NASA Contractor Report 178220

ICASE REPORT NO. 86-61

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Contract No. NAS1-18107

December 1986

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Operated by the Universities Space Research Association

National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665

Generalized Inf-Sup Condition for Chebyshev Approximation of the Navier-Stokes Equations

by Christine BERNARDI*, Claudio CANUTO# & Yvon MADAY°

Abstract : We study an abstract mixed problem and its approximation ; both are well-posed if and only if several inf-sup conditions are satisfied. These results are applied to a spectral Galerkin method for the Stokes problem in a square, when it is formulated in Chebyshev weighted Sobolev spaces. Finally, a collocation method for the Navier-Stokes equations at Chebyshev nodes is analyzed.

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Research was supported for the last two authors by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while they were in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

I. Introduction.

In the last few years, a number of algorithms using spectral collocation methods have been successfully implemented to solve the incompressible Navier-Stokes equations in a domain Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$

$$(I.1)\quad \begin{cases} -\nu\,\Delta\mathbf{u} + \mathrm{grad}\,p + (\mathbf{u}.\nabla)\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \quad , \\ \mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega \quad , \end{cases}$$

provided with appropriate boundary conditions : periodic, Dirichlet or mixed ones. We refer to [VGH] and to [CHQZ] for a detailed bibliography. In most of them, the collocation points in the nonperiodic directions are the nodes of a Gauss-Lobatto quadrature formula associated with the Chebyshev polynomials. Indeed the use of the Fast Fourier Transform allows a less expensive computation of the derivatives and the nonlinear terms. However, as far as we know, the only theoretical justifications of some of these algorithms are achieved with the Chebyshev nodes replaced by the Legendre ones (see e.g., [BMM1][BMM2]). The aim of this paper is the numerical analysis of a collocation method involving the Chebyshev nodes, for Dirichlet boundary conditions and when the domain Ω is the square $]-1,1[^2$; corresponding numerical experiments can be found in [Mo][Mé].

The analysis we present here has already been achieved for the Legendre collocation nodes. in [BMM2]. The extension to the Chebyshev methods presents essentially two difficulties :

1) In the variational formulation of both the continuous and the discrete problems, the classical Sobolev spaces have to be replaced by weighted Sobolev spaces, the weight of which is the Chebyshev one. Several trace theorems in these spaces, as well as some regularity results for the Dirichlet problem for the laplacian in a square, are needed by our study ; these are recent (see [BM]).

2) Due to the Chebyshev weight, the analysis of the Stokes problem - which one obtains by neglecting the nonlinear terms in (I.1) - and of its approximation involves a variational formulation of the form

$$(I.2)\quad \begin{cases} a(\mathbf{u},\mathbf{v}) + b_1(\mathbf{v},p) = \langle \mathbf{f},\mathbf{v}\rangle \quad , \\ b_2(\mathbf{u},q) = 0 \quad . \end{cases}$$

In this system, the bilinear forms b_1 and b_2 are distinct, in opposition to the classical saddle-point problem first studied in [Br]. We shall consider such a problem in an abstract framework and prove that it is well-posed if and only if several inf-sup conditions are satisfied : two for the form a and one for each of the forms b_1 and b_2 . We shall also analyze its discretization and state general error estimates which can be applied to a number of numerical methods. Our hope is that this abstract variational formulation will be used for other equations involving weighted spaces.

In the particular case of the discretization of the Stokes problem by spectral methods, in order to obtain a well-posed problem, we have to exhibit the spurious modes of the pressure, i.e. the modes which cancel the discrete gradient. Thus, we derive an appropriate choice for the space of discrete pressures and we can prove that the inf-sup conditions are satisfied. The results we obtain are very similar to the corresponding ones for the Legendre methods (see [BMM2]). The convergence estimates can be extended to the Navier-Stokes equations without difficulty.

An outline of the paper is as follows. Section II is devoted to an abstract variational problem with inf-sup conditions. In Section III, we state a variational formulation of the Stokes problem in weighted Sobolev spaces, the weight of which is the Chebyshev one. In Section IV, we study a Galerkin spectral method to discretize this problem. The spectral collocation method for the Stokes problem is thoroughly analyzed in Section V. The results are extended in Section VI to the full Navier-Stokes equations thanks to a fixed point theorem.

Notations : The norm of any Banach space E is denoted by $\|.\|_E$, while $\langle .,. \rangle$ is the duality pairing between E and its dual space E'. For any pair (E,F) of Banach spaces, $\mathcal{L}(E,F)$ represents the space of continuous linear mappings from E into F. We mean by $A \otimes B$ the tensorial product of any sets A and B in a Banach space, while $A^{\otimes 2}$ is the tensorial product of A by itself.

For any domain Δ in $\mathbb{R}^d$ and for any real number s, we use the classical hilbertian Sobolev spaces $H^s(\Delta)$, the norm of which is denoted by $\|.\|_s$.

In all that follows, c, c' ... are generic constants, independent of the discretization.

II. An abstract variational system, and its approximation.

II.1. The continuous case.

Let X_i and M_i $(i = 1, 2)$ be real reflexive Banach spaces. We assume we are given three continuous bilinear forms, $a : X_2 \times X_1 \to \mathbb{R}$ and $b_i : X_i \times M_i \to \mathbb{R}$ $(i = 1, 2)$. For any given f in X_1' and g in M_2', we consider the following problem : *Find* (u,p) *in* $X_2 \times M_1$ *such that*

$$\text{(II.1)} \qquad \left| \begin{array}{l} \forall\, v \in X_1 \;,\quad a(u,v) + b_1(v,p) = \langle f,v \rangle \quad, \\ \forall\, q \in M_2 \;,\quad b_2(u,q) = \langle g,q \rangle \quad. \end{array} \right.$$

In order to study this problem, let us introduce the linear operators $A \in \mathcal{L}(X_2, X_1')$ and $B_i \in \mathcal{L}(X_i, M_i')$ $(i = 1, 2)$ associated with the forms a and b_i by the relations

$$\text{(II.2)} \qquad \forall\, u \in X_2 \,,\ \forall\, v \in X_1 \,,\quad \langle Au,v \rangle = a(u,v) \quad,$$

$$\text{(II.3)} \qquad \forall\, u \in X_i \,,\ \forall\, q \in M_i \,,\quad \langle B_i u,q \rangle = b_i(u,q) \quad;$$

we denote by $B_i^T \in \mathcal{L}(M_i, X_i')$ $(i = 1, 2)$ the adjoint operator of B_i.

For any g in M_i' $(i = 1, 2)$, we define the closed affine space

$$\text{(II.4)} \qquad K_i(g) = \{\, v \in X_i \,;\, \forall\, q \in M_i \,,\ b_i(v,q) = \langle g,q \rangle \,\} \quad,$$

and we note that the subspace $K_i = K_i(0)$ is the kernel of the operator B_i. Moreover we introduce the linear mapping $\Pi : X_1' \to K_1'$ defined for each $f \in X_1'$ by

$$\text{(II.5)} \qquad \forall\, v \in K_1 \,,\quad \langle \Pi f,v \rangle = \langle f,v \rangle \quad.$$

Finally, we denote by K_1° the polar set of K_1 i.e., the kernel of the mapping Π.

Let us define the operator $\Lambda : X_2 \times M_1 \to X_1' \times M_2'$ by the relation

$$\text{(II.6)} \qquad \Lambda(u,p) = (Au + B_1^T p \,,\, B_2 u) \quad;$$

then (II.1) is equivalent to

$$\text{(II.7)} \qquad \Lambda(u,p) = (f,g) \quad.$$

We can prove the following theorem :

<u>Theorem II.1</u> : *The operator* Λ *is an isomorphism from* $X_2 \times M_1$ *onto* $X_1' \times M_2'$ *if and only if the following conditions* (C_0) *and* (C_i) $(i = 1, 2)$ *are satisfied*

(C_0) $\quad \Pi A : K_2 \to K_1'$ *is an isomorphism* ,

(C_i) $\quad$ *there exists a constant* $\beta_i > 0$ *, such that for any* $q \in M_i$, $\|B_i^T q\|_{X_i'} \geq \beta_i \|q\|_{M_i}$.

Proof : We assume first that conditions (C_0) and (C_i) $(i = 1, 2)$ are satisfied, and we prove that Λ is an isomorphism. Given f in X_1' and g in M_2', let us observe that by (C_2) and [Br, Thm 0.1] there exists $\bar{u}$ in X_2 such that $B_2\bar{u} = g$ and $\|\bar{u}\|_{X_2} \leqslant \beta_2^{-1}\|g\|_{M_2'}$. Moreover by (C_0) there exists a unique u_0 in K_2 such that $\Pi A u_0 = \Pi f - \Pi A \bar{u}$, and one has

$$\|u_0\|_{X_2} \leqslant c\,(\|f\|_{X_1'} + \|g\|_{M_2'}) \quad .$$

Let us set $u = \bar{u} + u_0$. The element $f - Au$ belongs to K_1°, hence by (C_1) and the Closed Range Theorem, there exists a unique p in M_1 such that $B_1^T p = f - Au$. Thus we have proved that Λ is onto. It is straightforward to check that Λ is one to one, and since it is continuous because so are A, B_1^T and B_2, we conclude that Λ is an isomorphism by using the Open Mapping Theorem.

Conversely, let us suppose that Λ is an isomorphism. For each q in M_1, we have $\Lambda(0,q) = (B_1^T q, 0)$; hence, from the continuity of Λ^{-1}, we obtain

$$\|q\|_{M_1} \leqslant \|\Lambda^{-1}\| . \|B_1^T q\|_{X_1'} \quad ,$$

which is (C_1). On the other hand, for each g in M_2', let us set $(w,q) = \Lambda^{-1}(0,g)$. Then the mapping : $g \to w$ is continuous from M_2' into X_1 and is such that $B_2 w = g$. Thus we have (C_2), using [Br, Thm 0.1]. Now we prove that ΠA is an isomorphism. Given f in K_1', let us set $(w,q) = \Lambda^{-1}(f,0)$. Then w belongs to K_2 and satisfies

$$\Pi A w = \Pi f - \Pi B_1^T q = \Pi f = f \quad ,$$

since ΠB_1^T is equal to 0. Thus ΠA is onto. Finally, let w in K_2 be such that $\Pi A w = 0$; by (C_1) and the Closed Range Theorem, there exists q in M_1 such that $B_1^T q = -Aw$. Thus $\Lambda(w,q)$ is equal to $(0,0)$, hence, w is equal to 0, i.e., ΠA is one to one. This concludes the proof of the theorem.

<u>Remark II.1</u> : By well-known results of functional analysis, we can express condition (C_0) in a variational form. Precisely, (C_0) is equivalent to the following condition (see, e.g. [Ba]) : there exists a constant $\alpha_1 > 0$ such that

$$\text{(II.8)} \qquad \forall u \in K_2 \;,\; \sup_{v \in K_1} \frac{a(u,v)}{\|v\|_{X_1}} \geqslant \alpha_1 \|u\|_{X_2}$$

and

$$\text{(II.9)} \qquad \forall v \in K_1 \setminus \{0\} \;,\; \sup_{u \in K_2} a(u,v) > 0 \;.$$

By the Open Mapping Theorem this is also equivalent to the existence of a constant $\alpha_2 > 0$ such that

$$\text{(II.10)} \qquad \forall v \in K_1 \;,\; \sup_{u \in K_2} \frac{a(u,v)}{\|u\|_{X_2}} \geqslant \alpha_2 \|v\|_{X_1}$$

and

$$\text{(II.11)} \qquad \forall u \in K_2 \setminus \{0\} \;,\; \sup_{v \in K_1} a(u,v) > 0 \;.$$

If K_1 (or K_2) are finite dimensional spaces, then the relation (II.9) or (II.11) can be replaced by the requirement that

(II.12) $\dim K_1 = \dim K_2$.

Similarly, we can write the condition (C_i) $(i = 1, 2)$ equivalently as follows : there exists a constant $\beta_i > 0$ such that

$$(II.13)_i \quad \forall q \in M_i , \quad \sup_{v \in X_i} \frac{b_i(v, q)}{\|v\|_{X_i} \|q\|_{M_i}} \geqslant \beta_i > 0 \quad .$$

These are the forms under which the conditions (C_0) and (C_i) $(i = 1, 2)$ are usually checked in the applications.

Following the proof of Theorem II.1 one can easily estimate the norm of the inverse isomorphism Λ^{-1} in term of the constants associated with the forms a and b_i.

Denoting by γ the norm of a :

$$(II.14) \qquad \gamma = \sup_{u \in X_2 , v \in X_1} \frac{a(u,v)}{\|u\|_{X_2} \|v\|_{X_1}} ,$$

we have the following result.

<u>Corollary II.1</u> : *Assume that hypotheses* (II.8), (II.9) *and* $(II.13)_i$ $(i = 1, 2)$ *hold. Then, the solution* (u,p) *of problem* (II.1) *satisfies the following estimates*

$$(II.15) \quad \|u\|_{X_2} \leqslant \alpha_1^{-1} \|f\|_{X_1'} + \beta_2^{-1} (1 + \alpha_1^{-1}\gamma) \|g\|_{M_2'} \quad ,$$

$$(II.16) \quad \|p\|_{M_1} \leqslant \beta_1^{-1} (1 + \alpha_1^{-1}\gamma) \|f\|_{X_1'} + \beta_1^{-1}\beta_2^{-1}\gamma (1 + \alpha_1^{-1}\gamma) \|g\|_{M_2'} \quad .$$

<u>II.2. The finite dimensional approximation.</u>

We want to approximate problem (II.1) by a finite dimensional system. To this end, we introduce a discretization parameter $\delta > 0$, and we assume we are given closed subspaces $X_{i\delta}$ and $M_{i\delta}$ $(i = 1, 2)$ contained in X_i and M_i respectively. The continuous bilinear forms a and b_i are approximated by three continuous bilinear forms $a_\delta : X_{2\delta} \times X_{1\delta} \to \mathbb{R}$ and $b_{i\delta} : X_{i\delta} \times M_{i\delta} \to \mathbb{R}$ $(i = 1, 2)$.

For any f_δ in X_1' and g_δ in M_2' , we consider the following approximation of problem (II.1) :*Find* (u_δ , p_δ) *in* $X_{2\delta} \times M_{1\delta}$ *such that*

$$(II.17) \quad \left| \begin{array}{l} \forall v_\delta \in X_{1\delta} , \quad a_\delta(u_\delta , v_\delta) + b_{1\delta}(v_\delta , p_\delta) = \langle f_\delta , v_\delta \rangle \quad , \\ \forall q_\delta \in M_{2\delta} , \quad b_{2\delta}(u_\delta , q_\delta) = \langle g_\delta , q_\delta \rangle \quad . \end{array} \right.$$

We assume that the forms a_δ and $b_{i\delta}$ satisfy the necessary and sufficient conditions for problem

(II.17) to be well-posed, as stated in Theorem II.1. More precisely, for any g in M_i' (i = 1, 2), we introduce the affine subspace

(II.18) $K_{i\delta}(g) = \{ v_\delta \in X_{i\delta} \ ; \forall q_\delta \in M_{i\delta} \ , \ b_{i\delta}(v_\delta, q_\delta) = \langle g, q_\delta \rangle \}$,

and we set $K_{i\delta} = K_{i\delta}(0)$. We make the following assumptions :

a) there exists a constant $\alpha_{1\delta} > 0$ such that

(II.19) $$\forall u_\delta \in K_{2\delta} \ , \quad \sup_{v_\delta \in K_{1\delta}} \frac{a_\delta(u_\delta, v_\delta)}{\|v_\delta\|_{X_1}} \geqslant \alpha_{1\delta} \|u_\delta\|_{X_2} \ ;$$

(II.20) $$\forall v_\delta \in K_{1\delta} \setminus \{0\} \ , \quad \sup_{u_\delta \in K_{2\delta}} a_\delta(u_\delta, v_\delta) > 0 \ ;$$

this second condition, if $K_{1\delta}$ and $K_{2\delta}$ are finite dimensional, can be equivalently replaced by

(II.21) $\dim K_{1\delta} = \dim K_{2\delta}$;

b) there exists a constant $\beta_{i\delta} > 0$ (i = 1, 2) such that

(II.22)$_i$ $$\forall q_\delta \in M_{i\delta} \ , \quad \sup_{v_\delta \in X_{i\delta}} \frac{b_{i\delta}(v_\delta, q_\delta)}{\|v_\delta\|_{X_i} \|q_\delta\|_{M_i}} \geqslant \beta_{i\delta} > 0 \ .$$

Moreover, we denote by

(II.23) $$\gamma_\delta = \sup_{u_\delta \in X_{2\delta},\, v_\delta \in X_{1\delta}} \frac{a_\delta(u_\delta, v_\delta)}{\|u_\delta\|_{X_2} \|v_\delta\|_{X_1}}$$

the norm of the form a_δ.

<u>Corollary II.2</u> : *Assume that hypotheses* (II.19), (II.20) *and* (II.22)$_i$ (i = 1, 2) *hold. Then problem* (II.17) *has a unique solution* (u_δ, p_δ). *Moreover the solution* (u_δ, p_δ) *satisfies the following estimates*

(II.24) $$\|u_\delta\|_{X_2} \leqslant \alpha_{1\delta}^{-1} \|f_\delta\|_{X_1'} + \beta_{2\delta}^{-1} (1 + \alpha_{1\delta}^{-1}\gamma_\delta) \|g_\delta\|_{M_2'} \ ,$$

(II.25) $$\|p_\delta\|_{M_1} \leqslant \beta_{1\delta}^{-1} (1 + \alpha_{1\delta}^{-1}\gamma_\delta) \|f_\delta\|_{X_1'} + \beta_{1\delta}^{-1} \beta_{2\delta}^{-1} \gamma_\delta (1 + \alpha_{1\delta}^{-1}\gamma_\delta) \|g_\delta\|_{M_2'} \ .$$

<u>II.3. Error estimates.</u>

We assume now that there exist a solution (u,p) to problem (II.1) and a solution (u_δ, p_δ) to problem (II.17). We want to derive a general error estimate between them.

First, we estimate how an element of $K_i(g)$ can be approximated by an element of $K_{i\delta}(g_\delta)$, for any g and g_δ given in M_i' .

<u>Proposition II.1</u> : *Assume that hypothesis* (II.22)$_i$ *holds. For any element* v *in* $K_i(g)$, *the following estimate is satisfied :*

$$
(\text{II}.26)\quad \left|\begin{aligned}
&\inf_{w_\delta \in K_{i\delta}(g_\delta)} \|v-w_\delta\|_{X_i} \\
&\quad \leqslant c\,(1+\beta_{i\delta}^{-1})\,[\inf_{v_\delta \in X_{i\delta}} \{\|v-v_\delta\|_{X_i} + \sup_{q_\delta \in M_{i\delta}} \frac{(b_i-b_{i\delta})(v_\delta\,,q_\delta)}{\|q_\delta\|_{M_i}}\} \\
&\qquad + \sup_{q_\delta \in M_{i\delta}} \frac{\langle g-g_\delta\,,q_\delta\rangle}{\|q_\delta\|_{M_i}}\ ]\ .
\end{aligned}\right.
$$

Proof : Let v_δ be any element in $X_{i\delta}$. By $(\text{II}.22)_i$ and [Br, Thm 0.1], there exists z_δ in $X_{i\delta}$ such that

$$(\text{II}.27)\quad \forall\, q_\delta \in M_{i\delta}\,,\quad b_{i\delta}(z_\delta\,,q_\delta) = b_{i\delta}(v_\delta\,,q_\delta) - \langle g_\delta\,,q_\delta\rangle\quad ,$$

$$(\text{II}.28)\quad \|z_\delta\|_{X_i} \leqslant \beta_{i\delta}^{-1} \sup_{q_\delta \in M_{i\delta}} \frac{b_{i\delta}(z_\delta\,,q_\delta)}{\|q_\delta\|_{M_i}}\ .$$

Clearly, the element $w_\delta = v_\delta - z_\delta$ belongs to $K_{i\delta}(g_\delta)$ and satisfies

$$(\text{II}.29)\qquad \|v-w_\delta\|_{X_i} \leqslant \|v-v_\delta\|_{X_i} + \|z_\delta\|_{X_i}\ .$$

Thanks to (II.27), we write

$$\forall\, q_\delta \in M_{i\delta}\,,\quad b_{i\delta}(z_\delta\,,q_\delta) = -\,b_i(v-v_\delta\,,q_\delta) - (b_i-b_{i\delta})(v_\delta\,,q_\delta) + \langle g-g_\delta\,,q_\delta\rangle\quad ;$$

hence, using (II.28) and the continuity of b_i yields

$$\|z_\delta\|_{X_i} \leqslant \beta_{i\delta}^{-1}\,\{\,c\,\|v-v_\delta\|_{X_i} + \sup_{q_\delta \in M_{i\delta}} \frac{(b_i-b_{i\delta})(v_\delta\,,q_\delta)}{\|q_\delta\|_{M_i}} + \sup_{q_\delta \in M_{i\delta}} \frac{\langle g-g_\delta\,,q_\delta\rangle}{\|q_\delta\|_{M_i}}\ \}\ .$$

The last inequality, together with (II.29), gives the proposition.

Next, we derive the main error estimate for $u-u_\delta$.

__Theorem II.2__ : *Assume that hypothesis* (II.19) *holds. Then the solutions* (u,p) *of* (II.1) *and* $(u_\delta\,,p_\delta)$ *of* (II.17), *satisfy the following estimate*

$$
(\text{II}.30)\quad \left|\begin{aligned}
&\|u-u_\delta\|_{X_2} \leqslant c\,(1+\alpha_{1\delta}^{-1})\,[\,(1+\gamma_\delta) \inf_{w_\delta \in K_{2\delta}(g_\delta)} \|u-w_\delta\|_{X_2} \\
&\qquad + \inf_{v_\delta \in X_{2\delta}} \{\,(1+\gamma_\delta)\,\|u-v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a-a_\delta)(v_\delta\,,z_\delta)}{\|z_\delta\|_{X_1}}\ \} \\
&\qquad + \inf_{q_\delta \in M_{1\delta}} \{\|p-q_\delta\|_{M_1} + \sup_{z_\delta \in X_{1\delta}} \frac{(b_1-b_{1\delta})(z_\delta\,,q_\delta)}{\|z_\delta\|_{X_1}}\ \} \\
&\qquad + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f-f_\delta\,,z_\delta\rangle}{\|z_\delta\|_{X_1}}\ ]\ .
\end{aligned}\right.
$$

Proof : Let w_δ be any element in $K_{2\delta}(g_\delta)$. By (II.19), we have

$$\text{(II.31)}\qquad \|u_\delta - w_\delta\|_{X_2} \leqslant \alpha_{1\delta}^{-1} \sup_{z_\delta \in K_{1\delta}} \frac{a_\delta(u_\delta - w_\delta\,,\,z_\delta)}{\|z_\delta\|_{X_1}} .$$

But, thanks to (II.1), (II.17) and (II.18), for any z_δ in $K_{1\delta}$, we can write

$$\begin{aligned} a_\delta(u_\delta - w_\delta\,,\,z_\delta) &= -\,a_\delta(w_\delta\,,\,z_\delta) + \langle f_\delta\,,\,z_\delta\rangle \\ &= a(u, z_\delta) - a_\delta(w_\delta\,,\,z_\delta) + b_1(z_\delta\,,\,p) - \langle f - f_\delta\,,\,z_\delta\rangle . \end{aligned}$$

Now, let v_δ be any element in $X_{2\delta}$ and q_δ be any element in $M_{1\delta}$. By (II.18), we obtain

$$a_\delta(u_\delta - w_\delta\,,\,z_\delta) = a(u, z_\delta) - a_\delta(w_\delta\,,\,z_\delta) + b_1(z_\delta\,,\,p - q_\delta) + (b_1 - b_{1\delta})(z_\delta\,,\,q_\delta) - \langle f - f_\delta\,,\,z_\delta\rangle ,$$

hence,

$$\text{(II.32)}\qquad \left|\begin{aligned} a_\delta(u_\delta - w_\delta\,,\,z_\delta) = a(u - v_\delta\,,\,z_\delta) + a_\delta(v_\delta - w_\delta\,,\,z_\delta) + (a - a_\delta)(v_\delta\,,\,z_\delta) + b_1(z_\delta\,,\,p - q_\delta) \\ + (b_1 - b_{1\delta})(z_\delta\,,\,q_\delta) - \langle f - f_\delta\,,\,z_\delta\rangle . \end{aligned}\right.$$

Using (II.23) and the continuity of a and b_1 in the last formula, we deduce

$$\begin{aligned} \|u_\delta - w_\delta\|_{X_2} \leqslant \alpha_{1\delta}^{-1} \Big[\gamma_\delta \|u - w_\delta\|_{X_2} &+ \Big\{ (\gamma + \gamma_\delta) \|u - v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a - a_\delta)(v_\delta\,,\,z_\delta)}{\|z_\delta\|_{X_1}} \Big\} \\ &+ \Big\{ c \|p - q_\delta\|_{M_1} + \sup_{z_\delta \in X_{1\delta}} \frac{(b_1 - b_{1\delta})(z_\delta\,,\,q_\delta)}{\|z_\delta\|_{X_1}} \Big\} + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f - f_\delta\,,\,z_\delta\rangle}{\|z_\delta\|_{X_1}} \Big] , \end{aligned}$$

which gives the theorem.

