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(NASA-CR-178215) THE NONLINEAR DEVELOPMENT OF GORTLER VORTICES IN GROWING BOUNDARY LAYERS Final Report (NASA) 53 p CSCL 20D

Contract No. NASI-18107
December 1986

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association
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ABSTRACT

The development of Gortler vortices in boundary layers over curved walls in the nonlinear regime is investigated. The growth of the boundary layer makes a parallel flow analysis impossible except in the high wavenumber regime so in general the instability equations must be integrated numerically. Here the spanwise dependence of the basic flow is described using a Fourier series expansion whilst the normal and streamwise variations are taken into account using finite differences. The calculations suggest that a given disturbance imposed at some position along the wall will eventually reach a local equilibrium state essentially independent of the initial conditions. In fact, the equilibrium state reached is qualitatively similar to the large amplitude high wave-number solution described asymptotically by Hall (1982b). In general, it is found that the nonlinear interactions are dominated by a 'mean field' type of interaction between the mean flow and the fundamental. Thus, even though higher harmonics of the fundamental are necessarily generated, most of the disturbance energy is confined to the mean flow correction and the fundamental. A major result of our calculations is the finding that the downstream velocity field develops a strongly inflectional character as the flow moves downstream. The latter result suggests that the major effect of Gortler vortices on boundary layers of practical importance might be to make them highly receptive to rapidly growing Rayleigh modes of instability.

This work was supported under the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.
Our concern is with the effect of nonlinearity on the growth of Taylor-Gortler vortices in developing boundary layers. The presence of such vortices in many flows of practical importance such as those which occur over turbine blades or over laminar flow aerofoils has recently stimulated much research aimed at understanding their structure in the linear regime. However the nonlinear problem has, apart from the high wavenumber analysis of Hall (1982b), received little attention because of the difficulty in taking care of non-parallel effect. At high wavenumbers Hall found that nonlinear effects have a stabilizing effect and prevent the exponential growth of the vortices predicted by linear theory.

In previous investigations Hall (1982a, 1983), hereafter referred to as I, II respectively, looked at the linear Gortler problem and showed that except at high wavenumbers parallel flow calculations for the Gortler problem are not valid because the streamwise and normal dependences of the vortices cannot be separated. In fact, in the only regime where the instability equations can be reduced to ordinary differential equations, the asymptotic theory of I provides trivially a neutral curve or growth rate at least as accurate as that produced by the parallel flow theories. Here by 'parallel' we simply mean any theory which ignores any term in the linear instability equations. Of course the parallel flow theories correspond to truncations of the instability equations of varying severity. Thus for example Gortler (1940), (later corrected numerically by Hammerlin (1956!)), retained only the terms which would be present in the corresponding Taylor-Couette flow calculation whilst Smith (1955) retained many terms associated with the growth of the boundary layer. Other truncations of the instability equations have been given by, for example, Floryan and Saric (1978) and Herbert (1976).
At 0(1) wavenumbers the various parallel flow theories give quite different most extreme results and in the/ cases predict instability at zero Görtler number or zero wavenumber. In II it was argued that at 0(1) wavenumbers these calculations are necessarily incorrect because their neglect of streamwise derivatives of the disturbance velocity field gives the wrong structure for the disturbance at the edge of the boundary layer. If these terms are retained it was shown that the vortices decay to zero at the edge of the boundary layer at a rate independent of the vortex wavenumber. In fact the linear instability equations are parabolic in the streamwise direction and can therefore be solved numerically by marching downstream from some initial location. A 'local' neutral position can then be defined to be the point where some disturbance flow quantity has a zero rate of change along the wall. This position depends on the location and form for the initial disturbance so that the notion of a unique neutral curve is not tenable for the Görtler problem. However, at high wavenumbers the numerical calculations of II converged to the unique asymptotic result of I.

