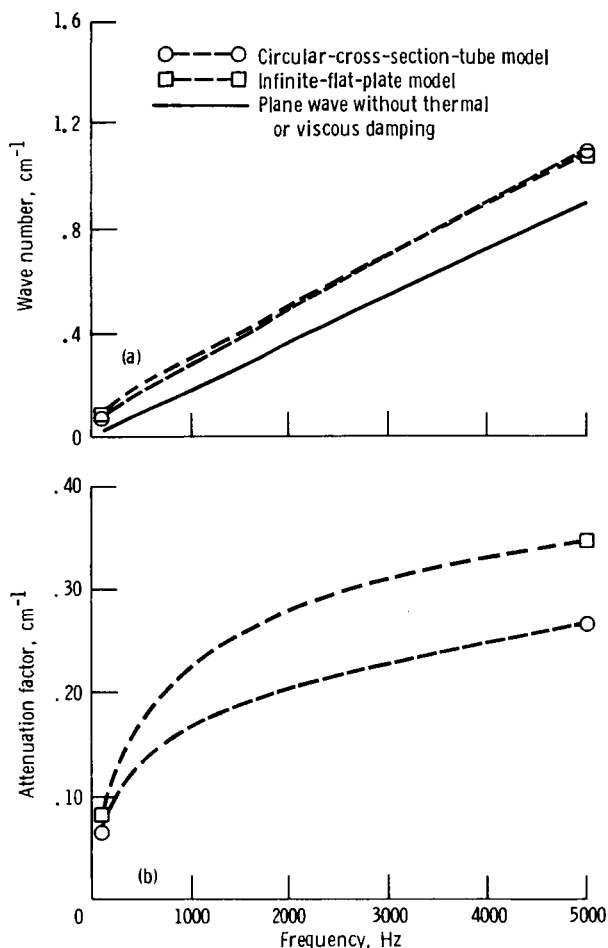
(a) Wave number for circular-cross-section tube with radius of 0.010 cm, wave between flat plates separated by 0.010 cm, and plane wave without thermal or viscous damping.

(b) Attenuation factor for circular-cross-section tube with radius of 0.010 cm, and wave between flat plates separated by 0.010 cm.

Figure 6.—Wave number and attenuation factor, imaginary and real parts, respectively, of propagation constant of equations (13) and (14) as functions of frequency.

from equations (C10), (C12), and (C19) or the V_t tube volume factors in the Bergh and Tijdeman recursion relation given here as equation (C57), where essentially the cross-sectional area S of the tube enters as a result of requiring conservation of mass flowing through the tube. The mass per unit time passing point x is given by

$$\frac{dm}{dt} = \rho_a S \overline{u}(x) \qquad (23)$$

In adapting this requirement to tubing of different shape, even if S is expressed as πr^2, set S equal to the actual cross-sectional area of the new tube, rather than changing r to g. Similarly, when calculating an appropriate end correction to the geometrical length of a tube (0.82 r for flanged tubing or
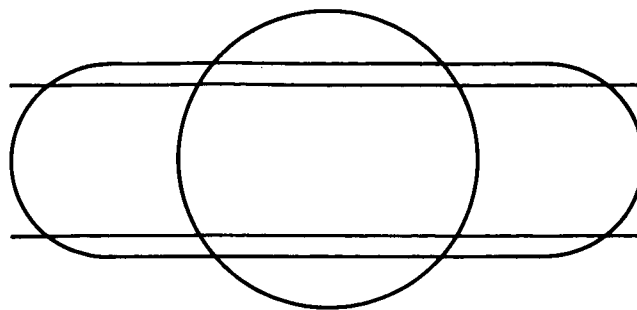
Figure 7.—Cross-sectional shapes of circular, flat-oval, and infinite-parallel-plate ducts, all having same size generalized or equivalent radius.

0.61 r for an unflanged opening), replace r by $\sqrt{S/\pi}$ for a noncircular tube (ref. 8, pp. 348–350).

Measurements on Flat-Oval Tubing

In order to evaluate the applicability of the circular-tube and/or flat-plate expressions to the case of flat-oval tubing, two samples of such tubing were mounted on the resonator chamber of the large Galton whistle sound source shown in figure 8 and described in reference 15. Silicon strain-gauge pressure transducers were used to measure the pressure in the whistle and in a small cavity at the end of the tubing. The geometrical length of one sample was 5.08 cm and of the other

Figure 8.—Galton whistle sound source for testing frequency response of flat-oval probe microphone tubing.

was 7.62 cm, while the cavity volumes were 0.01325 and 0.01630 cm^3, respectively. The cross-sectional area of the tubing was 0.01714 cm^2, and the length of the perimeter was 0.571 cm. Thus, $g = 0.060$. When the expected pressure-transfer properties of the tubing were calculated, an end correction of 0.07 cm was added to the geometrical length of each tube.

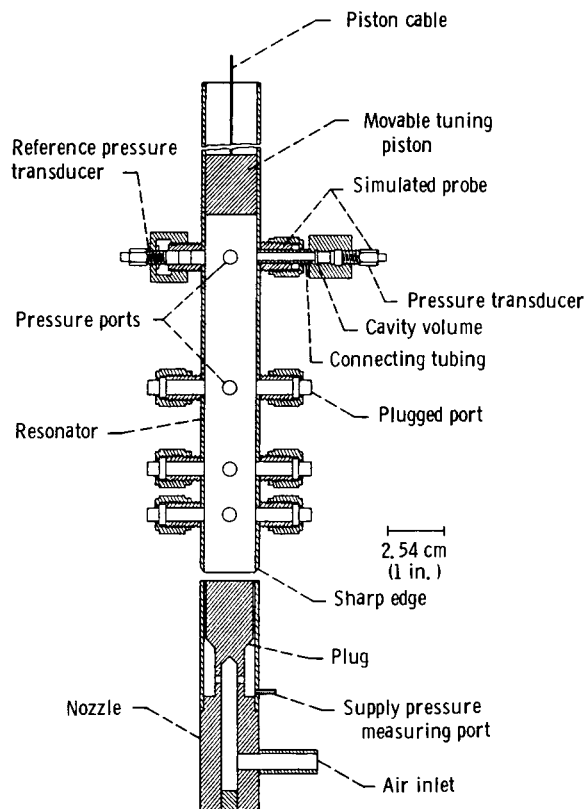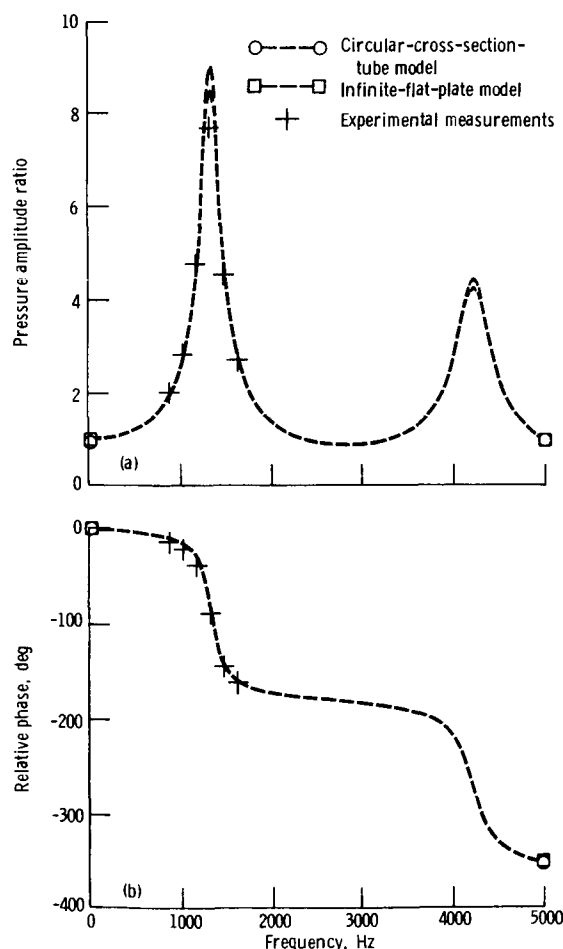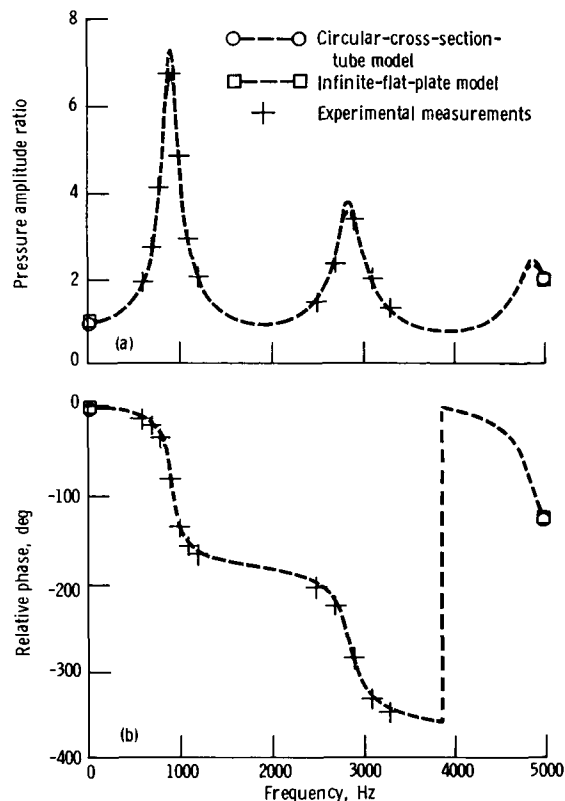The magnitudes of the pressure disturbance amplitudes were measured by using two Princeton Applied Research model 126 lockin amplifiers operated as ac voltmeters, and the ratio of the end-cavity pressure amplitude to the whistle pressure amplitude was plotted as a function of frequency. The relative phase of the two signals was also measured with one of the lockin amplifiers. Care was taken to subtract the phase shift in the signal channel input filter of the amplifier at each frequency.

The experimental data are shown as crosses superimposed on theoretical predictions in figures 9 (shorter tube) and 10

(a) Ratio of amplitude of transducer cavity pressure to amplitude of whistle pressure at probe tube inlet.
(b) Relative phase of transducer cavity pressure and whistle pressure at probe tube inlet.

