

NASA Contractor Report 178209
ICASE Report No. 86-74

ICASE

**CONVENIENT TOTAL VARIATION DIMINISHING CONDITIONS
FOR NONLINEAR DIFFERENCE SCHEMES**

(NASA-CR-178209) CONVENIENT TOTAL VARIATION
DIMINISHING CONDITIONS FOR NONLINEAR
DIFFERENCE SCHEMES Final Report (NASA)
33 p

N87-17453

CSCD 12A

G3/64

Unclas
43749

Eitan Tadmor

Contract No. NAS1-18107
November 1986

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association



National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

**CONVENIENT TOTAL VARIATION DIMINISHING CONDITIONS
FOR NONLINEAR DIFFERENCE SCHEMES**

Eitan Tadmor^{*}

School of Mathematical Sciences, Tel-Aviv University

and

Institute for Computer Applications in Science and Engineering

ABSTRACT

Convenient conditions for nonlinear difference schemes to be total-variation diminishing (TVD) are derived. It is shown that such schemes share the TVD property, provided their numerical fluxes meet a certain positivity condition at extrema values but can be arbitrary otherwise. Our conditions are invariant under different incremental representations of the nonlinear schemes, and thus provide a simplified generalization of the TVD conditions due to Harten and others [3] - [7], [13].

Abbreviated title: Convenient Total Diminishing Conditions

Key words: nonlinear difference schemes, total variation, extrema values

AMS(MOS) subject classification: 35L65, 65M10

Research was supported in part by NASA Contract No. NAS1-18107 while in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665-5225. Additional support was provided by NSF Grant No. DMS85-03294 and ARO Grant No. DAAG-85-K-0190 while in residence at UCLA, Los Angeles, CA 90024.

^{*}Bat-Sheva Foundation Fellow.

1. INTRODUCTION

We consider discrete approximations to the scalar conservation law

$$(1.1) \quad \frac{\partial}{\partial t} [u(x,t)] + \frac{\partial}{\partial x} [f(u(x,t))] = 0, \quad (x,t) \in \mathbb{R} \times [0,\infty).$$

Let $v(t) = \{v_v(t)\}$ be the approximate solution, and denote by

$$(1.2) \quad TV[v(t)] = \sum_v |\Delta v_{v+1/2}|, \quad \Delta v_{v+1/2} \equiv v_{v+1}(t) - v_v(t),$$

its total-variation at time level t . A desirable property for such an approximate solution to share with the exact one, is that its total variation should decrease in time¹. Difference scheme which give rise to such total-variation diminishing solutions--called TVD schemes after Harten [3]--is the subject of this paper.

TVD schemes prevent spurious oscillations in their solutions, and unlike monotone schemes, they can still allow for high-accuracy in most of the computational domain. Consequently, the TVD schemes can offer a substantial gain in computational efficiency as indeed was verified in a wide range of applications, e.g. [11] and the references therein.

Sufficient TVD criteria for explicit and implicit fully-discrete schemes were given by Harten in [3], [4], and analogously for semi-discrete schemes in [13], [7]. Necessity for three-point schemes was proved in [15, Lemma 2.2] and a general TVD characterization for multi-point stencils was provided in

¹We use the notion of order in its weak sense; thus, decrease means nonincrease, positive refers to nonnegative, etc.

[5], [9]. Roughly speaking, these criteria assert that a given scheme has the TVD property, provided it can be written in an appropriate incremental form which meets a certain positivity condition, augmented with a CFL restriction in the explicit case. A difference approximation of (1.1) can be equally represented by a variety of different incremental forms, yet the positivity condition mentioned above is not invariant under such different representations. Hence, the key step in seeking the TVD property according to the above criteria, requires us to find the correct incremental form so that this positivity condition could be met, e.g. [10, Part I].

In this paper, we provide alternative more convenient TVD characterizations, in the sense that they are uniformly valid for the various different incremental representations of a given scheme. To this end, we first note that the total variation of a grid function depends solely on its extrema values, see (1.2). It is therefore plausible to assert that in order for a difference scheme to share the TVD property, its incremental coefficients should be controlled only at critical neighborhoods where the approximate grid solution attains extrema values. Indeed, our sufficient TVD conditions have the flavor of this assertion, namely, they place a positivity restriction only on those incremental coefficients which are associated with such critical neighborhoods. Moreover, we are also able to express these local TVD conditions solely in terms of the numerical fluxes of the nonlinear schemes, rather than invoking any of their special incremental decompositions. Putting it in different words, we show that the TVD property holds for difference schemes whose numerical viscosity correspond to upwind differencing at extrema values but can be arbitrary otherwise. Thus, in contrast to the more restrictive global positivity conditions mentioned

earlier, our TVD criteria are localized to extrema values, and consequently can be equally applied to different incremental representations.

We begin by discussing the semi-discrete case in Section 2. Fully-discrete implicit and explicit schemes are treated in Section 3. To utilize our TVD criteria in the latter cases, the extrema values at the next time-level are to be known in advance. This necessitates additional ingredients in the fully-discrete cases, whose purpose is to provide us with such a priori control on the behavior of extrema at the next time level. In particular, standard recipes of constructing TVD schemes which make use of anti-diffusive correctors and limiters, are all shown to naturally follow in light of our above arguments.

2. SEMI-DISCRETE SCHEMES

We consider semi-discrete schemes in the conservative form

$$(2.1) \quad \frac{d}{dt} v_v(t) = - \frac{1}{\Delta x_v} [h_{v+1/2} - h_{v-1/2}],$$

with $\Delta x_v \equiv \frac{1}{2} (x_{v+1} - x_{v-1})$ being the variable meshsize and $h_{v+1/2}$ denotes the Lipschitz continuous numerical flux which is consistent with the differential one

$$(2.2) \quad h_{v+1/2} = h(v_{v-p+1}, \dots, v_{v+p}), \quad h(u, u, \dots, u) = f(u).$$

To study the TVD property of these schemes, we forward difference (2.1),

$$(2.3) \quad \frac{d}{dt} \Delta v_{v+1/2} = - \frac{1}{\Delta x_{v+1}} [h_{v+3/2} - h_{v+1/2}] + \frac{1}{\Delta x_v} [h_{v+1/2} - h_{v-1/2}],$$

multiply by $s_{v+1/2} \equiv s_{v+1/2}(t) = \text{sgn}[\Delta v_{v+1/2}(t)]$ and sum by parts, obtaining

$$(2.4) \quad \frac{d}{dt} TV[v(t)] = \sum_v \frac{1}{\Delta x_v} [s_{v+1/2} - s_{v-1/2}] \cdot [h_{v+1/2} - h_{v-1/2}].$$

The only contributions to the sum on the right came from extrema values where $s_{v+1/2} \neq s_{v-1/2}$, and the requirement of these contributions to be negative yields

Lemma 2.1: The semi-discrete scheme (2.1) is TVD, if we have

$$(2.5a) \quad h_{v+1/2} \geq h_{v-1/2} \quad \text{at maxima values} \quad v_v(t),$$

$$(2.5b) \quad h_{v+1/2} \leq h_{v-1/2} \quad \text{at minima values} \quad v_v(t).$$

In other words, Lemma 2.1 requires maxima values to decrease in time and minima values to increase in time. Moreover, if the distance between such extrema values exceeds the stencil width of $2p + 1$ cells, then the corresponding terms inside the summation on the right of (2.4) are independent and consequently (2.5) is also necessary for TVD in this case.

