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A Multigrid LU-SSOR Scheme for Approximate Newton Iteration Applied to the Euler Equations

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A MULTIGRID LU-SSOR SCHEME FOR APPROXIMATE NEWTON ITERATION APPLIED TO THE EULER EQUATIONS

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SUMMARY

A new efficient relaxation scheme in conjunction with a multigrid method is developed for the Euler equations. The LU-SSOR scheme is based on a central difference scheme and does not need flux splitting for Newton iteration. Application to transonic flow shows that the new method surpasses the performance of the LU implicit scheme.

INTRODUCTION

Recently several implicit schemes have been successfully developed in conjunction with a multigrid method for steady-state solution of the unsteady Euler equations (refs. 1 to 3). Although the alternating direction implicit scheme could be improved to achieve the expected efficiency of the multigrid method in two-dimensions (ref. 1), its inherent limitations in three-dimensions suggested alternative approaches (ref. 2). An alternative implicit scheme which is stable in any number of space dimensions was based on LU factorization. The LU implicit scheme was proved to be robust and efficient for high Mach number flows as well as transonic flows (ref. 4). It was also shown that a symmetric Gauss-Seidel relaxation method for solving the unfactored implicit scheme was a variation of the LU implicit scheme.

The Newton iteration method has been a subject of investigation for solution of the steady Euler equations (refs. 5 to 7). Because of the rapid growth of the operation count with the number of mesh cells, the system was solved indirectly. Jespersen (ref. 5) and Hemker and Spekreijse (ref. 6) used the symmetric Gauss-Seidel method while MacCormack (ref. 7) applied the line Gauss-Seidel method to the Navier-Stokes equations. In this paper an efficient relaxation scheme in conjunction with a multigrid method is developed for approximate Newton iteration. The new LU-SSOR scheme needs scalar diagonal inversions while the Gauss-Seidel method or the LU implicit scheme require block matrix inversions. It is desirable that the matrix should be diagonally dominant to assure the convergence of a relaxation method. The new method based on a central difference scheme achieves this without flux splitting which substantially increases the computational work per cycle.

GOVERNING EQUATIONS

The Euler equations are obtained from the Navier-Stokes equations by neglecting viscous terms. Let ρ, u, v, E, H, and p be the density, Cartesian velocity components, total energy, total enthalpy, and pressure, and let x and y be Cartesian coordinates. Then for a two-dimensional flow these equations can be written as

$$\frac{\partial W}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \tag{1}$$

where W is the vector of dependent variables, and F and G are convective flux vectors

$$W = (\rho, \rho u, \rho v, \rho E)^T$$

$$F = (\rho u, \rho u^2 + p, \rho vu, u(\rho E + p))^T \tag{2}$$

$$G = (\rho v, \rho uv, \rho v^2 + p, v(\rho E + p))^T$$

The pressure is obtained from the equation of state

$$p = \rho(\gamma - 1)\left\{E - \frac{1}{2}(u^2 + v^2)\right\} \tag{3}$$

These equations are to be solved for a steady state $\partial W/\partial t = 0$ where t denotes time.

SEMI-DISCRETE FINITE VOLUME METHOD

A convenient way to assure a steady state solution independent of the time step is to separate the space and time discretization procedures. In semi-discrete finite volume method one begins by applying a semi-discretization in which only the spatial derivatives are approximated. The use of a finite volume method for space discretization allows one to handle arbitrary geometries and helps one to avoid problems with metric singularities that are usually associated with finite difference methods. The scheme reduces to a central difference scheme on a Cartesian grid, and is second order accurate in space provided that the mesh is smooth enough. It also has the property that uniform flow is an exact solution of the difference equations.