Remark II.2 : Sometimes it can be useful to make a further assumption on the form $b_{1\delta}$, which will always be satisfied in the applications we have in mind. It will allow us to predict an optimal rate of decay for the error $u - u_\delta$ even in the case where $M_{i\delta}$ is a bad approximation of M_i. Thus we assume that there exists a subspace $\hat{M}_{1\delta}$ of M_1, such that the form $b_{1\delta}$ is in fact defined on $X_{2\delta} \times \hat{M}_{1\delta}$ and satisfies

$$\text{(II.33)}\qquad \forall\, v_\delta \in K_{1\delta}\,,\ \forall\, q_\delta \in \hat{M}_{1\delta}\,,\quad b_{1\delta}(v_\delta\,,\,q_\delta) = 0 .$$

(In applications, $\hat{M}_{1\delta}$ will be a subspace of M_1 containing $M_{1\delta}$ and such that the elements of M_1 can be approximated in an optimal way by elements of $\hat{M}_{1\delta}$). Then, we can replace $\displaystyle\inf_{q_\delta \in M_{1\delta}}$ by $\displaystyle\inf_{q_\delta \in \hat{M}_{1\delta}}$ in the right-hand side of (II.30).

Now, we indicate a remarkable case in which estimate (II.30) can be simplified. The result follows easily from (II.32).

Corollary II.3 : *Assume that hypothesis* (II.19) *holds and that*

(II.34) $K_{1\delta} \subset K_1$.

Then the solutions (u,p) *of* (II.1) *and* (u_δ, p_δ) *of* (II.17), *satisfy the following estimate*

$$\text{(II.35)}\quad \left\{ \begin{aligned} \|u-u_\delta\|_{X_2} \leqslant c\,(1+\alpha_{1\delta}^{-1})\,[\,(1+\gamma_\delta) & \inf_{w_\delta \in K_{2\delta}(g_\delta)} \|u-w_\delta\|_{X_2} \\ & + \inf_{v_\delta \in X_{2\delta}} \{(1+\gamma_\delta)\|u-v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a-a_\delta)(v_\delta, z_\delta)}{\|z_\delta\|_{X_1}}\} \\ & + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f-f_\delta, z_\delta\rangle}{\|z_\delta\|_{X_1}}\,] . \end{aligned} \right.$$

Finally, an error estimate for $p-p_\delta$ is provided by the following result.

<u>Theorem II.3</u> : *Assume that hypotheses* (II.19) *and* $(II.22)_1$ *hold. Then the solutions* (u,p) *of* (II.1) *and* (u_δ, p_δ) *of* (II.17), *satisfy the following estimate*

$$\text{(II.36)}\quad \left\{ \begin{aligned} \|p-p_\delta\|_{M_1} \leqslant c\,(1+\beta_{1\delta}^{-1})(1+\alpha_{1\delta}^{-1})(1+\gamma_\delta)\,[\,(1+\gamma_\delta) & \inf_{w_\delta \in K_{2\delta}(g_\delta)} \|u-w_\delta\|_{X_2} \\ & + \inf_{v_\delta \in X_{2\delta}} \{(1+\gamma_\delta)\|u-v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a-a_\delta)(v_\delta, z_\delta)}{\|z_\delta\|_{X_1}}\} \\ & + \inf_{q_\delta \in M_{1\delta}} \{\|p-q_\delta\|_{M_1} + \sup_{z_\delta \in X_{1\delta}} \frac{(b_1-b_{1\delta})(z_\delta, q_\delta)}{\|z_\delta\|_{X_1}}\} \\ & + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f-f_\delta, z_\delta\rangle}{\|z_\delta\|_{X_1}}\,] . \end{aligned} \right.$$

Proof : Let q_δ be any element in $M_{1\delta}$. By $(II.22)_1$, we have

$$\text{(II.37)}\quad \|p_\delta-q_\delta\|_{M_1} \leqslant \beta_{1\delta}^{-1} \sup_{z_\delta \in X_{1\delta}} \frac{b_{1\delta}(z_\delta, p_\delta-q_\delta)}{\|z_\delta\|_{X_1}} .$$

But, thanks to problems (II.1) and (II.17), for any z_δ in $X_{1\delta}$ and v_δ in $X_{2\delta}$ we can write

$$\begin{aligned} b_{1\delta}(z_\delta, p_\delta-q_\delta) &= -\,a_\delta(u_\delta, z_\delta) + \langle f_\delta, z_\delta\rangle - b_{1\delta}(z_\delta, q_\delta) \\ &= a(u, z_\delta) - a_\delta(u_\delta, z_\delta) + b_1(z_\delta, p) - \langle f-f_\delta, z_\delta\rangle - b_{1\delta}(z_\delta, q_\delta) \\ &= a(u-v_\delta, z_\delta) - a_\delta(u_\delta-v_\delta, z_\delta) + (a-a_\delta)(v_\delta, z_\delta) + b_1(z_\delta, p-q_\delta) \\ &\quad + (b_1-b_{1\delta})(z_\delta, q_\delta) - \langle f-f_\delta, z_\delta\rangle . \end{aligned}$$

Using (II.23) and the continuity of a and b_1 in the last formula, we deduce from (II.37)

$$\|p_\delta - q_\delta\|_{M_1} \leqslant \beta_{1\delta}^{-1} \,[\, \gamma_\delta \, \|u-u_\delta\|_{X_2} + \{ (\gamma+\gamma_\delta) \, \|u-v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a-a_\delta)(v_\delta \, , z_\delta)}{\|z_\delta\|_{X_1}} \, \}$$

$$+ \{ c \, \|p-q_\delta\|_{M_1} + \sup_{z_\delta \in X_{1\delta}} \frac{(b_1 - b_{1\delta})(z_\delta \, , q_\delta)}{\|z_\delta\|_{X_1}} \} + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f-f_\delta \, , z_\delta \rangle}{\|z_\delta\|_{X_1}} \,] \; .$$

This, together with (II.30), proves the theorem.

<u>Remark II.3</u> : Assume that hypotheses (II.19) and $(\text{II}.22)_i$ hold and that the discrete forms a_δ and $b_{i\delta}$ ($i = 1, 2$) are the restrictions to the spaces of discretization of the continuous forms a and b_i ($i=1, 2$) respectively, and that f and f_δ coincide. Then, the previous estimates can be substancially simplified as follows :

$$\text{(II.38)} \quad \|u-u_\delta\|_{X_2} \leqslant c\,(1+\alpha_\delta^{-1}) \inf_{w_\delta \in K_{2\delta}(g_\delta)} \|u-w_\delta\|_{X_2} \; ,$$

and

$$\text{(II.39)} \quad \|p-p_\delta\|_{M_1} \leqslant c\,(1+\beta_{1\delta}^{-1})(1+\alpha_{1\delta}^{-1}) \, (\inf_{w_\delta \in K_{2\delta}(g_\delta)} \|u-w_\delta\|_{X_2} + \inf_{q_\delta \in M_{1\delta}} \|p-q_\delta\|_{M_1}) \; .$$

III. Variational formulation of the Stokes problem in weighted Sobolev spaces.

We are interested in solving the Stokes problem in the domain $\Omega =]-1,1[^2$: thus, given a force field $\mathbf{f}$ in Ω and a viscosity $\nu > 0$, we look for a velocity field $\mathbf{u}$ and a pressure p (defined up to an additive constant) such that the following equations are satisfied

$$\text{(III.1)} \qquad \begin{cases} -\nu\,\Delta\mathbf{u} + \operatorname{grad} p = \mathbf{f} & \text{in } \Omega \quad , \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \quad , \end{cases}$$

together with a Dirichlet boundary condition on the boundary $\partial\Omega$ of Ω; we shall first study the homogeneous case

$$\text{(III.2)} \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \quad .$$

Since we want to study a spectral Chebyshev approximation of the Stokes problem, we first give a variational formulation of (III.1)(III.2) in terms of weighted Sobolev spaces, the weight being precisely the Chebyshev one in each variable. Then we use Theorem II.1 to prove the well-posedness of the variational problem. Finally, we extend the results to the case of non homogeneous boundary conditions.

III.1. The weighted spaces and the homogeneous Stokes problem.

Let us briefly recall some basic material about weighted spaces of Chebyshev type (for further details, see e.g. [CHQZ][Ma2]). If $\varrho(\zeta) = (1-\zeta^2)^{-1/2}$ denotes the Chebyshev weight on the interval $]-1,1[$, let

$$L^2_\varrho(-1,1) = \{\ \varphi :]-1,1[\to \mathbb{R}\ ;\ \int_\Omega \varphi^2(\zeta)\,\varrho(\zeta)\,d\zeta < +\infty\ \}$$

be the Lebesgue space associated with the measure $\varrho(\zeta)\,d\zeta$, provided with the inner product

$$\text{(III.3)} \qquad (\varphi,\psi)_\varrho = \int_\Omega \varphi(\zeta)\,\psi(\zeta)\,\varrho(\zeta)\,d\zeta$$

and the norm $\|.\|_{0,\varrho} = (.,.)_\varrho^{1/2}$.

A scale of weighted Sobolev spaces is defined as follows : for any integer $m \geqslant 0$, $H^m_\varrho(-1,1)$ is the subspace of $L^2_\varrho(-1,1)$ of the functions such that their distributional derivatives of order $\leqslant m$ belong to $L^2_\varrho(-1,1)$; it is a Hilbert space for the inner product associated with the norm

$$\text{(III.4)} \qquad \|\varphi\|_{m,\varrho} = \Big(\sum_{k=0}^{m} |\varphi|^2_{k,\varrho}\Big)^{1/2} \quad ,$$

where

$$\text{(III.5)} \qquad |\varphi|_{k,\varrho} = \|d^k\varphi/d\zeta^k\|_{0,\varrho} \quad .$$

For a real number $s = m + \alpha$, $0 < \alpha < 1$, we define $H^s_\varrho(-1,1)$ as the interpolation space between

$H_\rho^m(-1,1)$ and $H_\rho^{m+1}(-1,1)$ of index α (cf. [LM]); we denote its norm by $\|.\|_{s,\rho}$.

Finally, we can apply a rotation and a translation to define similar Sobolev spaces on any segment of length 2 in $\mathbb{R}^2$. We use the same notations as before to indicate them, as well as their norms.

The generic point in the square Ω will be denoted by $\mathbf{x} = (x,y)$. We introduce the vertices $\mathbf{a}_J$, $J \in \mathbb{Z}/4\mathbb{Z}$, of $\bar{\Omega}$ (where $\mathbf{a}_{J+1}$ follows $\mathbf{a}_J$ counterclockwise), and call Γ_J the edge with vertices $\mathbf{a}_{J-1}$ and $\mathbf{a}_J$; for any edge Γ_J , $J \in \mathbb{Z}/4\mathbb{Z}$, $\mathbf{n}_J$ is the unit outward normal to Ω on Γ_J and τ_J the unit vector orthogonal to $\mathbf{n}_J$, directed counterclockwise.

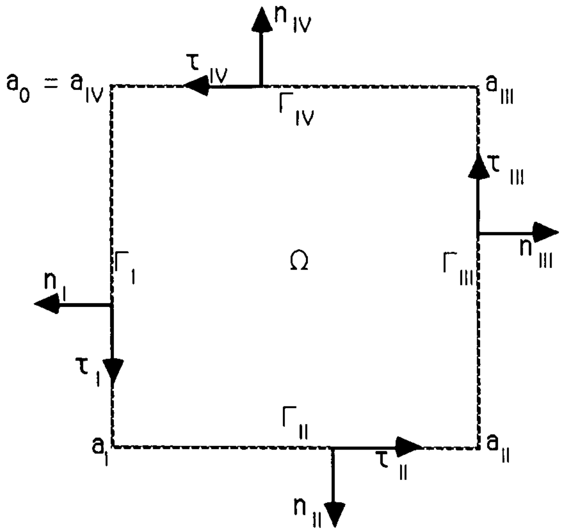

Figure III.1

The square Ω.

The Chebyshev weight on Ω is defined as $\omega(\mathbf{x}) = \rho(x)\,\rho(y)$. Let

$L^2_\omega(\Omega) = \{ v : \Omega \to \mathbb{R} \; ; \int_\Omega \varphi^2(\mathbf{x})\, \omega(\mathbf{x})\, d\mathbf{x} < +\infty \}$

be the Lebesgue space associated with the measure $\omega(\mathbf{x})\, d\mathbf{x}$, provided with the inner product

(III.6) $\quad (\varphi,\psi)_\omega = \int_\Omega \varphi(\mathbf{x})\, \psi(\mathbf{x})\, \omega(\mathbf{x})\, d\mathbf{x}$

and the norm $\|.\|_{0,\omega} = (.,.)_\omega^{1/2}$.

Next, a scale of weighted Sobolev spaces is defined as follows : for any integer $m \geqslant 0$, $H^m_\omega(\Omega)$ is the subspace of $L^2_\omega(\Omega)$ of the functions such that their distributional derivatives of order $\leqslant m$ belong to $L^2_\omega(\Omega)$; it is a Hilbert space for the inner product associated with the norm

(III.7) $\quad \|\varphi\|_{m,\omega} = \left(\sum_{k=0}^m |\varphi|^2_{k,\omega} \right)^{1/2}$,

where

(III.8) $\quad |\varphi|_{k,\omega} = (\sum_{j=0}^{k} \|\partial^k\varphi/\partial x^j \partial y^{k-j}\|_{0,\omega}^2)^{1/2}$.

For a real number $s = m + \alpha$, $0 < \alpha < 1$, we define $H_\omega^s(\Omega)$ as the interpolation space between $H_\omega^m(\Omega)$ and $H_\omega^{m+1}(\Omega)$ of index α; we denote its norm by $\|.\|_{s,\omega}$.

Being concerned with homogeneous Dirichlet boundary conditions, for any integer $m \geqslant 1$, we consider the closed subspace of the functions of $H_\omega^m(\Omega)$ which vanish on the boundary $\partial\Omega$ together with all their derivatives of order up to $m-1$ (the traces being defined in the sense of [LM]) ; this space, denoted by $H_{\omega,0}^m(\Omega)$, is the closure of $\mathcal{D}(\Omega)$ under the norm of $H_\omega^m(\Omega)$ (see [BM, Prop. II.9]). Due to the Poincaré inequality, an equivalent norm on $H_{\omega,0}^m(\Omega)$ is the seminorm $|.|_{m,\omega}$. The dual space of $H_{\omega,0}^m(\Omega)$ will be denoted by $H_\omega^{-m}(\Omega)$; whenever the space $L_\omega^2(\Omega)$ is identified to its dual space, we have for instance

(III.9) $\quad H_\omega^{-1}(\Omega) = \{ f + \partial g/\partial x + \partial h/\partial y, (f,g,h) \in [L_\omega^2(\Omega)]^3 \}$.

Now, we go back to the Stokes problem (III.1)(III.2). Assume that $\mathbf{f}$ belongs to $[H_\omega^{-1}(\Omega)]^2$. A weighted variational formulation of (III.1) is obtained by requiring that the first equation in (III.1) is satisfied in $[H_\omega^{-1}(\Omega)]^2$, and the second equation in $L_\omega^2(\Omega)$. In order to satisfy also (III.2), we define the space

(III.10) $\quad X = X_1 = X_2 = [H_{\omega,0}^1(\Omega)]^2$,

in which we look for the velocity. Since the pressure p is defined up to an additive constant and div $\mathbf{u}$ has zero average in Ω, we introduce the closed subspaces

(III.11) $\quad M_1 = \{ q \in L_\omega^2(\Omega) \,;\, \int_\Omega q(\mathbf{x})\, \omega(\mathbf{x})\, d\mathbf{x} = 0 \}$

and

(III.12) $\quad M_2 = \{ q \in L_\omega^2(\Omega) \,;\, \int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0 \}$,

in which respectively we look for the pressure and we choose the test functions to enforce the divergence-free condition ; this choice will be justified later.

Thus, for any $\mathbf{f}$ in X', we consider the following variational formulation of the Stokes problem (III.1)(III.2) : *Find* $(\mathbf{u},p)$ *in* $X \times M_1$ *such that*

(III.13)
$$\left| \begin{array}{l} \forall\, \mathbf{v} \in X, \quad \nu \int_\Omega \mathbf{grad}\, \mathbf{u} \,.\, \mathbf{grad}\, (\mathbf{v}\omega)\, d\mathbf{x} - \int_\Omega \mathrm{div}\, (\mathbf{v}\omega)\, p\, d\mathbf{x} = \langle \mathbf{f},\mathbf{v} \rangle \quad , \\ \forall\, q \in M_2 \,, \quad \int_\Omega \mathrm{div}\, \mathbf{u}\, q\, \omega\, d\mathbf{x} = 0 \quad . \end{array} \right.$$

This formulation will turn out to be a particular case of the abstract variational system (II.1), provided we define the bilinear forms $a : X \times X \to \mathbb{R}$ and $b_i : X \times M_i \to \mathbb{R}$ $(i = 1, 2)$ respectively by

(III.14) $\quad a(\mathbf{u},\mathbf{v}) = -\nu \langle \Delta\mathbf{u},\mathbf{v} \rangle = \nu \int_\Omega \mathbf{grad}\, \mathbf{u} \,.\, \mathbf{grad}\, (\mathbf{v}\omega)\, d\mathbf{x}$,

(III.15) $\quad b_1(\mathbf{v},q) = \langle \mathbf{v}, \mathbf{grad}\, q \rangle = - \int_\Omega \mathrm{div}\, (\mathbf{v}\omega)\, q\, d\mathbf{x}$, (III.16) $b_2(\mathbf{v},q) = -(\operatorname{div}\mathbf{v},q)_\omega = -\int_\Omega \operatorname{div}\mathbf{v}\, q\, \omega\, d\mathbf{x}$.

Note that we have

(III.17) $K_1 = \{ \mathbf{v} \in X \,;\, \operatorname{div}(\mathbf{v}\omega) = 0 \text{ in } \Omega \}$

and

(III.18) $K_2 = \{ \mathbf{v} \in X \,;\, \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega \}$.

We have at once the following result.

Proposition III.1 : *The forms* $a : X \times X \to \mathbb{R}$ *and* $b_i : X \times M_i \to \mathbb{R}$ $(i = 1, 2)$ *are continuous.*

Proof : It is a straightforward consequence of the continuity of the mapping : $\varphi \to \omega^{-1}\operatorname{grad}(\varphi\omega)$ from $H^1_{\omega,0}(\Omega)$ into $L^2_\omega(\Omega)$. In order to prove this result, we write

$$\omega^{-1}\operatorname{grad}(\varphi\omega) = \operatorname{grad}\varphi + \varphi \begin{pmatrix} x/(1-x^2) \\ y/(1-y^2) \end{pmatrix}$$

and, due to Hardy's inequality (see [N, Chap. 6, Lemme 2.1]), the terms $\varphi\, x/(1-x^2)$ and $\varphi\, y/(1-y^2)$ can be bounded by the norm of φ in $H^1_{\omega,0}(\Omega)$ (see [CQ2, Lemma 1.2] for details).

Thus, in order to apply Theorem II.1 to problem (III.13), we have to check the inf-sup conditions (II.8)(II.9) and $(\text{II.13})_i$ $(i = 1, 2)$.

III.2. The inf-sup condition for a.

Let us first deal with the form a. For some technical reasons, we introduce the weighted Sobolev spaces relative to the inverse of the Chebyshev weight : for any real number $s \geq 0$, the spaces $H^s_{1/\rho}(-1,1)$ are defined in the same way as the spaces $H^s_\rho(-1,1)$ with ρ replaced by $1/\rho$ and provided with the norm $\|.\|_{s,1/\rho}$; the spaces $H^s_{1/\omega}(\Omega)$ are defined in the same way as the spaces $H^s_\omega(\Omega)$ with ω replaced by $1/\omega$ and provided with the norm $\|.\|_{s,1/\omega}$; for any integer $m \geq 0$, we denote by $H^m_{1/\omega,0}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ under the norm $\|.\|_{m,1/\omega}$. We recall the following result due to [BM].

Lemma III.1 : *For any integer* $m \geq 0$, *the mapping :* $\varphi \to \omega^{1/2}\varphi$ *is an isomorphism from* $H^m_{\omega,0}(\Omega)$ *onto* $H^m_0(\Omega)$ *and from* $H^m_0(\Omega)$ *onto* $H^m_{1/\omega,0}(\Omega)$.

Now, we are in a position to prove the following proposition.

Proposition III.2 : *There exists a constant* $\alpha > 0$ *such that*

$$\text{(III.19)} \quad \forall\, \mathbf{u} \in K_2 , \quad \sup_{\mathbf{v} \in K_1} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{v}\|_{1,\omega}} \geq \alpha \|\mathbf{u}\|_{1,\omega} \quad ,$$

$$\text{(III.20)} \quad \forall\, \mathbf{v} \in K_1\,, \quad \sup_{\mathbf{u} \in K_2} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_{1,\omega}} \geqslant \alpha\, \|\mathbf{v}\|_{1,\omega} \quad .$$

Proof : Let us start with (III.19). If $\mathbf{u}$ belongs to K_2 , then there exists φ in $H_0^2(\Omega)$ such that $\mathbf{u} = \mathbf{curl}\ \varphi$ (see e.g. [GR, Chapter 1, Thm 3.1]); since $\mathbf{u}$ is in X, φ belongs in fact to $H^2_{\omega,0}(\Omega)$. Define $\mathbf{v} = \omega^{-1}\ \mathbf{curl}\ (\varphi\omega)$. By Lemma III.1, $\varphi\omega$ belongs to $H^2_{1/\omega,0}(\Omega)$, hence $\mathbf{v}$ belongs to $H^1_{\omega,0}(\Omega)$, with

$$\|\mathbf{v}\|_{1,\omega} \leqslant c\, \|\mathbf{u}\|_{1,\omega} \quad .$$

Moreover,

$$a(\mathbf{u},\mathbf{v}) = \nu \int_\Omega\ (\mathrm{curl}\, \mathbf{u})\ (\mathrm{curl}\, (\mathbf{v}\omega))\ d\mathbf{x} = \nu \int_\Omega\ \Delta\varphi\ \Delta(\varphi\omega)\ d\mathbf{x} \quad .$$

According to [MM, Lemma 3.2], there exists a constant $c > 0$ such that

$$\int_\Omega\ \Delta\varphi\ \Delta(\varphi\omega)\ d\mathbf{x} \geqslant c\, \|\varphi\|^2_{2,\omega} \quad .$$

We conclude that (III.19) holds.

In order to check (III.20), we use a similar argument. If $\mathbf{v}$ belongs to K_2 , then there exists ψ in $H^2_{1/\omega,0}(\Omega)$ such that $\mathbf{v}\omega = \mathbf{curl}\ \psi$. We set $\mathbf{u} = \mathbf{curl}\ (\psi\omega^{-1})$, so that $\mathbf{u}$ belongs to K_1 , and we conclude as before.

III.3. The inf-sup condition for b_i $(i = 1, 2)$.

Next, we check the inf-sup condition (II.13)_i for the forms b_i $(i = 1, 2)$. To this end, we recall some trace and regularity results in weighted Sobolev spaces, due to [BM].

With the notations of Figure III.1, we have the following lemmas.

Lemma III.2 [BM, Thm II.3] : *The trace operator* $R:\ v \to (v_{|\Gamma_J})_{J \in \mathbb{Z}/4\mathbb{Z}}$ *is linear continuous from the space* $H^1_\omega(\Omega)$ *onto the subspace of* $\prod_{J \in \mathbb{Z}/4\mathbb{Z}} H^{3/4}_\varrho(\Gamma_J)$ *of the functions* $(\varphi_J)_{J \in \mathbb{Z}/4\mathbb{Z}}$ *satisfying*

$$\text{(III.21)} \quad \forall\, J \in \mathbb{Z}/4\mathbb{Z}, \quad \varphi_J(\mathbf{a}_J) = \varphi_{J+1}(\mathbf{a}_J) \quad .$$

Moreover, the operator R *admits a continuous right inverse.*

Lemma III.3 [BM, Thm II.4] : *The trace operator* $S:\ v \to (v_{|\Gamma_J}\,,\ \partial v/\partial n_J)_{J \in \mathbb{Z}/4\mathbb{Z}}$ *is linear continuous from the space* $H^2_\omega(\Omega)$ *onto the subspace of* $\prod_{J \in \mathbb{Z}/4\mathbb{Z}} (H^{7/4}_\varrho(\Gamma_J) \times H^{3/4}_\varrho(\Gamma_J))$ *of the functions* $(\varphi_J\,,\ \chi_J)_{J \in \mathbb{Z}/4\mathbb{Z}}$ *satisfying*

$$(\text{III}.22) \quad \forall J \in \mathbb{Z}/4\mathbb{Z}, \quad \left| \begin{array}{l} \varphi_J(\mathbf{a}_J) = \varphi_{J+1}(\mathbf{a}_J) \quad , \\ (\partial\varphi_J/\partial\tau_J)(\mathbf{a}_J) = \chi_{J+1}(\mathbf{a}_J) \quad , \\ \chi_J(\mathbf{a}_J) = -(\partial\varphi_{J+1}/\partial\tau_{J+1})(\mathbf{a}_J) \quad . \end{array} \right.$$

Moreover, the operator S admits a continuous right inverse.

<u>Lemma III.4</u> [BM, Thm II.2] : *The trace operator T : $v \to (v_{|\Gamma_J}, \partial v/\partial n_J)_{J\in\mathbb{Z}/4\mathbb{Z}}$ is linear continuous from the space $H^2_{1/\omega}(\Omega)$ onto the subspace of $\prod_{J\in\mathbb{Z}/4\mathbb{Z}} (H^{5/4}_{1/\varrho}(\Gamma_J) \times H^{1/4}_{1/\varrho}(\Gamma_J))$ of the functions $(\varphi_J, \chi_J)_{J\in\mathbb{Z}/4\mathbb{Z}}$ satisfying* (III.21). *Moreover, the operator T admits a continuous right inverse.*

Finally, consider the Dirichlet problem

$$(\text{III}.23) \quad \left| \begin{array}{ll} -\Delta\varphi = \chi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \quad . \end{array} \right.$$

<u>Lemma III.5</u> [BM, Thm III.1] : *If χ belongs to $L^2_\omega(\Omega)$, then the solution φ belongs to $H^2_\omega(\Omega) \cap H^1_0(\Omega)$ and*

$$(\text{III}.24) \quad \|\varphi\|_{2,\omega} \leqslant c\, \|\chi\|_{0,\omega} \quad .$$

If χ belongs to $L^2_{1/\omega}(\Omega)$, then the solution φ belongs to $H^2_{1/\omega}(\Omega) \cap H^1_0(\Omega)$ and

$$(\text{III}.25) \quad \|\varphi\|_{2,1/\omega} \leqslant c\, \|\chi\|_{0,1/\omega} \quad .$$

Now, we are in a position to prove the following propositions.