We now turn to discuss the relation between the TVD criteria in Lemma 2.1 and a different kind of TVD conditions due to Harten [3], [4] and Osher [9]; see also [5], [13]. In order to implement the latter, one should start with nonlinear semi-discrete schemes which assume the incremental form

$$(2.6) \quad \frac{d}{dt} v_v(t) = C_{v+1/2}^+ \Delta v_{v+1/2} - C_{v-1/2}^- \Delta v_{v-1/2}, \quad \Delta v_{v+1/2} \equiv v_{v+1}(t) - v_v(t).$$

The nonlinearity is reflected here by the possible dependence of the coefficients $C_{v+1/2}^\pm$ on $v_{v-p+1}, \dots, v_{v+p}$. Forward differencing of (2.6) gives

$$(2.7) \quad \frac{d}{dt} \Delta v_{v+1/2} = (C_{v+3/2}^+ \Delta v_{v+3/2} - C_{v+1/2}^+ \Delta v_{v+1/2}) - (C_{v+1/2}^- \Delta v_{v+1/2} - C_{v-1/2}^- \Delta v_{v-1/2}).$$

Multiplying (2.7) by $s_{v+1/2}$ and summing by parts we find

$$(2.8) \quad \frac{d}{dt} \sum_v |\Delta v_{v+1/2}| = \sum_v s_{v+1/2} \cdot \frac{d}{dt} \Delta v_{v+1/2} = \\ - \sum_v [(s_{v+1/2} - s_{v-1/2}) \cdot C_{v+1/2}^+ + (s_{v+1/2} - s_{v+3/2}) \cdot C_{v+1/2}^-] \Delta v_{v+1/2},$$

and using the fact that² $\Delta v_{v+1/2} = s_{v+1/2} \cdot |\Delta v_{v+1/2}|$ where $s_{v+1/2}^2 \equiv 1$, we end up with

$$(2.9) \quad \frac{d}{dt} TV[v(t)] = - \sum_v [(1 - s_{v-1/2} s_{v+1/2}) \cdot C_{v+1/2}^+ + \\ (1 - s_{v+1/2} s_{v+3/2}) \cdot C_{v+1/2}^-] \cdot |\Delta v_{v+1/2}|.$$

The quantities inside the two parenthesis on the right are positive. Hence, the summation on the right is positive and consequently the scheme (2.6) is

²The signum function at zero is defined to be ± 1 , so that its square $\equiv 1$.

TVD, provided the incremental coefficients, $C_{v+1/2}^\bullet$, are positive

$$(2.10) \quad C_{v+1/2}^+ \geq 0, \quad C_{v+1/2}^- \geq 0.$$

The positivity requirement (2.10) is the usual condition which characterizes the TVD schemes (2.6), e.g., [3], [5], [7], [13]. Given a semi-discrete conservative approximation of (1.1), it can be equally represented in a variety of different incremental forms. The positivity condition is not invariant, however, under such different representations. Thus, a key step in seeking the TVD property for a given scheme, requires us to find the correct incremental form so that the positivity condition (2.10) could be met, e.g. [10].

Lemma 2.1 provides us with a local TVD criterion which makes no reference to the incremental representation of the scheme (2.1). How does this compare with the global positivity condition placed on the incremental coefficients in (2.10)? a second glance at (2.9) shows that whenever the grid values v_v and v_{v+1} are located in a monotone profile, i.e., when both $1 - s_{v-1/2} s_{v+1/2}$ and $1 - s_{v+1/2} s_{v+3/2}$ vanish, then the corresponding term in the summation on the right of (2.9) also vanishes independently of the incremental coefficients $C_{v+1/2}^\pm$. This tells us, therefore, that the positivity condition (2.10) can be localized to extrema values, bearing a close similarity to the local nature of the TVD criterion (2.5). To be more precise, let us abbreviate

$$(2.11) \quad \chi_v = 1 - s_{v-1/2} s_{v+1/2};$$

then (2.9) reads

$$(2.12) \quad \frac{d}{dt} TV[v(t)] = - \sum_v [\chi_v c_{v+1/2}^+ + \chi_{v+1} c_{v+1/2}^-] \cdot |\Delta v_{v+1/2}|,$$

and we are led to the following.

Lemma 2.2: The semi-discrete scheme (2.6) is TVD, if we have

$$(2.13) \quad \chi_v c_{v+1/2}^+ + \chi_{v+1} c_{v+1/2}^- \geq 0.$$

For smooth grid functions, we have almost everywhere (i.e., with the exception of critical neighborhoods), $\chi_v = \chi_{v+1} = 0$, hence the TVD condition (2.13) is automatically fulfilled in these cases.

Once the positivity condition (2.10) was localized to those incremental coefficients associated with extrema values (2.13), we can go one step further and complete the comparison with Lemma 2.1, dealing with the numerical fluxes instead.