NONLINEAR ADAPTIVE DISSIPATION

In typical calculation of flow with discontinuities by a central difference scheme, wiggles appear in the neighborhood of shock waves where pressure gradient is severe. In order to suppress the tendency for spurious odd and even point oscillations, and to prevent unsightly overshoots near shock waves, the scheme is augmented by artificial dissipative terms. The dissipative term, which is constructed so that it is of third order in smooth regions of the flow, is explicitly added to the residual. For the density equation, for example, the dissipation has the form

$$d_{i+1/2,j} - d_{i-1/2,j} + d_{i,j+1/2} - d_{i,j-1/2}$$

where

$$d_{i+1/2,j} = \varepsilon^{(2)}_{i+1/2,j}(\rho_{i+1,j} - \rho_{i,j})$$

$$-\varepsilon^{(4)}_{i+1/2,j}(\rho_{i+2,j} - 3\rho_{i+1,j} + 3\rho_{i,j} - \rho_{i-1,j}) \qquad (4)$$

Let S be the cell area which is equivalent to the inverse of the determinant of transformation Jacobian. Both coefficients include a normalizing factor $S_{i+1/2,j}/\Delta t$ proportional to the length of the cell side, and $\varepsilon^{(2)}_{i+1/2,j}$ is also made proportional to the normalized second difference of the pressure

$$\nu_{i,j} = \left| \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{p_{i+1,j} + 2p_{i,j} + p_{i-1,j}} \right| \qquad (5)$$

in the adjacent cells. The third order terms provide background damping of high frequency modes. The first order terms are needed to control oscillations in the neighborhood of shock waves, and are turned on by sensing strong pressure gradients in the flow. The dissipative terms for the other equations are constructed from similar formulas with the exception of the energy equation where the differences are of ρH rather than ρE. The purpose of this is to allow a steady state solution for which H remains constant. Increasing the amount of artificial viscosity improves the rate of convergence although too much dissipation can hurt it. However, it is desirable to make the amount be as small as possible in order not to degrade the accuracy of solution. Typical amount of the third order terms is almost negligible when compared to the physical viscosity.

LU-SSOR SCHEME

A prototype implicit scheme for a system of nonlinear hyperbolic equations such as the Euler equations can be formulated as

$$W^{n+1} = W^n - \beta\,\Delta t(D_x F(W^{n+1}) + D_y G(W^{n+1})) - (1-\beta)\,\Delta t(D_x F(W^n) + D_y G(W^n)) \qquad (6)$$

where D_x and D_y are difference operators that approximate $\partial/\partial x$ and $\partial/\partial y$. Here n denotes the time level. In this form the scheme is too expensive, since it calls for the solution of coupled nonlinear equations at each time step. Let the Jacobian matrices be

$$A = \frac{\partial F}{\partial W}, \quad B = \frac{\partial G}{\partial W} \qquad (7)$$

and let the correction be

$$\delta W = W^{n+1} - W^n$$

The scheme can be linearized by setting

$$F(W^{n+1}) = F(W^n) + A\delta W + O(\|\delta W\|^2)$$

$$G(W^{n+1}) = G(W^n) + B\delta W + O(\|\delta W\|^2)$$

and dropping terms of the second and higher order.

This yields

$$[I + \beta\, \Delta t(D_x A + D_y B)]\delta W + \Delta t\, R = 0 \tag{8}$$

where R is the residual

$$R = D_x F(W^n) + D_y G(W^n)$$

If $\beta = 1/2$ the scheme remains second order accurate in time, while for other values of β the time accuracy drops to first order.

The unfactored implicit scheme equation (8) produces a large block banded matrix which is very costly to invert and requires huge storage. If $\beta = 1$ the scheme reduces to a Newton iteration in the limit $\Delta t \to \infty$.

$$(D_x A + D_y B)\delta W + R = 0 \tag{9}$$

A diagonally dominant form of equation (9)

$$(D_x^- A^+ + D_x^+ A^- + D_y^- B^+ + D_y^+ B^-)\ \delta W + R = 0 \tag{10}$$

can be written as

$$A^+_{ij}\delta W_{ij} - A^+_{i-1,j}\,\delta W_{i-1,j} + A^-_{i+1,j}\,\delta W_{i+1,j} - A^-_{ij}\delta W_{ij}$$

$$+ B^+_{ij}\delta W_{ij} - B^+_{i,j-1}\delta W_{i,j-1} + B^-_{i,j+1}\,\delta W_{i,j+1} - B^-_{ij}\delta W_{ij} + R_{ij} = 0 \tag{11}$$