<u>Proposition III.3</u> : *There exists a constant $\beta_2 > 0$ such that*

$$(\text{III}.26) \quad \forall q \in M_2 , \quad \sup_{\mathbf{v}\in X} \frac{b_2(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}} \geqslant \beta_2 \|q\|_{0,\omega} \quad .$$

Proof : Let q be any function in M_2. We consider the unique solution φ of (III.23) with $\chi = q$, for which we have

$$\|\varphi\|_{2,\omega} \leqslant c\, \|q\|_{0,\omega} \quad .$$

We want to find a function ψ in $H^2_\omega(\Omega)$ such that

$$(\text{III}.27) \quad \mathbf{curl}\, \psi = -\,\mathbf{grad}\, \varphi \quad \text{on } \partial\Omega \quad ,$$

or equivalently

$$(\text{III}.28) \quad \forall J \in \mathbb{Z}/4\mathbb{Z}, \quad \partial\psi/\partial\tau_J = -\,\partial\varphi/\partial n_J \quad \text{and} \quad \partial\psi/\partial n_J = \partial\varphi/\partial\tau_J = 0 \quad \text{on } \Gamma_J \quad .$$

To this end, we set

$$(\text{III}.29) \quad \psi_I(\mathbf{a}_I) = 0 \quad ,$$

and, for $J = II, III, IV, I$ successively,

(III.30) $\quad \forall\, \mathbf{x} \in \Gamma_J\,, \quad \psi_J(\mathbf{x}) = \psi_{J-1}(\mathbf{a}_{J-1}) - \int_{\Gamma_J} (\partial\varphi/\partial n_J)\, \chi([\mathbf{a}_{J-1}, \mathbf{x}])\, d\sigma \quad ,$

where $\chi([\mathbf{a}_{J-1}, \mathbf{x}])$ is the characteristic function of $[\mathbf{a}_{J-1}, \mathbf{x}]$ in Γ_J. By Lemma III.3, $\partial\varphi/\partial n_J$ belongs to $H_\varrho^{3/4}(\Gamma_J)$ and satisfies

(III.31) $\quad \forall J \in \mathbb{Z}/4\mathbb{Z}, \quad \left| \begin{array}{l} (\partial\varphi/\partial n_{J+1})(\mathbf{a}_J) = (\partial\varphi/\partial\tau_J)(\mathbf{a}_J) = 0 \quad , \\ (\partial\varphi/\partial n_J)(\mathbf{a}_J) = -\,(\partial\varphi/\partial\tau_{J+1})(\mathbf{a}_J) = 0 \quad . \end{array} \right.$

Hence, ψ_J is in $L_\varrho^2(\Gamma_J)$ and $\partial\psi_J/\partial\tau_J = -\,\partial\varphi/\partial n_J$ is in $H_\varrho^{3/4}(\Gamma_J)$, so that ψ_J belongs to $H_\varrho^{7/4}(\Gamma_J)$ (see [BM, Lemma II.4]). Moreover, since q is in M_2, we have

$$\textstyle\sum_{J \in \mathbb{Z}/4\mathbb{Z}} \int_{\Gamma_J} (\partial\varphi/\partial n_J)\, d\sigma = \int_\Omega \Delta\varphi\, d\mathbf{x} = -\int_\Omega q\, d\mathbf{x} = 0 \quad ,$$

hence, the functions ψ_J satisfy

$$\forall\, J \in \mathbb{Z}/4\mathbb{Z}, \quad \psi_J(\mathbf{a}_J) = \psi_{J+1}(\mathbf{a}_J) \quad .$$

We also have by (III.31)

$$\forall\, J \in \mathbb{Z}/4\mathbb{Z}, \quad (\partial\psi_J/\partial\tau_J)(\mathbf{a}_{J-1}) = (\partial\psi_J/\partial\tau_J)(\mathbf{a}_J) = 0 \quad .$$

Thus, the functions $(\psi_J, 0)_{J \in \mathbb{Z}/4\mathbb{Z}}$ satisfy the compatibility conditions (III.22). By Lemma III.3, there exists a function ψ in $H_\omega^2(\Omega)$ such that

$$\psi = \psi_J \quad \text{and} \quad \partial\psi/\partial n_J = 0 \quad \text{on } \Gamma_J\,, \quad J \in \mathbb{Z}/4\mathbb{Z} \quad ,$$

and which satisfies

(III.32) $\quad \|\psi\|_{2,\omega} \leqslant c \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\psi_J\|_{7/4,\varrho} \leqslant c\, \|\varphi\|_{2,\omega} \quad .$

Finally, we set $\mathbf{v} = \mathbf{grad}\, \varphi + \mathbf{curl}\, \psi$. Thanks to (III.27) and (III.32), the function $\mathbf{v}$ belongs to X and satisfies

(III.33) $\quad \|\mathbf{v}\|_{1,\omega} \leqslant c\, \|q\|_{0,\omega} \quad .$

Moreover, we have div $\mathbf{v} = q$, so that

$$b_2(\mathbf{v}, q) = \int_\Omega q^2\, \omega(\mathbf{x})\, d\mathbf{x} \quad .$$

This formula, together with (III.33), gives the proposition.

<u>Remark III.1</u> : Note that the proof of Proposition III.2 utilizes the condition $\int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0$ in the definition (III.12) of M_2.

<u>Proposition III.4</u> : *There exists a constant* $\beta_1 > 0$ *such that*

(III.34) $\quad \forall\, q \in M_1\,, \quad \sup\limits_{\mathbf{v} \in X} \dfrac{b_1(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,\omega}} \geqslant \beta_1\, \|q\|_{0,\omega} \quad .$

Proof : The proof is similar to the previous one. Given any function q in M_1, denote by φ the solution of the Dirichlet problem (III.23) with $\chi = q\omega$. Since $q\omega$ is in $L^2_{1/\omega}(\Omega)$, Lemma III.5 implies that φ belongs to $H^2_{1/\omega}(\Omega)$, with

$$\|\varphi\|_{2,1/\omega} \leqslant c\,\|q\|_{0,\omega} \quad .$$

Next, we build up a function ψ in $H^2_{1/\omega}(\Omega)$ such that again (III.27) holds. First, we define its trace on $\partial\Omega$ by (III.29) and (III.30). By Lemma III.4, $\partial\varphi/\partial n_J$ belongs to $H^{1/4}_{1/\varrho}(\Gamma_J) \subset L^1(\Gamma_J)$, thus ψ_J is well-defined and belongs to $H^{5/4}_{1/\varrho}(\Gamma_J)$. Moreover, since q is in M_1 , ψ_J satisfies

$$\forall\, J \in \mathbb{Z}/4\mathbb{Z}, \quad \psi_J(\mathbf{a}_J) = \psi_{J+1}(\mathbf{a}_J) \quad .$$

Applying again Lemma III.4, we see that there exists a function ψ in $H^2_{1/\omega}(\Omega)$ satisfying

$$\psi = \psi_J \quad \text{and} \quad \partial\psi/\partial n_J = 0 \quad \text{on } \Gamma_J\,, \quad J \in \mathbb{Z}/4\mathbb{Z}$$

and such that

(III.35) $\quad \|\psi\|_{2,1/\omega} \leqslant c \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\psi_J\|_{5/4,1/\varrho} \leqslant c\,\|\varphi\|_{2,1/\omega} \quad .$

The function $\tilde{\mathbf{v}} = \mathbf{grad}\,\varphi + \mathbf{curl}\,\psi$ belongs to X and satisfies

(III.36) $\quad \|\tilde{\mathbf{v}}\|_{1,1/\omega} \leqslant c\,\|q\|_{0,\omega} \quad .$

Finally, we set $\mathbf{v} = \tilde{\mathbf{v}}\omega^{-1}$. By Lemma III.1,

$$\|\mathbf{v}\|_{1,\omega} \leqslant c\,\|\tilde{\mathbf{v}}\|_{1,1/\omega} \quad ,$$

while on the other hand $\operatorname{div}(\mathbf{v}\omega) = \operatorname{div}\tilde{\mathbf{v}} = -\,q\omega$, so that

$$b_1(\mathbf{v}, q) = \int_\Omega q^2\,\omega(\mathbf{x})\,d\mathbf{x} \quad .$$

This proves the proposition.

<u>Remark III.2</u> : Here, we have used the condition $\int_\Omega q(\mathbf{x})\,\omega(\mathbf{x})\,d\mathbf{x} = 0$ in the definition (III.11) of M_1 .

<u>III.4. The existence and uniqueness results.</u>

Thanks to Propositions III.2, III.3 and III.4, we can apply Theorem II.1 to the variational problem (III.13), and obtain the main result of this section.

<u>Theorem III.1</u> : *For each* $\mathbf{f}$ *in* X', *there exists a unique variational solution* $(\mathbf{u},p)$ *in* $X \times M_1$ *to the Stokes problem* (III.1)(III.2). *Moreover, the following inequality is satisfied for a constant* $c > 0$

(III.37) $\quad \|\mathbf{u}\|_{1,\omega} + \|p\|_{0,\omega} \leqslant c\,\|\mathbf{f}\|_{X'} \quad .$

<u>Remark III.3</u> : In the Stokes problem (III.1)(III.2), the pressure is defined up to an additive constant. In the variational problem (III.13), this constant is fixed by the condition

(III.38) $\quad \int_\Omega p(\mathbf{x})\,\omega(\mathbf{x})\,d\mathbf{x} = 0$

(which is not the usual one – see [GR, Chapter I, Thm 5.1][BMM2, (V.2)] for instance).

We conclude this section with the case of non homogeneous Dirichlet boundary conditions, which we write in the form

(III.39) $\mathbf{u} = \varphi_J$ on Γ_J, $J \in \mathbb{Z}/4\mathbb{Z}$.

The following result holds.

Theorem III.2 : *For each* $\mathbf{f}$ *in* X' *and for each* $(\varphi_J)_{J\in\mathbb{Z}/4\mathbb{Z}}$ *in* $\prod_{J\in\mathbb{Z}/4\mathbb{Z}} H_\varrho^{3/4}(\Gamma_J)$ *satisfying*

(III.40) $\forall J \in \mathbb{Z}/4\mathbb{Z},\quad \varphi_J(\mathbf{a}_J) = \varphi_{J+1}(\mathbf{a}_J)$

(III.41) $\sum_{J\in\mathbb{Z}/4\mathbb{Z}} \int_{\Gamma_J} \varphi_J \cdot \mathbf{n}_J \, d\sigma = 0$,

there exists a unique variational solution $(\mathbf{u},p)$ *in* $[H_\omega^1(\Omega)]^2 \times M_1$ *to the Stokes problem* (III.1)(III.39). *Moreover, the following inequality is satisfied for a constant* $c > 0$

(III.42) $\|\mathbf{u}\|_{1,\omega} + \|p\|_{0,\omega} \leqslant c\,(\|\mathbf{f}\|_{X'} + \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{3/4,\varrho})$.

Proof : In order to apply Theorem III.1, we look for a function $\mathbf{u}_b$ in $H_\omega^1(\Omega)$ such that

(III.43) $\operatorname{div} \mathbf{u}_b = 0$ in Ω and $\mathbf{u}_b = \varphi_J$ on Γ_J, $J \in \mathbb{Z}/4\mathbb{Z}$,

or equivalently, setting $\mathbf{u}_b = \mathbf{curl}\,\psi$, for a function ψ in $H_\omega^2(\Omega)$ such that

(III.44) $\forall J \in \mathbb{Z}/4\mathbb{Z},\quad \partial\psi/\partial\tau_J = \varphi_J \cdot \mathbf{n}_J$ and $\partial\psi/\partial n_J = -\varphi_J \cdot \tau_J$ on Γ_J .

To this end, we set

(III.45) $\psi_I(\mathbf{a}_I) = 0$,

and, for $J = II, III, IV, I$ successively,

(III.46) $\forall \mathbf{x} \in \Gamma_J,\quad \psi_J(\mathbf{x}) = \psi_{J-1}(\mathbf{a}_{J-1}) + \int_{\Gamma_J} (\varphi_J \cdot \mathbf{n}_J)\, \chi([\mathbf{a}_{J-1}, \mathbf{x}])\, d\sigma$,

where $\chi([\mathbf{a}_{J-1}, \mathbf{x}])$ is the characteristic function of $[\mathbf{a}_{J-1}, \mathbf{x}]$ in Γ_J. For $J \in \mathbb{Z}/4\mathbb{Z}$, the pair $(\psi_J, -\varphi_J \cdot \tau_J)$ belongs to $H_\varrho^{7/4}(\Gamma_J) \times H_\varrho^{3/4}(\Gamma_J)$ and, due to (III.40) and (III.41), we have the conditions

$$\forall J \in \mathbb{Z}/4\mathbb{Z},\quad \left|\begin{array}{l} \psi_J(\mathbf{a}_J) = \psi_{J+1}(\mathbf{a}_J) \quad , \\ (\partial\psi_J/\partial\tau_J)(\mathbf{a}_J) = -(\varphi_{J+1} \cdot \mathbf{n}_{J+1})(\mathbf{a}_J) \quad , \\ (\varphi_J \cdot \mathbf{n}_J)(\mathbf{a}_J) = (\partial\psi_{J+1}/\partial\tau_{J+1})(\mathbf{a}_J) \quad . \end{array}\right.$$

It follows from Lemma III.3 that there exists ψ in $H_\omega^2(\Omega)$ satisfying

$$\forall J \in \mathbb{Z}/4\mathbb{Z},\quad \psi = \psi_J \quad \text{and} \quad \partial\psi/\partial n_J = -\varphi_J \cdot \tau_J \quad \text{on } \Gamma_J \quad ,$$

whence (III.44); moreover, we have

$$\|\psi\|_{2,\omega} \leqslant c \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{3/4,\varrho} \quad .$$

Next, we define $\mathbf{u}_b = \mathbf{curl}\,\psi$ and, using Theorem III.1, we consider the unique solution $(\tilde{\mathbf{u}},p)$ of problem (III.13) with $\langle \mathbf{f},\mathbf{v}\rangle$ replaced by $\langle \mathbf{f},\mathbf{v}\rangle - a(\mathbf{u}_b, \mathbf{v})$. Clearly, the pair $(\mathbf{u} = \tilde{\mathbf{u}}+\mathbf{u}_b, p)$ is solution of the Stokes problem (III.1)(III.39) and satisfies (III.42).

Remark III.4 : The previous result can also be proven by a direct application of the abstract Theorem II.1. As a matter of fact, if $(\varphi_J)_{J\in\mathbb{Z}/4\mathbb{Z}}$ belongs to $\prod_{J\in\mathbb{Z}/4\mathbb{Z}} H_\varrho^{3/4}(\Gamma_J)$ and satisfies (III.40), by Lemma III.2, there exists a vector field $\mathbf{u}_b'$ in $[H_\omega^1(\Omega)]^2$ such that

(III.47) $\quad \mathbf{u}'_{b|\Gamma_J} = \varphi_J \quad , J \in \mathbb{Z}/4\mathbb{Z}$

and

(III.48) $\quad \|\mathbf{u}_b'\|_{1,\omega} \leqslant c \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{3/4,\varrho} \quad .$

Let $(\tilde{\mathbf{u}}', p)$ be the solution in $X \times M_1$ of the variational problem

(III.49) $$\begin{cases} \forall\, \mathbf{v}\in X, \quad a(\tilde{\mathbf{u}}',\mathbf{v}) + b_1(\mathbf{v},p) = \langle \mathbf{f},\mathbf{v}\rangle - a(\mathbf{u}_b', \mathbf{v}) \quad , \\ \forall\, q\in M_2, \quad b_2(\tilde{\mathbf{u}}',q) = - b_2(\mathbf{u}_b', q) \quad . \end{cases}$$

Thanks to Theorem II.1, the continuity of the forms a and b_2 and (III.48), we have

$$\|\tilde{\mathbf{u}}'\|_{1,\omega} + \|p\|_{0,\omega} \leqslant c\,\{\, \|\mathbf{f}\|_{X'} + \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{3/4,\varrho} \,\} \quad .$$

Finally, let us set $\mathbf{u} = \tilde{\mathbf{u}}' + \mathbf{u}_b'$. Then, $(\mathbf{u},p)$ is the solution of the Stokes problem (III.1)(III.39). Indeed, the only nontrivial condition to check is the continuity equation. By (III.49) we have

(III.50) $\quad \forall\, q\in L^2_\omega(\Omega) \,/ \int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0, \quad \int_\Omega \operatorname{div}\mathbf{u}\, q\, \omega\, d\mathbf{x} = 0 \quad .$

Condition (III.41) implies that

$$\int_\Omega \operatorname{div}\mathbf{u}\, d\mathbf{x} = 0 \quad ,$$

which, together with (III.50), yields $\operatorname{div}\mathbf{u} = 0$ in Ω.

IV. A Galerkin method for the Stokes problem.

Henceforward, we fix an integer $N \geqslant 3$. In this section, we will study the convergence properties of a spectral Galerkin approximation to problem (III.13). Such a method is never used in practical computations - being less efficient than a tau or a collocation scheme. However, we consider it here because it is the simplest method which can be handled by the theory of Section II and in which the main difficulties are present.

IV.1. The discrete problem.

We look for an approximate solution of (III.13), the components of which are algebraic polynomials of degree $\leqslant N$ in each variable. This solution will be defined by a Galerkin method using the variational formulation of Section III. A suitable orthogonal basis in order to study the algorithm consists of the Chebyshev polynomials of the first kind.

Let us begin by recalling some basic material about Chebyshev expansions. We denote by $T_m(x)$, $m = 0, 1, \dots$ the Chebyshev polynomials of the first kind. They are defined by $T_m(x) = \cos(m \operatorname{Arcos} x)$, and satisfy the orthogonality relation

(IV.1) $\quad \int_{-1}^{1} T_m(\zeta)\, T_n(\zeta)\, (1-\zeta^2)^{-1/2}\, d\zeta = c_m\, (\pi/2)\, \delta_{mn} \quad, m \geqslant 0, n \geqslant 0 \quad,$

where c_0 is equal to 2 and c_m is equal to 1 for $m \geqslant 1$, and δ_{mn} denotes the Kronecker symbol. Moreover, for any $m \geqslant 1$, the Chebyshev polynomials satisfy the differential equation

(IV.2) $\quad \forall\, \zeta \in\,]-1,1[, \quad ((1-\zeta^2)^{1/2}\, T_m'(\zeta))' + m^2\, T_m(\zeta)\, (1-\zeta^2)^{-1/2} = 0$

and the relations

(IV.3) $\quad \forall\, \zeta \in\,]-1,1[, \quad T_{m+1}(\zeta) = 2\,\zeta\, T_m(\zeta) - T_{m-1}(\zeta) \quad,$

(IV.4) $\quad \forall\, \zeta \in\,]-1,1[, \quad T_m(\zeta) = T_{m+1}'(\zeta)/2(m+1) - T_{m-1}'(\zeta)/2(m-1) \quad.$

Let $P_N(-1,1)$ denote the space of the algebraic polynomials of degree $\leqslant N$ in one variable, restricted to the interval $]-1,1[$. Each φ in $P_N(-1,1)$ can be expanded as $\varphi(\zeta) = \sum_{m=0}^{N} \hat{\varphi}_m\, T_m(\zeta)$, with

(IV.5) $\quad \hat{\varphi}_m = (2/\pi c_m) \int_{-1}^{1} \varphi(\zeta)\, T_m(\zeta)\, (1-\zeta^2)^{-1/2}\, d\zeta \quad, 0 \leqslant m \leqslant N \quad.$

$\overset{\circ}{P}_N(-1,1)$ will be the subspace $P_N(-1,1) \cap H_0^1(-1,1)$ of the polynomials vanishing at the end points $\zeta = \pm 1$.

Next, we denote by $P_N(\Omega) = [P_N(-1,1)]^{\otimes 2}$ the space of the algebraic polynomials in $\mathbb{R}^2$ which are of degree $\leqslant N$ in each variable. Finally, we set $\overset{\circ}{P}_N(\Omega) = P_N(\Omega) \cap H_0^1(\Omega)$.

Let us now introduce the Galerkin approximation to problem (III.13). We look for the approximate velocity $\mathbf{u}^N$ in the space $X_N = [\overset{\circ}{P}_N(\Omega)]^2$ and for the approximate pressure p^N in $P_N(\Omega)$. We consider the following problem : *Find* $(\mathbf{u}^N, p^N)$ *in* $X_N \times P_N(\Omega)$ *such that*

(IV.6) $$\begin{cases} \forall\, \mathbf{v} \in X_N\,, \quad \nu \int_\Omega \operatorname{grad} \mathbf{u}^N \,.\, \operatorname{grad} (\mathbf{v}\omega)\, d\mathbf{x} - \int_\Omega \operatorname{div} (\mathbf{v}\omega)\, p^N\, d\mathbf{x} = \langle \mathbf{f},\mathbf{v}\rangle \quad , \\ \forall\, q \in P_N(\Omega)\,, \quad \int_\Omega \operatorname{div} \mathbf{u}^N\, q\, \omega\, d\mathbf{x} = 0 \quad . \end{cases}$$

However, as it will be discussed in the next subsection, in order to have a well-posed discrete problem, we must restrict the space of the pressures to a proper subspace M_{1N} of $P_N(\Omega)$. Similarly we restrict the space of test functions for the divergence free-condition to a proper subspace M_{2N} of $P_N(\Omega)$. Then, we obtain a particular case of the abstract approximate problem (II.17), if we set $X_{1\delta} = X_{2\delta} = X_N$ provided with the norm of X, $M_{1\delta} = M_{1N}$ and $M_{2\delta} = M_{2N}$ provided with the norm of $L^2_\omega(\Omega)$, and if the forms a_δ and $b_{i\delta}$ $(i = 1, 2)$ are respectively the forms a and b_i $(i = 1, 2)$ defined in (III.14) to (III.16).

In order to apply the abstract convergence results of Section II, we have to choose the finite dimensional spaces M_{1N} and M_{2N} for the pressure, in such a way that the inf-sup conditions for the above forms hold. In the next subsection, we characterize those pressures which cannot satisfy such a condition for the forms b_1 and b_2 .

IV.2. The spurious modes of the pressure.

Spurious or "parasitic" modes of the pressure are those components of the numerical pressure that are not controlled by the discrete equations which approximate the Stokes system. Parasitic modes in finite difference or finite element methods have longly been investigated (see e.g. [GR]). In spectral methods, attention on such a problem was first brought by Y. MORCHOISNE [Mo]. The characterization of spurious modes for various spectral methods of mixed Fourier-Legendre or fully Legendre type has been carried out by [BMM1][BMM2].

Our aim is to characterize the polynomials q in $P_N(\Omega)$ which satisfy, respectively for $i = 1$ or 2, the following condition

(IV.7)$_i$ $\quad \forall\, \mathbf{v} \in X_N\,, \quad b_i(\mathbf{v},q) = 0 \quad .$

Let Z_{iN} denote the subspace of all q in $P_N(\Omega)$ for which (IV.7)$_i$ holds. It is clear that such polynomials cannot verify (II.13)$_i$. Moreover, if M_{1N} contains a non trivial element of Z_{1N}, the pressure in the solution of the discrete problem is not unique.

Let us first deal with the form b_1 defined by (III.15). We recall that the polynomials T'_m, $m = 1, \dots, N+1$, form a basis of $P_N(-1,1)$ which, due to (IV.1) and (IV.2), is orthogonal with respect to the inner product

(IV.8) $\quad (u,v)_{1/\rho} = \int_{-1}^{1} u(\zeta)\, v(\zeta)\, (1-\zeta^2)^{1/2}\, d\zeta \quad .$

Precisely, one has

(IV.9) $\quad \int_{-1}^{1} T'_m(\zeta)\, T'_n(\zeta)\, (1-\zeta^2)^{1/2}\, d\zeta = (\pi/2)\, m^2\, \delta_{mn} \quad , m \geqslant 0, n \geqslant 0 \quad .$

Lemma IV.1 : *The dimension of the subspace* Z_{1N} *is equal to* 8.