To this end, the scheme (2.1) is rewritten in its canonical incremental representation (2.6) where, see [15, Section 2]

$$(2.14) \quad c_{v+1/2}^+ = \frac{1}{\Delta x_v} \cdot \frac{f(v_v) - h_{v+1/2}}{\Delta v_{v+1/2}}, \quad c_{v-1/2}^- = \frac{1}{\Delta x_v} \cdot \frac{f(v_v) - h_{v-1/2}}{\Delta v_{v-1/2}}, \quad \Delta v_{v+1/2} \neq 0.$$

Applying Lemma 2.2 to these coefficients, then (2.13) reads

$$(2.15a) \quad \frac{\chi_v}{\Delta x_v} \cdot \frac{f(v_v) - h_{v+1/2}}{\Delta v_{v+1/2}} + \frac{\chi_{v+1}}{\Delta x_{v+1}} \cdot \frac{f(v_{v+1}) - h_{v+1/2}}{\Delta v_{v+1/2}} \geq 0,$$

or, equivalently,

$$(2.15b) \quad \frac{\chi_v}{\Delta x_v} \cdot s_{v+1/2} (h_{v+1/2} - f(v_v)) + \frac{\chi_{v+1}}{\Delta x_{v+1}} \cdot s_{v+1/2} (h_{v+1/2} - f(v_{v+1})) \leq 0.$$

Hence, in case of an isolated extrema value where $\chi_v = 2$, $\chi_{v\pm 1} = 0$, the inequality (2.15) is fulfilled if and only if

$$(2.16a) \quad s_{v+1/2} (h_{v+1/2} - f(v_v)) \leq 0,$$

$$(2.16b) \quad s_{v-1/2} (h_{v-1/2} - f(v_v)) \leq 0.$$

In fact, (2.16) covers the general case of extrema values whether isolated or not. For, if $\chi_v = \chi_{v+1} = 2$, then, since v_{v+1} is also an extrema value, we have in view of (2.16b)

$$(2.17) \quad s_{v+1/2} (h_{v+1/2} - f(v_{v+1})) \leq 0,$$

and a weighted average of (2.16a), (2.17) yields (2.15).

We summarize this by stating

Corollary 2.3: The semi-discrete scheme (2.1) is TVD, if we have

$$(2.18a) \quad h_{v+1/2} \geq f(v_v) \geq h_{v-1/2} \quad \text{at maxima values} \quad v_v(t),$$

$$(2.18b) \quad h_{v+1/2} \leq f(v_v) \leq h_{v-1/2} \quad \text{at minima values} \quad v_v(t).$$

The TVD condition (2.18), which was derived on the basis of the incremental decomposition (2.14), is somewhat more stringent than our TVD criterion (2.5) in that the former requires $f(v_v)$ to separate between the numerical fluxes on both sides of extrema values. Incremental decompositions of (2.1) other than (2.14), may lead to slightly different local TVD conditions; yet, they all share a similar kind of a separation requirement at extrema values, which in view of the consistency relation (2.2) is a generic property of the TVD numerical fluxes.

Lemma 2.1 and Corollary 2.3 enable one to verify the TVD property of first as well as higher order accurate semi-discrete schemes, without making reference to any of their special incremental representations. To demonstrate this point, we turn to

Example 2.4: Consider the class of generalized MUSCL schemes [8], where

$$(2.19) \quad h_{v+1/2} = h^E(v_v + \frac{\Delta x}{2} d_v, v_{v+1} - \frac{\Delta x}{2} d_{v+1}).$$

Here, $\Delta x_v \equiv \Delta x$ is the uniform mesh spacing, $h^E(\cdot, \cdot)$ stands for any E-flux, satisfying

$$(2.20) \quad \text{sgn}(w_{v+1} - w_v) \cdot (h^E(w_v, w_{v+1}) - f(w)) \leq 0$$

for all w between w_v and w_{v+1} , and d_v is an approximate derivative at x_v which guarantee second-order accuracy if chosen so that

$$(2.21) \quad \Delta d_{v \pm 1/2} = \frac{1}{\Delta x} \cdot [\Delta v_{v+1/2} - \Delta v_{v-1/2}] + O(\Delta x)^2.$$

In [8, Lemma 2.3], Osher introduces a special incremental decomposition of these schemes in order to show that they meet the positivity condition (2.10) and hence share the TVD property, provided for each v we have

$$(2.22) \quad 0 \leq \frac{\Delta x}{\Delta v_{v \pm 1/2}} \cdot d_v \leq 1.$$

Note that in the particular case of v_v being an extrema value, (2.22) implies that d_v must vanish and, consequently, that accuracy degenerates to first-order at these points.

In contrast to the special positivity arguments made above, Lemma 2.1 suggests a straightforward TVD derivation in this case. Localized at extrema values, we set $d_v = 0$, so that in view of the E-condition (2.20), TVD is guaranteed if in addition we have

$$(2.23a) \quad \text{sgn}(v_{v+1} - v_v) = \text{sgn}(v_{v+1} - v_v - \frac{\Delta x}{2} d_{v+1})$$

$$(2.23b) \quad \text{sgn}(v_v - v_{v-1}) = \text{sgn}(v_v - v_{v-1} - \frac{\Delta x}{2} d_{v-1}),$$

i.e., if the neighboring discrete derivatives of extrema values satisfy

$$(2.24) \quad \frac{1}{2} \left| \frac{\Delta x}{\Delta v_{v-1/2}} d_{v-1} \right| \leq 1, \quad \frac{1}{2} \left| \frac{\Delta x}{\Delta v_{v+1/2}} d_{v+1} \right| \leq 1.$$

One possible choice for such discrete derivatives, d_v , could be

$$(2.25) \quad d_v = \frac{s_v}{\Delta x} \cdot \text{Min}[|\Delta v_{v-1/2}|, |\Delta v_{v+1/2}|], \quad s_v \equiv \frac{1}{2}[s_{v-1/2} + s_{v+1/2}].$$

The above TVD analysis relies on the conservative form of nonlinear difference schemes. We turn now to discuss another representation which is found useful to utilize TVD criteria for such schemes, making use of their viscosity form. To this end, recall the definition of the incremental coefficients, $C_{v+1/2}^{\pm}$ in (2.14). The identity

$$(2.26) \quad \Delta x_{v+1} C_{v+1/2}^{-} - \Delta x_v C_{v+1/2}^{+} = \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}}, \quad \Delta f_{v+1/2} \equiv f(v_{v+1}) - f(v_v),$$

shows that between these two incremental coefficients, $C_{v+1/2}^{\pm}$, there is only one degree of freedom, which could be expressed in terms of $Q_{v+1/2}$,

$$(2.27) \quad Q_{v+1/2} = \Delta x_{v+1} C_{v+1/2}^{-} + \Delta x_v C_{v+1/2}^{+}.$$

Eliminating $C_{v+1/2}^{\pm}$ from (2.26) and (2.27) we find

$$(2.28) \quad C_{v+1/2}^{-} = \frac{1}{2\Delta x_{v+1}} (Q_{v+1/2} + \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}}), \quad C_{v+1/2}^{+} = \frac{1}{2\Delta x_v} (Q_{v+1/2} - \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}}).$$

Our scheme (2.1) is then recast into the viscosity form [16]

$$(2.29) \quad \frac{d}{dt} v_v(t) = - \frac{1}{2\Delta x_v} [f(v_{v+1}) - f(v_{v-1})] + \\ + \frac{1}{2\Delta x_v} [Q_{v+1/2} \Delta v_{v+1/2} - Q_{v-1/2} \Delta v_{v-1/2}],$$

thus revealing the role Q plays as the numerical viscosity coefficient.