By simulating it with backward and forward relaxation sweeps, we obtain the symmetric successive over-relaxation (SSOR) method, which can be written in two steps as

$$(A^+_{ij} - A^-_{ij})\delta W^*_{ij} + A^-_{i+1,j}\delta W^*_{i+1,j} + (B^+_{ij} - B^-_{ij})\delta W^*_{ij} + B^-_{i,j+1}\delta W^*_{i,j+1} + R_{ij} = 0 \tag{12}$$

followed by

$$(A^+_{ij} - A^-_{ij})\delta W_{ij} - A^+_{i-1,j}\delta W_{i-1,j} + A^-_{i+1,j}\delta W^*_{i+1,j} + (B^+_{ij} - B^-_{ij})\delta W_{ij}$$

$$- B^+_{i,j-1}\delta W_{i,j-1} + B^-_{i,j+1}\delta W^*_{i,j+1} + R_{ij} = 0 \tag{13}$$

where D_x^- and D_y^- are backward difference operators and D_x^+ and D_y^+ are forward difference operators. Here, two-point operators are used for steady flow calculations. A^+, A^-, B^+, and B^- are constructed so that the eigenvalues of "+" matrices are non-negative and those of "-" matrices are non-positive.

$$A^+ = \frac{1}{2}(A + r_A I),\ A^- = \frac{1}{2}(A - r_A I)$$

$$B^+ = \frac{1}{2}(B + r_B I),\ B^- = \frac{1}{2}(B - r_B I) \tag{14}$$

where

$$r_A \geq \max(|\lambda_A|),\ r_B \geq \max(|\lambda_B|) \tag{15}$$

Here, λ_A and λ_B represent eigenvalues of Jacobian matrices. Subtract equation (12) from equation (13) to get

$$(A^+_{ij} - A^-_{ij})\,\delta W_{ij} - A^+_{i-1,j}\,\delta W_{i-1,j} + (B^+_{ij} - B^-_{ij})\delta W_{ij} - B^+_{i,j-1}\,\delta W_{i,j-1}$$

$$= (A^+_{ij} - A^-_{ij})\delta W^*_{ij} + (B^+_{ij} - B^-_{ij})\delta W^*_{ij} \tag{16}$$

This may be written as

$$(D_x^- A^+ + D_y^- B^+ - A^- - B^-)\,\delta W = (A^+ + B^+ - A^- - B^-)\,\delta W^* \tag{17}$$

where

$$\delta W^* = (D_x^+ A^- + D_y^+ B^- + A^+ + B^+)^{-1}(-R) \tag{18}$$

If we take "+" and "-" matrices as given in equation (14), then

$$A^+ - A^- = r_A I,\ B^+ - B^- = r_B I$$

Thus equation (17) becomes the LU-SSOR scheme for approximate Newton iteration

$$(D_x^- A^+ + D_y^- B^+ - A^- - B^-)(D_x^+ A^- + D_y^+ B^- + A^+ + B^+)\,\delta W = -(r_A + r_B)\,R \tag{19}$$

The equation (19) can be inverted in two steps.

MULTIGRID METHOD

In order to adapt the LU-SSOR scheme for a multigrid algorithm, auxiliary meshes are introduced by doubling the mesh spacing. Values of the flow variables are transferred to a coarser grid by the rule

$$\omega_{2h} = \frac{\sum S_h \omega_h}{S_{2h}} \tag{20}$$

where the subscripts denote values of the mesh spacing parameter, S is the cell area, and the sum is over the four cells on the fine grid composing each cell on the coarser grid. The rule conserves mass, momentum, and energy. The solution on a coarse grid is updated as follows.

(1) Calculate the correction and update the solution on the fine grid
(2) Transfer the values of the variables to the coarse grid
(3) Collect the residual on the fine grid for the coarse grid. A forcing function is then defined as

$$P_{2h} = \sum R_h^c - R_{2h}^t \tag{21}$$

where R is the residual. Superscripts c and t mean the collected and the transferred values respectively. The residual on the coarse grid is given by

$$R_{2h} = R_{2h}^t + P_{2h} = \sum R_h^c \tag{22}$$

(4) Calculate the correction and update the solution on the coarse grid.