Proof : Let us expand each q in Z_{1N} as $q(\mathbf{x}) = \sum_{m,n=0}^{N} \hat{q}_{mn}\, T_m(x) T_n(y)$. Condition $(\text{IV}.7)_1$ is equivalent to the conditions

(IV.10) $\quad \forall\, v \in \overset{\circ}{P}_N(\Omega), \quad \int_\Omega (\partial(v\omega)/\partial x)\, q\, d\mathbf{x} = 0$

and

(IV.11) $\quad \forall\, v \in \overset{\circ}{P}_N(\Omega), \quad \int_\Omega (\partial(v\omega)/\partial y)\, q\, d\mathbf{x} = 0 \quad .$

Let us consider first (IV.10). A basis in $\overset{\circ}{P}_N(\Omega)$ is given by the polynomials

(IV.12) $\quad v_{mn}(\mathbf{x}) = (1-x^2)\, T'_m(x)\, [T_n(y) - T_{\alpha(n)}(y)] \quad , 1 \leqslant m \leqslant N-1, 2 \leqslant n \leqslant N \quad ,$

where $\alpha(n)$ is equal to 0 if n is even and to 1 if n is odd (we use here the fact that $T_m(\pm 1) = (\pm 1)^m$). From equation (IV.2) and the orthogonality condition (IV.1), it is clear that (IV.10) is equivalent to the set of relations

(IV.13) $\quad \hat{q}_{mn} - c_{\alpha(n)}\, \hat{q}_{m\alpha(n)} = 0 \quad , 1 \leqslant m \leqslant N-1, 2 \leqslant n \leqslant N \quad .$

Working out condition (IV.11) in a perfectly symmetric way, we end up with another set of relations, namely

(IV.14) $\quad \hat{q}_{mn} - c_{\alpha(m)}\, \hat{q}_{\alpha(m)n} = 0 \quad , 2 \leqslant m \leqslant N, 1 \leqslant n \leqslant N-1 \quad .$

The relations (IV.13) and (IV.14) provide an orthogonal basis for Z_{1N}. This is given by the four modes $T_0(x)T_0(y)$, $T_N(x)T_0(y)$, $T_0(x)T_N(y)$ and $T_N(x)T_N(y)$ (since the corresponding coefficients do not appear in (IV.13) and (IV.14)), and by four other polynomials, the non-zero Chebyshev coefficients of which are respectively $\hat{q}_{01}$, $\hat{q}_{10}$, $\hat{q}_{11}$ and $\hat{q}_{02}$ plus the coefficients defined from these by the relations (IV.13) and (IV.14). This proves the lemma.

The next lemma allows us to find another basis for Z_{1N}, which is easier to handle.

Lemme IV.2 : *The set of all elements in* $P_N(-1,1)$ *which are orthogonal with respect to the inner product* $(.,.)_\rho$ *to the space* $\overset{\circ}{P}_N(-1,1)$ *is the subspace of dimension* 2 *spanned by* $\{T'_N, T'_{N+1}\}$.

Proof : Each polynomial φ in $P_N(-1,1)$ satisfying $\varphi(\pm 1) = 0$ can be written as $\varphi(\zeta) = (1-\zeta^2)\,\tilde{\varphi}(\zeta)$, where $\tilde{\varphi}$ belongs to $P_{N-2}(-1,1)$. Expanding $\tilde{\varphi}$ according to the basis $\{T'_m\}_{1 \leq m \leq N-1}$ and using (IV.9), we see that $\tilde{\varphi}$ is orthogonal to T'_N and T'_{N+1}. On the other hand, the subspace $\overset{\circ}{P}_N(-1,1) = \{ \varphi \in P_N(-1,1) \; ; \varphi(\pm 1) = 0 \}$ has codimension 2, hence the lemma is proved.

Corollary IV.1 : *The set of all elements in* $P_N(\Omega)$ *which are orthogonal with respect to the inner product* $(.,.)_\omega$ *to the space* $\{ v \in P_N(\Omega) \; ; v(\pm 1, \pm 1) = 0 \}$ *is the subspace of dimension* 4 *spanned by* $\{ T'_N, T'_{N+1} \}^{\otimes 2}$.

Proof : Each polynomial v in $P_N(\Omega)$ satisfying $v(\pm 1, \pm 1) = 0$ can be written as $v = w + z$, with

$$\forall\, (x,y) \in \overline{\Omega}, \quad w(\pm 1, y) = z(x, \pm 1) = 0$$

(take for instance $w(x,y) = ((1+y)/2)\, v(x,1) + ((1-y)/2)\, v(x,-1)$ and $z = v - w$). Hence, it follows from Lemma IV.2 that $\{ T'_N, T'_{N+1} \}^{\otimes 2}$ is orthogonal to $\{ v \in P_N(\Omega) \; ; v(\pm 1, \pm 1) = 0 \}$. On the other hand, since this space has codimension 4, the corollary is proved.

Noting that, if $\mathbf{v}$ belongs to X_N, then ω^{-1} div $(\mathbf{v}\omega)$ belongs to $P_N(\Omega)$ and is equal to 0 in $(\pm 1, \pm 1)$, and recalling the proof of Lemma IV.1, we obtain the following characterization.

Proposition IV.1 : *The subset* Z_{1N} *is the vector space of dimension* 8 *spanned by* $\{ T_0, T_N \}^{\otimes 2}$ *and by* $\{ T'_N, T'_{N+1} \}^{\otimes 2}$.

Remark IV.1 : Note that the basis in $\{ T'_N, T'_{N+1} \}^{\otimes 2}$ is a simple linear combination of the orthogonal basis of Z_{1N} defined in the proof of Lemma IV.1, as it can be seen by using (IV.4).

We consider now the form b_2 defined by (III.10). First, we recall the following result, due to [BMM2, Corollaire V.1]. Let us for a while denote by L_m, $m = 0, 1, \ldots$ the Legendre polynomials.

Lemma IV.3 : *The range of* X_N *by the divergence operator is the subspace* D_N *of* $P_N(\Omega)$ *of codimension* 8 *defined by*

(IV.15) $$D_N = \{ r \in P_N(\Omega) \; ; r(\pm 1, \pm 1) = 0 \text{ and } \forall\, q \in \{ L_0, L_N \}^{\otimes 2}, \int_\Omega r(\mathbf{x})\, q(\mathbf{x})\, d\mathbf{x} = 0 \} ,$$

Remark IV.2 : The lemma can also be established by a proof similar to the one of Lemma IV.2, by using the properties of the Legendre polynomials.

We can now characterize the space Z_{2N}. Let us introduce the polynomial q_N in $P_N(-1,1)$ defined by

(IV.16) $\quad \forall\, \varphi \in P_N(-1,1), \quad \int_{-1}^{1} q_N(\zeta)\, \varphi(\zeta)\, \varrho(\zeta)\, d\zeta = \int_{-1}^{1} \varphi(\zeta)\, d\zeta$

(i.e. q_N is the orthogonal projection of $1/\varrho$ onto $P_N(-1,1)$ with respect to the inner product $(.,.)_\varrho$).

<u>Proposition IV.2</u> : *The subset* Z_{2N} *is the vector space of dimension* 8 *spanned by* $\{ q_N , T_N \}^{\otimes 2}$ *and by* $\{ T'_N , T'_{N+1} \}^{\otimes 2}$.

Proof : The subspace Z_{2N} is precisely the orthogonal space to D_N in $P_N(\Omega)$ with respect to the inner product $(.,.)_\omega$. It follows that Z_{2N} has dimension 8. Next, from (IV.15) and Lemma IV.2, we deduce that Z_{2N} contains the subspace spanned by $\{ T'_N , T'_{N+1} \}^{\otimes 2}$. In order to obtain the remaining components, we have to translate the orthogonality relation in (IV.16) in terms of the inner product $(.,.)_\omega$. To this purpose, we observe that, for any φ in $P_N(-1,1)$, if λ is the coeficient of ζ^N in r, we have (see [DR, § 1.13])

$$\int_{-1}^{1} \varphi(\zeta)\, L_N(\zeta)\, d\zeta = \lambda \int_{-1}^{1} \zeta^N L_N(\zeta)\, d\zeta = \lambda\, 2^N (N!)^2 / (2N)!\,(N+1/2)$$

$$(\varphi, T_N)_\omega = \lambda\, (\zeta^N, T_N)_\omega = \lambda\, \pi\, 2^{-N} \quad ,$$

so that

(IV.17) $\quad \forall\, \varphi \in P_N(-1,1), \quad \int_{-1}^{1} \varphi(\zeta)\, L_N(\zeta)\, d\zeta = [2^N (N!)^2 / (2N)!\,(N+1/2)]\, (\varphi, T_N)_\omega \quad .$

Using this property and the definition (IV.16) of q_N, we deduce from (IV.15) that Z_{2N} contains the subspace spanned by $\{ q_N , T_N \}^{\otimes 2}$.

Finally, let us check that the polynomials T'_N, T'_{N+1}, q_N and T_N are linearly independent (which will imply that the 8 elements of $\{ q_N , T_N \}^{\otimes 2} \cup \{ T'_N , T'_{N+1} \}^{\otimes 2}$ are linearly independent). Assume that

$$\lambda_1 q_N + \lambda_2 T_N + \mu_1 T'_N + \mu_2 T'_{N+1} = 0 \quad .$$

Using (IV.4), we have

$$\lambda_1 q_N = (\lambda_2/2(N-1))\, T'_{N-1} - \mu_1 T'_N - (\mu_2 + \lambda_2/2(N+1))\, T'_{N+1} \quad .$$

Since N is $\geqslant 3$, $T'_1 = 1$ is orthogonal to T'_{N-1}, T'_N and T'_{N+1} with respect to $(.,.)_{1/\varrho}$, so that

$$0 = \lambda_1 \int_{-1}^{1} q_N(\zeta)\, (1-\zeta^2)^{1/2}\, d\zeta = \lambda_1 \int_{-1}^{1} (1-\zeta^2)\, d\zeta = 4\lambda_1/3 \quad ,$$

whence $\lambda_1 = 0$. Since the T'_k 's are mutually orthogonal with respect with $(.,.)_{1/\varrho}$, we obtain at once $\lambda_2 = \mu_1 = \mu_2 = 0$.

Remark IV.3 : In this work, we are not concerned with the Stokes problem in the cube $]-1, 1[^3$. However, note that, in the three-dimensional case, the dimension of the space Z_{iN} $(i = 1, 2)$ depends on N : it is equal to 12N+4. Indeed, by the same techniques as in [BMM2, §5], it is an easy matter to prove that Z_{1N} (resp. Z_{2N}) is spanned by $\{T_0, T_N\}^{\otimes 3}$ (resp. $\{q_N, T_N\}^{\otimes 3}$) and by the (12N−4) independent modes of

$$[\{T'_N, T'_{N+1}\}\otimes\{T'_N, T'_{N+1}\}\otimes P_N(-1,1)] \cup [\{T'_N, T'_{N+1}\}\otimes P_N(-1,1)\otimes\{T'_N, T'_{N+1}\}]$$
$$\cup\, [P_N(-1,1)\otimes\{T'_N, T'_{N+1}\}\otimes\{T'_N, T'_{N+1}\}]$$

IV.3. An inf-sup condition for the forms b_i $(i = 1, 2)$.

The characterization carried out in the previous section suggests the most direct choice of the test and trial spaces M_{1N} and M_{2N} for the pressure. Precisely, we choose for M_{iN} the orthogonal complement to Z_{iN} in $P_N(\Omega)$ $(i = 1, 2)$, i.e.

(IV.18)$_i$ $M_{iN} = \{ q \in P_N(\Omega) \; ; \forall\, r \in Z_{iN}, (q,r)_\omega = 0 \}$.

Due to (IV.7)$_i$, M_{iN} is characterized as

(IV.19) $M_{1N} = \{ \omega^{-1} \operatorname{div} (\mathbf{v}\omega), \mathbf{v} \in X_N \}$

and

(IV.20) $M_{2N} = \{ \operatorname{div} \mathbf{v}, \mathbf{v} \in X_N \}$.

Note also that M_{iN} is contained in M_i $(i = 1, 2)$.

In order to check the inf-sup conditions (II.22)$_i$ for b_i over the spaces $X_N \times M_{iN}$ $(i = 1, 2)$, we apply an abstract result of tensor algebra due to [Be, Chapter V, Appendix B], which we recall here. Let H be a Hilbert space, and denote by $(.,.)$ its inner product and by $\|.\|$ the corresponding norm. For any pair of planes A and B in H, we define the gap between A and B as the quantity

(IV.21) $\delta(A,B) = \inf \{ \|\varphi-\psi\| \; ; \varphi \in A, \psi \in B \text{ and } \|\varphi\| = \|\psi\| = 1 \}$.

Setting

(IV.22) $\eta(A,B) = \sup \{ (\varphi,\psi) \; ; \varphi \in A, \psi \in B \text{ and } \|\varphi\| = \|\psi\| = 1 \}$,

one has

(IV.23) $\delta^2(A,B) = 2\,(1-\eta(A,B))$.

Lemma IV.4 : *Let (A,B) be a pair of planes in H, the intersection of which is $\{0\}$. Any element q in $H^{\otimes 2}$ which belongs to $(A^{\otimes 2})^\perp$ and to $(B^{\otimes 2})^\perp$ can be written*

(IV.24) $q = r + s$,

where r and s belong to $A^\perp \otimes B^\perp$ and $B^\perp \otimes A^\perp$ respectively and satisfy

(IV.25) $\sup\{\|r\|, \|s\|\} \leqslant (1+2/\delta(A,B))^2 \|q\|$.

<u>Proposition IV.3</u> : *Let the space* M_{1N} *be defined by* (IV.18)$_1$. *There exists a constant* $\tilde{\beta}_1 > 0$ *independent of* N *such that*

(IV.26) $$\forall q \in M_{1N}, \quad \sup_{\mathbf{v}\in X_N} \frac{b_1(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}} \geqslant \tilde{\beta}_1 N^{-2} \|q\|_{0,\omega} \quad .$$

Proof : In the space $H = P_N(-1,1)$ provided with the inner product $(.,.)_\varrho$, define $A = \{T_0, T_N\}$ and $B = \{T'_N, T'_{N+1}\}$, and recall that, due to Lemma IV.2, $B^\perp$ is the space $(1-\zeta^2)\,P_{N-2}(-1,1)$. According to Proposition IV.1, M_{1N} is equal to $(A^{\otimes 2})^\perp \cap (B^{\otimes 2})^\perp$. Since N is $\geqslant 3$, $A \cap B$ is $\{0\}$, hence, by the previous lemma, each q in M_{1N} can be written as in (IV.24), with

(IV.27) $$r(x,y) = (1-y^2)\sum_{m=1}^{N-1} a_m(y)\,T_m(x) \quad , a_m \in P_{N-2}(-1,1) \quad ,$$

and

(IV.28) $$s(x,y) = (1-x^2)\sum_{n=1}^{N-1} b_n(x)\,T_n(y) \quad , b_n \in P_{N-2}(-1,1) \quad .$$

Using (IV.2), we write

$$\begin{aligned} r\omega &= (1-y^2)^{1/2}\sum_{m=1}^{N-1} a_m(y)\,T_m(x)\,(1-x^2)^{-1/2} \\ &= (\partial/\partial x)[(1-x^2)^{1/2}(1-y^2)^{1/2}\sum_{m=1}^{N-1} a_m(y)\,T'_m(x)/m^2] = \partial(v\omega)/\partial x \quad , \end{aligned}$$

where v, defined by

$$v(\mathbf{x}) = -(1-x^2)(1-y^2)\sum_{m=1}^{N-1} a_m(y)\,T'_m(x)/m^2 \quad ,$$

belongs to $\overset{\circ}{P}_N(\Omega)$. Moreover, we have

$$\|\partial v/\partial x\|_{0,\omega} \leqslant c\,\|\partial(v\omega)/\partial x\|_{0,1/\omega} = c\,\|r\omega\|_{0,1/\omega} \leqslant c'\,\|r\|_{0,\omega} \quad ,$$

and, by the Poincaré inequality, $\|v\|_{0,\omega} \leqslant c\,\|r\|_{0,\omega}$. Using the following inverse inequality [CQ1, Lemma 2.1], valid for any real numbers r and s, $0 \leqslant r \leqslant s$,

(IV.29) $$\forall \varphi \in P_N(\Omega), \quad \|\varphi\|_{s,\omega} \leqslant c\,N^{2(s-r)}\,\|\varphi\|_{r,\omega} \quad ,$$

we can control the norm $\|.\|_{0,\omega}$ of $\partial v/\partial y$ by

$$\|\partial v/\partial y\|_{0,\omega} \leqslant c\,N^2\,\|r\|_{0,\omega} \quad .$$

Working out in a symmetric way for the component s of q, we end up with an element $\mathbf{v} = (v,w)$ of X_N satisfying

(IV.30) $$\omega^{-1}\,\mathrm{div}\,(\mathbf{v}\omega) = -q$$

(so that $b_1(\mathbf{v},q) = \|q\|^2_{0,\omega}$) and

(IV.31) $$\|\mathbf{v}\|_{1,\omega} \leqslant c\,N^2\,(\|r\|_{0,\omega} + \|s\|_{0,\omega}) \quad .$$

It remains to estimate the gap (IV.21) between A and B. To this end, set $\varphi = \lambda_1 T_0 + \lambda_2 T_N$ in A and $\psi = \mu_1 T'_N + \mu_2 T'_{N+1}$ in B. From (IV.1) and (IV.9), it follows that

(IV.32) $\quad \|\varphi\|^2_{0,\varrho} = \pi\,(\lambda_1^2 + \lambda_2^2/2)$ and $\|\psi\|^2_{0,\varrho} = \pi\,(N^3\,\mu_1^2 + (N+1)^3\,\mu_2^2)$.

Moreover, we have

$$(T_0\,,T'_m)_\varrho = 0 \quad \text{if m is even} \quad \text{and} \quad (T_0\,,T'_m)_\varrho = \pi\, m \quad \text{if m is odd} \quad ,$$

$$(T_N\,,T'_N)_\varrho = 0 \quad \text{and} \quad (T_N\,,T'_{N+1})_\varrho = \pi\,(N+1) \quad ,$$

so that

(IV.33) $\quad |(\varphi,\psi)_\varrho| \leqslant \pi\,(\,|\lambda_1|\,|\mu_1|\,N + |\lambda_1|\,|\mu_2|\,(N+1) + |\lambda_2|\,|\mu_2|\,(N+1)\,)$

From (IV.32) and (IV.33), we conclude that

(IV.34) $\quad \forall\,\varphi \in A, \forall\,\psi \in B, \quad |(\varphi,\psi)_\varrho| \leqslant c\,N^{-1/2}\,\|\varphi\|_{0,\varrho}\,\|\psi\|_{0,\varrho}$.

Since for $N \geqslant 3$ the polynomials T_0 , T_N , T'_N and T'_{N+1} are linearly independent, $\delta(A,B)$ is greater than 0. Then, due to (IV.22), (IV.23) and (IV.34), it is bounded from below independently of N. Hence, the proposition follows from (IV.30), (IV.31) and (IV.25).

<u>Proposition IV.4</u> : *Let the space* M_{2N} *be defined by* $(\text{IV}.18)_2$. *There exists a constant* $\tilde{\beta}_2 > 0$ *independent of* N *such that*

(IV.35) $\quad \forall\, q \in M_{2N}\,, \quad \sup\limits_{\mathbf{v} \in X_N} \dfrac{b_2(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}} \geqslant \tilde{\beta}_2\,N^{-2}\,\|q\|_{0,\omega}$.

Proof : In the space $H = P_N(-1,1)$ provided with the inner product $(.,.)_\varrho$, we set now $A = \{\,q_N\,,T_N\,\}$ and $B = \{\,T'_N\,,T'_{N+1}\,\}$; due to the definition (IV.16) of q_N , $A^\perp$ is the subspace of $P_{N-1}(-1,1)$ of the polynomials having zero average on $(-1,1)$. From Proposition IV.2, we see that M_{2N} is equal to $(A^{\otimes 2})^\perp \cap (B^{\otimes 2})^\perp$. Thus, by Lemma IV.4, each q in M_{2N} can be expanded as in (IV.24) with

(IV.36) $\quad r(x,y) = (1-y^2)\sum_{m=1}^{N-1} a_m(y)\,L_m(x) \quad , a_m \in P_{N-2}(-1,1) \quad ,$

and

(IV.37) $\quad s(x,y) = (1-x^2)\sum_{n=1}^{N-1} b_n(x)\,L_n(y) \quad , b_n \in P_{N-2}(-1,1)$

(here, we find more appropriate to use a Legendre expansion for r and s). Following the proof of Proposition IV.3, but using the differential equation satisfied by the Legendre polynomials (see also [BMM2, § V]), we define an element $\mathbf{v} = (v,w)$ in X_N such that

(IV.38) $\quad \operatorname{div} \mathbf{v} = -\,q$

(so that $b_2(\mathbf{v},q) = \|q\|^2_{0,\omega}$) and

(IV.39) $\quad \|\mathbf{v}\|_{1,\omega} \leqslant c\,N^2\,(\,\|r\|_{0,\omega} + \|s\|_{0,\omega}\,)$.

In order to estimate the gap between A and B, let $\varphi = \lambda_1\,q_N + \lambda_2\,T_N$ and $\psi = \mu_1\,T'_N + \mu_2\,T'_{N+1}$ be arbitrary elements in A and B respectively. By (IV.16) and (IV.4) we have

$$(q_N\,,T_m)_\varrho = \int_{-1}^{1} T_m(\zeta)\,d\zeta = [T_{m+1}/2(m+1) - T_{m-1}/2(m-1)]_{-1}^{1} \quad ,$$

so that

$$(q_N\,,T_m)_\varrho = 0 \quad \text{if m is odd} \quad \text{and} \quad (q_N\,,T_m)_\varrho = 2/(1-m^2) \quad \text{if m is even} \quad .$$

Thus $\|\varphi\|_{0,\omega}^2 \geqslant c\,(\lambda_1^2 + \lambda_2^2)$. Moreover,

$$(q_N\,,T_m')_\varrho = \int_{-1}^{1} T_m'(\zeta)\,d\zeta = 0 \quad \text{if m is even} \quad \text{and} \quad (q_N\,,T_m')_\varrho = 2 \quad \text{if m is odd} \quad .$$

As in the proof of Proposition IV.3, we have again (IV.34), hence $\delta(A,B)$ is bounded from below independently of N. Thus the proposition is proved.

Remark IV.4 : Unfortunately, the constants involved in the inf-sup conditions (IV.26) and (IV.35) are not independent of N. However, we shall see that this does not corrupt the accuracy of the approximation of the velocity in the homogeneous case.

IV.4. The inf-sup condition for the form a.

Let us denote here by K_{iN} ($i = 1, 2$) the discrete kernels

(IV.40) $\quad K_{iN} = \{\, \mathbf{v} \in X_N \,;\, \forall\, \mathbf{v} \in M_{iN}\,,\, b_i(\mathbf{v},q) = 0 \,\}$.

According to $(IV.7)_i$ and $(IV.18)_i$, we actually have

$$K_{1N} = \{\, \mathbf{v} \in X_N \,;\, \operatorname{div}(\mathbf{v}\omega) = 0 \text{ in } \Omega \,\} \quad ,$$

$$K_{2N} = \{\, \mathbf{v} \in X_N \,;\, \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \,\} \quad ,$$

or equivalently

(IV.41) $\quad K_{iN} = K_i \cap X_N \quad (i = 1, 2)$,

where continuous kernels K_i are defined in (III.17) and (III.18). Moreover, by Propositions IV.1 and IV.2, we have

(IV.42) $\quad \dim K_{1N} = \dim K_{2N}$.

Thus, by Remark II.1, it is enough to check the inf-sup condition (II.19) for the form a.

Proposition IV.5 : *There exists a constant* $\tilde{\alpha} > 0$ *independent of* N *such that*

(IV.43) $\quad \forall\, \mathbf{u} \in K_{2N}\,, \quad \sup\limits_{\mathbf{v} \in K_{1N}} \dfrac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{v}\|_{1,\omega}} \geqslant \tilde{\alpha}\, \|\mathbf{u}\|_{1,\omega}$.

Proof : Let $\mathbf{u} = (u,v)$ be an element of K_{2N} . By (IV.41), there exists φ in $H^2_{\omega,0}(\Omega)$ such that $\mathbf{u} = \mathbf{curl}\;\varphi$. Since

$$\varphi(x,y) = -\int_{-1}^{x} v(\xi,y)\,d\xi = \int_{-1}^{y} u(x,\eta)\,d\eta \quad ,$$

necessarily φ belongs to $P_N(\Omega)$ (see also [S]). Thus the element $\mathbf{v} = \omega^{-1}\,\mathbf{curl}\,(\varphi\omega)$ used in the proof of Proposition III.2 to check the continuous inf-sup condition is indeed an element of $K_1 \cap X_N = K_{1N}$, whence the result.

Remark IV.5 : The constant $\tilde{\alpha}$ is the constant α of Proposition III.2, hence it is independent of N.

IV.5. A convergence estimate.

We have proved above that the abstract assumptions (II.19), (II.21) and $(II.22)_i$ $(i = 1, 2)$ are fulfilled with the present choice of spaces for the discrete velocity and pressure. Applying Corollary II.2, we derive the following result.

Theorem IV.1 : *For each integer* $N \geqslant 3$, *the Galerkin approximation* (IV.6) *to the Stokes problem* (III.1)(III.2) *has a unique solution* $(\mathbf{u}^N, p^N)$ *in* $X_N \times M_{1N}$, *where* M_{1N} *is defined by* $(IV.18)_1$. *Moreover, the following inequality is satisfied*

(IV.44) $\quad \|\mathbf{u}^N\|_{1,\omega} + N^{-2}\|p^N\|_{0,\omega} \leqslant c\,\|\mathbf{f}\|_{X'}$

for a constant $c > 0$ *independent of* N.

We can obtain a convergence estimate for the velocity using the abstract error estimate (II.35). To this end, note that by (IV.41), the inclusion (II.34) holds. Thus we get

(IV.45) $$\|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} \leqslant c \inf_{\mathbf{v}_N \in K_{2N}} \|\mathbf{u}-\mathbf{v}_N\|_{1,\omega} .$$

We now recall a general result of polynomial approximation theory in the Chebyshev norms. This result is due to [Ma2], except for the case $m = 0$ (where $H^0_{\omega,0}(\Omega)$ is the space $L^2_\omega(\Omega)$ and Π^0_N is the truncation of the Chebyshev series) for which we refer to [CQ1].

Lemma IV.5 : *For each integer* $m \geqslant 0$, *there exists a projection operator* Π^m_N : $H^m_{\omega,0}(\Omega) \to P_N(\Omega) \cap H^m_{\omega,0}(\Omega)$ *such that, for* $0 \leqslant r \leqslant m \leqslant s$,

(IV.46) $\quad \forall\, \varphi \in H^s_{\omega 0}(\Omega), \quad \|\varphi - \Pi^m_N \varphi\|_{r,\omega} \leqslant c(r,m,s)\, N^{r-s}\, \|\varphi\|_{s,\omega}$.