Applying (2.13) to $C_{v+1/2}^{\bullet}$ given in (2.28), then Lemma 2.2 tells us

Lemma 2.5: The scheme (2.29) is TVD, if its viscosity satisfies

$$(2.30) \quad (\chi_v \Delta x_{v+1} + \chi_{v+1} \Delta x_v) Q_{v+1/2} \geq (\chi_v \Delta x_{v+1} - \chi_{v+1} \Delta x_v) \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}}.$$

In the case of equally spaced meshpoints, $\Delta x_v \equiv \Delta x$, we conclude that the scheme

$$(2.31) \quad \frac{d}{dt} v_v(t) = - \frac{1}{2\Delta x} [f(v_{v+1}) - f(v_{v-1})] + \\ + \frac{1}{2\Delta x} [Q_{v+1/2} \Delta v_{v+1/2} - Q_{v-1/2} \Delta v_{v-1/2}],$$

is TVD, provided the following simple inequality is fulfilled at neighborhood of extrema values

$$(2.32) \quad (\chi_v + \chi_{v+1}) Q_{v+1/2} \geq (\chi_v - \chi_{v+1}) \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}}.$$

Example 2.6: Consider a first-order accurate TVD scheme of the form

$$(2.33) \quad \frac{d}{dt} v_v(t) = - \frac{1}{\Delta x_v} [h_{v+1/2} - h_{v-1/2}] = \\ = \frac{1}{2\Delta x_v} [f(v_{v+1}) - f(v_{v-1})] + \frac{1}{2\Delta x_v} [Q_{v+1/2} \Delta v_{v+1/2} - Q_{v-1/2} \Delta v_{v-1/2}].$$

In order to convert it into a second-order accurate TVD scheme, we add to it an anti-diffusive conservative difference

$$(2.34a) \quad \frac{d}{dt} v_v(t) = - \frac{1}{\Delta x_v} [h_{v+1/2} - h_{v-1/2}] - \frac{1}{\Delta x_v} [\tilde{h}_{v+1/2} - \tilde{h}_{v-1/2}].$$

The numerical flux correction, $\tilde{h}_{v+1/2}$, is chosen to be of the form [10, Corollary 4.9]

$$(2.34b) \quad \tilde{h}_{v+1/2} = \frac{1}{2} [\tilde{g}_v + \tilde{g}_{v+1} - s_{v+1/2} |\tilde{g}_{v+1} - \tilde{g}_v|],$$

where the so called modified flux correction, \tilde{g}_v , should satisfy

$$(i) \quad \text{sgn}(\tilde{g}_v) = s_{v-1/2} = s_{v+1/2} \quad \text{at nonextrema values } v_v(t),$$

$$(ii) \quad \tilde{g}_v = 0, \quad \text{at extrema values } v_v(t).$$

Now, if $v_v(t)$ or $v_{v+1}(t)$ is an extrema value, then by (ii) we have that $\tilde{g}_v = 0$ or $\tilde{g}_{v+1} = 0$, and consequently $\tilde{h}_{v+1/2}$ vanishes in both cases since by (i)

$$\tilde{h}_{v+1/2} = \frac{1}{2} [\tilde{g}_{v+1} - s_{v+1/2} |\tilde{g}_{v+1}|] = 0 \quad \text{or} \quad \tilde{h}_{v+1/2} = \frac{1}{2} [\tilde{g}_v - s_{v+1/2} |\tilde{g}_v|] = 0.$$

Hence, $\tilde{h}_{v+1/2} = \tilde{h}_{v-1/2} = 0$ at extrema values $v_v(t)$ and in view of Lemma 2.1, the modified scheme (2.34) inherits the TVD property of (2.33). Next we observe that the modification of (2.33) into (2.34) has the net effect of decreasing the original first-order viscosity, $Q_{v+1/2}$ into

$$Q_{v+1/2} - 2 \frac{\tilde{h}_{v+1/2}}{\Delta v_{v+1/2}}; \quad \text{for second order accuracy [17, Lemma 4.4], the latter}$$

should be a Lipschitz continuous grid function of order $O(|\Delta v_{v+1/2}|)$, i.e.,

$$\tilde{h}_{v+1/2} = \frac{1}{2} Q_{v+1/2} \Delta v_{v+1/2} + O(\Delta v_{v+1/2})^2.$$

To this end, one could choose the modified flux correction, \tilde{g}_v , as

$$(2.34c) \quad \tilde{g}_v = \frac{s_v}{2} \cdot \text{Min}[|Q_{v+1/2} \Delta v_{v+1/2}|, |Q_{v-1/2} \Delta v_{v-1/2}|], \quad s_v \equiv \frac{1}{2} [s_{v-1/2} + s_{v+1/2}].$$

In this way, second-order accuracy is achieved away from extrema values, noting that $\tilde{h}_{v+1/2}$ takes there the value \tilde{g}_v or \tilde{g}_{v+1} which modulo quadratic error terms are equal $\frac{1}{2} Q_{v+1/2} \Delta v_{v+1/2}$.

Example 2.7: A simple recipe suggested by (2.32), for constructing a TVD scheme with second-order accuracy away from extrema values, is to set the numerical viscosity Q to be

$$(2.35) \quad Q_{v+1/2} = (\chi_v - \chi_{v+1}) \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}}.$$

The resulting scheme (2.31), (2.35), amounts to the usual second-order central differencing augmented with first-order conservative connection at extrema values

$$(2.36) \quad \frac{d}{dt} v_v(t) = - \frac{1}{2\Delta x} [f(v_{v+1}) - f(v_{v-1})] - \frac{1}{2\Delta x} [\Delta \chi_{v+1/2} \Delta f_{v+1/2} - \Delta \chi_{v-1/2} \Delta f_{v-1/2}].$$

The last two examples dealt with TVD schemes which are second-order accurate away from extrema values. In our final example for this section, we demonstrate a simple recipe of enforcing the TVD property on arbitrary conservative schemes while maintaining their high accuracy away from extrema values, see also [9].

Example 2.8: Let $h_{v+1/2}$ be any highly accurate consistent flux. For example, the $2p$ -th order accurate central differencing are identified with the numerical flux, e.g., [18]

$$h_{v+1/2} = \sum_{k=1}^p d_{kp} [f(v_{v+k}) + \dots + f(v_{v-k+1})],$$

where

$$d_{11} = \frac{1}{2} \quad \text{and} \quad d_{12} = \frac{2}{3}, \quad d_{22} = -\frac{1}{12}$$

for second and fourth order accuracy, or

$$d_{kN} = \frac{(-1)^k \Delta x}{2 \sin(\frac{k \Delta x}{2})}, \quad \Delta x \equiv \frac{2\pi}{2N+1}$$

for spectral accuracy occupying periodic stencils of $2N+1$ meshpoints.