For the next coarser grid the residual is recalculated as

$$R_{4h} = R_{4h}^t + P_{4h} = \sum (R_{2h}^u + P_{2h}) = \sum (R_{2h}^u + \sum R_h^c - R_{2h}^t) \tag{23}$$

Similarly, the residual for the next grid is

$$R_{8h} = \sum \left\{ (R_{4h}^u + \sum (R_{2h}^u + \sum R_h^c - R_{2h}^t) - R_{4h}^t \right\} \tag{24}$$

where the superscript u means the updated value. On the first coarse grid, R_{2h} is replaced by $\sum R_h^c$ with the result that the evolution on the coarse grid is driven by the residuals on the fine grid. The evolution on the next coarser grid is driven by an estimate of what the fine grid residuals would have been as a result of the correction on the first coarse grid. The process is repeated on successively coarser grids.

(5) Finally, the correction calculated on each grid is passed back to the next finer grid by bilinear interpolation.

Since the evolution on a coarse grid is driven by residuals collected from the next finer grid, the final solution on the fine grid is independent of the choice of boundary conditions on the coarse grids. The surface boundary condition is treated in the same way on every grid, by using the normal pressure gradient to extrapolate the surface pressure from the pressure in the cells adjacent to the wall. Values are extrapolated to the fictitious cells inside the body surface for the second difference dissipation on the coarse grids. The far field conditions can either be transferred from the fine grid, or recalculated.

RESULTS

The new LU-SSOR scheme in conjunction with a multigrid method is applied to transonic airfoil calculation. The test case is the NACA 0012 airfoil at Mach 0.8 and 1.25° angle of attack. Figures 1 and 2 show the plot of Mach number contours and the surface pressure distribution respectively. These solutions are obtained on a 128 by 32 C-mesh. The convergence histories are shown in figures 4 and 6. A four-level multigrid is used without grid sequencing. The lift history is shown in figure 4 while the density residual is shown in figure 6. The results obtained by the LU implicit scheme are presented in figures 3 and 5 for comparison. As the results show, the convergence rate of the LU-SSOR scheme is about 30 percent faster than that of the LU implicit scheme. Moreover, the computational work for the LU-SSOR scheme is about 30 percent less than that for the LU implicit scheme. The overall computational work is reduced by a factor of two. The convergence for the engineering accuracy is usually achieved in less than 10 CPU seconds with the Cray XMP computer.

REFERENCES

1. Jameson, A., and Yoon, S., "Multigrid Solution of the Euler Equations using Implicit Schemes," AIAA Paper 85-0293, AIAA 23rd Aerospace Sciences Meeting, Jan. 1985. AIAA Journal, (to appear).

2. Jameson, A., and Yoon, S., "LU Implicit Schemes with Multiple Grids for the Euler Equations," AIAA Paper 86-0105, AIAA 24th Aerospace Sciences Meeting, Jan. 1986. AIAA Journal, (to appear).

3. Anderson, W.K., Thomas, J.L., and Whitfield, D.L., "Multigrid Acceleration of the Flux Split Euler Equations," AIAA Paper 86-0274, AIAA 24th Aerospace Sciences Meeting, Jan. 1986.

4. Yoon, S., and Jameson, A., "An LU Implicit Scheme for High Speed Inlet Analysis," NASA CR-175098, Apr. 1986. AIAA Paper 86-1520, AIAA/ASME 22nd Joint Propulsion Conference, June 1986.

5. Jespersen, D.C., "A Multigrid Method for the Euler Equations," AIAA Paper 83-0124, AIAA 21st Aerospace Sciences Meeting, Jan. 1983.

6. Hemker, P.W., and Spekreijse, S.P., "Multiple Grid and Osher's Scheme for the Efficient Solution of the Steady Euler Equations," Journal of Numerical Mathematics, Dec. 1986, (to appear).

7. MacCormack, R.W., "Current Status of Numerical Solutions of the Navier Stokes Equations," AIAA Paper 85-0032, AIAA 23rd Aerospace Sciences Meeting, Jan. 1985.