Using the previous lemma one can show that divergence-free vector fields can be approximated by divergence-free polynomial fields with an optimal error estimate in the weighted Sobolev scale.

Lemma IV.6 : *For each* $\mathbf{v}$ *in* K_2, *there exists an element* $Q_N\mathbf{v}$ *of* X_N *satisfying* $\mathrm{div}\,(Q_N\mathbf{v}) = 0$ in Ω *such that, if* $\mathbf{v}$ *belongs to* $H^s_\omega(\Omega)$ *for a real number* $s \geqslant 1$, (IV.47) $\quad \|\mathbf{v}-Q_N\mathbf{v}\|_{1,\omega} \leqslant c\, N^{1-s}\, \|\mathbf{v}\|_{s,\omega}$.

Proof : A general proof, which applies to vector fields in $\mathbb{R}^3$ as well, has been given in [SV]. For convenience of the reader we give here a simpler proof, which however holds in the 2-dimensional case only. Each $\mathbf{v}$ satisfying the assumptions of the lemma can be written $\mathbf{v} = \mathrm{curl}\ \varphi$, with φ in $H^2_{\omega,0}(\Omega) \cap H^{s+1}_{\omega}(\Omega)$. Define φ_N in $P_N(\Omega) \cap H^2_{\omega,0}(\Omega)$ as $\varphi_N = \Pi^2_N\varphi$ and set $Q_N\mathbf{v} = \mathrm{curl}\ \varphi_N$. Then (IV.47) is a direct consequence of (IV.46).

Using (IV.47), we derive from (IV.45) the following convergence result.

<u>Theorem IV.2</u> : *Assume that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (III.1)(III.2) *belongs to* $[H^s_\omega(\Omega)]^2 \times H^{s-1}_\omega(\Omega)$ *for a real number* $s \geqslant 1$. *Then the approximate velocity* $\mathbf{u}^N$, *as defined in Theorem* IV.1, *satisfies the convergence estimate*

(IV.48) $\quad \|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} \leqslant c\, N^{1-s}\, \|\mathbf{u}\|_{s,\omega}$

for a constant $c > 0$ *independent of* N.

<u>IV.6 Computation of the pressure.</u>

Unfortunately, it is not possible to establish a similar optimal result for the pressure, if the discrete space of pressures is defined by $(\mathrm{IV}.18)_1$. As a matter of fact, it follows from Proposition IV.1 and Corollary IV.1 that the elements of M_{1N} are polynomials which vanish at the four corners of the domain, while the exact pressure needs not satisfy this condition. Thus, spectral accuracy in the pressure cannot be achieved.

The remedy consists in choosing as discrete space of pressures another supplementary space to Z_{1N} in the space $P_N(\Omega)$, which exhibits better approximation properties to the functions of M_1 and, at the same time, which fulfills an inf-sup condition asymptotically not worse than (IV.26).

From a general point of view, the new space $\tilde{M}_{1N}$ of discrete pressures can be defined by introducing a new subspace $\tilde{Z}_{1N}$ of dimension 8 (which will be a suitable perturbation of Z_{1N}) and then setting

(IV.49) $\quad \tilde{M}_{1N} = \{\, q \in P_N(\Omega)\ ;\ \forall\, r \in \tilde{Z}_{1N}\ ,\ (q,r)_\omega = 0 \,\}$.

The space $\tilde{M}_{1N}$ will allow the exact pressure to be approximated at a spectral rate if there exists a real number λ, $0 < \lambda < 1$, such that

(IV.50) $\quad P_{[\lambda N]}(\Omega) \cap M_1 \subset \tilde{M}_{1N}$.

($[\lambda N]$ is the integral part of λN). This means that the elements of $\tilde{Z}_{1N}$ should be orthogonal to

$P_{[\lambda N]}(\Omega) \cap M_1$.

On the other hand, as far as the inf-sup condition for b_{1N} is concerned, let $\pi_N : \tilde{M}_{1N} \to M_{1N}$ denote the orthogonal projection onto M_{1N} with respect to the inner product $(.,.)_\omega$. We make the assumption that there exists a constant $c > 0$ independent of N such that

(IV.51) $\quad \forall\, q \in \tilde{M}_{1N}, \quad \|q\|_{0,\omega} \leqslant c\, \|\pi_N q\|_{0,\omega}$.

Proposition IV.6 : *Let* $\tilde{M}_{1N}$ *be a subspace of* $P_N(\Omega)$ *such that hypothesis* (IV.51) *holds. There exists a constant* $\tilde{\beta}_1 > 0$ *independent of* N *such that*

(IV.52) $$\forall\, q \in \tilde{M}_{1N}, \quad \sup_{\mathbf{v} \in X_N} \frac{b_1(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}} \geqslant \tilde{\beta}_1\, N^{-2}\, \|q\|_{0,\omega} .$$

Proof : Due to the definition $(IV.18)_1$ of M_{1N}, it follows that one has for all q in $\tilde{M}_{1N}$

(IV.53) $\quad \forall\, \mathbf{v} \in X_N, \quad b_1(\mathbf{v},q) = b_1(\mathbf{v},\pi_N q)$,

from which we deduce

$$\sup_{\mathbf{v} \in X_N} \frac{b_1(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}} \geqslant \sup_{\mathbf{v} \in X_N} \frac{b_1(\mathbf{v},\pi_N q)}{\|\mathbf{v}\|_{1,\omega}} \geqslant \tilde{\beta}_1\, N^{-2}\, \|\pi_N q\|_{0,\omega} .$$

The result follows by using (IV.51).

Let $\tilde{K}_{1N}$ denote now the kernel defined as in (IV.40), with M_{1N} replaced by $\tilde{M}_{1N}$. If (IV.51) holds, $\tilde{K}_{1N}$ still satisfies (IV.41), thanks to (IV.53). Hence, both (IV.42) and Proposition IV.5 are valid. Applying Corollary II.2, we obtain the same result as in Theorem IV.1.

Theorem IV.3 : *For each integer* $N \geqslant 3$, *the Galerkin approximation* (IV.6) *to the Stokes problem* (III.1)(III.2) *has a unique solution* $(\mathbf{u}^N, p^N)$ *in* $X_N \times \tilde{M}_{1N}$, *where* $\tilde{M}_{1N}$ *satisfies the hypothesis* (IV.51). *Moreover, the following inequality is satisfied*

(IV.54) $\quad \|\mathbf{u}^N\|_{1,\omega} + N^{-2}\, \|p^N\|_{0,\omega} \leqslant c\, \|\mathbf{f}\|_{X'}$

for a constant $c > 0$ *independent of* N.

Remark IV.6 : It follows from (IV.53) that, if $(\mathbf{u}^N, p^N)$ is the solution of (IV.6) in $X_N \times \tilde{M}_{1N}$, then $(\mathbf{u}^N, \pi_N p^N)$ is the solution of (IV.6) in $X_N \times M_{1N}$. In particular, the discrete velocity is independent of the choice of the space of pressures.

We can obtain a convergence estimate for both the velocity and the pressure.

Theorem IV.4 : *Assume that hypotheses* (IV.50) *and* (IV.51) *hold and that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (III.1)(III.2) *belongs to* $[H^s_\omega(\Omega)]^2 \times H^{s-1}_\omega(\Omega)$ *for a real number* $s \geqslant 1$. *Then* *the approximate solution* $(\mathbf{u}^N, p^N)$ *in* $X_N \times \tilde{M}_{1N}$, *as defined in Theorem* IV.3, *satisfies the convergence estimates*

(IV.55) $\quad \|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} \leqslant c\, N^{1-s}\, \|\mathbf{u}\|_{s,\omega}$

(IV.56) $\quad \|p-p^N\|_{0,\omega} \leqslant c\, N^{3-s}\, (\, \|\mathbf{u}\|_{s,\omega} + \|p\|_{s-1,\omega}\,)$

for a constant $c > 0$ *independent of* N.

Proof : It is a straightforward consequence of (II.38) and (II.39), if we note that, due to (IV.50),

$$\inf_{q_N \in \tilde{M}_{1N}} \|p-q_N\|_{1,\omega} \leqslant \|p-\Pi^0_{[\lambda N]}p\|_{0,\omega} \leqslant c\, N^{1-s}\, \|p\|_{s-1,\omega} \quad .$$

Now, we give an example of choice of the space $\tilde{M}_{1N}$ (or, equivalently, of $\tilde{Z}_{1N}$) for which (IV.50) and (IV.51) hold. The argument is an adaptation of [BMM2, § V.3].

Let us recall that Z_{1N} is spanned by $\{\, T_0\,, T_N\,\}^{\otimes 2}$ and $\{\, T'_N\,, T'_{N+1}\,\}^{\otimes 2}$. By (IV.4), the polynomials $s_N = T'_N$ and $t_N = T'_{N+1}$ admit the Chebyshev expansions

(IV.57) $\quad s_N = \sum_{m=0}^{N-1} \alpha_m T_m = 2N\,(\, T_{N-1} + T_{N-3} + \dots + T_{N-(2j+1)} + \dots\,)$

(IV.58) $\quad t_N = \sum_{n=0}^{N} \beta_n T_n = 2(N+1)\,(\, T_N + T_{N-2} + \dots + T_{N-2k} + \dots\,) \quad .$

Let us define the polynomials

(IV.59) $\quad \tilde{s}_N = \sum_{\lambda N < m < N} \alpha_m T_m = 2N\,(\, T_{N-1} + T_{N-3} + \dots + T_{m_0}\,) \quad ,$

where m_0 is the smallest integer $> \lambda N$ for which α_{m_0} is $\neq 0$, and

(IV.60) $\quad \tilde{t}_N = \sum_{\lambda N < n < N} \beta_n T_n = 2(N+1)\,(\, T_{N-2} + T_{N-4} + \dots + T_{n_0}\,) \quad ,$

where n_0 is the smallest integer $> \lambda N$ for which β_{n_0} is $\neq 0$. Finally, let us set

(IV.61) $\quad \tilde{Z}_{1N} = \mathrm{Span}\,[\, \{\, T_0\,, T_N\,\}^{\otimes 2} \cup \{\, \tilde{s}_N\,, \tilde{t}_N\,\}^{\otimes 2}\,]$

and define $\tilde{M}_{1N}$ by (IV.49).

<u>Proposition IV.7</u> : *The space* $\tilde{M}_{1N}$ *defined by* (IV.49) *and* (IV.61) *satisfy the hypotheses* (IV.50) *and* (IV.51).

Proof : The inclusion (IV.50) holds by construction of $\tilde{Z}_{1N}$, due to the definitions (IV.59) and (IV.60) of $\tilde{s}_N$ and $\tilde{t}_N$ .Let us check (IV.51). To this end, we shall exhibit the inverse mapping of the projection π_N . For each element ψ in $\{\, s_N\,, t_N\,\}^{\otimes 2}$, we denote by $\tilde{\psi}$ the corresponding element in $\{\, \tilde{s}_N\,, \tilde{t}_N\,\}^{\otimes 2}$, and by $\bar{\psi}$ its projection onto the orthogonal complement to $\{\, T_0\,, T_N\,\}^{\otimes 2}$ in $L^2_\omega(\Omega)$. It is easily seen that $\bar{\psi}$ has the form

$$\bar{\psi} = \psi - \sum_{\varphi \in \{T_0, T_N\}^{\otimes 2}} \varepsilon_\varphi \varphi \quad ,$$

for suitable coefficients ε_φ. Thus, $\bar{\psi}$ belongs to Z_{1N}, according to Proposition IV.1. For each q in M_{1N}, let us define

$$\text{(IV.62)} \qquad \tau_N q = q - \sum_{\psi \in \{s_N, t_N\}^{\otimes 2}} \gamma_\psi \bar{\psi} \quad ,$$

where the coefficients γ_ψ are such that $\tau_N q$ belongs to $\tilde{M}_{1N}$, i.e.,

$$\text{(IV.63)} \qquad \gamma_\psi = \frac{(q, \tilde{\psi})}{\|\tilde{\psi}\|^2_{0,\omega}}$$

(we have used the fact that, if ψ_1 and ψ_2 are two different elements of $\{s_N, t_N\}^{\otimes 2}$, then $\bar{\psi}_1$ is orthogonal to $\tilde{\psi}_2$). Since $\bar{\psi}$ belongs to Z_{1N}, we have $b_1(\mathbf{v}, q-\tau_N q) = 0$ for all $\mathbf{v}$ in X_N, or equivalently, recalling the definition of M_{1N},

$$\forall\, r \in M_{1N}, \quad (q-\tau_N q, r)_\omega = 0 \quad .$$

Thus q is the orthogonal projection of $\tau_N q$ onto M_{1N}, i.e., τ_N is the inverse of π_N. Finally, let us estimate the norm of τ_N. By (IV.62) and (IV.63), we have for all q in M_{1N}

$$\|\tau_N q\|_{0,\omega} \leqslant \|q\|_{0,\omega} + \sum_{\psi \in \{s_N, t_N\}^{\otimes 2}} |\gamma_\psi| \, \|\bar{\psi}\|_{0,\omega} \quad ,$$

$$\leqslant \|q\|_{0,\omega} \left(1 + \sum_{\psi \in \{s_N, t_N\}^{\otimes 2}} \frac{\|\psi\|_{0,\omega}}{\|\tilde{\psi}\|_{0,\omega}} \right) \quad .$$

Taking into account (IV.1), (IV.57) and (IV.58), one obtains

$$\|s_N\|_{0,\varrho} + \|t_N\|_{0,\varrho} \leqslant c\, N^{3/2} \quad ,$$

while by (IV.1), (IV.59) and (IV.60), one has

$$\|\tilde{s}_N\|_{0,\varrho} \geqslant c\, N^{3/2} \quad \text{and} \quad \|\tilde{t}_N\|_{0,\varrho} \geqslant c\, N^{3/2} \quad .$$

It follows that, for any ψ in $\{s_N, t_N\}^{\otimes 2}$, $\|\psi\|_{0,\omega} / \|\tilde{\psi}\|_{0,\omega}$ is bounded independently of N, thus for each q in M_{1N}

$$\|\tau_N q\|_{0,\omega} \leqslant c\, \|q\|_{0,\omega} \quad ,$$

which is precisely (IV.51).

Due to Proposition IV.7, we can apply our algorithm with the space $\tilde{M}_{1N}$ defined by (IV.49) and (IV.61), and we do obtain the estimates (IV.55) and (IV.56).

V. A collocation method for the Stokes problem.

In this section, we will study the convergence properties of a spectral collocation approximation to the Stokes problem (III.1). This method can be extended to an efficient collocation scheme for the approximation of the full Navier-Stokes equations (see [Mé]) ; the error analysis for this latter approximation will be performed in the following section.

V.1. The discrete problem.

We consider the simplest Chebyshev collocation approximation of the Stokes problem (III.1). This scheme uses a single grid for both the momentum and the continuity equation, the grid being given by the cartesian product of the Gauss-Lobatto points in one space variable for the Chebyshev mesure $\varrho(\zeta)\,d\zeta$.

More precisely, for a fixed $N \geqslant 3$, let us set for $0 \leqslant j \leqslant N$

(V.1) $\quad \zeta_j = \cos(j\pi/N)$ and $\varrho_j = \pi/\bar{c}_j N$,

(with $\bar{c}_0 = \bar{c}_N = 2$ and $\bar{c}_j = 1$ for $1 \leqslant j \leqslant N-1$). These are respectively the knots and the weights of the Gauss-Lobatto integration formula for the Chebyshev weight, exact for polynomials of degree $\leqslant 2N-1$, i.e.,

(V.2) $\quad \forall\, \varphi \in P_{2N-1}(-1,1),\ \int_{-1}^{1} \varphi(\zeta)\,\varrho(\zeta)\,d\zeta = \sum_{j=0}^{N} \varphi(\zeta_j)\,\varrho_j$.

It will be useful in the sequel to recall that the interior quadrature nodes are the zeros of the polynomial T_N', i.e.,

(V.3) $\quad \forall\, j,\ 1 \leqslant j \leqslant N-1\ ,\ T_N'(\zeta_j) = 0$.

We recall also that, as a consequence of (V.2), the bilinear form

(V.4) $\quad \forall\, (\varphi,\psi) \in [\mathcal{C}^0([-1,1])]^2,\ (\varphi,\psi)_{\varrho,N} = \sum_{j=0}^{N} \varphi(\zeta_j)\,\psi(\zeta_j)\,\varrho_j$,

is indeed a scalar product on $P_N(-1,1)$ and the associated norm is uniformly equivalent to the norm $\|.\|_{0,\varrho}$ since we have (see [CQ1, §3])

(V.5) $\quad \forall\, \varphi \in P_N(-1,1),\ \|\varphi\|_{0,\varrho} \leqslant (\varphi,\varphi)_{\varrho,N}^{1/2} \leqslant \sqrt{2}\,\|\varphi\|_{0,\varrho}$.

Finally, for each function φ in $\mathcal{C}^0(\bar{\Omega})$, we shall denote by $\boldsymbol{I}_N\varphi$ the unique polynomial in $P_N(\Omega)$ which interpolates φ at the nodes defined in (V.1), i.e., such that

(V.6) $\quad \forall\, j,\ 0 \leqslant j \leqslant N,\ \ \boldsymbol{I}_N\varphi(\zeta_j) = \varphi(\zeta_j)$.

Next we introduce the cartesian product of the points defined in (V.1)

(V.7) $\quad \Xi_N = \{\, \mathbf{x} = (\zeta_j\,,\zeta_k),\ 0 \leqslant j,k \leqslant N \,\}$, as well as the corresponding weights

(V.8) $\quad \forall\, \mathbf{x} = (\zeta_j\,,\zeta_k) \in \Xi_N\,, \quad \omega_{\mathbf{x}} = \varrho_j\, \varrho_k$.

The bilinear form

(V.9) $\quad \forall\, (\varphi,\psi) \in [\mathcal{C}^0(\overline{\Omega})]^2, \quad (\varphi,\psi)_{\omega,N} = \sum_{\mathbf{x} \in \Xi_N} \varphi(\mathbf{x})\, \psi(\mathbf{x})\, \omega_{\mathbf{x}}$,

is an inner product on $P_N(\Omega)$. The associated norm, which we denote by $\|.\|_{\omega,N}$, is uniformly equivalent to the norm $\|.\|_{0,\omega}$ (see [CQ1, §3]), more precisely we have,

(V.10) $\quad \forall\, \varphi \in P_N(\Omega), \quad \|\varphi\|_{0,\omega} \leqslant \|\varphi\|_{\omega,N} \leqslant 2\, \|\varphi\|_{0,\omega}$.

Finally, for each function f in $\mathcal{C}^0(\overline{\Omega})$, we shall denote by $\mathcal{J}_N\varphi$ the unique polynomial in $P_N(\Omega)$ which interpolates φ at the nodes defined in (V.7), i.e.,

(V.11) $\quad \forall\, x \in \Xi_N\,, \quad \mathcal{J}_N\varphi(x) = \varphi(x)$.

Note that we have

(V.12) $\quad \forall\, \psi \in P_N(\Omega), \quad (\varphi - \mathcal{J}_N\varphi, \psi)_{\omega,N} = 0$.

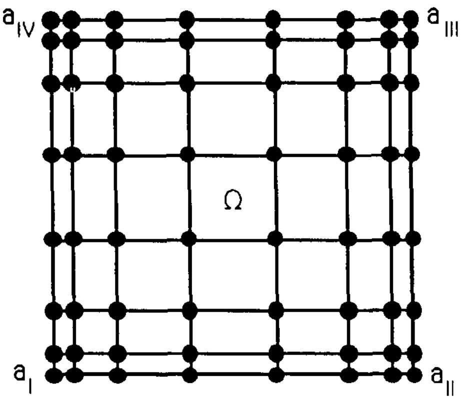

Figure V.1

The set Ξ_N of collocation nodes (for N = 7).

Let us now introduce the collocation approximation to problem (III.1), first in the homogeneous case (III.2). We look for the approximate velocity $\mathbf{u}^N$ in the space $X_N = [P_N^0\,(\Omega)]^2$ and for the approximate pressure p^N in $P_N(\Omega)$. We assume here that $\mathbf{f}$ belongs to $[\mathcal{C}^0(\overline{\Omega})]^2$. We consider the following problem : *Find* $(\mathbf{u}^N,p^N)$ *in* $X_N \times P_N(\Omega)$ *such that*

(V.13) $$\left|\begin{array}{l} -\,\nu\, \Delta\mathbf{u}^N(\mathbf{x}) + (\mathrm{grad}\; p^N)(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Xi_N \cap \Omega \quad , \\ (\mathrm{div}\; \mathbf{u}^N)(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \Xi_N \quad . \end{array}\right.$$

In order to discuss the well-posedness of this problem, as well as its convergence properties, we now give a variational formulation of (V.13) which fits into the abstract scheme (II.17). Thus let us introduce the following bilinear forms $a_N : X_N \times X_N \to \mathbb{R}$ and b_{iN} : $X_N \times P_N(\overline{\Omega}) \to \mathbb{R}$ $(i = 1, 2)$ defined by

(V.14) $\quad a_N(\mathbf{u},\mathbf{v}) = -\nu\,(\Delta\mathbf{u},\mathbf{v})_{\omega,N}$,

(V.15) $\quad b_{1N}(\mathbf{v},q) = (\mathbf{v},\mathbf{grad}\,q)_{\omega,N}$,

and

(V.16) $\quad b_{2N}(\mathbf{v},q) = -(\mathrm{div}\,\mathbf{v},q)_{\omega,N}$.

Let us note that, if q belongs to $P_N(\Omega)$, **grad** q belongs to $[P_{N-1}(-1,1)\otimes P_N(-1,1)] \times [P_N(-1,1)\otimes P_{N-1}(-1,1)]$. Hence, from (V.2) we deduce that for any $\mathbf{v}=(v,w)$ in X_N

$$b_{1N}(\mathbf{v},q) = \int_{-1}^{1} \varrho(x)\,dx \sum_{k=0}^{N} (\partial q/\partial x)(x,\zeta_k)\,v(x,\zeta_k)\,\varrho_k + \sum_{j=0}^{N} \varrho_j \int_{-1}^{1} (\partial q/\partial y)(\zeta_j,y)\,w(\zeta_j,y)\,\varrho(y)\,dy \ .$$

Let us integrate by parts the first term in the x-direction and the second one in the y-direction. Noting that $\omega^{-1}\,\mathrm{div}(\mathbf{v}\omega)$ belongs to $[P_{N-1}(-1,1)\otimes P_N(-1,1)] + [P_N(-1,1)\otimes P_{N-1}(-1,1)]$ and recalling (V.2) we obtain

(V.17) $\quad b_{1N}(\mathbf{v},q) = (\omega^{-1}\,\mathrm{div}(\mathbf{v}\omega),q)_{\omega,N}$.

Using the same argument we also have

(V.18) $\quad a_N(\mathbf{u},\mathbf{v}) = \nu\,(\mathbf{grad}\,\mathbf{u}, \omega^{-1}\,\mathbf{grad}\,(\mathbf{v}\omega))_{\omega,N}$.

Thus the bilinear forms a_N and b_{iN} $(i = 1, 2)$ are discrete approximations of the forms a and b_i $(i = 1, 2)$ defined in (III.14) to (III.16).

<u>Proposition V.1</u> : *Problem* (V.13) *is equivalent to the following variational one : Find* $(\mathbf{u}^N,p^N)$ *in* $X_N \times P_N(\Omega)$ *such that*

(V.19)
$$\begin{cases} \forall\,\mathbf{v}\in X_N, & a_N(\mathbf{u}^N,\mathbf{v}) + b_{1N}(\mathbf{v},p^N) = (\mathbf{f},\mathbf{v})_{\omega,N} \ , \\ \forall\,q\in P_N(\Omega), & b_{2N}(\mathbf{u}^N,q) = 0 \ . \end{cases}$$

Proof : The first equation in (V.19) is obtained by taking the inner product in $\mathbb{R}^2$ of the first equation in (V.13) by $\mathbf{v}(\mathbf{x})\,\omega_{\mathbf{x}}$, for $\mathbf{x}$ in $\Xi_N \cap \Omega$, and summing up over the points of Ξ_N (let us recall that, since $\mathbf{v}$ belongs to X_N, it is equal to $\mathbf{0}$ at any point of Ξ_N of the boundary of Ω). Similarly, the second equation in (V.19) is obtained by multiplying the second equation in (V.13) by $q(\mathbf{x})\,\omega_{\mathbf{x}}$ and summing up over all $\mathbf{x}$ in Ξ_N. Conversely (V.13) follows from (V.19) using as test functions the Lagrange basis in X_N and $P_N(\Omega)$ associated to the set of points Ξ_N.

In analogy to the Galerkin approximation, one expects that the spaces of trial and test pressures have to be restricted to proper subspaces M_{1N} and M_{2N} of $P_N(\Omega)$ so that the bilinear forms a_N and b_{iN} $(i = 1, 2)$ satisfy the inf-sup conditions (II.19), (II.20) and $(II.22)_i$. Then, we obtain another particular case of the abstract approximate problem (II.17), if we set, as before, $X_{1\delta} = X_{2\delta} \square X_N$ provided with the norm of X, $M_{1\delta} = M_{1N}$ and $M_{2\delta} = M_{2N}$ provided with the norm $\|.\|_{0,\omega}$, and if the forms a_δ and $b_{i\delta}$ $(i = 1, 2)$ are respectively the forms a_N and b_{iN} $(i = 1, 2)$ defined in (V.14), (V.15) and (V.16).

Hereafter we shall characterize the spurious modes for the pressure and we shall indicate again one choice of the spaces M_{1N} and M_{2N} which leads to a "spectral" rate of convergence for both the velocity field and the pressure.

