Real-Time Combiner Loss

M. K. Simon and A. Mileant
Telecommunications Systems Section

Telemetry signals from several channels are aligned in time and combined by the Real-Time Combiner (RTC) in order to increase the strength of the total signal. In this article, the impact of the timing jitter in the RTC on the bit/symbol error rate is investigated. Equations are derived for the timing jitter loss associated with the coded and uncoded channels. Included are curves that depict the bit/symbol error rate vs. E_b/N_0 *and* E_s/N_0 *for some typical telemetry conditions. The losses are typically below 0.1 dB.*

I. Introduction

The Real-Time Combiner (RTC) is part of the Baseband Assembly (BBA). Its purpose is to time align baseband signals from different antennas in order to increase the strength of the total combined signal. Figure 1 represents a block diagram of the overall downlink telemetry process of signal combining, subcarrier demodulation, symbol synchronization and data recovery.

If there were no timing jitter in the RTC, the maximum signal-to-noise ratio at the output of the RTC would be the sum of the SNRs at its input. However, timing jitter will decrease the output SNR from its maximum value. This decrease in combined SNR will in turn increase the bit/symbol error rate of the recovered data. In Ref. 1 the degradation in the RTC's output SNR was estimated. In this analysis, the degradation in bit/symbol error rate performance will be estimated for the rate 1/2, constraint length 7 convolutional code and the uncoded data cases. Expressions are derived for the timing jitter loss.

II. Impact of the Timing Jitter on the RTC's SNR

As shown in Fig. 2, the operation of the RTC can be visualized as a weighted sum of L streams of binary data. At the discrete time instant i, the RTC produces at its output

$$x_i = \sum_{j=1}^{L} \alpha_j (s_{ji} + n_{ji}) \tag{1}$$

where α_j is the jth weighting factor, s_{ji} is the signal amplitude of the ith sample of the jth signal with $E\{s_{ji}\} = \sqrt{S_j}$ and n_{ji} is the amplitude of the additive thermal noise in the jth channel. This noise is modeled as a zero-mean Gaussian process with variance $\sigma_j^2 = N_{0j}B_j$, where B_j is the noise bandwidth at the jth input to the RTC. The noise processes in each of the channels are statistically independent of each other.

The transmitted data symbols are recovered by adding up all the samples belonging to each individual symbol. Let N_s be the number of Nyquist samples per symbol, then, as shown in Fig. 2, at the end of each symbol the RTC produces

$$y_k = \sum_{i=1}^{N_s} x_i \tag{2}$$

Assuming *perfect time alignment* of the combined signals, inserting Eq. (1) in Eq. (2) and using the above definitions, the first two moments of the kth symbol, y_k, will be

$$E\{y_k\} = N_s \sum_{j=1}^{L} \alpha_j \sqrt{S_j} \tag{3}$$

$$E\{(y_k)^2\} = N_s^2 \sum_{j=1}^{L} (\alpha_j \sqrt{S_j})^2 + N_s \sum_{j=1}^{L} \alpha_j^2 \sigma_j^2 \tag{4}$$

The first term in Eq. (4) represents the expected value of the signal energy in the kth symbol, and the second term, the corresponding noise variance.

Imperfect timing in the RTC loops decreases the effective energy of the detected symbol. Since the RTC's loop update time is much larger than the symbol time, the alignment error is constant during many symbols. Let ϵ_j be the normalized alignment error defined as follows

$$\epsilon_j \doteq 4|\delta_j|/N_{sc} \tag{5}$$

where δ_j is the alignment error of the jth channel relative to the master channel ($j = 1$) and N_{sc} is the period of the subcarrier waveform (both δ_j and N_{sc} are in units of Nyquist samples and δ_j is a random variable with the probability density function (pdf) given by Eq. (75) of Ref. 1). Using that equation, the pdf of ϵ_j will be

$$p(\epsilon_j) = [1/(2\pi\sigma_{\epsilon j}^2)^{1/2}]\,[\exp\{-(\epsilon_j - \mu_j)^2/(2\sigma_{\epsilon j}^2)\} + \exp\{-(\epsilon_j + \mu_j)^2/(2\sigma_{\epsilon j}^2)\}]U(\epsilon_j) \tag{6}$$

where $\mu_j = E\{4\delta_j/N_{sc}\}$, $\sigma_{\epsilon j}^2 = (4/N_{sc})^2\ \sigma_{Nj}^2$ and $U(\cdot)$ is the unit step function. The closed-loop variance of the timing jitter, σ_{Nj}^2, is given by Eq. (53) of Ref. 1. It can be shown that for data modulating a squarewave subcarrier, the variance of ϵ_j will be

$$\sigma_{\epsilon j}^2 = B_L\{[\mathrm{erf}(\eta_1)\mathrm{erf}(\eta_j)]^{-2} - 1\}/(2B_1) \tag{7}$$

where B_L is the bandwidth of the RTC loop and $\eta_j \doteq (S_j/2\sigma_j^2)^{1/2}$.