Figure 10.—Ratio of cavity to whistle pressure amplitude and relative phase of two signals as functions of frequency for flat-oval tubing of geometrical length 7.62 cm and equivalent radius $g = 0.060$.

(longer tube). The lengths of the bars on the crosses in these figures do not signify experimental uncertainty. There are, in principle, two theoretical curves on each graph, one from the circular tube formulas and one for infinite flat plates, both evaluated for $g = 0.060$ cm. The two curves are essentially the same, however, except near the peaks in the pressure-amplitude ratios. There the slightly greater damping in the flat-plate case leads to a slightly lower peak. The relative phase curves are totally indistinguishable.

The agreement between the experimental data and the theoretical curves is excellent. Since figure 7 suggests that the flat-oval tubing may resemble a pair of flat plates more than a circular tube, one might expect the experimental points to lie closer to the curves of the flat-plate model than those of the circular one. In fact they do, but the possibility of slight extra, uncontrolled damping or air leakage in the apparatus implies that such fine distinctions would be hard to defend.

Concluding Remarks

Very satisfactory predictions of the acoustical properties of pressure-transmission systems incorporating flat-oval tubing

(a) Ratio of amplitude of transducer cavity pressure to amplitude of whistle pressure at probe tube inlet.
(b) Relative phase of transducer cavity pressure and whistle pressure at probe tube inlet.

Figure 9.—Ratio of cavity to whistle pressure amplitude and relative phase of two signals as functions of frequency for flat-oval tubing of geometrical length 5.08 cm and equivalent radius $g = 0.060$.

Figure 11.—Calculated magnitude of ratio of transducer cavity pressure to pressure at probe tube inlet as a function of frequency for flat-oval tubing of length 7.7 cm and equivalent radius $g = 0.020$ cm.
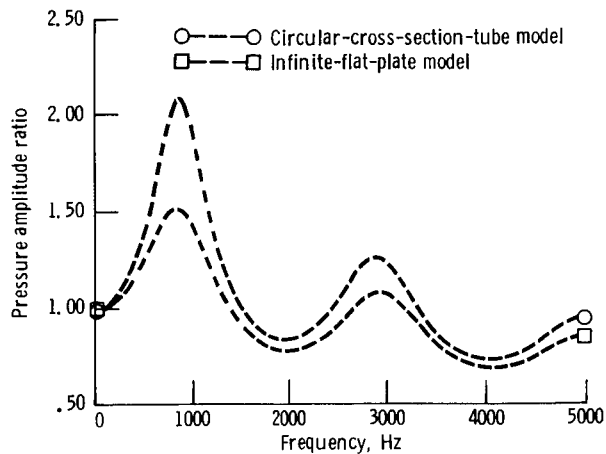

Figure 12.—Calculated magnitude of ratio of transducer cavity pressure to pressure at probe tube inlet as a function of frequency for flat-oval tubing of length 7.7 cm and equivalent radius $g = 0.010$ cm.

with an equivalent radius $g = 0.060$ cm can be made by using equations derived either for circular tubes or a pair of infinite flat plates. This result was anticipated from the surprising similarity of the theoretically calculated propagation constants for waves in circular tubes and waves between flat plates, if both types of ducts possess the same equivalent radius as defined in equation (22). Experimental measurements of the pressure-transfer properties of samples of flat-oval tubing with $g = 0.060$ over a frequency range from 500 to 3500 Hz agreed very well with the theoretical calculations, confirming the applicability of the theoretical models investigated.

Further calculations show, however, that for only somewhat smaller values of g the greater damping in the flat-plate model leads to significant differences between the flat-plate and circular-tube predictions. Figures 11 and 12 display predicted pressure-amplitude ratios for 7.7-cm lengths of tubing with

$g = 0.020$ and 0.010 cm, respectively. The damping is higher, and predictions from the two models are significantly different. This suggests a lower bound of about $g = 0.040$ cm, below which circular-tube properties may no longer be adapted with confidence to tubing of other cross-sectional shapes with the equivalent radius formalism. Further experiments would be required to determine if either of the models considered here gives a good representation of very small flat-oval tubing.

National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio, March 13, 1986

Appendix A
Symbols

A,B complex constants of integration (eq. (13))

C_1,C_2 complex constants of integration (eq. (B13))

$\mathcal{C}$ capacitance per unit length of transmission line (eq. (C1))

c adiabatic phase velocity of sound (eq. (11))

c_p specific heat at constant pressure (eq. (6))

c_v specific heat at constant volume

D/Dt differential operator (eq. (2))

F geometry-dependent function for flat plates (eq. (16)) or ratio of Bessel functions for circular tube (eq. (C10))

$F_1(x)$ arbitrary function of integration (eq. (B27))

$f(x)$ function for separation of variables (eq. (B10))

$\mathcal{G}$ shunt leakage conductance per unit length of transmission line (eq. (C2)

g equivalent radius (generalized radius) (eq. (22))

h distance separating parallel plates (eq. (17))

i position-dependent harmonic amplitude of i' (eq. (C4))

i' electric current in transmission line (eq. (C6))

J_0,J_2 integer-order Bessel function of first kind (eq. (19))

j imaginary unit

K complex constant of integration (eq. (B11))

k polytropic constant for cavity (eq. (C39))

L length of section of tube or duct (eq. (C23))

$\mathcal{L}$ inductance per unit length of transmission line (eq. (C6))

ℓ subscript labeling tube section (eq. (C46))

m mass of parcel of gas (eq. (23))

n complex effective polytropic factor for tube (eq. (B31))

Pr Prandtl number (eq. (B6))

p position-dependent harmonic amplitude of excess pressure (eq. (7))

p_a average absolute pressure (eq. (7))

p' absolute pressure (eq. (1))

$q(z)$ function for separation of variables (eq. (B10))

q' electric charge (eq. (C1))

R electric resistance

R_0 ideal gas constant (eq. (5))

$\mathcal{R}$ electric resistance per unit length of transmission line (eq. (C6))

r inside radius of tube with circular cross section

S cross-sectional area of tube (eq. (23))

s general real parameter (eq. (19))

T position-dependent harmonic amplitude of temperature variations (eq. (7))

T_a average or ambient absolute temperature (eq. (7))

T' absolute temperature (eq. (5))

t time (eq. (7))

U harmonic volume velocity amplitude (flow amplitude) (eq. (C8))

u axial component of $\mathbf{u}$ (eq. (8))

$\bar{u}$ average value of u over cross section of tube (eq. (18))

$\mathbf{u}$ position-dependent harmonic amplitude of fluid velocity (eq. (7))

$\mathbf{u}'$ fluid velocity (eq. (1))

V position-dependent harmonic amplitude of V' (eq. (C4))

V volume of cavity (eq. (C39))

V_t volume of tube section (eq. (C56))

V' electric potential difference (eq. (C1))

v,w transverse rectangular components of $\mathbf{u}$ (eq. (10))

x axial rectangular coordinate (eq. (8))

$\mathcal{Y}$ total shunt electric or acoustic admittance per unit length (eqs. (C5) and (C12))

y,z transverse rectangular coordinates (eq. (8))

$Z(x)$ acoustic impedance of tube at x (eq. (C17))

Z_c characteristic impedance of tube (eq. (C19))

Z_{in} acoustic input impedance of tube (eq. (C32))

Z_T acoustic terminating impedance of tube (eq. (C33))

$\mathcal{Z}$ total series electric or acoustic impedance per unit length (eqs. (C7) and (C10))

z dimensionless scaled transverse coordinate (eqs. (B8) and (B20))

α shear wave number (eq. (B7))

γ ratio of specific heats, c_p/c_v (eq. (11))

ζ general complex argument of function (eq. (16))

λ thermal conductivity (eq. (6))

μ	coefficient of shear viscosity (eq. (1))	ρ'	absolute fluid density (eq. (1))
ρ	position-dependent harmonic amplitude of density variations (eq. (7))	σ	diaphragm deflection factor (eq. (C39))
		φ	complex propagation constant (eq. (13))
ρ_a	average fluid density (eq. (7))	ω	angular frequency (eq. (7))

Appendix B

Plane Waves Propagating Between Infinite Planes

Consider a rectangular coordinate system with two infinite rigid plates located at $y = \pm h/2$. Suppose a wave propagates in the space between them, and choose the x-axis to point in the direction of propagation. The linearized equations describing the gas are those given as equations (8) through (12) in the section Solving Fluid Equations for Noncircular Tubing, except that since any z-location is equivalent to any other, the z-derivative terms and hence the z-component fluid velocity amplitude w disappear from the equations. Thus we seek to find the small deviations from equilbrium by solving

$$j\omega u = -\frac{1}{\rho_a}\frac{\partial p}{\partial x} + \frac{\mu}{\rho_a}\frac{\partial^2 u}{\partial y^2} \tag{B1}$$

$$0 = -\frac{\partial p}{\partial y} \tag{B2}$$

$$j\omega\rho = -\rho_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{B3}$$

$$\rho = \frac{\gamma}{c^2}\left(p - \rho_a R_0 T\right) \tag{B4}$$

$$j\omega\rho_a c_p T = \lambda\frac{\partial^2 T}{\partial y^2} + j\omega p \tag{B5}$$

We start with equation (B5). When the Prandtl number which relates viscous and thermal effects is defined as

$$Pr \equiv \frac{\mu c_p}{\lambda} \tag{B6}$$

the shear wave number as

$$\alpha \equiv j^{3/2}h\sqrt{\frac{\rho_a\omega}{\mu}} \tag{B7}$$

and a new dimensionless variable z as

$$z = \alpha\left(\frac{y}{h}\right)\sqrt{Pr} \tag{B8}$$

equation (B5) may be written as

$$T + \frac{\partial^2 T}{\partial z^2} = \frac{p}{\rho_a c_p} \tag{B9}$$

This may be solved by separation of variables. Let

$$T(x, z) = f(x)q(z) \tag{B10}$$

Substituting this into equation (B9) yields

$$q(z) + \frac{d^2 q}{dz^2} = \frac{1}{\rho_a c_p}\frac{p(x)}{f(x)} = K \tag{B11}$$

with the conclusions

$$f(x) = \frac{1}{\rho_a c_p K}p(x) \tag{B12}$$

and

$$q(z) = C_1 \sin z + C_2 \cos z + K \tag{B13}$$

If the plates are isothermal, so that the temperature fluctations vanish at the walls, the two boundary conditions are

$$T = 0 \tag{B14}$$

for

$$y = \pm\frac{h}{2}$$

or

$$q = 0 \tag{B15}$$

for

$$z = \pm\frac{\alpha\sqrt{Pr}}{2}$$

Thus we find that in equation (B13)

$$C_1 = 0 \qquad C_2 = -\frac{K}{\cos\left(\dfrac{\alpha\sqrt{Pr}}{2}\right)} \tag{B16}$$

yielding

$$T(x,z) = f(x)q(z)$$

$$= \left[1 - \frac{\cos z}{\cos\left(\dfrac{\alpha\sqrt{Pr}}{2}\right)} \right] \frac{p(x)}{\rho_a c_p} \qquad \text{(B17)}$$