V.2. The spurious modes of the pressure.

The characterization of the parasitic modes for the discrete pressure can be carried out by suitably modifying the proofs given in Section IV.2 in the case of the Galerkin approximation. However, we prefer to follow a different strategy, which consists in transferring the results therein obtained into the context of a collocation approximation. To this end, let us define the operators $R_N : P_N(-1,1) \to P_N(-1,1)$ by

(V.20) $\quad \forall \psi \in P_N(-1,1), \quad (R_N\varphi,\psi)_{\varrho,N} = (\varphi,\psi)_\varrho \quad ,$

then $S_N : P_N(\Omega) \to P_N(\Omega)$ by

(V.21) $\quad \forall \psi \in P_N(\Omega), \quad (S_N\varphi,\psi)_{\omega,N} = (\varphi,\psi)_\omega \quad .$

By (V.5) and (V.10), both R_N and S_N are isomorphisms, the norms of which can be bounded independently of N if we endow $P_N(-1,1)$ and $P_N(\Omega)$ respectively with the norms $\|.\|_{0,\varrho}$ and $\|.\|_{0,\omega}$.

Now, let Z_{iN} denote the subspace of all q in $P_N(\Omega)$ for which we have

$(V.22)_i \quad \forall \mathbf{v} \in X_N, \quad b_{iN}(\mathbf{v},q) = 0 \quad .$

In order to characterize the spaces Z_{iN}, we introduce the polynomials r_0 and r_N of $P_N(-1,1)$ satisfying

(V.23) $\quad \forall j, 0 \leqslant j \leqslant N \ , \quad r_0(\zeta_j) = \delta_{0j} \quad$ and $\quad r_N(\zeta_j) = \delta_{Nj} \quad ,$

and a polynomial

(V.24) $\quad q_N^* \square R_N q_N \quad ,$

where q_N is defined in (IV.16). It is an easy matter to check that q_N^* satisfies the relation

(V.25) $\quad \forall \varphi \in P_N(-1,1), \quad (q_N^*,\varphi)_{\varrho,N} = \int_{-1}^{1} \varphi(\zeta)\, d\zeta \quad .$

In analogy to Proposition IV.1, we can state the following result.

<u>Proposition V.2</u> : *The subset* Z_{1N} *is the vector space of dimension* 8 *spanned by* $\{ T_0 , T_N \}^{\otimes 2}$ *and by* $\{ r_0 , r_N \}^{\otimes 2}$.

Proof : Let q in P_N satisfy $(V.22)_1$. Then $S_N^{-1}q$ is such that

$$\forall \mathbf{v} \in X_N , \quad b_1(\mathbf{v}, S_N^{-1} q) = 0 \quad ,$$

hence, by proposition IV.1, q belongs to the space spanned by $\{ R_N T_0 , R_N T_N \}^{\otimes 2}$ and $\{ R_N T_N' , R_N T_{N+1}' \}^{\otimes 2}$. By (V.2) we have $R_N T_0 = T_0$ and $R_N T_N' = T_N'$, while a direct computation shows that (see e.g. [CQ1, §3])

(V.26) $\quad (T_N , T_N)_{\varrho,N} = 2 (T_N , T_N)_\varrho \quad ,$

so that $R_N T_N = 2 T_N$. Let us check that $R_N T_{N+1}'(\zeta) = ((N+1)/N) \zeta T_N'(\zeta)$. Differentiating (IV.3) and using (IV.4) yields

(V.27) $\quad T_{N+1}'(\zeta) = ((N+1)/N) (\zeta T_N'(\zeta) + N T_N(\zeta)) \quad .$

Thus the relation

$$((N+1)/N) (\zeta T_N' , \psi)_{\varrho,N} = (T_{N+1}' , \psi)_\varrho \quad ,$$

holds for all ψ in $P_{N-1}(-1,1)$, as a consequence of (V.2) and (IV.1). For $\psi = T_N$, the identity follows from (IV.4), (V.2), (V.26) and (V.27).

Hence we have proven that $Z_{1N} = \text{Span}\,[\{ T_0 , T_N \}^{\otimes 2} \cup \{ T_N' , \zeta T_N' \}^{\otimes 2}]$. Finally we easily derive from (IV.2) and (V.3) that

(V.28) $\quad r_0(\zeta) = (-1)^N ((1+\zeta)/2N^2) T_N'(\zeta) \quad$ and $\quad r_N(\zeta) = ((1-\zeta)/2N^2) T_N'(\zeta) \quad ,$

which ends the proof.

Following the same lines we can prove the analogous result to Proposition IV.2.

<u>Proposition V.3</u> : *The subspace* Z_{2N} *is the vector space of dimension* 8 *spanned by* $\{ q_N^* , T_N \}^{\otimes 2}$ *and by* $\{ r_0 , r_N \}^{\otimes 2}$.

	$i = 1$	$i = 2$
Form b_i	$\{T_0, T_N\}^{\otimes 2} \cup \{T'_N, T'_{N+1}\}^{\otimes 2}$	$\{q_N, T_N\}^{\otimes 2} \cup \{T'_N, T'_{N+1}\}^{\otimes 2}$
Form b_{iN}	$\{T_0, T_N\}^{\otimes 2} \cup \{r_0, r_N\}^{\otimes 2}$	$\{q_N^*, T_N\}^{\otimes 2} \cup \{r_0, r_N\}^{\otimes 2}$

Figure V.2

The parasitic modes for the forms b_i and b_{iN} ($i = 1, 2$).

V.3. An inf-sup condition for the forms b_{iN} ($i = 1, 2$).

In order to satisfy the discrete inf-sup condition $(II.22)_i$ for the forms b_{iN} ($i=1, 2$), according to Propositions V.2 and V.3, we can choose as spaces M_{iN} ($i = 1, 2$) any supplementary space to Z_{iN} in $P_N(\Omega)$, i.e. any subspace of $P_N(\Omega)$ such that

$$(V.29) \quad \left| \begin{array}{l} \operatorname{codim} M_{iN} = 8 \quad , \\ M_{iN} \cap Z_{iN} = \{0\} \quad . \end{array} \right.$$

If this condition is fulfilled, the existence of a constant β_{iN} for which $(II.22)_i$ holds is ensured by the finite dimension of the spaces M_{iN} . As in the previous case of approximation by Galerkin spectral method, the choice

$$(V.30)_i \quad M_{iN} = \{q \in P_N(\Omega) \; ; \; \forall\, r \in Z_{iN} \, , \, (q,r)_{\omega,N} = 0 \} \, ,$$

leads to a minimal constant and is useful to prepare a better choice. Alternative characterizations for $(V.30)_i$ follows from Propositions V.2 and V.3 : we have as in the Galerkin approximation (see (IV.19) and (IV.20))

$$(V.31) \quad \left| \begin{array}{l} M_{1N} = \{ \omega^{-1} \operatorname{div} (\mathbf{v}\omega), \mathbf{v} \in X_N \} \\ \quad = \{ q \in P_N(\Omega) \; ; \; \forall\, r \in \{T_0, T_N\}^{\otimes 2}, \; (q,r)_\omega = 0 \text{ and } q(\pm 1, \pm 1) = 0 \} \quad , \end{array} \right.$$

and

$$(V.32) \quad \left| \begin{array}{l} M_{2N} = \{ \operatorname{div} \mathbf{v}, \mathbf{v} \in X_N \} \\ \quad = \{ q \in P_N(\Omega) \; ; \; \forall\, r \in \{q_N, T_N\}^{\otimes 2}, \; (q,r)_\omega = 0 \text{ and } q(\pm 1, \pm 1) = 0 \} \quad . \end{array} \right.$$

We are now able to precise the constant β_{iN} for this choice of spaces M_{iN} ($i = 1, 2$).

Proposition V.4 : *Let the space* M_{iN} *be defined by* $(V.30)_i$. *There exists a constant* $\tilde{\beta}_i > 0$ *independent of* N *such that*

$$(V.33)_i \quad \forall\, q \in M_{iN} \; , \quad \sup_{\mathbf{v} \in X_N} \frac{b_{iN}(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}} \geq \tilde{\beta}_i N^{-2} \|q\|_{0,\omega} \quad .$$

Proof : Consider the case $i = 1$ first. We know from the proof of Proposition IV.3 that, for each q in M_{1N}, there exists $\mathbf{v}$ in X_N such that $-\omega^{-1}\operatorname{div}(\mathbf{v}\omega) = q$ and $\|\mathbf{v}\|_{1,\omega} \leqslant c\,N^2\,\|q\|_{0,\omega}$ (see (IV.30) and (IV.31)). Since the discrete inner product induces a norm uniformly equivalent to the norm $\|.\|_{0,\omega}$ on $P_N(\Omega)$ (see (V.10)), we have

$$b_{1N}(\mathbf{v},q) = (q,q)_{\omega,N} \geqslant c\,\|q\|^2_{0,\omega} \quad ,$$

whence $(V.33)_1$ holds. The case $i = 2$ follows similarly using now the proof of Proposition IV.4 .

As established in Section IV.6, the choice $(V.30)_i$ is not well suited to approximate the space of pressures. Here again an alternative choice $\tilde{M}_{1N}$ for the pressure space can be given. In order to ensure the exact pressure in M_1 to be approximated within spectral accuracy by an element in $\tilde{M}_{1N}$, we require that there exists a real number λ, $0 < \lambda < 1$, such that

(V.34) $\quad P_{[\lambda N]}(\Omega) \cap M_1 \subset \tilde{M}_{1N} \quad .$

Furthermore, in order to retain the compatibility between $\tilde{M}_{1N}$ and X_N, we also require that the orthogonal projection $\pi^*_N : \tilde{M}_{1N} \to M_{1N}$ onto M_{1N} with respect to the inner product $(.,.)_{\omega,N}$, satisfies

(V.35) $\quad \forall\, q \in \tilde{M}_{1N}\,,\quad \|q\|_{0,\omega} \leqslant c\,\|\pi^*_N q\|_{0,\omega} \quad .$

This yields, as in Section IV, the following inf-sup condition :

<u>Proposition V.5</u> : *Let* $\tilde{M}_{1N}$ *be a subspace of* $P_N(\Omega)$ *such that hypothesis* (V.35) *holds. There exists a constant* $\tilde{\beta}_1 > 0$ *independent of* N *such that*

(V.36) $$\forall\, q \in \tilde{M}_{1N}\,,\quad \sup_{\mathbf{v} \in X_N} \frac{b_{1N}(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}} \geqslant \tilde{\beta}_1\, N^{-2}\,\|q\|_{0,\omega} \quad .$$

We refer to Section IV to the proof of Proposition V.5 and for the discussion of the existence of spaces $\tilde{M}_{1N}$ satisfying the hypotheses (V.34) and (V.35); indeed the space $\tilde{M}_{1N}$ defined by (IV.49) and (IV.61) works (see also Section V.6 for some considerations on the implementation of the method).

<u>V.4.The inf-sup condition for the form a_N.</u>

Let us assume here that M_{iN} $(i = 1, 2)$ is any supplementary space in $P_N(\Omega)$ of the space Z_{iN} defined in $(V.22)_i$, i.e. satisfies (V.29). Setting

(V.37) $\quad K_{iN} = \{\, \mathbf{v} \in X_N \;;\; \forall\, q \in M_{iN}\,,\; b_{iN}(\mathbf{v},q) = 0 \,\} \quad ,$

by the definition of Z_{iN}, we actually have for all $\mathbf{v}$ in K_{iN}

$$\forall q \in P_N(\Omega), \quad b_{iN}(\mathbf{v},q) = 0 \quad ,$$

thus, as in the Galerkin method,

(V.38) $\quad K_{iN} = K_i \cap X_N \quad (i = 1, 2)$.

It is readily seen that in this case we have also $\dim K_{1N} = \dim K_{2N}$. Thus, by Remark II.1, it is enough to check the inf-sup condition (II.19) for the form a_N over $K_{2N} \times K_{1N}$.

Proposition V.6 : *There exists a constant* $\tilde{\alpha} > 0$ *independent of* N *such that*

(V.39) $$\forall \mathbf{u} \in K_{2N}, \quad \sup_{\mathbf{v} \in K_{1N}} \frac{a_N(\mathbf{u},\mathbf{v})}{\|\mathbf{v}\|_{1,\omega}} \geq \tilde{\alpha} \|\mathbf{u}\|_{1,\omega} .$$

Proof : Let $\mathbf{u} = (u,v)$ be an element of K_{2N}. As in the proof of Proposition IV.5, we deduce from (V.38) that there exists φ in $P_N(\Omega) \cap H^2_{\omega,0}(\Omega)$ such that

(V.40) $\quad \mathbf{u} = \operatorname{curl} \varphi$.

Let us set again $\mathbf{v} = \omega^{-1} \operatorname{curl} (\varphi\omega)$ and recall (see the proof of Proposition III.2) that

(V.41) $\quad \|\mathbf{v}\|_{1,\omega} \leq c \|\mathbf{u}\|_{1,\omega}$.

Setting now, as in the continuous case, $\chi = \Delta\varphi$, we have by the definition (V.14) of the form a_N

$$a_N(\mathbf{u},\mathbf{v}) = - \nu (\Delta\mathbf{u},\mathbf{v})_{\omega,N} = - \nu (\operatorname{curl} \chi, \omega^{-1} \operatorname{curl} (\varphi\omega))_{\omega,N}$$

or more explicitely

(V.42) $\quad a_N(\mathbf{u},\mathbf{v}) = - \nu (\partial\chi/\partial x, \omega^{-1} \partial(\varphi\omega)/\partial x)_{\omega,N} - (\partial\chi/\partial y, \omega^{-1} \partial(\varphi\omega)/\partial y)_{\omega,N}$.

On the other hand, recalling that the Chebyshev weight ω is the product of the Chebyshev weights in each direction, i.e. $\omega(\mathbf{x}) = \varrho(x)\varrho(y)$, we have

$$- (\partial\chi/\partial x, \omega^{-1} \partial(\varphi\omega)/\partial x)_{\omega,N}$$
$$= - \textstyle\sum_{k=0}^{N} \varrho_k \{ \sum_{j=0}^{N} (\partial\chi/\partial x)(\zeta_j,\zeta_k)[(\partial\varphi/\partial x)(\zeta_j,\zeta_k) + \varphi(\zeta_j,\zeta_k)\, \varrho'(\zeta_j)/\varrho(\zeta_j)]\, \varrho_j \} .$$

By (V.2) and (V.3) it follows that

$$\begin{aligned} - (\partial\chi/\partial x, \omega^{-1} \partial(\varphi\omega)/\partial x)_{\omega,N} &= - \textstyle\sum_{k=0}^{N} \varrho_k \left[\int_{-1}^{1} (\partial\chi/\partial x)(x,\zeta_k)\, (\partial(\varphi\varrho)/\partial x)(x,\zeta_k)\, dx\right] , \\ &= \textstyle\sum_{k=0}^{N} \varrho_k \left[\int_{-1}^{1} (\partial^2\chi/\partial x^2)(x,\zeta_k)\, \varphi(x,\zeta_k)\, \varrho(x)\, dx\right] , \\ &= \textstyle\sum_{j=0}^{N} \sum_{k=0}^{N} (\partial^2\chi/\partial x^2)(\zeta_j,\zeta_k)\, \varphi(\zeta_j,\zeta_k)\, \varrho_j\varrho_k] . \end{aligned}$$

The term in (V.42) containing the y-derivative can be handled similarly. Thus, we have proved that

(V.43) $\quad a_N(\mathbf{u},\mathbf{v}) = (\Delta^2\varphi,\varphi)_{\omega,N}$.

By [MM, Lemma 3.2], there exists a constant $c > 0$ independent of N such that

$$\forall \varphi \in P_N(\Omega) \cap H^2_{\omega,0}(\Omega), \quad (\Delta^2\varphi,\varphi)_{\omega,N} \geq c \|\varphi\|^2_{2,\omega} .$$

Then, the proposition follows from (V.40), (V.41) and (V.43).

V.5. A convergence estimate in the homogeneous case.

In this section we consider the collocation approximation (V.13) to the homogeneous Stokes problem (III.1)(III.2). Let us recall that $X_N = [P_N^0(\Omega)]^2$.

For an appropriate choice of the spaces of pressures and of test functions, we know from Propositions V.6, V.4 and V.5 that the bilinear forms a_N, b_{1N} and b_{2N} restricted to these finite dimensional subspaces satisfy the inf-sup conditions (II.19), (II.21) and $(II.22)_i$ $(i = 1, 2)$ for suitable constants. Moreover, it follows from (V.16), (V.17), (V.18) and (V.10) that a_N and b_{iN} $(i = 1, 2)$ are uniformly continuous over $X_N \times X_N$ and $X_N \times M_{iN}$ respectively.

Corollary II.2 yields the following stability result.

Theorem V.1 : *For each integer* $N \geqslant 3$, *the collocation approximation* (V.13) *to the Stokes problem* (III.1)(III.2) *has a unique solution* $(\mathbf{u}^N, p^N)$ *in* $X_N \times M_{1N}$, *where* M_{1N} *is defined by* $(V.30)_1$ *(resp. in* $X_N \times \tilde{M}_{1N}$, *where* $\tilde{M}_{1N}$ *satisfies the hypothesis* (V.35)*). Moreover, the following inequality is satisfied*

(V.44) $\quad \|\mathbf{u}^N\|_{1,\omega} + N^{-2}\|p^N\|_{0,\omega} \leqslant c\,\|\mathbf{f}\|_{[C^\circ(\bar{\Omega})]^2}$,

for a constant $c > 0$ *independent of* N.

Remark V.1 : In this case as for the Galerkin method (see Remark IV.6), the discrete velocity is independent of the space of pressures.

Let us now consider the convergence of the approximation.

Theorem V.2 : *Assume that the solution* $(\mathbf{u}, p)$ *of the Stokes problem* (III.1)(III.2) *belongs to* $[H^s_\omega(\Omega)]^2 \times H^{s-1}_\omega(\Omega)$ *for a real number* $s \geqslant 1$, *and the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma > 1$ *. Then the approximate velocity* $\mathbf{u}^N$*, as defined in Theorem* V.1 , *satisfies the convergence estimate*

(V.45) $\quad \|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} \leqslant c\,(N^{1-s}\|\mathbf{u}\|_{s,\omega} + N^{-\sigma}\|\mathbf{f}\|_{\sigma,\omega})$

for a constant c *independent of* N.

Proof : Let us first remark that

(V.46) $\quad \forall\, \mathbf{v} \in [\overset{\circ}{P}_{N-1}(\Omega)]^2, \forall\, \mathbf{z} \in X_N , \quad (a-a_N)(\mathbf{v}, \mathbf{z}) = 0 \quad ;$

indeed, the product $\mathbf{v}\,\mathbf{z}$ is an element of $P_{2N-1}(\Omega)$ and the discrete integration formula in the definition (V.14) of a_N is exact (see (V.2)). By (V.38), we can apply Corollary II.3 with $\mathbf{v}_\delta = \mathbf{w}_\delta = Q_{N-1}\mathbf{u}$ (the divergence-free polynomial approximation to $\mathbf{u}$, the existence of which is guaranteed by Lemma IV.6) to get the following error estimate for the velocity :

(V.47) $\quad \|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} \leq c\,(\|\mathbf{u}- Q_{N-1}\mathbf{u}\|_{1,\omega} + \sup\limits_{\mathbf{z}\in X_N} \dfrac{(\mathbf{f},\mathbf{z})_\omega - (\mathbf{f},\mathbf{z})_{\omega,N}}{\|\mathbf{z}\|_{1,\omega}})$.

Note that the term $a-a_N$ has disappeared in this estimate due to (V.46).

Due to Lemma IV.6, it is sufficient to bound the error on the data $\mathbf{f}$. We have for any $\mathbf{z}$ in X_N

$$|(\mathbf{f},\mathbf{z})_\omega - (\mathbf{f},\mathbf{z})_{\omega,N}| \leq |(\mathbf{f},\mathbf{z})_\omega - (\Pi^0_{N-1}\mathbf{f},\mathbf{z})_\omega| + |(\Pi^0_{N-1}\mathbf{f},\mathbf{z})_{\omega,N} - (\mathcal{J}_N\mathbf{f},\mathbf{z})_{\omega,N}| \quad ,$$

where Π^0_{N-1} is the orthogonal projection operator onto $[P_{N-1}(\Omega)]^2$ with respect to $(.,.)_\omega$, and $\mathcal{J}_N$ is the interpolation operator at the collocation nodes defined in (V.7). Hence we get by (V.10)

(V.48) $\quad \forall\, \mathbf{z}\in X_N\ ,\quad |(\mathbf{f},\mathbf{z})_\omega - (\mathbf{f},\mathbf{z})_{\omega,N}| \leq (\|\mathbf{f}-\Pi^0_{N-1}\mathbf{f}\|_{0,\omega} + \|\mathbf{f}-\mathcal{J}_N\mathbf{f}\|_{0,\omega})\, \|\mathbf{z}\|_{0,\omega}$.

The first term on the right-hand side can be estimated by (IV.46), while the interpolation operator satisfies the following inequality ([CQ1 , Thm 3.1]), valid for any real numbers r and s, $s > 1$ and $0 \leq r \leq s$,

(V.49) $\quad \forall\, \varphi \in H^s_\omega(\Omega),\quad \|\varphi-\mathcal{J}_N\varphi\|_{r,\omega} \leq c\, N^{2r-s}\, \|\varphi\|_{s,\omega}$.

This ends the proof of the theorem.

<u>Remark V.2</u> : Estimate (V.45) is optimal with respect to the regularity of the solution and of the data.

Let us turn now to the error estimate for the pressure.

<u>Theorem V.3</u> : *Assume that hypotheses* (V.34) *and* (V.35) *hold and that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (III.1)(III.2) *belongs to* $[H^s_\omega(\Omega)]^2 \times H^{s-1}_\omega(\Omega)$ *for a real number* $s \geq 1$ *, and the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma > 1$*. Then the approximate pressure* p^N *in* $\tilde{M}_{1N}$ *, as defined in Theorem* V.1 *, satisfies the convergence estimate*

(V.50) $\quad \|p-p^N\|_{0,\omega} \leq c\,\{ N^{3-s}\,(\|\mathbf{u}\|_{s,\omega} + \|p\|_{s-1,\omega}) + N^{2-\sigma}\, \|\mathbf{f}\|_{\sigma,\omega} \}$

for a constant c *independent of* N.

Proof : We use Theorem II.3 and, thanks to (V.46), we have

$$\|p-p^N\|_{0,\omega} \leqslant c\,N^2 \Big(\|\mathbf{u}-Q_{N-1}\mathbf{u}\|_{1,\omega} + \inf_{q_N \in \tilde{M}_{1N}} \Big\{ \|p-q_N\|_{0,\omega} + \sup_{\mathbf{z} \in X_N} \frac{(b_1-b_{1N})(\mathbf{z},q_N)}{\|\mathbf{z}\|_{1,\omega}} \Big\} + \sup_{\mathbf{z} \in X_N} \frac{(\mathbf{f},\mathbf{z})_\omega - (\mathbf{f},\mathbf{z})_{\omega,N}}{\|\mathbf{z}\|_{1,\omega}} \Big) .$$

Due to Lemma IV.6 and to (V.48), we only have to estimate the terms on the right-hand side concerning the pressure. Let us recall that the space $\tilde{M}_{1N}$ we have chosen satisfies (V.34) for a fixed $\lambda < 1$. Thus, we can take $q_N = \Pi^0_{[\lambda N]}p$. The truncation error is again estimated by (IV.46). Moreover, using the exactness (V.2) of the quadrature formula as described in the previous proof, we have

$$\forall\, \mathbf{z} \in X_N , \quad (b_1-b_{1N})(\mathbf{z},\Pi^0_{[\lambda N]}p) = 0 ,$$

hence the result.

<u>Remark V.3</u> : Let us for a while consider the problem : *Find* $(\mathbf{u}^N,p^N)$ *in* $X_N \times M_{1N}$ *(resp. in* $X_N \times \tilde{M}_{1N}$*) such that*

$$\text{(V.51)} \quad \begin{cases} \forall\, \mathbf{v} \in X_N , & a_N(\mathbf{u}^N,\mathbf{v}) + b_{1N}(\mathbf{v},p^N) = (\mathbf{f},\mathbf{v})_\omega , \\ \forall\, q \in P_N(\Omega), & b_{2N}(\mathbf{u}_N,q) = 0 \end{cases}$$

(which is problem (V.19) with $(\mathbf{f},\mathbf{v})_{\omega,N}$ replaced by $(\mathbf{f},\mathbf{v})_\omega$ or equivalently with $S_N\mathbf{f}$ replaced by $\mathbf{f}$). Then, Theorems V.1 to V.3 are still valid. Furthermore, since the second term in the right-hand side of (V.47) disappears, the estimates (V.45) and (V.50) can be replaced respectively by

$$\text{(V.52)} \quad \|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} \leqslant c\,N^{1-s}\,\|\mathbf{u}\|_{s,\omega} ,$$

$$\text{(V.53)} \quad \|p-p^N\|_{0,\omega} \leqslant c\,N^{3-s}\,(\,\|\mathbf{u}\|_{s,\omega} + \|p\|_{s-1,\omega}\,) .$$

This will be used in Section VI.

<u>V.6. Concluding remarks in the homogeneous case.</u>

We complete Section V.5 with some considerations on the practical implementation of the Chebyshev collocation scheme (V.13). As a matter of fact, we have indicated a pair of spaces M_{1N} and M_{2N} for which the collocation scheme is well-posed and guarantees spectral accuracy. Now, it remains to exhibit a precise set of algebraic equations, as well as of unknowns for the pressure, which correspond to the scheme and which are efficiently implementable.