With *imperfect timing* in the RTC loops, the first two moments of the kth symbol conditioned on $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \cdots \epsilon_L)$ will be

$$E\{y_k|\boldsymbol{\epsilon}\} = N_s \sum_{j=1}^{L} \alpha_j \sqrt{S_j}(1-\epsilon_j) \tag{8}$$

$$E\{(y_k)^2|\boldsymbol{\epsilon}\} = N_s^2 \sum_{j=1}^{L} [\alpha_j \sqrt{S_j}(1-\epsilon_j)]^2 + N_s \sum_{j=1}^{L} \alpha_j^2 \sigma_j^2 \tag{9}$$

$$\doteq E_k + N_k \tag{10}$$

Note that $\epsilon_1 = 0$ by definition.

It can be shown (see Ref. 1, Eq. [66]) that optimum signal combining is obtained when the weighting factors α_j are related as follows.

$$\alpha_j \sigma_j^2/\sqrt{S_j} = \alpha_1 \sigma_1^2/\sqrt{S_1}, \quad j = 2, \ldots, L \tag{11}$$

Since scaling all α's by a constant factor does not change the SNR at the output of the RTC, we set $\alpha_1 = 1.0$ and readily obtain the optimum values for all the remaining weighting coefficients. Namely, we make

$$\alpha_j = \frac{\sqrt{S_j}\ \sigma_1^2}{\sqrt{S_1}\ \sigma_j^2} = \gamma_j/\sqrt{m_j}, \quad j = 2, \ldots, L \tag{12}$$

where $m_j \doteq S_1/S_j$ and $\gamma_j \doteq \sigma_1^2/\sigma_j^2$.

With these preliminary steps completed, we are now ready to evaluate the conditional SNR at the RTC's output. Let $R_k \doteq E_k/N_k$ be the SNR at the end of the kth symbol (see Eq. (10)). Then, using Eq. (9), R_k conditioned on $\boldsymbol{\epsilon}$ is

$$R_k(\boldsymbol{\epsilon}) = \frac{N_s[(\Sigma\alpha_j\sqrt{S_j})^2 - 2(\Sigma\alpha_j\sqrt{S_j})(\Sigma\alpha_j\sqrt{S_j}\epsilon_j)]}{\Sigma\alpha_j^2 N_0 B_j} + \frac{N_s(\Sigma\alpha_j\sqrt{S_j}\epsilon_j)^2}{\Sigma\alpha_j^2 N_0 B_j} \tag{13}$$

where all summations are from $j = 1$ to $j = L$ and, again, $\epsilon_1 = 0$ by definition.

Using the definitions of m_j and γ_j, the optimum values of α_j (Eq. (12)) and simplifying, the conditional symbol SNR can be expressed in terms of S_1 and B_1, namely

$$R_k(\epsilon) = \frac{N_s S_1 \left[(\Sigma \gamma_j / m_j)^2 - 2(\Sigma \gamma_j / m_j)(\Sigma \gamma_j \epsilon_j / m_j) \right]}{N_0 B_1 (\Sigma \gamma_j / m_j)} + \frac{N_s (\Sigma \gamma_j \epsilon_j / m_j)^2}{N_0 B_1 (\Sigma \gamma_j / m_j)} \tag{14}$$

$$= R_b \left[\beta - 2 \Sigma \gamma_j \epsilon_j / m_j + (\Sigma \gamma_j \epsilon_j / m_j)^2 / \beta \right] \tag{15}$$

where

$$R_b \doteq 2E_{s1}/N_0 = 2R_s \tag{16a}$$

and

$$\beta \doteq \sum_{j=1}^{L} \gamma_j / m_j \tag{16b}$$

Note that $\beta > 1$ when $L > 1$. Equation (15) was obtained using the following relations: $N_s = 2B_1/r$ and $S_1/r = E_{s1}$ where E_{s1} is the symbol energy in the first channel and r is the symbol rate. Note that with no timing jitter in the RTC, the output SNR is

$$R_k = R_b \beta \quad \text{for all } k \tag{17}$$

So, β represents the factor by which the ideal combined SNR is larger than the SNR in channel 1.