Here we shall extend the parallel flow calculations of I to the nonlinear regime appropriate to disturbances with wavenumber of 0(1). At higher wavenumbers the asymptotic high wavenumber theory of Hall (1982b), hereafter referred to as III, showed that in this regime the nonlinear problem is dominated by a 'mean field' type of interaction rather than one typical of a Stuart-Watson approach. It was shown in III that the mean flow correction driven by a finite amplitude vortex ultimately becomes larger than the vortices driving it. At sufficiently large amplitude the mean flow correction described in III would cause the basic state to develop an inflection point and therefore possibly make the boundary layer susceptible to rapidly growing Rayleigh instabilities. A primary aim of the calculation is to confirm the latter result at large wavenumbers and investigate the situation at 0(1) wavenumbers. Our calculations will also enable linear instability calculations of finite amplitude Görtler vortices to be ultimately carried out along the lines of the recent calculation of Bennett and Hall (1986). The latter authors were concerned
with the corresponding internal full, developed flow between concentric cylinders and showed that even small amplitude vortices cause a massive destabilization of the undisturbed flow to Tollmien-Schlichting waves.

Since there is no rational way to reduce the nonlinear nonparallel Gortler problem to a series of ordinary differential equations using the Stuart-Watson method we solve the equations governing finite amplitude vortices using a numerical method based on the finite difference formulation of II together with a Fourier expansion to take care of the spanwise dependence of the flow. The vortices are assumed to be steady so that the equations governing their development can be marched downstream from the initial location where the disturbance is imposed. This is done using the implicit scheme of II together with an iteration procedure to take care of the nonlinear terms which are now present in the calculation. At each downstream location the energy in each Fourier mode can be calculated in order to monitor the development of the instability. We shall see that nonlinear effects prevent the exponential growth of the disturbances predicted by linear theory so that, at least in the limited number of cases we have investigated, nonlinear effects are stabilizing. We shall also see that any given vortex will sufficiently far downstream develop a structure consistent with the nonlinear theory of III. The latter result is not surprising since the effective vortex wavenumber increases in the streamwise direction until the asymptotic theory of III applies.

Apart from the arbitrariness associated with the linear problem described in II the nonlinear problem introduces further complications because of the further freedom we now have when imposing the initial disturbance. Our calculations are, of course, restricted to a finite number of situations but nevertheless the similarity between the results enables us to make some tentative conclusions about the role of nonlinear effects in the Gortler problem. The procedure adopted in the rest of the paper is as follows: in Section 2 we formulate the nonlinear instability equations and describe a
numerical scheme which can be used to integrate them. In section 3 we describe
the results we have obtained and use them to draw some conclusions about non-
linear Gortler vortices.

2. FORMULATION OF THE INSTABILITY EQUATIONS AND THEIR SOLUTION.

Consider the flow of a viscous fluid of kinematic viscosity $\nu$ over a wall
of curvature $a^{-1}k(x/l)$. Here $l$ and $a$ are typical length scales associated
with the downstream development of the flow and the local radius of curvature of
the wall. We take $U_0$ to be a typical flow speed and define a Reynolds number
$Re$ by

$$ Re = \frac{U_0 l}{\nu}, \quad (2.1) $$

and consider the limit of $Re \to \infty$ with the Gortler number $G$, defined by

$$ G = \frac{2l}{a} Re^{\frac{1}{2}}, \quad (2.2) $$

held fixed. Let us take $(X,Y,Z)$ to be dimensionless variables in the
streamwise, normal and spanwise directions scaled on $l, Re^\frac{1}{2} l, Re^\frac{3}{2} l$
 respectively. The velocity field is taken to be of the form

$$ \tilde{u} = U_0 (\tilde{u}(X,Y) + U(X,Y,Z)), \ Re^{-\frac{1}{2}} (\tilde{v}(X,Y) + V(X,Y,Z)), \ Re^{-\frac{3}{2}} W(X,Y,Z) \quad (2.3) $$

where $(\tilde{u}(X,Y), \tilde{v}(X,Y))$ corresponds to a Blasius boundary layer and
$(U,V,W)$ and the corresponding pressure perturbation $P$ are functions of
$X,Y,Z$. Following the procedure outlined in III it is an easy matter to
show from the Navier-Stokes equations that, correct to order $Re^{-\frac{1}{2}}$,
$U,V,W,P$ satisfy
\[ U_X + V_Y + W_Z = 0, \]

\[ U_{YY} + U_{ZZ} - \ddot{V}_Y = \ddot{U}_X + \ddot{U}_Y + \ddot{V}_Y + \ddot{U}_Y + Q_1, \]

\[ V_{YY} + V_{ZZ} - G\ddot{U} - P_Y = \ddot{U}_Y + \ddot{V}_X + \ddot{V}_Y + \ddot{V}_Y + Q_2, \]

\[ W_{YY} + W_{ZZ} - P_Z = \ddot{U}_W + \ddot{V}_W + Q_3, \quad (2.4a,b,c) \]

where \( Q_1, Q_2, Q_3 \) are defined by

\[ Q_1 = UU_X + UV_Y + UW_Z, \]

\[ Q_2 = UV_X + VV_Y + WV_Z + \frac{1}{2}G\ddot{U}^2, \]

\[ Q_3 = UW_X + VW_Y + WV_Z. \quad (2.5a,b,c) \]