Next, we denote by

$$(2.37) \quad h_{v+1/2}^U = \frac{1}{2} [f(v_v) + f(v_{v+1}) - s_{v+1/2} \cdot |f(v_{v+1}) - f(v_v)|]$$

the usual first-order accurate upwind numerical flux, and let us consider the semi-discrete conservative scheme

$$(2.38a) \quad \frac{d}{dt} v_v(t) = -\frac{1}{\Delta x_v} [H_{v+1/2} - H_{v-1/2}],$$

where $H_{v+1/2}$ is defined as follows

$$(2.38b) \quad H_{v+1/2} = |s_v s_{v+1}| \cdot h_{v+1/2} + (1 - |s_v s_{v+1}|) h_{v+1/2}^U,$$

$$s_v \equiv \frac{1}{2} (s_{v-1/2} + s_{v+1/2}).$$

Away from extrema values $|s_v| = |s_{v\pm 1}| = 1$, and the original high accuracy of $H_{v+1/2} = h_{v+1/2}$ is retained in those regions. At extrema values $s_v = 0$, hence $H_{v+1/2}$ coincides with the upwind flux $h_{v+1/2}^U$, satisfying

$$h_{v+1/2}^U - h_{v-1/2}^U = 2\text{Max}[f(v_v), f(v_{v+1})] - 2\text{Min}[f(v_{v-1}), f(v_v)] \geq 0,$$

at maxima values, and the inverse inequality

$$h_{v+1/2}^U - h_{v-1/2}^U = 2\text{Min}[f(v_v), f(v_{v+1})] - 2\text{Max}[f(v_{v-1}), f(v_v)] \leq 0,$$

at minima values. Consequently, the scheme (2.38) is TVD by Lemma 2.1.

We note that the numerical flux $H_{v+1/2}$ in (2.38b) is in general not smooth, except for the second-order case, $p = 1$, where the scheme (2.38) coincides with the previous example (2.36) and its global second order accuracy is maintained, e.g., [3], [4], [9]. The highly accurate stencils, $p > 1$, require further numerical and analytical investigation with regard to their accumulated accuracy in extrema free regions.

Remarks: (i) It is instructive to see why the necessary and sufficient conservative TVD criterion in Lemma 2.1 is reduced to the sufficient incremental TVD conditions derived from (2.12). To this end, let us insert the incremental coefficients (2.14) into (2.12) obtaining

$$\begin{aligned} \frac{d}{dt} TV[v(t)] = \\ = - \sum_v \left[\frac{\chi_v}{\Delta x_v} \cdot \frac{f(v_v) - h_{v+1/2}}{\Delta v_{v+1/2}} + \frac{\chi_{v+1}}{\Delta x_{v+1}} \cdot \frac{f(v_{v+1}) - h_{v+1/2}}{\Delta v_{v+1/2}} \right] \cdot |\Delta v_{v+1/2}|. \end{aligned}$$

Now, Lemma 2.2 and Corollary 2.3 were derived by requiring a termwise positivity of the brackets inside the summation on the right, see (2.15). Instead, if we first reindex this summation writing it as

$$- \sum_v \frac{\chi_v}{\Delta x_v} \cdot [s_{v+1/2}(f(v_v) - h_{v+1/2}) + s_{v-1/2}(f(v_v) - h_{v-1/2})],$$

we then end up with the necessary and sufficient TVD criterion (2.5). This makes apparent the difference between the two derivations due to the nonlinearity of the schemes.

(ii) The inequality (2.32) shows that the scheme (2.31) has the TVD property with an arbitrary amount of viscosity, except for intervals containing isolated extrema values where we need at least

$$(2.39) \quad Q_{v+1/2} \geq \left| \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}} \right|.$$

The quantity on the right corresponds to upwind differencing, and is responsible for the familiar first-order 'clipping' phenomenon at the extrema of TVD schemes, e.g., [3], [7], [10].

(iii) A classical argument which involves Helly's theorem, Lipschitz continuity of $|v(\cdot, t)|_{L^1}$ and the diagonal process implies the convergence of TVD schemes to a weak solution of (1.1), e.g., [4]. In particular, this is true for central differencing, $Q_{v+1/2} = 0$, augmented with extrema upwind

differencing (2.36). However, the limit solution may still be a physically irrelevant one, e.g., [8]. To avoid the latter, say in the convex case where $f''(u) > 0$, it is enough to have viscosity at the amount which exceeds [17]

$$(2.40) \quad Q_{v+1/2} \geq \frac{1}{6} \cdot f''(v_{v+1}) \cdot \Delta v_{v+1/2}^+, \quad \Delta v_{v+1/2}^+ = \frac{1}{2} (\Delta v_{v+1/2} + |\Delta v_{v+1/2}|).$$

Thus, central differencing will do along the monotone decreasing profiles, and additional $O(|\Delta v_{v+1/2}|)$ amount of viscosity is required along the monotone increasing ones.

3. FULLY-DISCRETE SCHEMES

We consider two-level fully-discrete explicit or implicit schemes in the conservative form

$$(3.1) \quad v_v(t + \Delta t) = v_v(t) - \lambda_v [h_{v+1/2} - h_{v-1/2}].$$

Here, $\Delta x_v \equiv \frac{1}{2} (x_{v+1} - x_{v-1})$ and Δt are the variable meshsize and time step such that $\lambda_v \equiv \frac{\Delta t}{\Delta x_v}$, and $h_{v+1/2}$ is the consistent Lipschitz continuous numerical flux which depend on $2p+1$ neighboring gridvalues from both time levels, t and $t + \Delta t$.