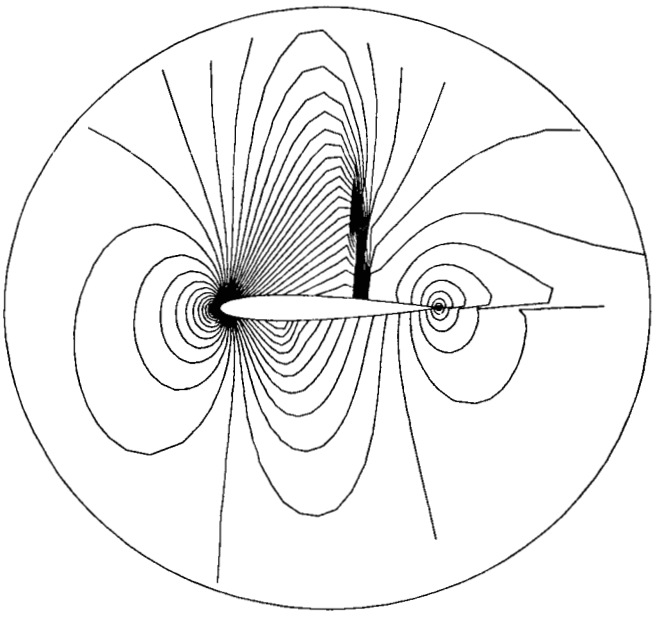

FIGURE 1. - MACH NUMBER CONTOURS FOR TRANSONIC FLOW, MACH 0.8 AND 1.25^{0} ANGLE OF ATTACK.

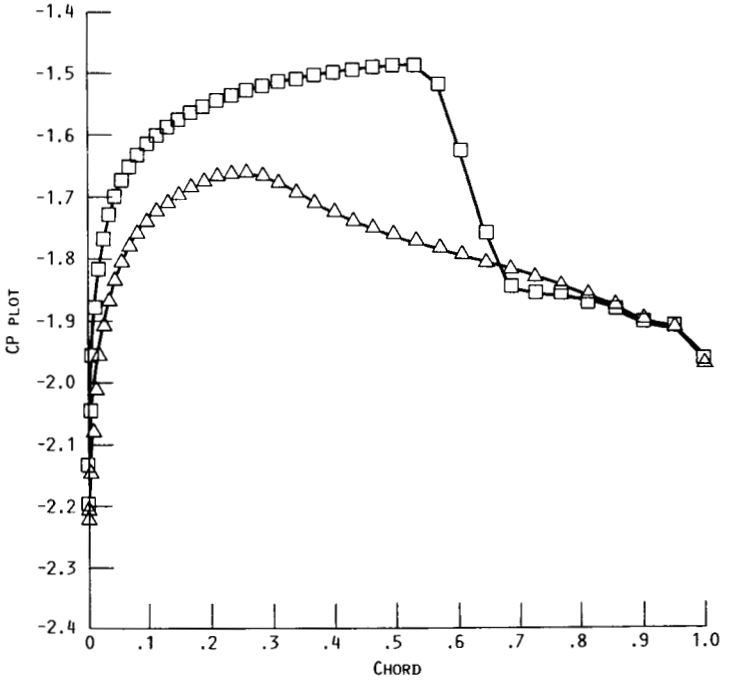

FIGURE 2. - SURFACE PRESSURE DISTRIBUTION.

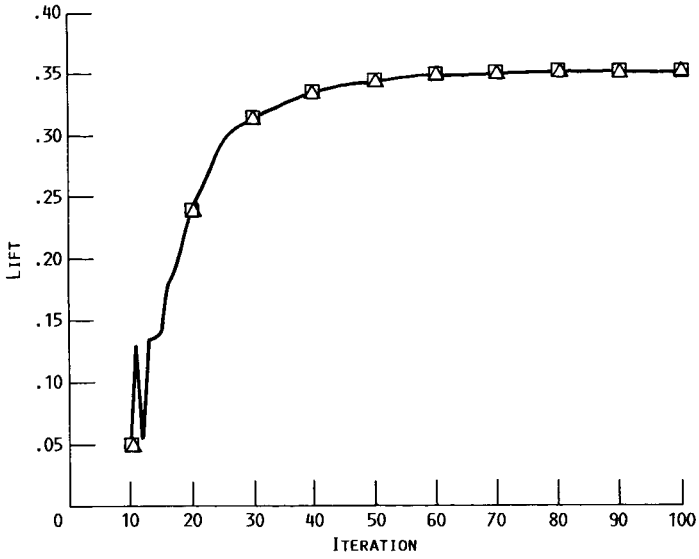

FIGURE 3. - LIFT HISTORY (LU IMPLICIT SCHEME).