To this end, let S_c denote the set of the four corners of the square Ω, and let $\hat{S}$ be a set of four collocation points in $\Xi_N \setminus S_c$ satisfying the following property :

$$\text{(V.54)} \quad \det(\, q_L(\mathbf{x}_J)\,) \neq 0 , \quad 1 \leqslant J, L \leqslant 4 ,$$

where $\mathbf{x}_J$ runs through $\hat{S}$ and q_L runs through $\{ q_N^*, T_N \}^{\otimes 2}$ (the polynomial q_N^* is defined in (V.25)).

Proposition V.7 : *Assume that hypothesis* (V.54) *holds. To apply the collocation scheme* (V.13) *is equivalent to solve the following linear system : Find* $(\mathbf{u}^N, p^N)$ *in* $X_N \times M_{1N}$, *where* M_{1N} *is defined by* $(V.30)_1$ *(or in* $X_N \times \tilde{M}_{1N}$, *where* $\tilde{M}_{1N}$ *satisfies the hypothesis* (V.35)*) such that*

$$\text{(V.55)} \quad \left| \begin{array}{l} -\nu\, \Delta \mathbf{u}^N(\mathbf{x}) + (\mathrm{grad}\, p^N)(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Xi_N \cap \Omega \quad , \\ (\mathrm{div}\, \mathbf{u}^N)(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \Xi_N \setminus \{ S_c \cup \hat{S} \} \quad . \end{array} \right.$$

Proof : Of course, any solution of (V.13) is a solution of (V.55). Conversely, let $(\mathbf{u}^N, p^N)$ satisfy (V.55). Since $\mathbf{u}^N$ vanishes on $\partial\Omega$, div $\mathbf{u}^N$ is equal to 0 at the four corners of Ω. Moreover, by Proposition V.3 , we know that $b_{2N}(\mathbf{u}^N, q)$ is equal to 0 for all q in $\{ q_N^*, T_N \}^{\otimes 2}$. By (V.55) this relation becomes

$$\forall\, q \in \{ q_N^*, T_N \}^{\otimes 2}, \quad \sum_{\mathbf{x} \in \hat{S}} (\mathrm{div}\, \mathbf{u}^N)(\mathbf{x})\, q(\mathbf{x})\, \omega_{\mathbf{x}} = 0 \quad .$$

Thanks to (V.54), we obtain $(\mathrm{div}\, \mathbf{u}^N)(\mathbf{x}) = 0$ for any $\mathbf{x} \in \hat{S}$. We conclude that $\mathbf{u}^N$ is divergence-free in Ω, hence $(\mathbf{u}^N, p^N)$ satisfies (V.13) (and (V.19)).

As far as the choice of degrees of freedom for the pressure is concerned, it seems unpractical to find out a subset of collocation points in the domain, which uniquely determine the polynomials of $\tilde{M}_{1N}$ (i.e., which form a unisolvent set for $\tilde{M}_{1N}$). It is more convenient to determine the discrete pressure through the complete set of collocation points in the domain (or, equivalently, to retain all the modes for the pressure). This means that the algebraic system to be actually solved is underspecified. Once a solution of this system is obtained in some way, it will yield a "good" velocity field and a "good" pressure gradient at the collocation points (and only there !). In order to get a "good" pressure, i.e. the pressure satisfying an estimate like (V.50), one has to extract from the computed pressure its component along $\tilde{M}_{1N}$. This can be done by taking the orthogonal projection of the computed pressure onto $\tilde{M}_{1N}$ with respect to $(.,.)_\omega$. Other techniques of filtering the spurious modes have been successfully applied (see, e.g. [Mé]).

V.7. The non homogeneous case.

Let us now consider the approximation of the non homogeneous Stokes problem (III.1)(III.39) by a collocation method. We shall suppose that this problem is well-posed, i.e.

that (III.40) and (III.41) hold.

Hereafter, we assume that the space M_{iN} is defined by $(V.30)_i$ and that the space $\tilde{M}_{1N}$ satisfies (V.35). After the analysis of the homogeneous case, we propose the following formulation of the discrete problem : *Find* $(\mathbf{u}^N, p^N)$ *in* $[P_N(\Omega)]^2 \times M_{1N}$ *(resp. in* $[P_N(\Omega)]^2 \times \tilde{M}_{1N}$*) such that*

$$\text{(V.56)} \quad \begin{cases} \forall\, \mathbf{v} \in X_N, \quad a_N(\mathbf{u}^N, \mathbf{v}) + b_{1N}(\mathbf{v}, p^N) = (\mathbf{f}, \mathbf{v})_{\omega,N} \quad, \\ \forall\, q \in M_{2N}, \quad b_{2N}(\mathbf{u}^N, q) = 0 \quad, \\ \mathbf{u}^N(\mathbf{x}) = \varphi_J(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Xi_N \cap \Gamma_J, \quad J \in \mathbb{Z}/4\mathbb{Z} \quad. \end{cases}$$

<u>Remark V.4</u> : This formulation is not as direct as the formulation (V.13) of the homogeneous case. Indeed, we have in mind to discretize the equations in a collocation way, hence we would like to satisfy each of the equations at a suitable set of points of Ξ_N. More precisely we would like to solve the following problem : *Find* $(\mathbf{u}^N, p^N)$ *in* $[P_N(\Omega)]^2 \times M_{1N}$ *(resp. in* $[P_N(\Omega)]^2 \times \tilde{M}_{1N}$*) such that*

$$\text{(V.57)} \quad \begin{cases} -\nu\, \Delta \mathbf{u}^N(\mathbf{x}) + (\operatorname{grad} p^N)(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Xi_N \cap \Omega \quad, \\ (\operatorname{div} \mathbf{u}^N)(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \Xi_N \quad, \\ \mathbf{u}^N(\mathbf{x}) = \varphi_J(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Xi_N \cap \Gamma_J, \quad J \in \mathbb{Z}/4\mathbb{Z} \quad. \end{cases}$$

The last problem is clearly equivalent to the variational formulation : *Find* $(\mathbf{u}^N, p^N)$ *in* $[P_N(\Omega)]^2 \times M_{1N}$ *(resp. in* $[P_N(\Omega)]^2 \times \tilde{M}_{1N}$*) such that*

$$\text{(V.58)} \quad \begin{cases} \forall\, \mathbf{v} \in X_N, \quad a_N(\mathbf{u}^N, \mathbf{v}) + b_{1N}(\mathbf{v}, p^N) = (\mathbf{f}, \mathbf{v})_{\omega,N} \quad, \\ \forall\, q \in P_N(\Omega), \quad b_{2N}(\mathbf{u}^N, q) = 0 \quad, \\ u^N(\mathbf{x}) = \varphi_J(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Xi_N \cap \Gamma_J, \quad J \in \mathbb{Z}/4\mathbb{Z} \quad. \end{cases}$$

However, it follows from Section V.3 that the relation $b_{2N}(\mathbf{u}^N, q) = 0$ can only be satisfied for all q in M_{2N} and generally not for all q in $P_N(\Omega)$. Indeed, since $\mathbf{u}^N$ is a polynomial, we would derive from (V.58) : $\operatorname{div} \mathbf{u}^N = 0$ exactly. In particular this imposes five conditions for $\mathbf{u}^N$ at the boundary :

$$\text{(V.59)} \quad (\operatorname{div} \mathbf{u}^N)(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \text{ corner of } \Omega \quad,$$

$$\text{(V.60)} \quad \textstyle\sum_{J \in \mathbb{Z}/4\mathbb{Z}} \int_{\Gamma_J} \mathbf{u}^N \cdot \mathbf{n}_J \, d\sigma = 0 \quad.$$

These equations solely depend upon the values of $\mathbf{u}^N$ at the boundary, hence upon $I_N \varphi_J$. In general these relations are not satisfied.

<u>Example</u> : Assume that m is even, $2 \leqslant m \leqslant N$, and let us choose as boundary conditions

$$\text{(V.61)} \quad \forall\, J \in \mathbb{Z}/4\mathbb{Z}, \quad \varphi_J = (\, L_m(\zeta), L_m(\zeta) \,)$$

(note that the Legendre polynomial of degree m has by definition a zero-average). These conditions fulfill the hypotheses (III.40) and (III.41). Thus, if $\mathbf{f}$ is chosen in X', the continuous problem is well-posed. But, condition (V.59) does not hold since $\mathcal{I}_N \varphi_J$ coincide with φ_J for any $J \in \mathbb{Z}/4\mathbb{Z}$ and div $\mathbf{u}^N$ at $\mathbf{a}_l$ for instance is equal to $L_m'(-1) - L_m'(1) = -m(m+1)$. In this example, the continuous problem has a solution in the weak sense only, since the divergence of the velocity is not zero at the four corners. However, even if the exact velocity field is divergence free at the four corners, condition (V.59) need not be satisfied : in fact, the Lagrange interpolation operator will not generally preserve the boundary values of the first derivative.

Hence, problem (V.57) is generally unsolvable and the formulation (V.56) is used instead. Besides, we can check the following proposition.

<u>Proposition V.8</u> : *Any solution* $(\mathbf{u}^N, p^N)$ *of problem* (V.56) *in* $[P_N(\Omega)]^2 \times M_{1N}$ *(resp. in* $[P_N(\Omega)]^2 \times \tilde{M}_{1N}$*) satisfies the collocation equation*

(V.62) $\quad -\nu\, \Delta \mathbf{u}^N(\mathbf{x}) + (\text{grad}\, p^N)(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad$ *for* $\mathbf{x} \in \Xi_N \cap \Omega$.

<u>Remark V.5</u> : Due to (V.32), solving the equation

$$\forall q \in M_{2N}, \quad b_{2N}(\mathbf{u}^N, q) = 0$$

in (V.56) is equivalent to the minimization of $\|\text{div}\, \mathbf{u}^N\|_{\omega,N}$; this condition is implemented in practice. We refer to [Mé] for details and numerical results in the non homogeneous case.

As in the proof of Theorem III.2 we shall need an element in the space of trial functions (i.e. $[P_N(\Omega)]^2$) that satisfies, in a discrete sense, the boundary condition (III.39). To this purpose we recall the following approximation result that can be found in [BM].

<u>Lemma V.1</u> : *There exists an operator* $\Pi_{N,b}$ *from* $H^1_\omega(\Omega)$ *into* $P_N(\Omega)$ *such that, for any function* φ *in* $H^1_\omega(\Omega)$

(V.63) $\quad \forall \mathbf{x} \in \Xi_N \cap \partial\Omega, \quad \Pi_{N,b}\varphi(\mathbf{x}) = \varphi(\mathbf{x}) \quad ,$

(V.64) $\quad \|\varphi - \Pi_{N,b}\varphi\|_{1,\omega} \leq c\, \Big(\inf\limits_{\varphi_N \in P_N(\Omega)} \|\varphi - \varphi_N\|_{1,\omega} + \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\varphi_{|\Gamma_J} - \mathcal{I}_N \varphi_{|\Gamma_J}\|_{3/4,\varrho} \Big) \quad .$

First, we check that the discrete problem (V.56) is well-posed.

<u>Theorem V.4</u> : *For each integer* $N \geq 3$, *the collocation approximation* (V.56) *to the Stokes problem* (III.1)(III.39) *has a unique solution* $(\mathbf{u}^N, p^N)$ *in* $X_N \times M_{1N}$, *where* M_{1N} *is defined by* $(V.30)_1$ *(resp. in* $X_N \times \tilde{M}_{1N}$, *where* $\tilde{M}_{1N}$ *satisfies the hypothesis* (V.35)*). Moreover, the*

following inequality is satisfied

(V.65) $\quad \|\mathbf{u}^N\|_{1,\omega} + N^{-2}\|p^N\|_{0,\omega} \leqslant c\,(\|\mathbf{f}\|_{[C^\circ(\bar\Omega)]^2} + N^{3/4} \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{3/4,\varrho})$,

for a constant $c > 0$ *independent of* N.

Proof : If we define $\tilde{\mathbf{u}}^N = \mathbf{u}^N - \Pi_{N,b}\mathbf{u}$ in X_N , then the pair $(\mathbf{u}^N,p^N)$ is a solution of (V.56) if and only if the pair $(\tilde{\mathbf{u}}^N,p^N)$ satisfies :

$$\text{(V.66)}\quad \left|\begin{array}{l} \forall\, \mathbf{v} \in X_N\,,\quad a_N(\tilde{\mathbf{u}}^N,\mathbf{v}) + b_{1N}(\mathbf{v},p^N) = (\mathbf{f},\mathbf{v})_{\omega,N} - a_N(\Pi_{N,b}\mathbf{u},\mathbf{v}) \quad , \\ \forall\, q \in M_{2N}\,,\quad b_{2N}(\tilde{\mathbf{u}}^N,q) = -\, b_{2N}(\Pi_{N,b}\mathbf{u},q) \quad . \end{array}\right.$$

We derive the result from Theorem V.1 and (V.64), together with the following estimate for the interpolation error ([CQ1, Thm 3.1]), valid for any real numbers r and s, $s > 1/2$ and $0 \leqslant r \leqslant s$,

(V.67) $\quad \forall\, \varphi \in H^s_\varrho(-1,1),\quad \|\varphi - I_N\varphi\|_{r,\varrho} \leqslant c\, N^{2r-s}\|\varphi\|_{s,\varrho}$.

In order to investigate the convergence properties,we shall introduce a slightly different approximation that satisfies condition (V.60). We first define the constant

(V.68) $\quad e_N = (1/8) \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \int_{\Gamma_J} I_N\varphi_J \cdot \mathbf{n}_J \, d\sigma$.

Due to (III.41) and (V.67), if φ_J , $J \in \mathbb{Z}/4\mathbb{Z}$, belongs to $H^\tau_\varrho(\Gamma_J)$ for $\tau > 1/2$, we have

(V.69) $\quad |e_N| \leqslant c\, N^{-\tau} \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho}$.

Let us denote now by $\mathbf{e}_N$ the vector (e_N , e_N) and remark that now $I_N\mathbf{u} - \mathbf{e}_N$ is a boundary condition that satisfies (III.40) and (III.41). Hence we can define the two auxiliary problems

a) *Find* $(\hat{\mathbf{u}},\hat{p})$ *in* $[H^1_\omega(\Omega)]^2 \times M_1$ *such that*

$$\text{(V.70)}\quad \left|\begin{array}{l} \forall\, \mathbf{v} \in X,\quad a(\hat{\mathbf{u}},\mathbf{v}) + b_1(\mathbf{v},\hat{p}) = (\mathbf{f},\mathbf{v})_\omega \quad , \\ \forall\, q \in M_2\,,\quad b_2(\hat{\mathbf{u}},q) = 0 \quad , \\ \hat{\mathbf{u}} = \Pi_{N,b}\mathbf{u} - \mathbf{e}_N = \mathcal{J}_N\mathbf{u} - \mathbf{e}_N \quad \text{on } \partial\Omega \quad ; \end{array}\right.$$

b) *Find* $(\hat{\mathbf{u}}^N,\hat{p}^N)$ *in* $[P_N(\Omega)]^2 \times M_{1N}$ *(resp. in* $[P_N(\Omega)]^2 \times \tilde{M}_{1N}$*) such that*

$$\text{(V.71)}\quad \left|\begin{array}{l} \forall\, \mathbf{v} \in X_N\,,\quad a_N(\hat{\mathbf{u}}^N,\mathbf{v}) + b_{1N}(\mathbf{v},\hat{p}^N) = (\mathbf{f},\mathbf{v})_{\omega,N} \quad , \\ \forall\, q \in M_{2N}\,,\quad b_{2N}(\hat{\mathbf{u}}^N,q) = 0 \quad , \\ \hat{\mathbf{u}}^N(\mathbf{x}) = (\Pi_{N,b}\mathbf{u})(\mathbf{x}) - \mathbf{e}_N = (\mathcal{J}_N\mathbf{u})(\mathbf{x}) - \mathbf{e}_N \quad \text{for } \mathbf{x} \in \Xi_N \cap \partial\Omega \quad . \end{array}\right.$$

The error bound between the solutions of problem (III.1)(III.39) and (V.56) will be obtained by studying the differences between $\mathbf{u}$ and $\hat{\mathbf{u}}$, $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}^N$, $\hat{\mathbf{u}}^N$ and $\mathbf{u}^N$.

<u>Proposition V.9</u> : *Assume that the boundary data* φ_J , $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H^\tau_\varrho(\Gamma_J)$ *for a real number* $\tau \geqslant 3/4$. *The following estimate is satisfied*

(V.72) $\|\mathbf{u}-\hat{\mathbf{u}}\|_{1,\omega} \leqslant c\, N^{3/2-\tau} \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho}$

for a constant $c > 0$ *independent of* N.

Proof : It is a consequence of Theorem III.2 and of the approximation estimates (V.67) and (V.69).

Similarly we can obtain an error bound between $\mathbf{u}^N$ and $\hat{\mathbf{u}}^N$. Indeed, note that $\mathbf{u}^N-\hat{\mathbf{u}}^N$ is constant, equal to $\mathbf{e}_N$. Hence, by(V.69), we obtain the following result.

Proposition V.10 : *Assume that the boundary data* φ_J , $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H^\tau_\varrho(\Gamma_J)$ *for a real number* $\tau \geqslant 3/4$. *The following estimate is satisfied*

(V.73) $\|\mathbf{u}^N-\hat{\mathbf{u}}^N\|_{1,\omega} \leqslant c\, N^{-\tau} \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho}$

for a constant $c > 0$ *independent of* N.

In order to get now an error bound between $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}^N$ we first note that problem (V.71) is a discrete approximation of problem (V.70) . Hence the abstract results of Section II can be applied. We have the following error estimate.

Proposition V.11 : *Assume that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (III.1)(III.39) *belongs to* $[H^s_\omega(\Omega)]^2 \times H^{s-1}(\Omega)$ *for a real number* $s \geqslant 1$, *that the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma > 1$ *and that the boundary data* φ_J, $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H^\tau_\varrho(\Gamma_J)$ *for a real number* $\tau \geqslant 3/4$. *The following estimate is satisfied*

(V.74) $\|\hat{\mathbf{u}}-\hat{\mathbf{u}}^N\|_{1,\omega} \leqslant c\,(N^{3-s}\|\mathbf{u}\|_{s,\omega} + N^{-\sigma}\|\mathbf{f}\|_{\sigma,\omega} + N^{7/2-\tau} \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho})$

for a constant $c > 0$ *independent of* N.

Proof : Let us set $\tilde{\mathbf{u}} = \hat{\mathbf{u}} - \Pi_{N,b}\mathbf{u} + \mathbf{e}_N$. Since $\mathbf{e}_N$ is constant, $(\tilde{\mathbf{u}},\hat{p})$ is the solution in $X \times M_1$ of

(V.75) $$\begin{cases} \forall\, \mathbf{v}\in X, \quad a(\tilde{\mathbf{u}},\mathbf{v}) + b_1(\mathbf{v},\hat{p}) = (\mathbf{f},\mathbf{v})_\omega - a(\Pi_{N,b}\mathbf{u},\mathbf{v}) \quad , \\ \forall\, q\in M_2\,, \quad b_2(\tilde{\mathbf{u}},q) = -\, b_2(\Pi_{N,b}\mathbf{u},q) \quad . \end{cases}$$

Next, $\tilde{\mathbf{u}}^N = \hat{\mathbf{u}}^N - \Pi_{N,b}\mathbf{u} + \mathbf{e}_N$ is the solution of (V.66). Let us define the forms g and g_N by the relations

$$\forall\, q\in M_2\,,\ \langle g,q\rangle = -\, b_2(\Pi_{N,b}\mathbf{u},q) \quad \text{and} \quad \forall\, q\in M_{2N}\,,\ \langle g_N\,,q\rangle = -\, b_{2N}(\Pi_{N,b}\mathbf{u},q) \quad .$$

Since we already noticed that the hypotheses of Corollary II.3 are fulfilled, we derive from (II.35)

$$\text{(V.76)}\quad \left\{ \begin{aligned} \|\hat{\mathbf{u}}-\hat{\mathbf{u}}^N\|_{1,\omega} \leqslant c\,[&\inf_{\mathbf{w}\in K_{2N}(g_N)} \|\tilde{\mathbf{u}}-\mathbf{w}\|_{1,\omega} + \inf_{\mathbf{v}\in[\overset{\circ}{P}_{N-1}(\Omega)]^2} \|\tilde{\mathbf{u}}-\mathbf{v}\|_{1,\omega} \\ &+ \sup_{\mathbf{z}\in X_N} \frac{(\mathbf{f},\mathbf{z})_\omega - (\mathbf{f},\mathbf{z})_{\omega,N}}{\|\mathbf{z}\|_{1,\omega}} + \sup_{\mathbf{z}\in X_N} \frac{a(\Pi_{N,b}\mathbf{u},\mathbf{z}) - a_N(\Pi_{N,b}\mathbf{u},\mathbf{z})}{\|\mathbf{z}\|_{1,\omega}} \;] \;. \end{aligned} \right.$$

In opposition to what we were able to do in the homogeneous case, it seems that the only way to bound the term $\inf_{\mathbf{w}\in K_{2N}(g_N)} \|\tilde{\mathbf{u}}-\mathbf{w}\|_{1,\omega}$ is to use Proposition II.1. By Proposition V.4 we have

$$\text{(V.77)}\quad \left\{ \begin{aligned} \inf_{\mathbf{w}\in K_{2N}(g_N)} \|\tilde{\mathbf{u}}-\mathbf{w}\|_{1,\omega} \leqslant c\,N^2\,[&\inf_{\mathbf{v}\in[\overset{\circ}{P}_{N-1}(\Omega)]^2} \|\tilde{\mathbf{u}}-\mathbf{v}\|_{1,\omega} \\ &+ \sup_{q\in M_{2N}} \frac{\langle g-g_N\,,q\rangle}{\|q\|_{0,\omega}} \;] \;. \end{aligned} \right.$$

1) From the definitions of g and g_N and (V.2) we write for any q in M_{2N}

$$\begin{aligned} \langle g-g_N\,,q\rangle &= b_{2N}(\Pi_{N,b}\mathbf{u},q) - b_2(\Pi_{N,b}\mathbf{u},q) \quad, \\ &= b_{2N}(\Pi_{N,b}\mathbf{u} - \Pi_{N-1,b}\mathbf{u},q) - b_2(\Pi_{N,b}\mathbf{u} - \Pi_{N-1,b}\mathbf{u},q) \;. \end{aligned}$$

Using now the uniform continuity of b_2 and b_{2N} we derive

$$|\langle g-g_N\,,q\rangle| \leqslant c\,\|\Pi_{N,b}\mathbf{u} - \Pi_{N-1,b}\mathbf{u}\|_{1,\omega}\,\|q\|_{0,\omega} \quad,$$

hence, by (V.64) we obtain

$$\text{(V.78)}\quad \sup_{q\in M_{2N}} \frac{|\langle g-g_N\,,q\rangle|}{\|q\|_{0,\omega}} \leqslant c\,(\,N^{1-\sigma}\|\mathbf{u}\|_{\sigma,\omega} + N^{3/2-\tau}\textstyle\sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho}\,) \;.$$

2) On the other hand, we compute

$$\begin{aligned} \inf_{\mathbf{v}\in[\overset{\circ}{P}_{N-1}(\Omega)]^2} \|\tilde{\mathbf{u}}-\mathbf{v}\|_{1,\omega} &\leqslant \|\tilde{\mathbf{u}}\|_{1,\omega} = \|\hat{\mathbf{u}} - \Pi_{N,b}\mathbf{u} + \mathbf{e}_N\|_{1,\omega} \\ &\leqslant \|\mathbf{u}-\hat{\mathbf{u}}\|_{1,\omega} + \|\mathbf{u}-\Pi_{N,b}\mathbf{u}\|_{1,\omega} + c\,|e_N| \;. \end{aligned}$$

Thus, by (V.72), (V.64) and (V.69) we obtain

$$\text{(V.79)}\quad \inf_{\mathbf{v}\in[\overset{\circ}{P}_{N-1}(\Omega)]^2} \|\tilde{\mathbf{u}}-\mathbf{v}\|_{1,\omega} \leqslant c\,(\,N^{1-s}\|\mathbf{u}\|_{s,\omega} + N^{3/2-\tau}\textstyle\sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho}\,) \;.$$

3) Due to (V.48), (IV.46) and (V.49), we have

$$\text{(V.80)}\quad \sup_{\mathbf{z}\in X_N} \frac{(\mathbf{f},\mathbf{z})_\omega - (\mathbf{f},\mathbf{z})_{\omega,N}}{\|\mathbf{z}\|_{1,\omega}} \leqslant c\,N^{-\sigma}\,\|\mathbf{f}\|_{\sigma,\omega} \;.$$

4) Let us now estimate the last term in (V.76). We have by definition of the forms a, a_N and by (V.2)

$$\begin{aligned} |a(\Pi_{N,b}u,z) - a_N(\Pi_{N,b}u,z)| &= |a(\Pi_{N,b}u - \Pi_{N-1,b}u,z) - a_N(\Pi_{N,b}u - \Pi_{N-1,b}u,z)| \\ &\leqslant c\,\|\Pi_{N,b}u - \Pi_{N-1,b}u\|_{1,\omega}\,\|z\|_{1,\omega} \;. \end{aligned}$$

Thus

$$\text{(V.81)} \qquad \sup_{\mathbf{z} \in X_N} \frac{a(\Pi_{N,b}\mathbf{u},\mathbf{z}) - a_N(\Pi_{N,b}\mathbf{u},\mathbf{z})}{\|\mathbf{z}\|_{1,\omega}} \leqslant c\,(N^{1-s}\,\|\mathbf{u}\|_{s,\omega} + N^{3/2-\tau} \textstyle\sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho}) \;.$$

Finally, estimate (V.74) follows from (V.76) to (V.81).