To study the TVD properties of these schemes, we forward difference (3.1)

$$(3.2) \quad \Delta v_{v+1/2}(t + \Delta t) = \Delta v_{v+1/2}(t) - \lambda_{v+1} [h_{v+3/2} - h_{v+1/2}] + \lambda_v [h_{v+1/2} - h_{v-1/2}],$$

multiply by $s_{v+1/2}(t + \Delta t) \equiv \text{sgn}[\Delta v_{v+1/2}(t + \Delta t)]$ and sum by parts, obtaining

$$(3.3) \quad \begin{aligned} \text{TV}[v(t + \Delta t)] &= \sum_v s_{v+1/2}(t + \Delta t) \cdot \Delta v_{v+1/2}(t) + \\ &+ \sum_v \lambda_v \cdot [s_{v+1/2}(t + \Delta t) - s_{v-1/2}(t + \Delta t)] \cdot [h_{v+1/2} - h_{v-1/2}]. \end{aligned}$$

The first summation on the right does not exceed $\text{TV}[v(t)]$, and the requirement from the second one to be negative yields

Lemma 3.1: The fully-discrete scheme (3.1) is TVD, if we have

$$(3.4a) \quad h_{v+1/2} \geq h_{v-1/2} \quad \text{at maxima values} \quad v_v(t + \Delta t),$$

$$(3.4b) \quad h_{v+1/2} \leq h_{v-1/2} \quad \text{at minima values} \quad v_v(t + \Delta t).$$

Lemma 3.1 is a manifestation of our previous assertion, namely, that the TVD properties of conservative schemes are determined solely by the behavior of their numerical fluxes at extrema values. Yet, unlike the semi-discrete case we had before, here there is the additional difficulty of tracing these unknown extrema values at the next time level, $t + \Delta t$.

A similar situation occurs with the incremental representations of fully-discrete nonlinear schemes. Consider for example two-level explicit schemes in the incremental form

$$(3.5) \quad v_v(t + \Delta t) = v_v(t) + C_{v+1/2}^+ \Delta v_{v+1/2}(t) - C_{v-1/2}^- \Delta v_{v-1/2}(t),$$

with $C_{v+1/2}^{\pm} \equiv C_{v+1/2}^{\pm}(v_{v-p+1}(t), \dots, v_{v+p}(t))$. To study the TVD properties of such schemes, we forward difference (3.5), multiply by $s_{v+1/2}(t + \Delta t)$ and sum by parts, obtaining

$$(3.6) \quad TV[v(t + \Delta t)] = \sum_v s_{v+1/2}(t + \Delta t) \cdot \Delta v_{v+1/2}(t) + \\ - \sum_v s_{v+1/2}(t + \Delta t) s_{v+1/2}(t) [\chi_v(t + \Delta t) C_{v+1/2}^+ + \chi_{v+1}(t + \Delta t) C_{v+1/2}^-] \cdot |\Delta v_{v+1/2}(t)|.$$

Since the first summation on the right does not exceed $TV[v(t)]$, we arrive at the following sufficient TVD condition.

Lemma 3.2: The explicit scheme (3.5) is TVD, if we have

$$(3.7a) \quad \chi_v(t + \Delta t) C_{v+1/2}^+ + \chi_{v+1}(t + \Delta t) C_{v+1/2}^- \geq 0, \text{ when } s_{v+1/2}(t + \Delta t) = s_{v+1/2}(t),$$

$$(3.7b) \quad \chi_v(t + \Delta t) C_{v+1/2}^+ + \chi_{v+1}(t + \Delta t) C_{v+1/2}^- \leq 2, \text{ when } s_{v+1/2}(t + \Delta t) \neq s_{v+1/2}(t).$$

Applying the last result to the incremental coefficients, compare (2.28),

$$(3.8) \quad C_{v+1/2}^{\pm} = \frac{1}{2} (Q_{v+1/2} \pm \lambda a_{v+1/2}), \quad a_{v+1/2} \equiv \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}},$$

we find that for equally spaced explicit schemes given in the viscosity form

$$(3.9) \quad v_v(t + \Delta t) = v_v(t) - \frac{\lambda}{2} [f(v_{v+1}) - f(v_{v-1})] + \\ + \frac{1}{2} [Q_{v+1/2} \Delta v_{v+1/2} - Q_{v-1/2} \Delta v_{v+1/2}], \quad \lambda_v \equiv \lambda,$$

the following TVD characterization holds.

Lemma 3.3: The explicit scheme (3.9) is TVD, if its viscosity coefficient satisfies

$$(3.10a) \quad [\chi_v(t + \Delta t) + \chi_{v+1}(t + \Delta t)] \cdot Q_{v+1/2} \geq [\chi_v(t + \Delta t) - \chi_{v+1}(t + \Delta t)] \cdot \lambda a_{v+1/2},$$

and the following CFL-like condition is fulfilled

$$(3.10b) \quad \text{Max}[Q_{v+1/2}, \lambda |a_{v+1/2}|] \leq 1.$$

Thus, we conclude that the TVD property of either scheme, (3.5) or (3.9), is determined by the behavior of their incremental and viscosity coefficients at extrema values, but as before, the difficulty lies in obtaining a priori knowledge about these values at time level $t + \Delta t$.

The inequality $0 \leq \chi \leq 2$ suggests one way of avoiding this difficulty, namely, to replace (3.7) by the simpler positivity condition

$$(3.11a) \quad c_{v+1/2}^+ \geq 0, \quad c_{v+1/2}^- \geq 0,$$

together with a CFL restriction

$$(3.11b) \quad c_{v+1/2}^+ + c_{v+1/2}^- \leq 1.$$

Yet, the simplicity of this sufficient TVD condition, which is originally due to Harten [3], [5], [9], [15], is obtained at the expense of its global dependence on the special incremental form being used.

Another attractive approach to circumvent the difficulty of tracing the next-time level extrema values is to view the scheme (3.1) just as a first predictor step. Then, the resulting spatial variation at time-level $t + \Delta t$ can be made the basis for an augmenting corrector step which will preserve the monotonicity of the predictor step and which will comply with (3.4). Such argument was used in connection with the FCT algorithm [1] and the ACM method [2]. In the following example, borrowed from [10, Corollary 4.9], we work out another corrective-type recipe of this kind, which highlights the essential features distinguishing the fully-discrete explicit case from the semi-discrete one.

Example 3.4: Consider an explicit first-order accurate TVD scheme of the form

$$(3.12a) \quad \begin{aligned} v_v^*(t + \Delta t) &= v_v(t) - \lambda [h_{v+1/2} - h_{v-1/2}] = \\ &= v_v(t) - \frac{\lambda}{2} [f(v_{v+1}) - f(v_{v-1})] + \frac{1}{2} [Q_{v+1/2} \Delta v_{v+1/2} - Q_{v-1/2} \Delta v_{v-1/2}]. \end{aligned}$$

The star indicates the predicted values at time level $t + \Delta t$. In order to convert this scheme into a second-value accurate TVD one, these values are corrected to second order accuracy, by augmenting an anti-diffusive corrector

step of the form

$$(3.12b) \quad v_v(t + \Delta t) = v_v^*(t + \Delta t) - [\tilde{h}_{v+1/2} - \tilde{h}_{v-1/2}].$$