In terms of the original variables, this is

$$T(x,y) = \left[1 - \frac{\cos\left(\dfrac{\alpha\sqrt{Pr}\,y}{h}\right)}{\cos\left(\dfrac{\alpha\sqrt{Pr}}{2}\right)} \right] \frac{p(x)}{\rho_a c_p} \qquad \text{(B18)}$$

Inserting the result of equation (B18) in equation (B4) and using the fact that $c_p - c_v = R_0$, we find

$$\rho(x,y) = \frac{\gamma}{c^2}\left\{ 1 - \frac{\gamma - 1}{\gamma} \right.$$

$$\left. \times \left[1 - \frac{\cos\left(\dfrac{\alpha\sqrt{Pr}\,y}{h}\right)}{\cos\left(\dfrac{\alpha\sqrt{Pr}}{2}\right)} \right] \right\} p(x) \qquad \text{(B19)}$$

Next, the Navier-Stokes equation, equation (B1), may be solved by separation of variables as was the energy equation. If we let

$$z = \alpha\,\frac{y}{h} \qquad \text{(B20)}$$

equation (B1) becomes

$$u + \frac{\partial^2 u}{\partial z^2} = -\frac{1}{j\omega\rho_a}\frac{dp}{dx} \qquad \text{(B21)}$$

Letting $u(x,z) = f(x)q(z)$ and requiring that u vanish at the plates because of viscous drag, we find

$$u(x,y) = \frac{1}{j\omega\rho_a}\left[\frac{\cos\left(\dfrac{\alpha y}{h}\right)}{\cos\left(\dfrac{\alpha}{2}\right)} - 1 \right]\frac{dp}{dx} \qquad \text{(B22)}$$

Notice that at any particular x, the average longitudinal fluid velocity is

$$\bar{u}(x) = \frac{1}{h}\int_{-h/2}^{h/2} u(x,y)\,dy$$

$$= \frac{1}{j\omega\rho_a}\left[\frac{\tan\left(\dfrac{\alpha}{2}\right)}{\left(\dfrac{\alpha}{2}\right)} - 1 \right]\frac{dp}{dx} \qquad \text{(B23)}$$

The quantity in square brackets appears in several expressions, and it is given a special function notation:

$$F(\alpha) \equiv \left(\frac{2}{\alpha}\right)\tan\left(\frac{\alpha}{2}\right) - 1 \qquad \text{(B24)}$$

Finally we look at the equation of continuity, equation (B3), which may be written

$$\frac{\partial v}{\partial y} = -j\omega\frac{\rho}{\rho_a} - \frac{\partial u}{\partial x} \qquad \text{(B25)}$$

Inserting equation (B19) for ρ into equation (B25) and equation (B22) for u, we get

$$\frac{\partial v}{\partial y} = \frac{1}{j\omega\rho_a}\left\langle \frac{\omega^2\gamma}{c^2}\left\{ 1 - \frac{\gamma - 1}{\gamma} \right.\right.$$

$$\left.\left. \times \left[1 - \frac{\cos\left(\dfrac{\alpha\sqrt{Pr}\,y}{h}\right)}{\cos\left(\dfrac{\alpha\sqrt{Pr}}{2}\right)} \right] \right\} p(x) \right.$$

$$\left. - \left[\frac{\cos\left(\dfrac{\alpha y}{h}\right)}{\cos\left(\dfrac{\alpha}{2}\right)} - 1 \right]\frac{d^2 p}{dx^2} \right\rangle \qquad \text{(B26)}$$

If we then integrate with respect to y, the result is

$$v(x,y) = \frac{1}{j\omega\rho_a}\left\langle \frac{\omega^2\gamma}{c^2}\left\{ y - \frac{\gamma - 1}{\gamma} \right.\right.$$

$$\left. \times \left[y - \frac{h}{\alpha\sqrt{Pr}}\frac{\sin\left(\frac{\alpha\sqrt{Pr}\,y}{h}\right)}{\cos\left(\frac{\alpha\sqrt{Pr}}{2}\right)} \right] \right\} p(x)$$

$$\left. - \left[\frac{h}{\alpha}\frac{\sin\left(\frac{\alpha\,y}{h}\right)}{\cos\left(\frac{\alpha}{2}\right)} - y \right]\frac{d^2p}{dx^2} + F_1(x) \right\rangle \quad \text{(B27)}$$

At the rigid plates, v, the transverse component of the fluid velocity, must vanish. Setting equation (B27) equal to zero first at $y = +h/2$ and then at $y = -h/2$ and adding the two results show that $F_1(x) = 0$. Subtracting one equation from the other and cancelling h give

$$F(\alpha)\frac{d^2p}{dx^2} - \frac{\omega^2}{c^2}\gamma\left[1 + \frac{\gamma - 1}{\gamma}F\left(\alpha\sqrt{Pr}\right) \right]p = 0 \quad \text{(B28)}$$

This has the solution

$$p(x) = Ae^{\varphi x} + Be^{-\varphi x} \quad \text{(B29)}$$

where the propagation constant φ may be written

$$\varphi = \frac{\omega}{c}[F(\alpha)]^{-1/2}\sqrt{\frac{\gamma}{n}} \quad \text{(B30)}$$

by using the notation

$$n \equiv \left[1 + \frac{\gamma - 1}{\gamma}F\left(\alpha\sqrt{Pr}\right) \right]^{-1} \quad \text{(B31)}$$

The quantity n may be regarded as a complex effective polytropic factor for pressure changes in the tube (ref. 4).

Since φ is a complex constant with positive real part, equation (B29) together with equation (7) represents a superposition of damped traveling plane waves, one of amplitude A traveling in the $-x$-direction and one of amplitude B traveling in the $+x$-direction. The constants A and B are determined by boundary conditions.

Inserting equation (B29) and its derivative into the expressions for v (eq. (B27)), u (eq. (B22)), ρ (Eq. (B19)), and T (eq. (B18)) completes the solution of the problem.

Appendix C

Transmission-Line Formalism

Once the general expressions for the pressure wave in a tube have been obtained, it remains to calculate the transfer properties of particular arrangements of probe tubing and transducer cavities. Bergh and Tijdeman (ref. 4) derived a workable but cumbersome recursion formula for this purpose. Another approach, which is fairly commonly used for one-dimensional acoustic waves propagating in ducts, is based on equations developed to describe electrical transmission lines. Since the transmission-line formalism with its *ABCD* matrices is much more compact and flexible than the Bergh and Tijdeman recursion relation of reference 4 (the matrices enable one to easily calculate pressure transfer relations for a greater variety of combinations of tubing and to evaluate other quantities of interest such as input impedances), it seems worthwhile to show how the acoustical equations are analogous to the transmission-line ones. The acoustical analysis may then be carried forward in a form suggested by the formalism developed for transmission lines.

Equations for Electrical Transmission Lines

Consider a coaxial cable transmission line for definiteness, although the same concepts apply to parallel-wire and other types of transmission line (ref. 16). Any transmission line is characterized by a distributed electric capacitance per unit length $\mathcal{C}$, inductance per unit length $\mathcal{L}$, resistance per unit length $\mathcal{R}$ (including both conductors), and leakage conductance per unit length $\mathcal{G}$ of the gas or other dielectric between the conductors ($\mathcal{G} \Delta x$ is the reciprocal of the inter-conductor leakage-resistance in length Δx). If a potential difference between the two conductors is set up at one end of the cable, charge will start to flow. Let us find differential equations for the current i' in the central wire and potential V' of the central wire relative to the outer one, as functions of the distance x along the wire and of the time.

As suggested in reference 16, choose an element of the transmission line of infinitesimal length Δx for analysis. There are four physical relations or quantities of interest: the capacitive relation between the voltage V' and the charge on the central wire in Δx, the inductive emf associated with the rate of change of the current in Δx, the $i'R$ drop in potential along the conductors, and the leakage current between the conductors due to conductance through the intervening medium.

The charge q' on length Δx of the central conductor is

$$q' = (\mathcal{C} \Delta x) \, V' \qquad \text{(C1)}$$

The rate at which charge leaves Δx, traveling down the conductor, differs from the rate at which it enters by an amount

equal to the leakage current through the dielectric plus the rate at which the local capacitance is being charged. That is,

$$\Delta i' = - V'(\mathcal{G} \Delta x) - \frac{\partial q'}{\partial t} \qquad \text{(C2)}$$

Substituting from equation (C1) and dividing by Δx give

$$\frac{\partial i'}{\partial x} = - \mathcal{G} V' - \mathcal{C} \frac{\partial V'}{\partial t} \qquad \text{(C3)}$$

If harmonic time dependence is assumed in the form

$$V' = V e^{j\omega t} \qquad \text{(C4a)}$$

$$i' = i e^{j\omega t} \qquad \text{(C4b)}$$

then the first basic transmission-line relation between complex amplitudes is

$$\frac{di}{dx} = - (\mathcal{G} + j\omega\mathcal{C})V \equiv - \mathcal{Y} V \qquad \text{(C5)}$$

Here $\mathcal{Y}$ is the total distributed shunt electric admittance per unit length of line.

The potential drop along Δx is

$$\Delta V' = - i'(\mathcal{R} \Delta x) - (\mathcal{L} \Delta x)\frac{\partial i'}{\partial t} \qquad \text{(C6)}$$

Dividing through by Δx and assuming harmonic time dependence result in the other transmission-line relation:

$$\frac{dV}{dx} = - (\mathcal{R} + j\omega\mathcal{L})i \equiv - \mathcal{Z} i \qquad \text{(C7)}$$

In this equation, $\mathcal{Z}$ is the total distributed series electric impedance per unit length of the line.