If the nonlinear functions \( Q_1, Q_2, Q_3 \) are set equal to zero in the above equations we recover the equations of II. The nonlinear theory of III gives an asymptotic solution of (2.4) valid in the limit of \( \frac{a}{a_z} \gg 1 \).

This limit is more relevant than it might appear to be at first sight since it corresponds to the large \( X \) state of any initial disturbance imposed on the flow. Thus in our numerical calculations we expect to recover qualitatively the results of III sufficiently far downstream from where the critical disturbance is introduced.

In order to reduce (2.4) to a form more suitable for computational purposes we can eliminate \( P \) and \( W \) from the linear terms in (2.4c,d) to give
\[
\begin{align*}
\frac{\partial^2}{\partial z^2} \left( \frac{\partial^2}{\partial x^2} \right) \hat{u} + \frac{\partial^2}{\partial z^2} \left( \frac{\partial^2}{\partial y^2} \right) \hat{v} + \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2}{\partial y^2} \right) \hat{w} & = \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2}{\partial z^2} \right) - \frac{\partial^2}{\partial y^2} \left( \frac{\partial^2}{\partial z^2} \right) \hat{u} + \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2}{\partial y^2} \right) \hat{v} + \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2}{\partial z^2} \right) \hat{w} \\
+ \left( \frac{\partial^2}{\partial y^2} \hat{u} - \frac{\partial^2}{\partial y^2} \hat{v} - \frac{\partial^2}{\partial x^2} \hat{w} \right) \hat{v} + 2 \left( \frac{\partial^2}{\partial x^2} \hat{u} + \frac{\partial^2}{\partial x^2} \hat{v} \right) \hat{u} \\
& = Q_{1XY} + Q_{2ZZ} - Q_{3YZ} 
\end{align*}
\]

where \( Q_1, Q_2 \) and \( Q_3 \) are given by \((2.5a,b,c)\) respectively.

Suppose that \( U, V \) and \( W \) are then expanded in the form

\[
\begin{align*}
U &= U_0 + \sum_{n=1}^{\infty} U_n(X,Y) \cos n \alpha z \\
V &= V_0 + \sum_{n=1}^{\infty} V_n(X,Y) \cos n \alpha z \\
W &= \sum_{n=1}^{\infty} W_n(X,Y) \sin n \alpha z 
\end{align*}
\]

where we have anticipated the well-known result that the nonlinear interactions which occur in the Taylor-Gortler problem do not generate a mean flow in the spanwise direction. We then substitute for \((U,V,W)\) from \((2.7)\) into \((2.4a)\) and \((2.6)\) and equate like Fourier coefficients. This procedure shows that the mean flow correction satisfies

\[
U_{0YY} - V_0 \frac{\partial^2}{\partial x^2} \hat{u} - U_0 \frac{\partial^2}{\partial y^2} \hat{u} - \hat{v} U_{0Y} = U_0 \frac{\partial^2}{\partial x^2} \hat{u} + V_0 \frac{\partial^2}{\partial y^2} \hat{v} + F_0
\]

where \( F_0 = \frac{1}{2} \sum_{m=1}^{\infty} \left( V_m \frac{\partial^2}{\partial m Y} - U_m \frac{\partial^2}{\partial m Y} - 2 m a u \frac{\partial^2}{\partial x^2} \hat{u} \right) \),

and \( V_0 \) is determined by

\[
\frac{\partial U_0}{\partial X} + \frac{\partial V_0}{\partial Y} = 0.
\]
For computational purposes we must of course truncate the infinite sums in (2.7) at some suitably large value for the upper limit. We therefore replace the upper limit in (2.7) by $N$.