The numerical flux correction, $\tilde{h}_{v+1/2}$, is chosen to be, compare Example 2.6,

$$(3.12c) \quad \tilde{h}_{v+1/2} = \frac{1}{2} [\tilde{g}_v + \tilde{g}_{v+1} - s_{v+1/2}^* |\tilde{g}_{v+1} - \tilde{g}_v|], \quad s_{v+1/2}^* \equiv \text{sgn}[\Delta v_{v+1/2}^*(t + \Delta t)],$$

where \tilde{g}_v should satisfy the two properties

$$(i) \quad \text{sgn}(\tilde{g}_v) = s_{v-1/2}^* = s_{v+1/2}^*, \quad \text{at nonextrema values } v_v^*(t + \Delta t)$$

$$(ii) \quad \tilde{g}_v = 0, \quad \text{at extrema values } v_v^*(t + \Delta t).$$

In addition, we require that the predicted monotonicity should be preserved, i.e.,

$$(iii) \quad s_{v+1/2}(t + \Delta t) = s_{v+1/2}^*(t + \Delta t),$$

so that by the usual summation by parts we have

$$\begin{aligned} \text{TV}[v(t + \Delta t)] &= \sum_v s_{v+1/2}^*(t + \Delta t) \Delta v_{v+1/2}^*(t + \Delta t) + \\ &+ \sum_v [s_{v+1/2}^*(t + \Delta t) - s_{v-1/2}^*(t + \Delta t)] \cdot [\tilde{h}_{v+1/2} - \tilde{h}_{v-1/2}]. \end{aligned}$$

Since the first summation on the right equals $TV[v^*(t + \Delta t)] \leq TV[v(t)]$, while the second is nonnegative, consult Example 2.6, the scheme (3.12) is TVD. Next, its second order accuracy is achieved if

$$\tilde{h}_{v+1/2} = \frac{1}{2} [Q_{v+1/2} - \lambda^2 a_{v+1/2}^2] \cdot \Delta v_{v+1/2}^*(t + \Delta t) + O[\Delta v_{v+1/2}^*(t + \Delta t)]^2.$$

To satisfy this (away from extrema values), and the first two properties listed above, we choose

$$(3.12d) \quad g_v = \frac{s_v^*}{2} B[(Q_{v+1/2} - \lambda^2 a_{v+1/2}^2) \cdot |\Delta v_{v+1/2}^*|, (Q_{v-1/2} - \lambda^2 a_{v-1/2}^2) \cdot |\Delta v_{v-1/2}^*|]$$

where the Lipschitz continuous form, $B[\cdot, \cdot]$, is yet to be determined so that the third property of monotonicity preserving will be satisfied. To this end, we note that

$$|\tilde{h}_{v+1/2}| \leq \text{Min}[|\tilde{g}_v|, |\tilde{g}_{v+1}|]$$

and therefore, since $\tilde{h}_{v+1/2}$ and $\tilde{h}_{v-1/2}$ must agree in sign,

$$|\Delta v_{v+1/2}^*(t + \Delta t) - \Delta v_{v+1/2}^*(t + \Delta t)| = |\tilde{h}_{v+3/2} - 2\tilde{h}_{v+1/2} + \tilde{h}_{v-1/2}| \leq |\tilde{g}_v| + |\tilde{g}_{v+1}|.$$

Hence, the sum on the right does not exceed $|\Delta v_{v+1/2}^*(t + \Delta t)|$ and consequently the monotonicity preserving property holds, provided $B[\cdot, \cdot]$ is chosen as the bilinear limiter form

$$(3.12e) \quad B[w_1, w_2] = \text{Min}[|w_1|, |w_2|].$$

The last example demonstrates the typical situation with explicit schemes, where the TVD property necessitates one kind or another of a Minmod

limiter in order to prevent new extrema values other than those which propagate from time level t .

An implicit version of the above corrective procedure is given in

Example 3.5: Consider an implicit first order accurate TVD scheme of the form

$$(3.13a) \quad v_v^*(t + \Delta t) + \frac{\lambda}{2} [f(v_{v+1}^*) - f(v_{v-1}^*)] + \\ + \frac{1}{2} [Q_{v+1/2} \Delta v_{v+1/2}^* - Q_{v-1/2} \Delta v_{v-1/2}^*] = v_v(t).$$

We augment it with an antidiffusive fully-implicit corrector step of the form

$$(3.13b) \quad v_v(t + \Delta t) + [\tilde{h}_{v+1/2} - \tilde{h}_{v-1/2}] = v_v^*(t + \Delta t),$$

where

$$(3.13c) \quad \tilde{h}_{v+1/2} = \frac{1}{2} [\tilde{g}_v + \tilde{g}_{v+1} - s_{v+1/2}(t + \Delta t) \cdot |\tilde{g}_{v+1} - \tilde{g}_v|].$$

Then (3.13b) serves as a second-order accurate solvable correction, if we set

$$(3.13d) \quad \tilde{g}_v = \frac{1}{2} \cdot s_v(t + \Delta t) \cdot \text{Min}[(Q_{v\pm 1/2} + \lambda^2 a_{v\pm 1/2}^2(t + \Delta t)) \cdot |\Delta v_{v\pm 1/2}(t + \Delta t)|].$$

The resulting scheme (3.13) is TVD under the original (possibly unlimited) CFL condition. Indeed, we have

$$\begin{aligned} \text{TV}[v(t + \Delta t)] &= \sum_v s_{v+1/2}(t + \Delta t) \Delta v_{v+1/2}^*(t + \Delta t) + \\ &+ \sum_v [s_{v+1/2}(t + \Delta t) - s_{v-1/2}(t + \Delta t)] \cdot [\tilde{h}_{v+1/2} - \tilde{h}_{v-1/2}]; \end{aligned}$$

the first summation on the right does not exceed $\text{TV}[v^*(t + \Delta t)] \leq \text{TV}[v(t)]$, while the second vanishes since $\tilde{h}_{v \pm 1/2}$ do at extrema values where $s_{v+1/2}(t + \Delta t) \neq s_{v-1/2}(t + \Delta t)$.

Other recipes for constructing implicit TVD schemes which are second-order accurate away from extrema values, are suggested by the following analogue of Lemma 3.3.