III. Evaluation of the Loss Due to Timing Jitter in the RTC

Timing jitter in the RTC loops has the effect of decreasing the effective SNR of the combined signal. This decrease of SNR in turn increases the bit/symbol error rate of the recovered data. In order to avoid confusion of terms, we define *degradation factor* as the factor by which the combined SNR is decreased due to timing jitter in the RTC, whereas we define *jitter loss* (like "radio loss") as the factor by which the SNR has to be increased in order to achieve a specified bit/symbol error rate. The evaluation of the degradation factor (decrease in combined SNR) requires averaging over ϵ of the conditional SNR in Eq. (15). This evaluation was done in Ref. 1. In this analysis, the jitter loss, namely, the degradation in bit error rate performance, is determined.

The evaluation of the jitter loss requires two steps. First, the particular coding scheme has to be specified. Then, averaging on the bit/symbol error rate is performed over the timing jitter pdf, Eq. (6). In our analysis two telemetry cases will be considered: (a) the rate 1/2, constraint length 7 convolutionally coded and (b) uncoded data.

A. Convolutional Code

For the rate 1/2, constraint length 7 convolutional code, an expression for bit error rate has been empirically found and is given as follows:

$$P_b = f_b(E_b/N_0) = \begin{cases} C_1 \exp(C_2 \, E_b/N_0), & E_b/N_0 > 0.8989 \\ 1/2, & E_b/N_0 < 0.8989 \end{cases} \tag{18}$$

where $C_1 = 85.7469$ and $C_2 = -5.7230$. The term E_b/N_0 is the bit energy to noise spectral density ratio at the input to the Maximum Likelihood Decoder (MCD). Note that $E_b/N_0 = 2E_s/N_0 = R_b$ of Eq. (15).

B. Uncoded Data

For uncoded data the symbol error rate is given by

$$P_s = f_s(E_s/N_0) = (1/2)\mathrm{erfc}\left[(E_s/N_0)^{1/2}\right] \tag{19}$$

where erfc(x) is the complementary error function. Note that now $E_s/N_0 = R_b/2$ of Eq. (15).

Since the data rates considered in our analysis are higher than the RTC's bandwidth, or, in other words, the loop timing error process δ_j is slow and essentially constant over a bit/symbol time interval, the average bit/symbol error rate is estimated using the so called "high-rate model," namely

$$P_i = \int_0^1 \cdots \int_0^1 f_i(R_k(\epsilon)) \prod_{j=2}^{L} p(\epsilon_j)\, d\epsilon,$$

$$i = \begin{cases} b, & \text{for coded channel} \\ s, & \text{for uncoded channel} \end{cases} \tag{20}$$

where $R_k(\epsilon)$ is given by Eq. (15) and $p(\epsilon_j)$ by Eq. (6).

For the coded channel, the average bit error rate as given above can be obtained in a closed form as follows. Assuming that the RTC has doppler compensation, μ_j in the pdf of the normalized timing jitter (Eq. [6]) will be zero. This is our first simplification in the evaluation of Eq. (20). A second simplification results in assuming that the contribution of the rightmost term in the numerator of Eq. (15) is much smaller than that of the middle term (i.e., $E\{\epsilon_j\} > E\{(\epsilon_j)^2\}$). By discarding the right-most term in the numerator of Eq. (15) we will obtain the upper bound of P_b. Inserting now these simplified Eqs. (6) and (15) with (18) in Eq. (20), integrating and simplifying, we finally obtain the average bit error rate at the output of the MCD, namely

$$P_b = C_1 \exp(C_2 R_b \beta) \prod_{j=2}^{L} I_j \tag{21}$$

where

$$I_j = \exp(\nu_j^2)\operatorname{erfc}(\nu_j) \tag{22}$$

$$\nu_j \doteq \sqrt{2} C_2 R_b \gamma_j \sigma_{\epsilon j}/m_j \tag{23}$$

and R_b is given by Eq. (16a).

For the uncoded case, the average degraded symbol error rate can be obtained by first approximating the complementary error function of Eq. (18) by $\operatorname{erfc}(x) \approx \exp(-x^2)/\sqrt{\pi}x$. Then we expand this function in a Taylor series about $\epsilon_j = 0$ ($j = 2, \ldots, L$). Finally, we take the expected value of the series expansion and keep the first few terms. Performing the above steps we find after some algebra the desired P_s, which for $L = 3$ becomes

$$P_s = (1/2)\operatorname{erfc}(\sqrt{R_s \beta})[1 + \sqrt{8/\pi}(R_s + 1/\beta)(\lambda_2 + \lambda_3) + 2(R_s^2 + R_s/\beta + 3/4\beta^2)(\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3)] \tag{24}$$

where again $R_s \doteq E_{s1}/N_0$ (Eq. [16a]) and $\lambda_i \doteq \gamma_i \sigma_i / m_i$. For $L = 2$, we simply set $\lambda_3 = 0$ in the above equation.