Similarly we find that $U_n$ satisfies

$$U_{nXY} - a^2 U_n - b U_n X - U_n X - V_n Y - 6U_n Y$$

$$= F_n = \sum_{m=1}^{N-1} V_{n-m} U_{mY} - U_{n-m} V_{mY} + ma W_{n-m} U_n - ma U_{n-m} W_m$$

$$+ \sum_{m \neq N}^{N-n} V_{n+m} U_{mY} - U_{n+m} V_{mY} - ma U_{m+n} W_m - ma U_{n+m} W_m$$

$$+ \sum_{m = n+1}^{N} V_{m-n} U_{mY} - U_{m-n} V_{mY} - ma W_{m-n} U_m - ma U_{m-n} W_m$$

(2.10)

where $\delta = \delta + U_0$ and $\delta = \delta + V_0$.

An equation of the same form can be derived from (2.6) by equating to zero the coefficient of $\cos n_\alpha x$. Suppose that $U_0$, $V_0$, $U_n$, $V_n$, $W_n$, for $n = 1, 2, 3, \ldots$ are known at $X$, we now describe how (2.8) and (2.10) can be stepped forward to $X + \varepsilon$. The scheme used is essentially that described in II together with an iteration procedure to take care of the nonlinear terms now present. Thus for example the mean flow equation (2.8) is discretized using finite differences in the $X$ and $Y$ directions to give
Here indices $n, m$ refer to the grid point $X = X_0 + m$, $Y = nh$.

The nonlinear terms on the right hand side of (2.11) are initially evaluated with $k = m - 1$ and the resulting tridiagonal system can be solved to give $U_0$ at $X = X_0 + (m + 1)\epsilon$. The equation (2.10) can be stepped forward in a similar manner to give $U_m, m = 1, \ldots, N$ at $X = X_0 + (m + 1)\epsilon$. Likewise the $V$ equation can be stepped forward by solving a pentadiagonal system. At this stage the nonlinear terms can be expressed in terms of the velocity field now calculated at $X = X_0 + (m + 1)\epsilon$. The equation can then be solved again for the flow quantities at $X = X_0 + (m + 1)\epsilon$ and the iteration procedure continued until the change in $U_{m+1}, U_m$ etc is sufficiently small. Thus (2.11) and the corresponding equations for $U_m, V_m$ are effectively solved with $k = m$ by iterating on the nonlinear terms on the right hand side.

3. RESULTS AND DISCUSSION

We shall firstly describe some results obtained in order to verify the numerical scheme used. These calculations were carried out at various values of the parameters of the problem but here we shall concentrate on the case

\[
a = .2, \ G = .0288, \ \kappa(X) = \frac{X}{20}.
\]

This choice for the curvature function $\kappa$ means that the effective local Gortler number varies like $X^{5/2}$ whilst the local wavenumber varies like $X^{3}$. The asymptotic theory of I showed that the neutral curve which can be uniquely
defined at high wavenumbers has the Gortler number proportional to the fourth power of the wavenumber and therefore (3.1) corresponds to a disturbance which remains in the unstable region when \( X \) increases.

The basic state was disturbed at \( X = 55 \) by imposing the condition

\[
U_1(\eta) = \eta^6 e^{-\eta^2}, \quad V_1(\eta) = 0
\]  

(3.2)

and integrating the linearized equations to \( X = 100 \). At this stage the disturbance is almost locally neutral stable according to the criterion of II and the linear velocity field was given an amplitude \( \Delta \) equal to the maximum \( X \)-disturbance velocity component. The nonlinear equations were then integrated for \( X > 100 \) and the local growth rates and energies of the different harmonics were calculated. We defined the energy of the nth harmonic to be

\[
E_n = \int_0^1 \{U_n^2(X,Y) + V_n^2(X,Y) + W_n^2(X,Y)\} dY, \quad n = 1,2,\ldots
\]  

(3.3)

and the energy of the mean flow distortion was defined by

\[
E_0 = \int_0^1 \{U_0^2(X,Y)\} dY.
\]  

(3.4)

Here we have omitted the contribution from \( V_0 \) since \( V_0 \to \text{constant} \) when \( Y \to \infty \). The growth rate \( \theta_n(X) \) of the nth mode was defined by

\[
\theta_n(X) = \frac{dE_n}{dX} E_n^{-1}
\]  