Analogous Quantities for Acoustic Tubes

In the natural analogy with an acoustical duct, the excess pressure p is associated with the electric potential and the

volume velocity is associated with the electric current. At any cross section the volume velocity or flow U is defined as

$$U = S\bar{u} \qquad (C8)$$

where S is the cross-sectional area of the duct carrying the wave, and $\bar{u}$ is the fluid velocity averaged over the cross section. Multiplying the volume velocity by the average fluid density gives the mass passing any point per unit time, so that conservation of mass can be conveniently expressed directly in terms of conservation of volume velocity.

Notice from equations (B23) and (B24) that

$$\frac{dp}{dx} = \frac{j\omega\rho_a}{SF(\alpha)} U \qquad (C9)$$

so by analogy with equation (C7), the total distributed series acoustic impedance per unit length of the duct may be said to be

$$\mathcal{Z} = -j\left(\frac{\omega}{c}\right)\frac{\rho_a c}{S}\frac{1}{F(\alpha)} \qquad (C10)$$

As a function of α, $\mathcal{Z}$ depends on the viscosity but not on the thermal conductivity of the gas. The imaginary part of $\mathcal{Z}/\omega$ is the distributed inertance of the fluid per unit length, while the real part of $\mathcal{Z}$ is the distributed series acoustic resistance of the duct per unit length, due to viscous damping. In the section Comparison with Flat-Plate Geometry, it is pointed out that the expressions for pressure and longitudinal fluid velocity waves in a circular duct deduced by Bergh and Tijdeman in reference 4 have the same form as the ones derived here for parallel flat plates. All the circular duct results may be obtained from the ones for parallel flat plates by replacing the function $F(\alpha)$ defined in equation (16) by the Bessel function ratio $J_2(\alpha)/J_0(\alpha)$. For that reason, it is possible to use the equations of this appendix to analyze either flat plates or circular ducts, by letting $F(\alpha)$ refer to the function of equation (16) or to the Bessel function ratio, depending on the case of interest. In addition, experiments described in this report show that, within limits, tubes with other cross-sectional shapes may be treated with either flat-plate or circular-tube functions if the tube is described in terms of its equivalent radius.

Differenting equation (C9) with respect to x and using equation (B28), we find

$$\frac{dU}{dx} = -j\frac{S}{\rho_a c}\left(\frac{\omega}{c}\right)\gamma\left[1 + \frac{\gamma-1}{\gamma}F\left(\alpha\sqrt{Pr}\right)\right]p \qquad (C11)$$

which is the other transmission-line relation for the acoustical wave in a duct. The total distributed-shunt acoustic admittance per unit length is

$$\mathcal{Y} = j\left(\frac{\omega}{c}\right)\frac{S}{\rho_a c}\gamma\left[1 + \frac{\gamma-1}{\gamma}F\left(\alpha\sqrt{Pr}\right)\right]$$

$$= j\left(\frac{\omega}{c}\right)\frac{S}{\rho_a c}\frac{\gamma}{n} \qquad (C12)$$

which depends on thermal conductivity but not viscosity. The imaginary part of $\mathcal{Y}/\omega$ is the distributed compliance of the fluid per unit length, while the real part of $\mathcal{Y}$ is the reciprocal of a resistance describing energy loss to the walls through thermal conduction per unit length.

The two quantities $\mathcal{Z}$ and $\mathcal{Y}$ define the fundamental physical properties of the duct. When multiplied by the appropriate length Δx, they give the series impedance and shunt admittance of an infinitesimal element of tube — a tube which is very short compared to the wavelength of any acoustical disturbance in the tube. Since an elastic fluid in a long tube sustains wave motion in which a harmonic disturbance at one point is not in phase with the disturbance at other locations, $\mathcal{Z}$ and $\mathcal{Y}$ cannot be used to directly find the impedances of a finite length of tube. Rather, the two quantities $\mathcal{Z}$ and $\mathcal{Y}$ are first used to calculate the characteristic impedance and propagation constant of the tube. It is then possible to analyze finite tubes with various terminations, or combinations of tubes.

We may proceed as follows: Starting from the transmission-line type equations

$$\frac{dp}{dx} = -\mathcal{Z}U \qquad (C13a)$$

$$\frac{dU}{dx} = -\mathcal{Y}p \qquad (C13b)$$

we differentiate equation (C13a) and substitute equation (C13b) to get

$$\frac{d^2p}{dx^2} = -\mathcal{Z}\frac{dU}{dx} = (\mathcal{Z}\mathcal{Y})p \qquad (C14)$$

This wave equation for p implies the result of equation (B29), with propagation constant

$$\varphi = \sqrt{\mathcal{Z}\mathcal{Y}} \qquad (C15)$$

From equations (C13a) and (C13b) again, the corresponding volume velocity wave is

$$U = -\frac{1}{\mathcal{Z}}\frac{dp}{dx} = -\frac{\varphi}{\mathcal{Z}}\left(Ae^{\varphi x} - Be^{-\varphi x}\right)$$

$$= \sqrt{\frac{\mathcal{Y}}{\mathcal{Z}}}\left(Be^{-\varphi x} - Ae^{\varphi x}\right) \qquad (C16)$$

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The combination of $\mathcal{Z}$ and $\mathcal{Y}$ appearing in the last expression also has physical significance. The acoustic impedance of the duct at any point is defined as

$$Z(x) \equiv \frac{p(x)}{U(x)} \qquad (C17)$$

Given our expressions for the waves,

$$Z(x) = \sqrt{\frac{\mathcal{Z}}{\mathcal{Y}}} \frac{Be^{-\varphi x} + Ae^{\varphi x}}{Be^{-\varphi x} - Ae^{\varphi x}} \qquad (C18)$$

If a wave is introduced at $x = 0$ into a semi-infinite tube along the positive x-axis there will be no reflected wave traveling back toward the origin. This implies that $A = 0$. The impedance of the duct under these conditions is independent of position and is called the characteristic impedance of the duct Z_c. Thus

$$Z_c = \sqrt{\frac{\mathcal{Z}}{\mathcal{Y}}} = \frac{\mathcal{Z}}{\varphi} = \frac{\varphi}{\mathcal{Y}} \qquad (C19)$$

At any x, the relation between $Z(x)$ and the characteristic impedance of the duct Z_c at that point indicates the relative amplitude and phase of leftward- and rightward-traveling pressure waves in the duct. Rearranging equation (C18) shows specifically that the ratio of leftward to rightward wave amplitudes is

$$\frac{Ae^{\varphi x}e^{j\omega t}}{Be^{-\varphi x}e^{j\omega t}} = \frac{Z(x) - Z_c}{Z(x) + Z_c} \qquad (C20)$$

Transmission-Line Matrices

The stipulation of the values of p and its derivative dp/dx at some point determines a unique solution to the wave equation, as the constants A and B in equation (B29) are thereby determined. Since the volume velocity U is proportional to dp/dx (see eq. (C13)), the choice of pressure and volume velocity at any point in a duct determines the pressure and volume velocity at any other point. That is, the pressure and volume velocity at any point in a duct can always be expressed as a function which is a linear combination of the pressure and volume velocity at a reference point. This fact leads to the transmission-line equations and $ABCD$ matrices (ref. 17).

For example, let the wave amplitudes at $x = 0$ in a long duct be called p_0 and U_0, with the corresponding quantities at $x = L$ being p_L and U_L. From equations (B29) and (C16) we obtain

$$p_0 = A + B \qquad (C21)$$

$$U_0 = \frac{(B - A)}{Z_c} \qquad (C22)$$

$$p_L = Ae^{\varphi L} + Be^{-\varphi L} \qquad (C23)$$

$$U_L = \frac{(Be^{-\varphi L} - Ae^{\varphi L})}{Z_c} \qquad (C24)$$

Equations (C23) and (C24) may be solved for A and B as

$$A = \frac{(p_L - Z_c U_L)e^{-\varphi L}}{2} \qquad (C25)$$

$$B = \frac{(p_L + Z_c U_L)e^{\varphi L}}{2} \qquad (C26)$$

Inserting equations (C25) and (C26) into equations (C21) and (C22) yields

$$p_0 = p_L \cosh \varphi L + U_L Z_c \sinh \varphi L \qquad (C27)$$

$$U_0 = p_L \frac{1}{Z_c} \sinh \varphi L + U_L \cosh \varphi L \qquad (C28)$$

which may be written as a matrix equation

$$\begin{bmatrix} p_0 \\ \\ U_0 \end{bmatrix} = \begin{bmatrix} \cosh \varphi L & Z_c \sinh \varphi L \\ \\ \frac{1}{Z_c} \sinh \varphi L & \cosh \varphi L \end{bmatrix} \begin{bmatrix} p_L \\ \\ U_L \end{bmatrix} \qquad (C29)$$

The matrix is called the transmission-line matrix, or the $ABCD$ matrix, the latter name coming from expressing the generic 2 by 2 matrix as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (C30)$$

There is no connection between the name $ABCD$ matrix and the A and B in equations (C25) and (C26).

The transmission-line matrix can easily be used to calculate the input impedance of a duct of length L which is closed at

the far end. The boundary condition is expressed by setting $U_L = 0$. Then

$$\begin{bmatrix} p_0 \\ \\ U_0 \end{bmatrix} = \begin{bmatrix} \cosh \varphi L & Z_c \sinh \varphi L \\ \\ \dfrac{1}{Z_c} \sinh \varphi L & \cosh \varphi L \end{bmatrix} \begin{bmatrix} p_L \\ \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} p_L \cosh \varphi L \\ \\ \dfrac{p_L}{Z_c} \sinh \varphi L \end{bmatrix} \tag{C31}$$

and

$$Z_{\text{in}} \equiv \frac{p_0}{U_0} = Z_c \coth \varphi L \tag{C32}$$

The input impedance can be calculated just as easily for an arbitrary terminating impedance Z_T. The result is

$$Z_{\text{in}} = Z_c \frac{Z_T \cosh \varphi L + Z_c \sinh \varphi L}{Z_T \sinh \varphi L + Z_c \cosh \varphi L} \tag{C33}$$

Other events in the life of a transmission line may also be expressed in terms of *ABCD* matrices including the effects of various kind of discontinuties in the line. Suppose that at some point in a duct there are several side branches so that not all the mass arriving at that point from the upstream duct leaves by the downstream duct. Using subscripts u to describe the situation just upstream of the discontinuity, d for the downstream side of the discontinuity, and 1, 2, 3, ... for the entrances to the side branches, we have

$$p_u = p_d = p_1 = p_2 = p_3 = \dots \tag{34}$$

and

$$U_u = U_d + U_1 + U_2 + U_3 + \dots \tag{C35}$$

Expressing U_1 in terms of the input impedance to branch 1, denoted Z_1, as

$$Z_1 \equiv \frac{p_1}{U_1} = \frac{p_d}{U_1} \tag{C36}$$

and expressing the flow into the other branches in a similar way, we obtain

$$U_u = U_d + \left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \dots \right) p_d \tag{C37}$$

or

$$\begin{bmatrix} p_u \\ \\ U_u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \\ \dfrac{1}{Z_1} + \dfrac{1}{Z_2} + \dfrac{1}{Z_3} + \dots & 1 \end{bmatrix} \begin{bmatrix} p_d \\ \\ U_d \end{bmatrix} \tag{C38}$$

As pointed out again in the section Applications of *ABCD* Matrices to Compound Systems, this matrix is equal to the product of several matrices of the same form, one for each of the individual side branches.