Lemma 3.6: The implicit scheme given in the viscosity form

$$\begin{aligned} (3.14) \quad v_v(t + \Delta t) &+ \frac{\lambda}{2} [f(v_{v+1}(t + \Delta t)) - f(v_{v-1}(t + \Delta t))] + \\ &- \frac{1}{2} [Q_{v+1/2} \Delta v_{v+1/2}(t + \Delta t) - Q_{v-1/2} \Delta v_{v-1/2}(t + \Delta t)] = v_v(t) \end{aligned}$$

is TVD, if we have

$$\begin{aligned} (3.15) \quad [\chi_v(t + \Delta t) + \chi_{v+1}(t + \Delta t)] Q_{v+1/2} &\geq \\ [\chi_v(t + \Delta t) - \chi_{v+1}(t + \Delta t)] \cdot \lambda a_{v+1/2}(t + \Delta t). \end{aligned}$$

We omit the proof and turn to our final

Example 3.7: The viscosity of the second-order accurate implicit Lax-Wendroff scheme is modified at extrema values, by setting

$$(3.16) \quad Q_{v+1/2} = -\lambda^2 a_{v+1/2}^2 + \frac{1}{2} (\chi_v + \chi_{v+1}) \cdot [\lambda |a_{v+1/2}| + \lambda^2 a_{v+1/2}^2],$$

where the quantities on the right are evaluated at time level $t + \Delta t$. The resulting scheme (3.14), (3.16) can be easily checked to satisfy (3.15) and hence is TVD. However, the linearized implicit LW scheme is unconditionally unstable -- the amplification factors of its nonconstant modes, they all lie outside the unit disc [19]. Consequently, the TVD property of (3.16) is achieved by switching to upwind differencing at the extrema of these unstable oscillatory modes, at the expense of lowering the effective overall accuracy.

Remarks: We note that the TVD characterization in Lemma 3.1 does not assume the CFL condition; it enters, indirectly, through the requirement of controlling extrema values at the next time-level. Substitution of the canonical incremental decomposition (2.14) into (3.7) reveals that the same is true with respect to the TVD conditions in Lemma 3.2 and 3.3, where the CFL limitation is implicitly contained already in (3.7a) and (3.10a).

ACKNOWLEDGEMENT

I would like to thank Ami Harten for his constructive remarks concerning this work.

REFERENCES

- [1] J. P. Boris and D. L. Book, "Flux-corrected transport, I. SHASTA, a fluid transport algorithm that works." J. Comp. Phys., Vol. 11, 1973, pp. 38-69.
- [2] A. Harten, "The artificial compression method for computation of shocks and contact discontinuities: III. Self-adjusting hybrid schemes," Math. Comp., Vol. 32, 1978, pp. 363-389.
- [3] A. Harten, "High resolution schemes for hyperbolic conservation laws," J. Comp. Phys., Vol. 49, 1983, pp. 357-393.
- [4] A. Harten, "On a class of high resolution total variation stable finite difference schemes," SINUM, Vol. 21, 1984, pp. 1-23.
- [5] A. Jameson and P. D. Lax, "Conditions for the construction of multi-point total variation diminishing difference schemes," in Advances in Numerical and Applied Mathematics (J. C. South and M. Y. Hussaini, ed.), ICASE Report No. 86-18, NASA Langley Research Center, 1986, pp. 361-381.
- [6] B. van Leer, "Towards the ultimate conservative difference scheme, II. Monotonicity and conservation combined in a second-order scheme," J. Comp. Phys., Vol. 14, 1974, pp. 361-376.

- [7] S. Osher, "Riemann solvers, the entropy condition and difference approximations," SINUM, Vol. 21, 1984, pp. 217-238.
- [8] S. Osher, "Convergence of generalized MUSCL schemes," SINUM, Vol. 22, 1984, pp. 947-961.
- [9] S. Osher and S. R. Chakravarthy, "Very high order accurate TVD schemes," ICASE Report No. 84-44, NASA Langley Research Center, 1984.
- [10] S. Osher and E. Tadmor, "On the convergence of difference approximations to scalar conservation laws," ICASE Report No. 85-28, NASA Langley Research Center, 1985.
- [11] P. L. Roe, "Some contributions to the modelling of discontinuous flows," in Lectures in Applied Mathematics, Vol. 22, Providence, RI.
- [12] P. L. Roe and M. J. Baines, "Asymptotic behavior of some nonlinear schemes for linear advection," in Notes on Numerical Fluid Mechanics (M. Pandolfi and R. Piva, ed.), Vol. 7, 1984, pp. 283-290.
- [13] R. Sanders, "On convergence of monotone finite difference schemes with variable spatial differencing," Math. Comp., Vol. 40, 1983, pp. 91-106.
- [14] P. K. Sweby, "High resolution schemes using flux limiters for hyperbolic conservation laws," SINUM, Vol. 21, 1984, pp. 995-1011.

- [15] E. Tadmor, "The large time behavior of the scalar, genuinely nonlinear Lax-Friedrichs scheme," Math. Comp., Vol. 43, 1984, pp. 353-368.

- [16] E. Tadmor, "Numerical viscosity and the entropy condition for conservative difference schemes," Math. Comp., Vol. 43, 1984, pp. 369-382.

- [17] E. Tadmor, "The numerical viscosity of entropy stable schemes for systems of conservation laws. II," to be published as NASA Contractor/ICASE report.

- [18] E. Tadmor, "Stability analysis of finite-difference, pseudospectral, and Fourier-Galerkin approximations for time-dependent problems," SIAM Review, to appear.

- [19] A. Lerat, "Implicit methods of second-order accuracy for the Euler equations," AIAA, Vol. 23, 1985, pp. 33-40.

Standard Bibliographic Page

1. Report No. NASA CR-178209 ICASE Report No. 86-74		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle CONVENIENT TOTAL VARIATION DIMINISHING CONDITIONS FOR NONLINEAR DIFFERENCE SCHEMES				5. Report Date November 1986	
				6. Performing Organization Code	
7. Author(s) Eitan Tadmor				8. Performing Organization Report No. 86-74	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225				10. Work Unit No.	
				11. Contract or Grant No. NAS1-18107	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546				13. Type of Report and Period Covered Contractor Report	
				14. Sponsoring Agency Code 505-90-21-01	
15. Supplementary Notes Langley Technical Monitor: Submitted to SIAM Journal of J. C. South Numerical Analysis Final Report					
16. Abstract Convenient conditions for nonlinear difference schemes to be total-variation diminishing (TVD) are derived. It is shown that such schemes share the TVD property, provided their numerical fluxes meet a certain positivity condition at extrema values but can be arbitrary otherwise. Our conditions are invariant under different incremental representations of the nonlinear schemes, and thus provide a simplified generalization of the TVD conditions due to Harten and others [3] - [7], [13].					
17. Key Words (Suggested by Authors(s)) nonlinear difference schemes, total variation, extrema values			18. Distribution Statement 64 - Numerical Analysis Unclassified - unlimited		
19. Security Classif.(of this report) Unclassified		20. Security Classif.(of this page) Unclassified		21. No. of Pages 32	
				22. Price A03	