(3.5)

so that for a parallel boundary layer in the linear regime \( \sigma_n \) would be twice the linear spatial amplification rate.
we know from the nonparallel calculations of I that \( \theta_1(x) \) initially
depends sensitively on the form and location of the initial disturbance.
Here the situation is more complex because we can specify each Fourier mode
and the mean flow distribution. In Figure 1 we have shown the dependence
of \( \theta_1, c > 1 \) on \( X \) for five different values of \( \Delta \) the disturbance flow
amplitude. Apart from the fundamental all the Fourier components of the
disturbance were set equal to zero at the initial location. The calculations
shown were carried out with \( N = 4, \epsilon = .025, y_m = 150 \). Similar calculations
were carried out by changing \( N \) to 8, \( \epsilon \) to .05 and \( y_m \) to 100 in turn. The
results agreed with those of Figure 1 to the graphical accuracy of that Figure.

We see in Figure 1 that for a 0.5% disturbance the growth rate over the
interval shown is indistinguishable from linear theory. At higher values of \( \Delta \)
the growth rate is initially increased above the linear value and then falls
below it when \( X \) increases. The amount by which the growth rate is decreased
from the linear value increases with \( \Delta \) and we conclude that in this situation
nonlinear effects are stabilizing. We attribute the initial increase in the
growth rate to the relatively quick change in flow structure which must necessarily
occur when nonlinear effects are operational.

In Figure 2 we have shown the corresponding growth rates for the first
harmonic, again we see that after the initial period of growth the disturbance
growth decreases with \( \Delta \). We note that the growth rates of Figures 1, 2 are
comparable even though the first harmonic is locally neutrally stable at a
higher Gortler Number than is the fundamental. This growth of the first
harmonic is of course driven by nonlinear effects. Though the calculations
represented in Figure 2 clearly indicate the stabilizing effect of nonlinearity
they do not indicate the emergence of any local equilibrium state as the
vortices develop downstream.
Further calculations were carried out for the same initial condition (3.2) but with different curvature distributions \( \kappa(X) \). The three curvature distributions which we examined in detail and the values of \( a \) and \( G \) used in the calculations were:

(a) \[ \kappa(X) = \frac{1}{1 + (0.02X - 2.4)}, \] \( a = 0.16, \ G = 0.1 \),

(b) \[ \kappa(X) = \frac{\sqrt{X}}{10}, \ a = 0.2, \ G = 0.23, \]

and (c) \[ \kappa(X) = \frac{\sqrt{X}}{10} (1 + 0.2 \sin \frac{X}{40}), \ a = 0.2, \ G = 0.23. \]

The first curvature distribution was chosen because it corresponds to a flow over a hump such that the flow is only unstable over a finite interval. The second distribution was chosen since, as in the asymptotic theory of I, it gives a local Gortler number proportional to the fourth power of the local wavenumber. At relatively large values of \( X \) the local growth rate changes little with \( X \) and in the nonlinear regime we might expect to recover the results of III. The case (c) was chosen in order to obtain information about the possible influence of a small amplitude wall waviness on nonlinear Gortler vortices. The linear growth rate curves corresponding to (a), (b) and (c) and the initial conditions (3.1) are shown in Figure 4. It is interesting to see that the small amplitude waviness causes a significant difference between (b) and (c). We note that, whereas (a) is stable beyond \( X = 140 \), (b) and (c) remain unstable up to \( X = 200 \) beyond which the growth rates increase slowly.

In Figure 3a we have shown the energy functions \( E_0 \) and \( E_1 \) corresponding to \( \Delta = 0.1, 0.2 \) together with \( a = 0.16, G = 0.0288, \) and \( \kappa(X) = \frac{X}{20} \). We see that the differences between the values of \( E_0, E_1 \) for \( \Delta = 0.1 \) and \( \Delta = 0.2 \) decreases with \( X \). This is presumably because when \( X \) is large the effective
wavenumber is also large and the analysis of III suggests that in this regime there exists a unique finite amplitude solution independent of its initial upstream form. However the calculation of III cannot be applied directly to the calculations reported here since they are restricted to an asymptotically small interval near to the neutral location. Nevertheless the short wavelength nonlinear theory of III does suggest that in this regime the origin of the disturbance is unimportant.