When, in the region of specified (desired) bit/symbol error rate, both curves of P_i vs. SNR (the ideal and that degraded by the RTC's timing jitter) are parallel, the *timing jitter loss* can be obtained from the following expressions.

For the coded channel

$$L_b = 10 \log_{10} [R_b \beta/(E_b/N_0)\text{ref}], \text{dB} \tag{25}$$

where $(E_b/N_0)\text{ref} = \ln(P_b/C_1)/C_2$, ($\ln(\cdot)$ denotes natural logarithm) and P_b is the degraded bit error rate obtained from Eq. (21).

For the uncoded channel

$$L_s = 10 \log_{10} [R_s \beta/(E_s/N_0)\text{ref}], \text{dB} \tag{26}$$

where $(E_s/N_0)\text{ref} = [\operatorname{erfc}^{-1}(2P_s)]^2$, erfc^{-1} is the inverse complementary error function and P_s is the degraded symbol error rate obtained from Eq. (24).

IV. Conclusion

In the above analysis the impact of the timing jitter in the RTC on the bit/symbol error rate was evaluated. Equations were obtained for the degraded average bit/symbol error rate vs. SNR in the reference channel for the rate 1/2, constraint length 7 convolutionally coded and uncoded data. The result of the above analysis are illustrated in Figs. 3 through 6 for several typical telemetry conditions. The losses are typically below 0.1 dB.

Reference

1. Simon, M. K., and Mileant, A., *Performance Analysis of the DSN Baseband Assembly (BBA) Real-Time Combiner (RTC)*, JPL Publication 84-94, Rev. 1, Jet Propulsion Laboratory, Pasadena, Calif., May 1, 1985.

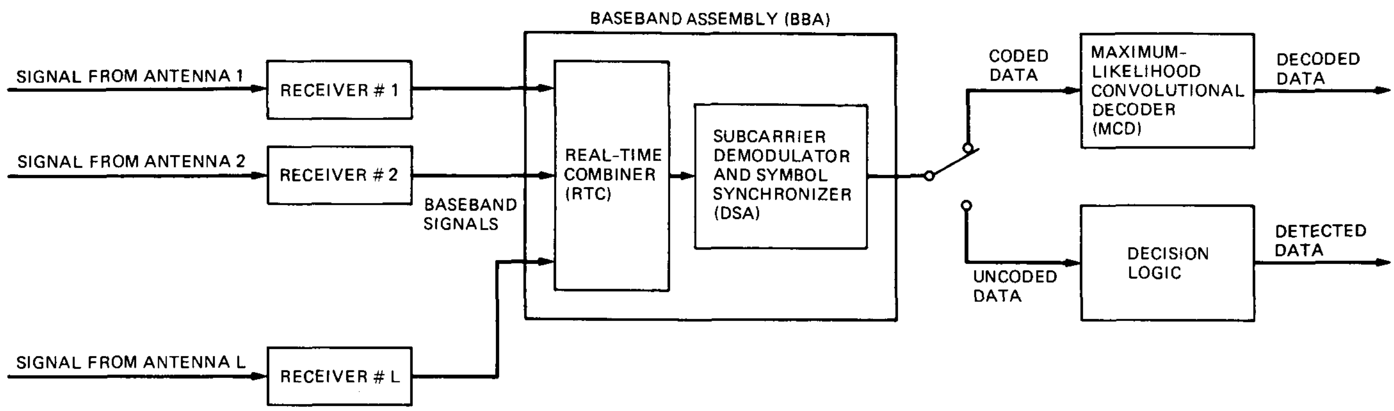

Fig. 1. Downlink process of signal combining, subcarrier demodulation, symbol synchronization and decoding of telemetry data