A particular side branch could be an actual side tube attached to the main duct to monitor the average total pressure, with input impedance calculated as above or with other *ABCD* matrices; or the side branch could be a local cavity, some of the flow into which pumps up the pressure rather than going out the other side. Bergh and Tijdeman (ref. 4) show, in the developments leading up to their appendix equation (53), for a cavity of volume V containing gas with polytropic constant k (value between 1 for an isothermal process and γ for an adiabatic one) and ddd (dimensionless diaphragm deflection) factor σ that the relation between the rate at which mass is flowing in and out of the volume and the excess pressure is

$$\frac{dm}{dt} = \frac{j\omega\gamma}{c^2} V \left(\sigma + \frac{1}{k} \right) p \tag{C39}$$

Since dm/dt can be considered equal to $\rho_a U$, the effective input impedance of such a cavity is

$$Z_{\text{in}} = \frac{p}{U} = \left[j \left(\frac{\omega}{c} \right) \frac{\gamma V}{\rho_a c} \left(\sigma + \frac{1}{k} \right) \right]^{-1} \tag{C40}$$

and the appropriate *ABCD* matrix for such a volume is

$$\begin{bmatrix} 1 & 0 \\ \\ j \left(\dfrac{\omega}{c} \right) \dfrac{\gamma V}{\rho_a c} \left(\sigma + \dfrac{1}{k} \right) & 1 \end{bmatrix} \tag{C41}$$

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Many other *ABCD* matrices can be derived. One can imagine a discontinuity in the form of a porous plug in a duct or an impervious, flexible membrane with mass, where there is a pressure discontinuity but the volume flow is continuous. This amounts to a lumped series impedance Z' in the duct, with matrix given by

$$\begin{bmatrix} 1 & Z' \\ 0 & 1 \end{bmatrix} \tag{C42}$$

There is actually a small series impedance associated with a step change in the radius of a circular duct (ref. 18), which can be placed in such a matrix.

An arbitrary discontinuity can be represented by a combination of series and shunt impedances. An *ABCD* matrix has been written for a section of duct with a conically tapered cross section (ref. 17).

Applications of *ABCD* Matrices to Compound Systems

Suppose one wishes to analyze the pressure-transfer properties or calculate the input impedance of a system of many interconnected parts, such as the general one shown in figure 13. All that is required is to write down the *ABCD* matrix for each separate part and then multiply them together. Since
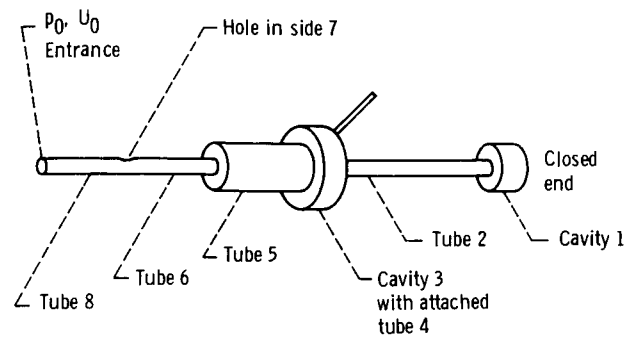

Figure 13.—General compound linear acoustic system.

$$\begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_1} + \dfrac{1}{Z_2} & 1 \end{bmatrix} \tag{C43}$$

a compound side branch discontinuity such as a cavity with an attached side tube can be represented by either one overall matrix or the product of two matrices, one for each of the branches at that point. The pressure and volume velocity at the left entrance to tube 2 in figure 13 are given by

$$\begin{bmatrix} p_2 \\ U_2 \end{bmatrix} = \begin{bmatrix} \text{matrix for} \\ \text{tube 2} \end{bmatrix} \begin{bmatrix} \text{matrix for} \\ \text{cavity 1} \end{bmatrix} \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \tag{C44}$$

while at the far left entrance to the complete system we have

$$\begin{bmatrix} p_0 \\ U_0 \end{bmatrix} = \underbrace{\begin{bmatrix} \text{tube} \\ 8 \end{bmatrix} \begin{bmatrix} \text{hole} \\ 7 \end{bmatrix} \begin{bmatrix} \text{tube} \\ 6 \end{bmatrix} \begin{bmatrix} \text{tube} \\ 5 \end{bmatrix} \begin{bmatrix} \text{side} \\ \text{tube} \\ 4 \end{bmatrix} \begin{bmatrix} \text{cavity} \\ 3 \end{bmatrix} \begin{bmatrix} \text{tube} \\ 2 \end{bmatrix} \begin{bmatrix} \text{cavity} \\ 1 \end{bmatrix}}_{ABCD \text{ matrices}} \begin{bmatrix} p_1 \\ 0 \end{bmatrix} \tag{C45}$$

It is quite easy to write computer programs to evaluate the matrices, multiply them, and print out input impedance $Z_{\text{in}} = p_0/U_0$ or pressure-transfer function p_1/p_0. (Since the equations are linear, for these calculations p_1 may be set equal to an arbitrary constant such as 1.)

Derivation of Bergh and Tijdeman Recursion Relation

While it is much easier to program and use the *ABCD* matrices than the Bergh and Tijdeman recursion relation (ref. 4), it is a satisfying exercise to derive the recursion relation from the matrix formalism and thus show they are equivalent. Consider the system of two tubes and a cavity shown in figure 14. At the entrance to the tube of length $L_{\ell+1}$, just after the cavity of volume V_ℓ, we have from equation (C29)
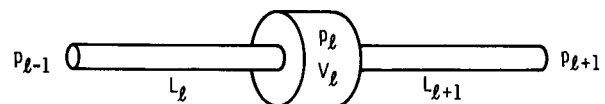
Figure 14.—Acoustic system consisting of cavity with input and output tubes, used in derivation of Bergh and Tijdeman recursion relation.

$$\begin{bmatrix} p_\ell \\ U_\ell \end{bmatrix} = \begin{bmatrix} \cosh \varphi_{\ell+1} L_{\ell+1} & Z_{c,\ell+1} \sin \varphi_{\ell+1} L_{\ell+1} \\ \dfrac{1}{Z_{c,\ell+1}} \sinh \varphi_{\ell+1} L_{\ell+1} & \cosh \varphi_{\ell+1} L_{\ell+1} \end{bmatrix}$$

$$\times \begin{bmatrix} p_{\ell+1} \\ U_{\ell+1} \end{bmatrix} \tag{C46}$$

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Multiplying this equation out, we find from the first component

$$p_\ell = p_{\ell+1} \cosh \varphi_{\ell+1}L_{\ell+1} + Z_{c,\ell+1}U_{\ell+1} \sinh \varphi_{\ell+1}L_{\ell+1} \quad \text{(C47)}$$

which implies

$$U_{\ell+1} = \frac{p_\ell - p_{\ell+1} \cosh \varphi_{\ell+1}L_{\ell+1}}{Z_{c,\ell+1} \sinh \varphi_{\ell+1}L_{\ell+1}} \quad \text{(C48)}$$

and from the second component

$$U_\ell = \frac{p_{\ell+1}}{Z_{c,\ell+1}} \sinh \varphi_{\ell+1}L_{\ell+1} + U_{\ell+1} \cosh \varphi_{\ell+1}L_{\ell+1} \quad \text{(C49)}$$

Inserting equation (C48) into equation (C49) gives as a result

$$\left[\begin{array}{c} p_\ell \\ \\ j\left(\dfrac{\omega}{c}\right)\dfrac{\gamma V_\ell}{\rho_a c}\left(\sigma + \dfrac{1}{k}\right)p_\ell + \dfrac{p_{\ell+1}}{Z_{c,\ell+1}} \sinh \varphi_{\ell+1}L_{\ell+1} + \left(\dfrac{p_\ell - p_{\ell+1} \cosh \varphi_{\ell+1}L_{\ell+1}}{Z_{c,\ell+1} \sinh \varphi_{\ell+1}L_{\ell+1}}\right) \cosh \varphi_{\ell+1}L_{\ell+1} \end{array} \right] \quad \text{(C52)}$$

So we finally calculate the pressure at the input of tube ℓ by multiplying equation (C52) by the *ABCD* matrix for a tube of length L_ℓ. The result is

$$p_{\ell-1} = p_\ell \cosh \varphi_\ell L_\ell + Z_{c,\ell} \sinh \varphi_\ell L_\ell$$

$$\times \left\{ j\left(\frac{\omega}{c}\right)\frac{\gamma V_\ell}{\rho_a c}\left(\sigma + \frac{1}{k}\right)p_\ell + \frac{p_{\ell+1}}{Z_{c,\ell+1}} \sinh \varphi_{\ell+1}L_{\ell+1} \right.$$

$$\left. + \left(\frac{p_\ell - p_{\ell+1} \cosh \varphi_{\ell+1}L_{\ell+1}}{Z_{c,\ell+1} \sinh \varphi_{\ell+1}L_{\ell+1}}\right) \cosh \varphi_{\ell+1}L_{\ell+1} \right\} \quad \text{(C53)}$$