In Figures 3b,c we have shown the total downstream velocity component \( u_T \) at the spanwise locations \( az = \pi/2, \pi, 2\pi \) together with the Blasius profile which exists in the absence of the vortices. The profiles shown correspond to \( X = 300 \) and we see that at this location there is very little difference between the profiles originating from \( \Delta = .1 \) and \( \Delta = .2 \). We further note that the values of \( u_T \) corresponding to \( az = \pi/2 \) are identical to those with \( az = 3\pi/2 \). Of particular interest is the fact that the \( az = \pi \) profile has a strongly inflectional profile which is probably locally unstable to highly amplified Rayleigh instabilities. The location \( az = \pi \) of course corresponds to the boundary between vortices where the motion of the fluid is away from the wall. We might therefore expect that such locations will be the most susceptible to the secondary instabilities which cause the onset of time dependence in the Gortler problem.

In Figures 3d,e,f,g we have shown the individual velocity components appropriate to the above situation with \( \Delta = .2 \). It can be seen that the disturbance is dominated by the fundamental and mean flow correction velocity components. We see that \( U_1 \) at \( X = 300 \) has a significantly different shape than the linear solution initially imposed on the flow at \( X = 100 \). We have no physical explanation of the nonlinear mechanism which produces this distortion.

In Figure 5 we have shown the energy functions appropriate to the curvature distribution (a). The linear eigenfunction was obtained by inserting (3.2) at \( X = 55 \) and integrating until \( X = 85 \), where the nonlinear terms were turned on.
The initial disturbance amplitudes were taken to be \( \alpha = 0.1 \) and \( \beta = 0.15 \). We see that the energy of the disturbance is again almost completely confined to the fundamental and mean flow correction. The maximum value of the disturbance energies \( E_1, E_2, E_3 \) occur close to the position where the linear growth rate \( \alpha \) of Figure 1 is zero. In contrast the maximum of \( E_0 \) occurs at a higher value of \( X \). This suggests that the results of Figure 5 are dominated by the interaction between the basic Blasius boundary layer and the fundamental component of the disturbance and never reach any 'local' nonlinear equilibrium state.

In Figure 6 the results corresponding to the case (b) are shown. The nonlinear terms were again turned on at \( X = 85 \). after integrating (3.2) from \( X = 55 \). and four calculations corresponding to \( \Delta = 0.05, 0.1, 0.15, 0.20 \) were carried out. Figures 6a, b, c show the evolution in \( X \) of the energy functions \( E_0, E_1 \) and \( E_2 \) for this situation. The functions \( E_0 \) and \( E_1 \) appear to approach limiting values essentially independent of \( \Delta \) whilst \( E_2 \) initially increases before decaying at sufficiently large values of \( X \). This suggests that as the vortices develop into a region where the effective Gortler number \( G_x \) and the effective wavenumber \( a^*_x \) satisfy \( G_x \sim a^*_x^4, a^*_x \gg 1 \) the asymptotic structure found in III is qualitatively recovered. In the latter calculation it was found that small wavelength Gortler vortices develop through a 'mean-field' interaction between the fundamental and mean flow correction. A quantitative comparison between our results and III is not possible since the asymptotics of III was restricted to a \( O(a^{-1}) \) neighbourhood of the neutral value of \( X \).

The downstream development of the individual velocity components in the above calculation is shown in Figure 7. It can be seen that the characteristic nonlinear shape of \( U_1 \) shown in Figure 3d for the \( \kappa = \frac{X}{20} \) calculation is reproduced at sufficiently large values of \( X \). The value of \( X \) required to produce this characteristic shape decreases with the size of the initial
amplitude. Similarly the mean flow corrections calculated have a similar shape to that shown in Figure 3d and that produced by the asymptotic theory of III. Indeed the mean flow corrections calculated for the cases (a), (b), and (c) together with the case \( k = \frac{X}{20} \) all had the same characteristic shape. Similarly the downstream velocity component was always found to increase away from the wall, reach a maximum and then decay to zero. Since this velocity component and the mean flow correction are always much larger than the other \( X \) velocity components it follows that the spanwise distribution of \( u_T \) the total downstream velocity component will always be similar to those shown in Figure 3b. In Figure 8 we have shown how these profiles develop in \( X \) for the case \( \Delta = .05 \) together with the corresponding undisturbed profile. Again the \( az = \pi \) profiles become highly inflectional and are presumably highly unstable to Rayleigh's instabilities of the type discussed by, for example, Cowley and Tufty (1986). We believe that the development of these highly inflectional profiles at the spanwise locations where upwelling occurs is the most likely source of the time-dependent secondary instabilities which steady Gortler vortices are known to suffer, see, for example, Aihara (1965).