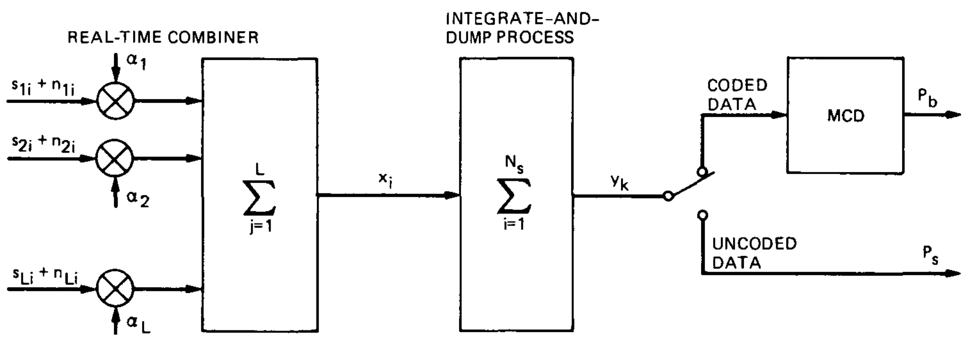

Fig. 2. Simplified representation of the signal combining and decoding process

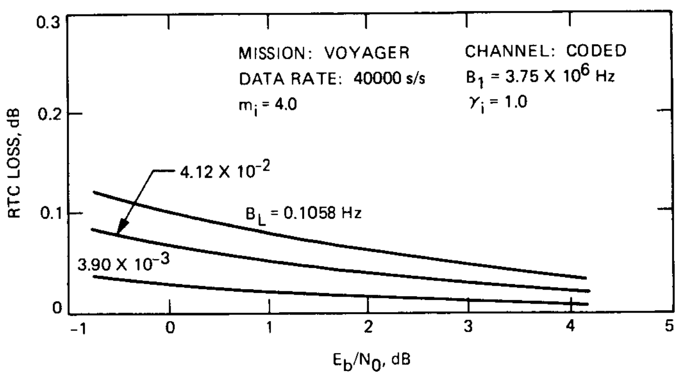

Fig. 3. The RTC loss vs master antenna E_b/N_0, L = 2

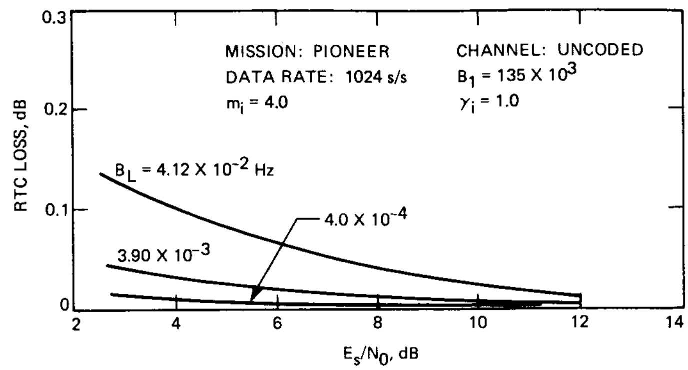

Fig. 5. The RTC loss vs master antenna E_s/N_0, L = 2

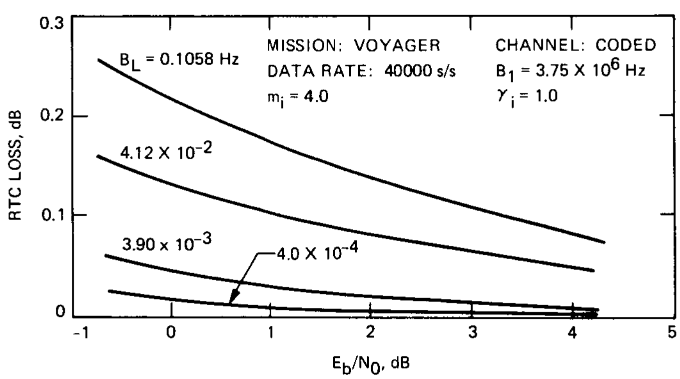

Fig. 4. The RTC loss vs master antenna E_b/N_0, L = 4

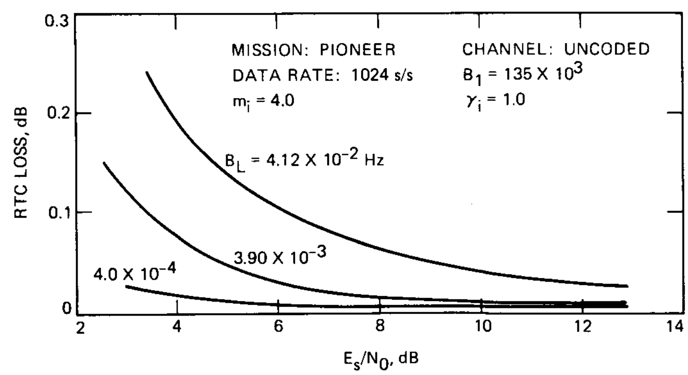

Fig. 6. The RTC loss vs master antenna E_s/N_0, L = 3