Dividing by p_ℓ and rearranging yield

$$\frac{p_{\ell-1}}{p_\ell} = \cosh \varphi_\ell L_\ell + j\left(\frac{\omega}{c}\right)\frac{\gamma V_\ell}{\rho_a c}\left(\sigma + \frac{1}{k}\right)Z_{c,\ell} \sinh \varphi_\ell L_\ell$$

$$+ \frac{Z_{c,\ell}}{Z_{c,\ell+1}} \frac{\sinh \varphi_\ell L_\ell}{\sinh \varphi_{\ell+1}L_{\ell+1}}\left(\cosh \varphi_{\ell+1}L_{\ell+1} - \frac{p_{\ell+1}}{p_\ell}\right) \quad \text{(C54)}$$

The second of the three terms summed on the right side of the equation can be transformed into the Bergh and Tijdeman form by noticing that the last equalities in equations (C12) and

(C19) imply (when the numerator and denominator are multiplied by tube length L)

$$U_\ell = \frac{p_{\ell+1}}{Z_{c,\ell+1}} \sinh \varphi_{\ell+1}L_{\ell+1}$$

$$+ \left(\frac{p_\ell - p_{\ell+1} \cosh \varphi_{\ell+1}L_{\ell+1}}{Z_{c,\ell+1} \sinh \varphi_{\ell+1}L_{\ell+1}}\right) \cosh \varphi_{\ell+1}L_{\ell+1} \quad \text{(C50)}$$

In view of equation (C41), just upstream (left) of the cavity, the pressure and volume flow are given by

$$\left[\begin{array}{cc} 1 & 0 \\ j\left(\dfrac{\omega}{c}\right)\dfrac{\gamma V_\ell}{\rho_a c}\left(\sigma + \dfrac{1}{k}\right) & 1 \end{array} \right] \left[\begin{array}{c} p_\ell \\ U_\ell \end{array} \right] \quad \text{(C51)}$$

where U_ℓ is expressed by equation (C50).

When the multiplication in equation (C51) is carried out, the pressure-velocity vector at the output of tube ℓ is

$$Z_c = \frac{\varphi}{\text{y}} = \frac{n\varphi L}{j\left(\dfrac{\omega}{c}\right)} \frac{\rho_a c}{\gamma} \frac{1}{SL} \quad \text{(C55)}$$

(As shown by the fact that it cancels out in equation (C55), the characteristic impedance Z_c does not properly depend on the length of a tube. However, Bergh and Tijdeman apparently wish to introduce it so that they can express their results in terms of the tube volume $V_t = SL$. Note that in all Bergh and Tijdeman expressions, the tube volumes V_t are somewhere divided by the corresponding tube length L.) Finally, when equation (C10) and the next-to-last part of equation (C19) are used, the ratio of characteristic impedances in the last term is equivalent to

$$\frac{Z_{c,\ell}}{Z_{c,\ell+1}} = \frac{Z_\ell \varphi_{\ell+1}}{\varphi_\ell Z_{\ell+1}} = \frac{V_{t,\ell+1}\varphi_{\ell+1}L_\ell F(\alpha_{\ell+1})}{V_{t,\ell}\varphi_\ell L_{\ell+1}F(\alpha_\ell)} \quad \text{(C56)}$$

So, regarding the F function as the ratio of Bessel functions J_2/J_0, we get the Bergh and Tijdeman recursion relation of reference 4, which is

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$$\frac{p_\ell}{p_{\ell-1}} = \left[\cosh \varphi_\ell L_\ell + \frac{V_\ell}{V_{t,\ell}} \left(\sigma_\ell + \frac{1}{k_\ell} \right) n_\ell \varphi_\ell L_\ell \sinh \varphi_\ell L_\ell + \frac{V_{t,\ell+1} \varphi_{\ell+1} L_\ell J_0(\alpha_\ell) J_2(\alpha_{\ell+1}) \sinh \varphi_\ell L_\ell}{V_{t,\ell} \varphi_\ell L_{\ell+1} J_2(\alpha_\ell) J_0(\alpha_{\ell+1}) \sinh \varphi_{\ell+1} L_{\ell+1}} \right.$$

$$\left. \times \left(\cosh \varphi_{\ell+1} L_{\ell+1} - \frac{p_{\ell+1}}{p_\ell} \right) \right]^{-1} \qquad \text{(C57)}$$

Examples of its use are found in reference 5.

Appendix D
Computer Programs

Several short FORTRAN routines have been written to apply the *ABCD* matrices to the analysis of probe tubes coupled to transducer cavities. The basic package consists of three general-purpose subroutines, which are called by a main program designed to reflect the structure of the system being analyzed. Listings for the subroutines and two sample main programs are given at the end of this appendix.

Subroutine TUBE evaluates the *ABCD* matrix of equation (C29) for a duct of generalized radius G, cross-sectional area S, and length L. It uses the formulas from either the circular-cylinder or the infinite-plate geometry, depending on whether the value of parameter IS is 1 or 2, respectively. The previous values of the components of the pressure-volume velocity vector are read in as the array PU, they are multiplied by the *ABCD* matrix, and the new vector is returned in the same variables. Information about the frequency, thermodynamic properties of the fluid, and mathematical constants is provided in COMMON blocks which must be initialized in the calling program. The propagation constant PHI calculated within TUBE is made available to the calling program in another COMMON block, to use if desired.

Function RFUN is used to evaluate either the F of equation (B24) or the ratio J_2/J_0, depending on the value of parameter IS. It is called by subroutine TUBE and uses subroutines in the NASA Lewis Research Center FORTRAN library to compute the Bessel functions. For large arguments the common limiting form given in equation (21) is used.

Subroutine VOL evaluates the *ABCD* matrix of equation (C41) for a small cavity in the line. The cavity is characterized by its volume V, ddd factor SIG, and polytropic constant POLYK. The appropriate values of POLYK vary from 1.0 for an isothermal process to 1.4 for an adiabatic one. This constant is set equal to 1.0 in the programs of reference 5; perhaps a better estimate could be made on the basis of the expression for n in equation (B31). Subroutine VOL also takes in and updates the current pressure-flow vector.

Two examples of calling programs are provided. The first was written to calculate, for frequencies from 25 to 5000 Hz, the pressure transfer function of a single tube and closed volume system, using both circular- and flat-geometry formulas for the tubing. The results are plotted on the same axes for comparison. The constants are defined in lines 1300 through 2300, 2800 through 2900, and 3600 through 3800. Dimensions of the tubing are read in by lines 2950 through 3190, and the transmission-line calculations are carried out at each frequency in the DO-loop in lines 3500 through 4960.

The pressure-volume velocity vector for the circular-tubing model is the array PUR, while that for the flat-plate model is PUF. The remainder of the program plots the results by using routines standard at the Lewis Research Center. Numerical values for circular geometry computed with this program agree with calculations from the programs of reference 5 and with graphs in Bergh and Tijdeman (ref. 4).

The output of this program for sample sections of flat-oval tubing leading to a small, closed cavity is shown in figures 9 through 12. In addition, to help an interested user determine that the computer listings have been copied correctly, table I gives selected numerical data calculated by this program.

The second program is an example of how the routines would be used to compute the pressure response of an infinite-line probe. Once again the calculations are done twice, once for each model geometry, and both results are plotted for comparison. The sample graphs in figure 15 show how similar the results are.

TABLE I.—SAMPLE NUMERICAL DATA FROM
COMPUTER PROGRAM

(a) Input Parameters

Example	Cavity volume, cm^3	Generalized radius of tube, cm	Area of tube, cm^2	Length of tube, cm	Frequency, Hz
1	0.01325	0.060	0.01714	5.15	1350
2	.01325			5.15	2700
3	.01325	↓	↓	5.15	4200
4	.01630			7.71	900
5	.01630			7.71	1900
6	.01630	↓	↓	7.71	2800

(b) Calculated Output

Example	Circular tube		Flat plates	
	Amplitude ratio	Relative phase, deg	Amplitude ratio	Relative phase, deg
1	8.98	−91.1	8.51	−91.1
2	.905	−179.2	.904	−179.2
3	4.46	−261.7	4.33	−261.7
4	7.29	−82.6	6.82	−82.6
5	.917	−180.3	.915	−180.3
6	3.62	−249.2	3.49	−249.2
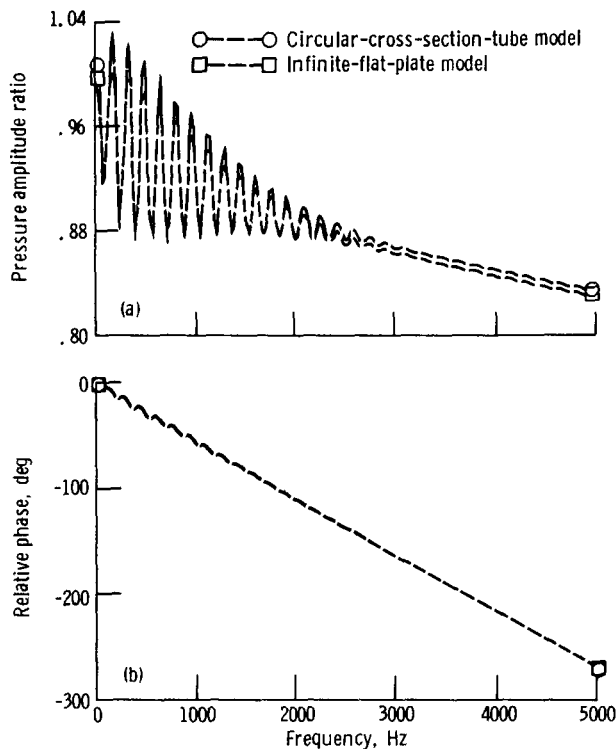
(a) Ratio of amplitude of transducer cavity pressure to amplitude of inlet pressure.

(b) Relative phase of transducer cavity pressure and inlet pressure.

Figure 15.—Calculated ratio of cavity to inlet pressure and relative phase of two pressures as function of frequency for infinite-line probe made from flat-oval tubing with $g = 0.061$ cm and $S = 0.01811$ cm^2. Distance from inlet end to transducer, 5 cm; distance from transducer to closed end, 100 cm.