Finally in Figure 9 we have shown the energy distributions for the case (c). The calculations were again performed by integrating (3.21) from \( X = 55. \) to \( X = 85. \) where the nonlinear terms were turned on. A comparison between Figures 9a,b,c and Figures 6a,b,c shows that the wall waviness does not have a significant effect on the energy distributions for \( \Delta = .05 \). However the solutions for the larger amplitude \( \Delta = .15 \) suggest that the waviness increases the energy of the disturbance up to \( X = 200. \) The computational expense of these calculations prevented us from determining whether a small amplitude waviness always leads to a destabilization of the boundary layer.
REFERENCES


Figure 1  The growth rate $\sigma_1$ for the wall $\kappa = \frac{X}{20}$, for $\Delta = .05, .1, .15, .2$

Figure 2  The growth rate $\sigma_2$ for the wall $\kappa = \frac{X}{20}$, for $\Delta = .05, .1, .15, .2$

Figure 3a  The energy distributions $E_0$ and $E_1$ for $\kappa = \frac{X}{20}$, and $\Delta = .1, .2$

Figure 3b  The total X velocity component at different spanwise locations for $X = 300., \kappa = \frac{X}{20}, \Delta = .1$

Figure 3c  The total X velocity component at different spanwise locations for $X = 300., \kappa = \frac{X}{20}, \Delta = .2$

Figure 3d  The X velocity components at $X = 300.$ for $\kappa = \frac{X}{20}, \Delta = .2$

Figure 3e  The Y velocity components at $X = 300.$ for $\kappa = \frac{X}{20}, \Delta = .2$

Figure 3f  The Z velocity components at $X = 300.$ for $\kappa = \frac{X}{20}, \Delta = .2$

Figure 3g  The mean flow correction at $X = 300.$ for $\kappa = \frac{X}{20}, \Delta = .2$

Figure 4  The growth rates for the cases (a), (b), and (c) respectively in the linear regime.

Figure 5a  The energy function $E_0$ for (a) with $\Delta = .1, .15$

Figure 5b  The energy function $E_1$ for (a) with $\Delta = .1, .15$

Figure 5c  The energy function $E_2$ for (a) with $\Delta = .1, .15$
Figure 5d  The energy function $E_3$ for (a) with $\Delta = .1, .15$.

Figure 6a  The energy function $E_0$ for (b) with $\Delta = .05, .1, .15, .2$.

Figure 6b  The energy function $E_1$ for (b) with $\Delta = .05, .1, .15, .2$.

Figure 6c  The energy function $E_2$ for (b) with $\Delta = .05, .1, .15, .2$.

Figure 7a  The $X$ velocity component $U_1$ for (b) with $\Delta = .05$.

Figure 7b  The $X$ velocity component $U_1$ for (b) with $\Delta = .1$.

Figure 7c  The $X$ velocity component $U_1$ for (b) with $\Delta = .15$.

Figure 7d  The $X$ velocity component $U_1$ for (b) with $\Delta = .20$.

Figure 7e  The $X$ velocity component $U_0$ for (b) with $\Delta = .05$.

Figure 7f  The $X$ velocity component $U_0$ for (b) with $\Delta = .1$.

Figure 7g  The $X$ velocity component $U_0$ for (b) with $\Delta = .15$.

Figure 7h  The $X$ velocity component $U_0$ for (b) with $\Delta = .2$.

Figure 8a  The total $X$ velocity component for (b) with $X = 125, 165, 205, 245$, and $\Delta = 0$.

Figure 8b  The total $X$ velocity component for (b) with $X = 125, 165, 205, 245$, and $\Delta = .05$, $az = \pi/2$. 
Figure 8c  The total X velocity component for (b) with \( X = 125, 165, 205, 245 \), and \( \Delta = .05 \), \( az = \pi \).

Figure 8d  The total X velocity component for (b) with \( X = 124, 165, 205, 245 \), and \( \Delta = .05 \), \( az = 2\pi \).

Figure 9a  The energy function \( E_0 \) for (c) with \( \Delta = .05, .15 \).

Figure 9b  The energy function \( E_1 \) for (c) with \( \Delta = .05, .15 \).