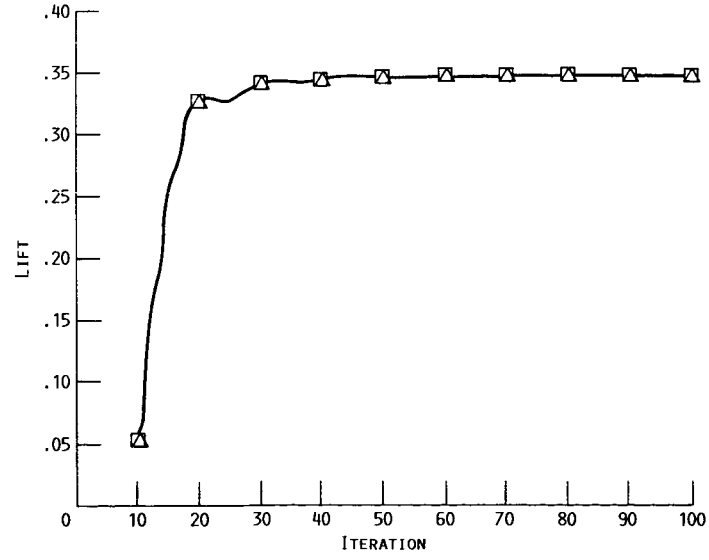

FIGURE 4. - LIFT HISTORY (PRESENT METHOD).

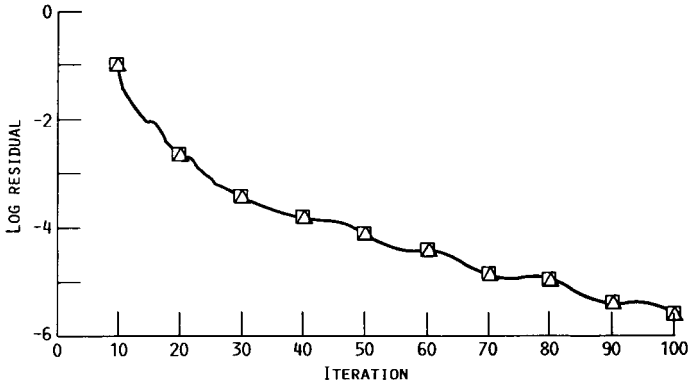

FIGURE 5. - CONVERGENCE HISTORY (LU IMPLICIT SCHEME).

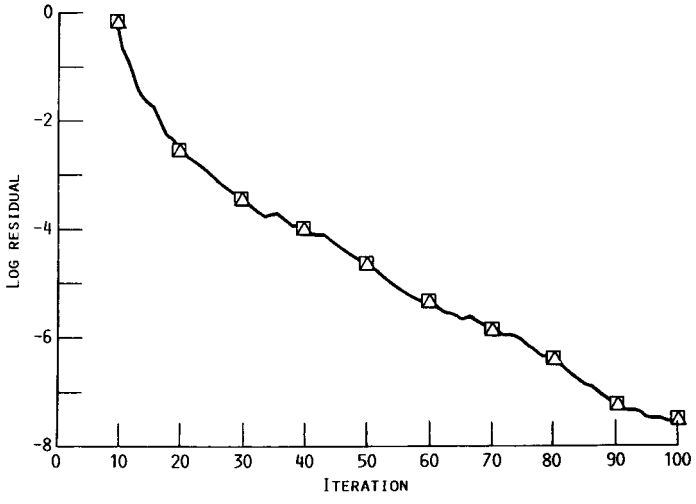

FIGURE 6. - CONVERGENCE HISTORY (PRESENT METHOD).

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