In general, names of variables are intended to suggest their significance. Definitions of variables placed in COMMON are as follows:

AMDENS   ambient density, ρ_a

AMPRES   ambient pressure, p_a

FREQ   ordinary frequency, f

GAMMA   ratio of specific heats, $\gamma = c_p/c_v$

J $\sqrt{-1}, j$

J32 j raised to 3/2 power

OMEGA   angular frequency, ω

PHI   propagation constant, φ

RHOC   the product $\rho_a c$

SPEED   adiabatic speed of sound, c

SRPRN   square root of Prandtl number, $\sqrt{Pr}$

VISC   coefficient of shear viscosity, μ

WAVENO   free-space wave number, $k = \omega/c$

Subroutine TUBE.—

```
0000100          SUBROUTINE TUBE(G,S,L,IS,PU)
0000200 C
0000300 C        SUBROUTINE TO EVALUATE ABCD MATRIX FOR TUBE OF
0000400 C        LENGTH L, CROSS SECTIONAL AREA S, AND
0000500 C        2(AREA)/PERIMETER EQUAL TO G.     THE PRESSURE-
0000600 C        VOLUME FLOW VECTOR PU IS UPDATED.
0000700 C
0000800 C        IS = 1 --> USE CYLINDRICAL DUCT FORMULAS
0000900 C        IS = 2 --> USE PARALLEL PLATE FORMULAS
0001000 C
0001100          COMPLEX PU(2),J,J32,PHI
0001200          COMMON /XFREQ/ FREQ,OMEGA,WAVENO
0001400          COMMON /THERMO/ GAMMA,SRPRN,AMPRES,AMDENS,SPEED,RHOC,VISC
0001500          COMMON /XTUBE/ PHI
0001600          COMMON /XCONST/ J,J32,PI
0001700 C
0001800          REAL L
0001900          COMPLEX ABCD(2,2),PUTEMP(2),Z,ZC,AL,ALP,CN,CD,CE,CS
0002000          COMPLEX RFUN
0002100 C
0002200          AL = J32*(G*SQRT(AMDENS*OMEGA/VISC))
0002300          ALP = AL*SRPRN
0002400          CN = GAMMA + (GAMMA - 1.)*RFUN(ALP,IS)
0002500          CD = RFUN(AL,IS)
0002600          PHI = WAVENO*CSQRT(CN/CD)
0002700 C
0002800          Z = -J*WAVENO*RHOC/(S*CD)
0002900 C
0003000 C        CHARACTERISTIC IMPEDANCE = SERIES IMPEDANCE/LENGTH /
0003100 C        PROPAGATION CONSTANT
0003200 C
0003300          ZC = Z/PHI
0003400 C
0003450          CE = CEXP(PHI*L)
0003500          ABCD(1,1) = (CE + CMPLX(1.,0.)/CE)/CMPLX(2.,0.)
0003600          ABCD(2,2) = ABCD(1,1)
0003700          CS = (CE - CMPLX(1.,0.)/CE)/CMPLX(2.,0.)
0003800          ABCD(1,2) = ZC*CS
0003900          ABCD(2,1) = CS/ZC
0004000 C
0004100          DO 1 K = 1, 2
0004200        1 PUTEMP(K) = PU(K)
0004300 C
0004400          DO 3 M = 1, 2
0004500          PU(M) = CMPLX(0.,0.)
0004600          DO 2 N = 1, 2
0004700        2 PU(M) = PU(M) + ABCD(M,N) * PUTEMP(N)
0004800        3 CONTINUE
0004900 C
0005000          RETURN
0005100          END
```

Function RFUN.—

```
0001900         COMPLEX FUNCTION RFUN(Z,IS)
0002000 C
0002100         COMPLEX Z,J0,J1,J2,Y0,Y1,N,D,B
0002200 C
0002220 C       IS = 1 --> EVALUATES J2(Z)/J0(Z)
0002240 C       IS = 2 --> EVALUATES (TAN(Z/2)/(Z/2)) - 1.
0002260 C
0002265         A = REAL(Z)
0002270         C = AIMAG(Z)
0002275         IF (A .EQ. 0.0 .AND. C .EQ. 0.0) GO TO 3
0002280         IF (ABS(C) .GE. 140.) GO TO 4
0002285 C
0002300         GO TO (1, 2),IS
0002400 C
0002500       1 CALL ZBESJY(Z,J0,J1,Y0,Y1)
0002600         J2 = CMPLX(2.0,0.0) * J1/Z - J0
0002700 C
0002800 C       SCALE NUMBERS BEFORE DIVIDING TO
0002900 C       AVOID EXPONENT OVERFLOW
0003000 C
0003100         AR0 = REAL(J0)
0003200         AI0 = AIMAG(J0)
0003300         AR2 = REAL(J2)
0003400         AI2 = AIMAG(J2)
0003500         A = AMAX1(ABS(AR0),ABS(AI0),ABS(AR2),ABS(AI2))
0003600         N = CMPLX(AR2/A, AI2/A)
0003700         D = CMPLX(AR0/A, AI0/A)
0003900         RFUN = N/D
0004000         GO TO 5
0004100 C
0004200       2 B = Z/CMPLX(2.0,0.0)
0004300         RFUN = CSIN(B)/(B*CCOS(B)) - CMPLX(1.0,0.0)
0004500         GO TO 5
0004600 C
0004700       3 RFUN = CMPLX(0.,0.)
0004800         GO TO 5
0004900 C
0005000       4 RFUN = CMPLX(-1.,0.) + CMPLX(0.,2.)/Z
0005100 C
0005200       5 RETURN
0005300         END
```