Figure 9c  The energy function \( E_2 \) for (c) with \( \Delta = .05, .15 \).
Figure 3a
Figure 3b

\[ \Delta = 0.1 \]

- Blasius flow
  - \( \alpha z = \pi \)
  - \( \alpha z = \pi/2 \)
  - \( \alpha z = 2\pi \)
Figure 3c

$\Delta = 0.2$

$az = \pi$

$az = 2\pi$

Blasius flow
Figure 3d

Y

-100

50

100

-1

.1

10U_2

10^3 U_4

10^2 U_3

U_1

Linear solution
Figure 3e

Linear solution

$V_1$

$10^3 V_4$

$10V_2$

$10^2 V_3$
Figure 3f
Figure 3g
Figure 5a

\[ E_j \]

\[ \Delta = 0.15 \]

\[ \Delta = 0.10 \]

\[ 100 \quad 200 \]
Figure 5b

\[ E_1 \]

\[ \Delta = 0.15 \]

\[ \Delta = 0.10 \]

\[ 100. \quad 200. \]

\[ x \]
Figure 5c

$E_2$

$\Delta = .15$

$\Delta = .10$
Figure 5d

$10^7 E_3$

$\Delta = .15$

$\Delta = .10$

$100.$ $200.$
Figure 6a

The graph shows several curves labeled with different values of $\Delta$: $\Delta = 0.2$, $\Delta = 0.1$, $\Delta = 0.05$. The $x$-axis is labeled from 100 to 200, and the $y$-axis is labeled with values from 0.2 to 0.8.
Figure 6b

$E_1$

$\Delta = 0.2$

$\Delta = 0.15$

$\Delta = 0.1$

$\Delta = 0.05$
Figure 6c

$10 E_2$

$\Delta = 0.2$

$\Delta = 0.15$

$\Delta = 0.1$

$\Delta = 0.05$

$X$
Figure 7a

$\Delta = 0.05$
Figure 7b

\[ \Delta = 0.1 \]

- $X = 245$
- $X = 205$
- $X = 165$
- $X = 125$
- $X = 85$
Figure 7c

\[ \Delta = 0.15 \]
Figure 7d

$\Delta = .2$

$X = 245,$

$X = 205,$

$X = 165$

$X = 125,$

$X = 85.$
Figure 7f
Figure 7a

\[ \Delta = 0.15 \]

X = 125, 165, 205, 245
Figure 8a

Blasius flow

\( X = 125, \quad X = 165, \quad X = 205, \quad X = 245 \)
Figure 8b

\[ a_z = \frac{\pi}{2}, \Delta = 0.05 \]

- Labels: \( X = 245, X = 205, X = 165, X = 125 \).
Figure 8c

$az = \pi$, $\Delta = .05$

$X = 245$
$X = 205$
$X = 165$
$X = 125$. 

$Y$ vs $U_T$
Figure 8d

\[ az = 2, \Delta = 0.05 \]
Figure 9a

\[ \Delta = 0.15 \]

\[ \Delta = 0.05 \]
Figure 9c

$10^3 E_2$

$\Delta = 0.15$

$\Delta = 0.05$
The development of Gortler vortices in boundary layers over curved walls in the nonlinear regime is investigated. The growth of the boundary layer makes a parallel flow analysis impossible except in the high wavenumber regime so in general the instability equations must be integrated numerically. Here the spanwise dependence of the basic flow is described using Fourier series expansion whilst the normal and streamwise variations are taken into account using finite differences. The calculations suggest that a given disturbance imposed at some position along the wall will eventually reach a local equilibrium state essentially independent of the initial conditions. In fact, the equilibrium state reached is qualitatively similar to the large amplitude high wave-number solution described asymptotically by Hall (1982b). In general, it is found that the nonlinear interactions are dominated by a 'mean field' type of interaction between the mean flow and the fundamental. Thus, even though higher harmonics of the fundamental are necessarily generated, most of the disturbance energy is confined to the mean flow correction and the fundamental. A major result of our calculations is the finding that the downstream velocity field develops a strongly inflectional character as the flow moves downstream. The latter result suggests that the major effect of Gortler vortices on boundary layers of practical importance might be to make them highly receptive to rapidly growing Rayleigh modes of instability.

Gortler vortices, boundary layers

Unclassified - unlimited