Subroutine VOL.—

```
0000100         SUBROUTINE VOL(V,SIG,POLYK,PU)
0000200 C
0000300 C       SUBROUTINE TO EVALUATE ABCD MATRIX FOR A LUMPED
0000400 C       VOLUME V.  PARAMETER SIG IS THE DIMENSIONLESS
0000500 C       INCREASE IN TRANSDUCER VOLUME DUE TO DIAPHRAGM
0000600 C       DEFLECTION WITH PRESSURE, AND POLYK IS THE
0000700 C       POLYTROPIC CONSTANT FOR THE VOLUME (SEE B & T).
0000720 C       THE PRESSURE-VOLUME VELOCITY VECTOR PU IS
0000740 C       UPDATED.
0000800 C
0000900         COMPLEX PU(2),J,J32
0001000         COMMON /XFREQ/ FREQ,OMEGA,WAVENO
0001200         COMMON /THERMO/ GAMMA,SRPRN,AMPRES,AMDENS,SPEED,RHOC,VISC
0001400         COMMON /XCONST/ J,J32,PI
0001500 C
0001600         COMPLEX ABCD(2,2),PUTEMP(2)
0001700 C
0001800         ABCD(1,1) = CMPLX(1.,0.)
0001900         ABCD(2,2) = ABCD(1,1)
0002000         ABCD(1,2) = CMPLX(0.,0.)
0002100         ABCD(2,1) = J*(WAVENO*GAMMA*V*(SIG + 1./POLYK)/RHOC)
0002200 C
0002300         DO 1 M = 1, 2
0002400 1 PUTEMP(M) = PU(M)
0002500 C
0002600         DO 3 M = 1, 2
0002700         PU(M) = CMPLX(0.,0.)
0002800         DO 2 N = 1, 2
0002900 2 PU(M) = PU(M) + ABCD(M,N) * PUTEMP(N)
0003000 3 CONTINUE
0003100 C
0003200         RETURN
0003300         END
```

```
0000100 C        PROGRAM TO PLOT COMPARISON OF CIRCULAR AND
0000200 C        FLATTENED TUBES, CALCULATIONS USE
0000300 C        TRANSMISSION LINE FORMALISM.
0000450 C        CGS UNITS USED THROUGHOUT.
0000500 C
0000600          COMPLEX J,J32,PHI
0000700          COMMON /XFREQ/ FREQ,OMEGA,WAVENO
0000900          COMMON /THERMO/ GAMMA,SRPRN,AMPRES,AMDENS,SPEED,RHOC,VISC
0001000          COMMON /XTUBE/ PHI
0001100          COMMON /XCONST/ J,J32,PI
0001200 C
0001300          PI = 3.141593
0001400          J = CMPLX(0.,1.)
0001500          J32 = CEXP(CMPLX(0.,0.75*PI))
0001600 C
0001700          GAMMA = 1.4017
0001800          SRPRN = 0.8414
0001900          ATM = 1.0129E6
0001925          AMPRES = ATM
0001950          TEMPC = 26.85
0002000          AMDENS = AMPRES/(2.8688E6*(TEMPC+273.15))
0002100          SPEED = SQRT(GAMMA*AMPRES/AMDENS)
0002200          RHOC = AMDENS * SPEED
0002300          VISC = 1.846E-4
0002400 C
0002500          COMPLEX PRATIO,PUR(2),PUF(2)
0002520          REAL XF(200),AR(200),AF(200),PR(200),PF(200)
0002530          INTEGER IVARS(10)
0002540          REAL XTIT(3)/'FREQ','UENC','Y   '/
0002560          REAL YTIT1(4)/'AMPL','ITUD','E RA','TIO '/
0002580          REAL YTIT2(2)/'PHAS','E   '/
0002600 C
0002800          SIG = 0.
0002900          POLYK = 1.2
0002930 C
0002950       3  WRITE (6, 1)
0002970       1  FORMAT (/' ENTER CAVITY VOLUME')
0002990          READ (5, 2, END=20) V
0003010       2  FORMAT (G10)
0003030          WRITE (6, 4)
0003050       4  FORMAT (/' ENTER GENERALIZED RADIUS OF DUCT')
0003070          READ (5, 2, END=20) G
0003090          WRITE (6, 6)
0003110       6  FORMAT (/' ENTER CROSS-SECTIONAL AREA OF DUCT')
0003130          READ (5, 2, END=20) S
0003150          WRITE (6, 7)
0003170       7  FORMAT (/' ENTER LENGTH OF DUCT')
0003190          READ (5, 2, END=20) XLEN
0003310 C
0003420          RMX = 0.
0003440          NPTS = 200
0003460 C
0003500          DO 10 KF = 1, NPTS
0003600          FREQ = FLOAT(25*KF)
0003650          XF(KF) = FREQ
0003700          OMEGA = 2. * PI * FREQ
0003800          WAVENO = OMEGA/SPEED
0003900 C
0004000          PUR(1) = CMPLX(1.,0.)
0004100          PUR(2) = CMPLX(0.,0.)
0004200          CALL VOL(V,SIG,POLYK,PUR)
0004220          PUF(1) = PUR(1)
0004240          PUF(2) = PUR(2)
0004300          CALL TUBE(G,S,XLEN,1,PUR)
0004350          CALL TUBE(G,S,XLEN,2,PUF)
0004400 C
0004500          PRATIO = CMPLX(1.,0.)/PUR(1)
0004600          AR(KF) = CABS(PRATIO)
0004650          RMX = AMAX1(RMX,AR(KF))
0004700          PHA = ATAN2(AIMAG(PRATIO),REAL(PRATIO))*180./PI
```

```
0004800          IF (PHA .GT. 0.) PHA = PHA - 360.
0004850          PR(KF) = PHA
0004900 C
0004910          PRATIO = CMPLX(1.,0.)/PUF(1)
0004920          AF(KF) = CABS(PRATIO)
0004930          RMX = AMAX1(RMX,AF(KF))
0004940          PHA = ATAN2(AIMAG(PRATIO),REAL(PRATIO))*180./PI
0004950          IF (PHA .GT. 0.) PHA = PHA - 360.
0004960       10 PF(KF) = PHA
0004970 C
0004973          IVARS(1) = 8
0004976          IVARS(2) = NPTS
0004979          IVARS(3) = 66
0004982          IVARS(4) = 62
0004985          IVARS(5) = NPTS - 1
0004988          IVARS(6) = 25
0004991          IVARS(7) = 0
0004994          IVARS(8) = NPTS
0004997          CALL GINTVL(0.,RMX,10,1,AMIN,AMAX)
0005000          CALL GINTVL(0.,5000.,10,0,AMIN,AMAX)
0005003          CALL GPLOT(XF,AR,IVARS)
0005006 C
0005009          IVARS(3) = 98
0005012          IVARS(4) = 65
0005015          CALL GPLOT(XF,AF,IVARS)
0005018 C
0005021          CALL TITLE(4,9,15,XTIT)
0005024          CALL TITLE(3,15,15,YTIT1)
0005027          CALL CORNER(1)
0005030          CALL COPY(1)
0005033          CALL DISPLA(1)
0005036 C
0005039          IVARS(3) = 66
0005042          IVARS(4) = 62
0005045          CALL GINTVL(-360.,0.,10,1,AMIN,AMAX)
0005048          CALL GINTVL(0.,5000.,10,0,AMIN,AMAX)
0005051          CALL GPLOT(XF,PR,IVARS)
0005054 C
0005057          IVARS(3) = 98
0005060          IVARS(4) = 65
0005063          CALL GPLOT(XF,PF,IVARS)
0005066 C
0005069          CALL TITLE(4,9,15,XTIT)
0005072          CALL TITLE(3,5,15,YTIT2)
0005075          CALL DISPLA(1)
0005200 C
0005220          GO TO 3
0005240 C
0005260       20 CALL TERM
0005300          STOP
0005400          END
```

```
0000100 C        PROGRAM TO PLOT RESPONSE OF INFINITE LINE
0000200 C        SYSTEMS.
0000300 C
0000400          COMPLEX J,J32,PHI
0000500          COMMON /XFREQ/ FREQ,OMEGA,WAVENO
0000600          COMMON /THERMO/ GAMMA,SRPRN,AMPRES,AMDENS,SPEED,RHOC,VISC
0000700          COMMON /XTUBE/ PHI
0000800          COMMON /XCONST/ J,J32,PI
0000900 C
0001000          PI = 3.141593
0001100          J = CMPLX(0.,1.)
0001200          J32 = CEXP(CMPLX(0.,0.75*PI))
0001300 C
0001400          GAMMA = 1.4017
0001500          SRPRN = 0.8414
0001600          ATM = 1.0129E6
0001700          AMPRES = ATM
0001800          TEMPC = 26.85
0001900          AMDENS = AMPRES/(2.8688E6*(TEMPC+273.15))
0002000          SPEED = SQRT(GAMMA*AMPRES/AMDENS)
0002100          RHOC = AMDENS * SPEED
0002200          VISC = 1.846E-4
0002300 C
0002400          COMPLEX PRATIO,PUR(2),PUF(2)
0002450          COMPLEX PREFR,PREFF
0002500          REAL XF(200),AR(200),AF(200),PR(200),PF(200)
0002600          INTEGER IVARS(10)
0002700          REAL XTIT(3)/'FREQ','UENC','Y   '/
0002800          REAL YTIT1(4)/'AMPL','ITUD','E RA','TIO '/
0002900          REAL YTIT2(2)/'PHAS','E   '/
0003000 C
0003100          SIG = 0.
0003200          POLYK = 1.2
0003300 C
0003400        3 WRITE (6, 1)
0003500        1 FORMAT (/' ENTER DISTANCE FROM SENSOR TO OUTLET')
0003600          READ (5, 2, END=20) XL1
0003700        2 FORMAT (G10)
0003800          WRITE (6, 4)
0003900        4 FORMAT (/' ENTER GENERALIZED RADIUS OF DUCT')
0004000          READ (5, 2, END=20) G
0004100          WRITE (6, 6)
0004200        6 FORMAT (/' ENTER CROSS-SECTIONAL AREA OF DUCT')
0004300          READ (5, 2, END=20) S
0004400          WRITE (6, 7)
0004500        7 FORMAT (/' ENTER LENGTH OF INFINITE TUBE')
0004600          READ (5, 2, END=20) XL2
0004700 C
0004800          RMX = 0.
0004900          NPTS = 200
0005000 C
0005100          DO 10 KF = 1, NPTS
0005200          FREQ = FLOAT(25*KF)
0005300          XF(KF) = FREQ
0005400          OMEGA = 2. * PI * FREQ
0005500          WAVENO = OMEGA/SPEED
0005600 C
0005700          PUR(1) = CMPLX(1.0E-6, 0.)
0005800          PUR(2) = CMPLX(0.,0.)
0005900          PUF(1) = PUR(1)
0006000          PUF(2) = PUR(2)
0006100 C
0006200          CALL TUBE(G,S,XL2,1,PUR)
0006300          CALL TUBE(G,S,XL2,2,PUF)
0006400 C
0006500          PREFR = PUR(1)
0006600          PREFF = PUF(1)
0006700 C
0006800          CALL TUBE(G,S,XL1,1,PUR)
0006900          CALL TUBE(G,S,XL1,2,PUF)
```

```
0007000 C
0007100        PRATIO = PREFR/PUR(1)
0007200        AR(KF) = CABS(PRATIO)
0007300        RMX = AMAX1(RMX,AR(KF))
0007400        PHA = ATAN2(AIMAG(PRATIO),REAL(PRATIO))*180./PI
0007500        IF (PHA .GT. 0.) PHA = PHA - 360.
0007600        PR(KF) = PHA
0007700 C
0007800        PRATIO = PREFF/PUF(1)
0007900        AF(KF) = CABS(PRATIO)
0008000        RMX = AMAX1(RMX,AF(KF))
0008100        PHA = ATAN2(AIMAG(PRATIO),REAL(PRATIO))*180./PI
0008200        IF (PHA .GT. 0.) PHA = PHA - 360.
0008300     10 PF(KF) = PHA
0008400 C
0008500        IVARS(1) = 8
0008600        IVARS(2) = NPTS
0008700        IVARS(3) = 66
0008800        IVARS(4) = 62
0008900        IVARS(5) = NPTS - 1
0009000        IVARS(6) = 25
0009100        IVARS(7) = 0
0009200        IVARS(8) = NPTS
0009300        CALL GINTVL(0.,RMX,10,1,AMIN,AMAX)
0009400        CALL GINTVL(0.,5000.,10,0,AMIN,AMAX)
0009500        CALL GPLOT(XF,AR,IVARS)
0009600 C
0009700        IVARS(3) = 98
0009800        IVARS(4) = 65
0009900        CALL GPLOT(XF,AF,IVARS)
0010000 C
0010100        CALL TITLE(4,9,15,XTIT)
0010200        CALL TITLE(3,15,15,YTIT1)
0010300        CALL CORNER(1)
0010400        CALL COPY(1)
0010500        CALL DISPLA(1)
0010600 C
0010700        IVARS(3) = 66
0010800        IVARS(4) = 62
0010900        CALL GINTVL(-360.,0.,10,1,AMIN,AMAX)
0011000        CALL GINTVL(0.,5000.,10,0,AMIN,AMAX)
0011100        CALL GPLOT(XF,PR,IVARS)
0011200 C
0011300        IVARS(3) = 98
0011400        IVARS(4) = 65
0011500        CALL GPLOT(XF,PF,IVARS)
0011600 C
0011700        CALL TITLE(4,9,15,XTIT)
0011800        CALL TITLE(3,5,15,YTIT2)
0011900        CALL DISPLA(1)
0012000 C
0012100        GO TO 3
0012200 C
0012300     20 CALL TERM
0012400        STOP
0012500        END
```

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1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TP-2601		

4. Title and Subtitle	5. Report Date
Propagation of Sound Waves in Tubes of Noncircular Cross Section	August 1986
	6. Performing Organization Code
	505-62-02

7. Author(s)	8. Performing Organization Report No.
W. Bruce Richards	E-2690
	10. Work Unit No.

9. Performing Organization Name and Address	
National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135	11. Contract or Grant No.

12. Sponsoring Agency Name and Address	13. Type of Report and Period Covered
National Aeronautics and Space Administration Washington, D.C. 20546	Technical Paper
	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

A study of plane-acoustic-wave propagation in small tubes with a cross section in the shape of a flattened oval is described. To begin, theoretical descriptions of a plane wave propagating in a tube with circular cross section and between a pair of infinite parallel plates, including viscous and thermal damping, are expressed in similar form. For a wide range of useful duct sizes, the propagation constant (whose real and imaginary parts are the amplitude attenuation rate and the wave number, respectively) is very nearly the same function of frequency for both cases if the radius of the circular tube is the same as the distance between the parallel plates. This suggests that either a circular-cross-section model or a flat-plate model can be used to calculate wave propagation in flat-oval tubing, or any other shape tubing, if its size is expressed in terms of an equivalent radius, given by $g = 2 \times$ (cross-sectional area)/(length of perimeter). Measurements of the frequency response of two sections of flat-oval tubing agree with calculations based on this idea. Flat-plate formulas are derived, the use of transmission-line matrices for calculations of plane waves in compound systems of ducts is described, and examples of computer programs written to carry out the calculations are shown.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Frequency response; Pressure transmission line; Noncircular tube; Viscous and thermal damping; Wave propagation in ducts; Acoustic impedance	Unclassified – unlimited STAR Category 02

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price*
Unclassified	Unclassified	32	A03