Using now the three previous propositions, we obtain the final error estimate

__Theorem V.5__ : *Assume that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (III.1)(III.39) *belongs to* $[H^s_\omega(\Omega)]^2 \times H^{s-1}_\omega(\Omega)$ *for a real number* $s \geqslant 1$, *that the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma > 1$ *and that the boundary data* φ_J, $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H^\tau_\varrho(\Gamma_J)$ *for a real number* $\tau \geqslant 3/4$. *Then the approximate velocity* u^N, *as defined in Theorem* V.4 , *satisfies the convergence estimate*

$$\text{(V.82)} \qquad \|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} \leqslant c\,(N^{3-s}\,\|\mathbf{u}\|_{s,\omega} + N^{-\sigma}\,\|\mathbf{f}\|_{\sigma,\omega} + N^{7/2-\tau} \textstyle\sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho})$$

for a constant c *independent of* N.

We conclude with an estimate for the pressure.

__Theorem V.6__ : *Assume that hypotheses* (V.34) *and* (V.35) *hold and that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (III.1)(III.39) *belongs to* $[H^s_\omega(\Omega)]^2 \times H^{s-1}_\omega(\Omega)$ *for a real number* $s \geqslant 1$, *that the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma > 1$ *and that the boundary data* φ_J, $J \in \mathbb{Z}/4\mathbb{Z}$, *belong to* $H^\tau_\varrho(\Gamma_J)$ *for a real number* $\tau \geqslant 3/4$. *Then the approximate pressure* p^N *in* $\tilde{M}_{1N}$, *as defined in Theorem* V.4, *satisfies the convergence estimate*

$$\text{(V.83)} \qquad \|p-p^N\|_{0,\omega} \leqslant c\,\{ N^{5-s}\,(\|\mathbf{u}\|_{s,\omega} + \|p\|_{s-1,\omega}) + N^{2-\sigma}\,\|\mathbf{f}\|_{\sigma,\omega} + N^{11/2-\tau} \textstyle\sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{\tau,\varrho} \}$$

for a constant c *independent of* N.

Proof : Using Proposition V.5 we derive from (V.56) that

$$\|p-p^N\|_{0,\omega} \leqslant c\,N^2\, \Big(\|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} + \inf_{q_N \in \tilde{M}_{1N}} \Big\{ \|p-q_N\|_{0,\omega} + \sup_{\mathbf{z} \in X_N} \frac{(b_1-b_{1N})(\mathbf{z},q_N)}{\|\mathbf{z}\|_{1,\omega}} \Big\}$$

$$+ \sup_{\mathbf{z} \in X_N} \frac{(\mathbf{f},\mathbf{z})_\omega - (\mathbf{f},\mathbf{z})_{\omega,N}}{\|\mathbf{z}\|_{1,\omega}} \Big) \;.$$

Still taking $q_N = \Pi^0_{[\lambda N]}p$, we obtain easily (V.83).

VI. A collocation method for the Navier-Stokes equations.

VI.1. The Navier-Stokes equations.

We are interested now in the approximation of the full Navier Stokes equations on the domain Ω by a collocation method. From now on, we shall denote by $\mathbf{x} = (x_1, x_2)$ the generic point of Ω, and by w_1 and w_2 the components of any vector $\mathbf{w}$ in $\mathbb{R}^2$. Given a force field $\mathbf{f}$ in Ω and a viscosity $\nu > 0$, the problem is to find a velocity field $\mathbf{u} = (u_1, u_2)$ and a pressure p solution of

(VI.1) $$\begin{cases} -\nu\,\Delta\mathbf{u} + \operatorname{grad} p + (\mathbf{u}.\nabla)\mathbf{u} = \mathbf{f} & \text{in } \Omega \quad , \\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega \quad , \end{cases}$$

such that $\mathbf{u}$ satisfies the following homogeneous boundary conditions

(VI.2) $$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \quad .$$

We mean by $(\mathbf{u}.\nabla)\mathbf{u}$ the sum $\sum_{1\le i\le 2} u_i(\partial\mathbf{u}/\partial x_i)$.

Let us consider the nonlinear term in equation (VI.1). We notice that, for any function $\mathbf{w}$ such that div $\mathbf{w} = 0$ in Ω, we have

(VI.3) $$\sum_{1\le i\le 2} w_i(\partial\mathbf{w}/\partial x_i) = \sum_{1\le i\le 2} \partial(w_i\mathbf{w})/\partial x_i \quad .$$

The two forms are equivalent for the continuous problem, but generally not for the discrete problems. For reasons of numerical stability as well as to reduce the computation cost, it seems more convenient to choose the second expression. Hence, we set

(VI.4) $$\mathbf{G}(\mathbf{w}) = \sum_{1\le i\le 2} \partial(w_i\mathbf{w})/\partial x_i - \mathbf{f} \quad .$$

In order to study the well-posedness of problem (VI.1)(VI.2) in the spaces X and M_1, we first state some properties of $\mathbf{G}$.

<u>Lemma VI.1</u> : *For any $\mathbf{f}$ in X', the mapping $\mathbf{G}$ is of class C^∞ from $[H^1(\Omega)]^2$ into X' and from X into X'. Furthermore, for any $\mathbf{w}$ in X, the operator $D\mathbf{G}(\mathbf{w})$ is compact from X into X'.*

Proof : For any $\mathbf{u}$ and $\mathbf{w}$ in X we have

$$\forall\, \mathbf{v} \in X, \quad \Big|\sum_{1\le i\le 2} \int_\Omega (\partial(u_i\mathbf{w})/\partial x_i)\,\mathbf{v}\,\omega\, d\mathbf{x}\Big| = \Big|\sum_{1\le i\le 2} \int_\Omega (u_i\mathbf{w})\,(\partial(\mathbf{v}\omega)/\partial x_i)\, d\mathbf{x}\Big|$$

whence, from Lemma III.1, we derive

(VI.5) $$\forall\, \mathbf{v} \in X, \quad \Big|\sum_{1\le i\le 2} \int_\Omega (\partial(u_i\mathbf{w})/\partial x_i)\,\mathbf{v}\,\omega\, d\mathbf{x}\Big| \le c \sum_{1\le i,j\le 2} \|u_i\, w_j\|_{0,\omega}\, \|\mathbf{v}\|_{1,\omega} \,.$$

We recall the imbedding of $H^{1/2}(\Omega)$ into $L^2_\omega(\Omega)$ (cf. [CQ3, Thm 4.1] or [BM]). Moreover, using the Calderón extension theorem (see [A, Thm 4.32]) together with [G, Thm 1.4.4.2], we know that the mapping : $(\varphi,\psi) \to \varphi\psi$ is bilinear continuous from $H^1(\Omega) \times H^1(\Omega)$ into $H^{1-\varepsilon}(\Omega)$ for any

$\varepsilon > 0$. Hence we have for $0 < \varepsilon \leqslant 1/2$

(VI.6) $\quad \|u_i\, w_j\|_{0,\omega} \leqslant c\, \|u_i\, w_j\|_{1/2} \leqslant c'\, \|u_i\, w_j\|_{1-\varepsilon} \leqslant c''\, \|u_i\|_1\, \|w_j\|_1$.

From (VI.5) and (VI.6), we deduce

(VI.7) $\quad \forall\, \mathbf{v} \in X, \quad |\sum_{1 \leqslant i \leqslant 2} \int_\Omega (\partial(u_i\mathbf{w})/\partial x_i)\, \mathbf{v}\, \omega\, d\mathbf{x}\, | \leqslant c\, \|\mathbf{u}\|_1\, \|\mathbf{w}\|_1\, \|\mathbf{v}\|_{1,\omega}$,

and

(VI.8) $\quad \forall\, \mathbf{v} \in X, \quad |\sum_{1 \leqslant i \leqslant 2} \int_\Omega (\partial(u_i\mathbf{w})/\partial x_i)\, \mathbf{v}\, \omega\, d\mathbf{x}\, | \leqslant c\, \|\mathbf{u}\|_{1,\omega}\, \|\mathbf{w}\|_{1,\omega}\, \|\mathbf{v}\|_{1,\omega}$.

Then, it is an easy matter to derive from (VI.4) that $\mathbf{G}$ is of class C^∞ from $[H^1(\Omega)]^2$ into X' and from X into X'. The compactness of $\mathbf{DG}(\mathbf{w})$ from X into X', is an easy consequence of (VI.6) and of the compactness of the imbedding : $H^{1-\varepsilon}(\Omega) \subset H^{1/2}(\Omega)$, $0 \leqslant \varepsilon < 1/2$.

Let us recall that the forms a and b_i ($i = 1, 2$) are defined in (III.14) to (III.16). From now on, we identify $L^2_\omega(\Omega)$ with its dual space and we denote by $(.,.)_\omega$ the duality pairing between X and X'. As in (III.13), for any $\mathbf{f}$ in X', the Navier-Stokes equations (VI.1)(VI.2) can be written in the following variational form : *Find* $(\mathbf{u},p)$ *in* $X \times M_1$ *such that*

(VI.9) $$\begin{cases} \forall\, \mathbf{v} \in X, \quad a(\mathbf{u},\mathbf{v}) + b_1(\mathbf{v},p) + (\mathbf{G}(\mathbf{u}),\mathbf{v})_\omega = 0 \quad , \\ \forall\, q \in M_2\, , \quad b_2(\mathbf{u},q) = 0 \quad . \end{cases}$$

Then, we derive

<u>Theorem VI.1</u> : *For any* $\mathbf{f}$ *in* X', *problem* (VI.9) *has at least a solution* $(\mathbf{u},p)$ *in* $X \times M_1$.

Proof : From (III.9) and the imbedding of $L^2_\omega(\Omega)$ into $L^2(\Omega)$ we first deduce that $\mathbf{f}$ belongs to $[H^{-1}(\Omega)]^2$. Using now [GR, Chapter IV, Thm 2.1] we obtain that there exists at least a pair $(\mathbf{u},p)$ in $[H^1_0(\Omega)]^2 \times L^2(\Omega)$ solution of (VI.1), where p is defined up to an additive constant. From Lemma VI.1, $\mathbf{G}(\mathbf{u})$ is an element of X'. Let $(\mathbf{u}',p')$ be the solution in $X \times M_1$ of the Stokes problem with data $-\,\mathbf{G}(\mathbf{u})$, as defined in Theorem III.1. Then, both $(\mathbf{u},p)$ and $(\mathbf{u}',p')$ are solutions in $[H^1_0(\Omega)]^2 \times L^2(\Omega)$ of the Stokes problem with data $-\,\mathbf{G}(\mathbf{u})$; the uniqueness of the solution of the Stokes problem implies that $\mathbf{u}$ and $\mathbf{u}'$ coincide and that $p-p'$ is constant. Now, fixing the constant over p so that (III.38) is satisfied, we see that $(\mathbf{u},p)$ belongs in fact to $X \times M_1$ and satisfies (VI.9).

Now, using Theorem III.1, we define the operator $\mathbf{T}$ from X' into X : $\mathbf{f} \to \mathbf{u} = \mathbf{T}\mathbf{f}$, where $(\mathbf{u},p)$ is the solution of problem (III.1)(III.2). Clearly, if $(\mathbf{u},p)$ is solution of the Navier-Stokes equations (VI.9) then $\mathbf{u}$ is a solution in X of

(VI.10) $\quad \mathbf{u} + \mathbf{T}\mathbf{G}(\mathbf{u}) = \mathbf{0}$.

This formulation will be very useful in the sequel, together with

Lemma VI.2 : *For any real number* $q > 2$, *there exists a constant* $c(q,\nu)$ *such that, if a solution* $(\mathbf{u},p)$ *of problem* (VI.9) *satisfies*

(VI.11) $\quad \|\mathbf{u}\|_{L^q(\Omega)} \leqslant c(q,\nu) \quad ,$

the operator $\mathbf{1} + \mathbf{TDG}(\mathbf{u})$ *is an isomorphism of* X.

Proof : By the compactness result of Lemma VI.1, the operator $\mathbf{1} + \mathbf{TDG}(\mathbf{u})$ is an isomorphism of X if and only if it is injective, i.e. the only solution $(\mathbf{w},r)$ of the following linearized Stokes problem

$$\left\{ \begin{array}{l} \forall\, \mathbf{v} \in X, \quad a(\mathbf{w},\mathbf{v}) + b_1(\mathbf{v},r) + (\mathbf{DG}(\mathbf{u}).\mathbf{w},\mathbf{v})_\omega = 0 \quad , \\ \forall\, q \in M_2 , \quad b_2(\mathbf{w},q) = 0 \quad , \end{array} \right.$$

is $(\mathbf{0},0)$. From (III.19) and (VI.5) we obtain

$$\nu\alpha\, \|\mathbf{w}\|_{1,\omega} \leqslant c \sum\nolimits_{1 \leqslant i,j \leqslant 2} \| u_i\, w_j \|_{0,\omega} \quad .$$

Next, using the imbedding of $H^1(\Omega)$ into any $L^s(\Omega)$, $s < +\infty$, and Lemma III.1, we have

$$\|u_i\, w_j\|_{0,\omega} \leqslant c\, \|u_i\, w_j\, \omega^{1/2}\|_0 \leqslant c(q)\, \|u_i\|_{L^q(\Omega)}\, \|w_j\, \omega^{1/2}\|_1 \leqslant c(q)\, c'\, \|u_i\|_{L^q(\Omega)}\, \|w_j\|_{1,\omega} \quad ,$$

so that

$$\nu\alpha\, \|\mathbf{w}\|_{1,\omega} \leqslant c'(q)\, \|\mathbf{u}\|_{L^q(\Omega)}\, \|\mathbf{w}\|_{1,\omega} \quad ,$$

and the lemma is proved with $c(q,\nu) < \nu\alpha/c'(q)$.

VI.2. A collocation method for the Navier-Stokes equations.

We are going to introduce a collocation problem to approximate the Navier-Stokes equations, by using the same nodes as in the linear case. Let us recall that, for a fixed integer $N \geqslant 3$, $X_N = [\overset{\circ}{P}_N(\Omega)]^2$. Henceforward, we still assume that M_{iN} $(i = 1, 2)$ is defined by $(\text{V.30})_i$ and that $\tilde{M}_{1N}$ satisfies the hypothesis (V.35).

Before writing the problem, let us consider the nonlinear terms in these equations, i.e. the function $\mathbf{G}$ defined by (VI.4). Obviously, if a function $\mathbf{w}$ of class $\mathcal{C}^\circ$ is known only by its values at the nodes $\mathbf{x}$ in Ξ_N (see (V.7) for the definition of Ξ_N), it is easy to derive the values of $w_i\mathbf{w}$ at the same nodes, whence $\mathcal{J}_N(w_i\mathbf{w})$. The pseudo-spectral approximation $\partial(w_i\mathbf{w})/\partial x_i$ consists in differentiating this interpolation function i.e., to compute $\partial\mathcal{J}_N(w_i\mathbf{w})/\partial x_i$.

Assume now that the force $\mathbf{f}$ is given in $[\mathcal{C}^\circ(\Omega)]^2$. Due to the previous remark, the collocation problem we are going to analyze is the following one : *Find* $(\mathbf{u}^N,p^N)$ *in* $X_N \times M_{1N}$ *such that*

$$
(\mathrm{VI}.12)\quad \left|\begin{array}{l} -\nu\, \Delta \mathbf{u}^N(\mathbf{x}) + (\mathrm{grad}\, p^N)(\mathbf{x}) + \sum_{1\leqslant i\leqslant 2} (\partial(\mathcal{J}_N(u_i^N \mathbf{u}^N))/\partial x_i)(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \mathbf{x} \in \Xi_N \cap \Omega \quad , \\ (\mathrm{div}\, \mathbf{u}^N)(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \Xi_N \quad . \end{array}\right.
$$

In order to study problem (VI.12), we set for any function $\mathbf{w}$ of $[\mathcal{C}^\infty(\overline{\Omega})]^2$

$$
(\mathrm{VI}.13)\quad \mathbf{G}_N(\mathbf{w}) = \sum_{1\leqslant i\leqslant 2} S_N(\partial(\mathcal{J}_N(w_{Ni}\mathbf{w}_N))/\partial x_i) - S_N\mathbf{f} \quad ,
$$

where S_N is defined in (V.21). Then, we have

$$
\forall\, \mathbf{v}_N \in X_N\,,\quad (\mathbf{G}_N(\mathbf{w}),\mathbf{v}_N)_\omega = (\sum_{1\leqslant i\leqslant 2} \partial(\mathcal{J}_N(w_{Ni}\mathbf{w}_N))/\partial x_i\,,\,\mathbf{v}_N)_{\omega,N} - (\mathbf{f},\mathbf{v}_N)_{\omega,N} \quad ,
$$

and due to (V.2)

$$
(\mathrm{VI}.14)\quad \forall\, \mathbf{v}_N \in X_N\,,\quad (\mathbf{G}_N(\mathbf{w}),\mathbf{v}_N)_\omega = -(\sum_{1\leqslant i\leqslant 2} w_{Ni}\mathbf{w}_N\,,\,\omega^{-1}\,\partial(\omega\mathbf{v}_N)/\partial x_i)_{\omega,N} - (\mathbf{f},\mathbf{v}_N)_{\omega,N} \quad .
$$

Hence, the next result can be proved exactly as for the linear case.

<u>Proposition VI.1</u> : *Problem* (VI.12) *is equivalent to the following variational one : find a pair* $(\mathbf{u}^N,p^N)$ *in* $X_N \times M_{1N}$ *such that*

$$
(\mathrm{VI}.15)\quad \left|\begin{array}{l} \forall\, \mathbf{v} \in X_N\,,\quad a_N(\mathbf{u}^N,\mathbf{v}) + b_{1N}(\mathbf{v},p^N) + (\mathbf{G}_N(\mathbf{u}^N),\mathbf{v})_\omega = 0 \quad , \\ \forall\, q \in M_{2N}\,,\quad b_{2N}(\mathbf{u}^N,q) = 0 \quad . \end{array}\right.
$$

Now, by Theorem V.1, we can define the operator $\mathbf{T}_N$ from X' into X_N : $\mathbf{f} \to \mathbf{u}^N = \mathbf{T}_N\mathbf{f}$, where $(\mathbf{u}^N,p^N)$ is the solution of problem (V.51)(recall that it is exactly problem (V.19) with $(\mathbf{f},\mathbf{v}_N)_{\omega,N}$ replaced by $(\mathbf{f},\mathbf{v}_N)_\omega$). Clearly, problem (VI.15) implies that $\mathbf{u}^N$ is a solution in X_N of

$$
(\mathrm{VI}.16)\quad \mathbf{u}^N + \mathbf{T}_N\,\mathbf{G}_N(\mathbf{u}^N) = \mathbf{0} \quad .
$$

We begin by stating some results about the operator $\mathbf{T}_N$.

<u>Proposition VI.2</u> : *For any* $\mathbf{f}$ *in* X', *the operator* $\mathbf{T}_N$ *satisfies*

$$
(\mathrm{VI}.17)\quad \|\mathbf{T}_N\mathbf{f}\|_{1,\omega} \leqslant c \sup_{\mathbf{v}_N \in X_N} \frac{(\mathbf{f},\mathbf{v}_N)_\omega}{\|\mathbf{v}_N\|_{1,\omega}} \quad ,
$$

and

$$
(\mathrm{VI}.18)\quad \lim_{N\to\infty} \|(\mathbf{T}-\mathbf{T}_N)\mathbf{f}\|_{1,\omega} = 0 \quad ;
$$

Moreover, if the solution $\mathbf{Tf}$ *belongs to* $[H^s_\omega(\Omega)]^2$ *for a real number* $s \geqslant 1$, *it satisfies the error estimate*

$$
(\mathrm{VI}.19)\quad \|(\mathbf{T}-\mathbf{T}_N)\mathbf{f}\|_{1,\omega} \leqslant c\, N^{1-s}\, \|\mathbf{Tf}\|_{s,\omega} \quad .
$$

Proof : By Proposition V.6, we obtain at once

$$\|\mathbf{T}_N\mathbf{f}\|_{1,\omega} \leq c \sup_{\mathbf{v}_N \in X_N} \frac{a_N(\mathbf{T}_N\mathbf{f}\,,\mathbf{v}_N)}{\|\mathbf{v}_N\|_{1,\omega}} = c \sup_{\mathbf{v}_N \in X_N} \frac{(\mathbf{f},\mathbf{v}_N)_\omega}{\|\mathbf{v}_N\|_{1,\omega}} \quad ,$$

which yields (VI.17). Next, we have already noticed (see Remark V.3) that the error estimate (VI.19) can be proved exactly by the same way as (V.45) for any $\mathbf{f}$ such that $\mathbf{Tf}$ belongs to $[H^s_\omega(\Omega)]^2$. Finally, (VI.18) holds by classical arguments using (VI.19) and the density of $[\mathcal{D}(\Omega)]^2$ in X.

To study problem (VI.16), we shall use a fixed point theorem due to M. CROUZEIX (see [C, Th. 2.2]), which is a refined form of the discrete implicit function theorem of [BRR]. For the reader's convenience, let us recall this theorem : we consider a C^1-mapping $\mathbf{F}_N$ from a Banach space X_N into itself and we assume that $\mathbf{u}^*_N$ is a point in X_N such that $D\mathbf{F}_N(\mathbf{u}^*_N)$ is an isomorphism of X_N. We denote by ε_N, γ_N and $L_N(\eta)$, $\eta \geq 0$, the quantities

(VI.20) $$\begin{cases} \varepsilon_N = \|\mathbf{F}_N(\mathbf{u}^*_N)\|_{X_N} \qquad \gamma_N = \|(D\mathbf{F}_N(\mathbf{u}^*_N))^{-1}\|_{\mathcal{L}(X_N,X_N)} \\ L_N(\eta) = \sup\{\, \|D\mathbf{F}_N(\mathbf{w}_N)-D\mathbf{F}_N(\mathbf{u}^*_N)\|_{\mathcal{L}(X_N,X_N)}\ ;\ \mathbf{w}_N \in X_N \text{ and } \|\mathbf{w}_N-\mathbf{u}^*_N\|_{X_N} \leq \eta \,\} \end{cases} .$$

<u>Theorem VI.2</u> : *Let us assume that* $2\gamma_N L_N(2\gamma_N\varepsilon_N) < 1$, *then for each* $\eta \geq 2\gamma_N\varepsilon_N$ *such that* $\gamma_N L_N(\eta) < 1$, *there exists a unique solution* $\mathbf{u}^N$ *of the equation* $\mathbf{F}_N(\mathbf{u}^N) = 0$ *in the ball* $B_N = \{\, \mathbf{w}_N \in X_N\ ;\ \|\mathbf{w}_N-\mathbf{u}^*_N\|_{X_N} \leq \eta \,\}$. *This solution satisfies*

(VI.21) $$\forall\, \mathbf{w}_N \in B_N\,, \quad \|\mathbf{u}^N-\mathbf{w}_N\|_{X_N} \leq [\gamma_N/(1-\gamma_N L_N(\eta))]\,.\,\|\mathbf{F}_N(\mathbf{w}_N)\|_{X_N} \quad .$$

We are going to apply this theorem to the mapping $\mathbf{F}_N = \mathbf{1} + \mathbf{T}_N\mathbf{G}_N$. In the sequel, we always assume that the function $\mathbf{f}$ is given in $[H^\sigma_\omega(\Omega)]^2$, $\sigma > 1$. We consider a solution $\mathbf{u}$ of the Navier-Stokes equations (VI.1)(VI.2) which is nonsingular in the following sense : the operator $\mathbf{1} + \mathbf{TDG}(\mathbf{u})$ is an isomorphism of X (by virtue of Lemma VI.2, such solutions exist for $\mathbf{f}$ small enough !). Even in the classical Sobolev spaces, regularity results of the solution $(\mathbf{u},p)$ as a consequence of the regularity of $\mathbf{f}$ are not easy to derive (see [G, §7.3]), whence we shall assume in the sequel that there exists a real number $s > 1$ such that $(\mathbf{u},p)$ belongs to $[H^s_\omega(\Omega)]^2 \times H^{s-1}_\omega(\Omega)$.

Let us denote by N' the integral part of (N-1)/2. We choose for $\mathbf{u}^*_N$ the image of $\mathbf{u}$ by the projection operator from $[H^1_{\omega,0}(\Omega) \cap H^s_\omega(\Omega)]^2$ onto $[\overset{\circ}{P}_{N'}(\Omega)]^2$ (this definition of $\mathbf{u}^*_N$ seems very complicated, but the fact that $u_{Ni}^*\mathbf{u}^*_N$ belongs to $[\overset{\circ}{P}_{N-1}(\Omega)]^2$ will make the estimates more straightforward, as it will appear later). It has been proved in [Ma2] that the following estimate holds for any real number r, $0 \leq r \leq s$,

(VI.22) $$\|\mathbf{u} - \mathbf{u}^*_N\|_{r,\omega} \leq c\, N^{r-s}\, \|\mathbf{u}\|_{s,\omega}$$

VI.3. Some properties of the mapping F_N.

In order to bound the constants γ_N, L_N and ε_N, we need several lemmas.

Lemma VI.3 : *For any real number $\varepsilon > 0$, there exists a constant c such that, for any φ and ψ in $\mathbb{P}^\circ_N(\Omega)$, the following estimate is satisfied*

(VI.23) $\quad \|(1-\mathcal{J}_N)(\varphi\psi)\|_{0,\omega} \leqslant c\, N^{\varepsilon-1}\, \|\varphi\|_{1,\omega}\, \|\psi\|_{1,\omega}$.

Proof : Recalling the definition of N' and from (V.12) we derive that

$$(1-\mathcal{J}_N)(\varphi\psi) = (1-\mathcal{J}_N)[(\varphi-\Pi^1_{N'}\varphi)(\psi-\Pi^1_{N'}\psi) + \Pi^1_{N'}\varphi(\psi-\Pi^1_{N'}\psi) + (\varphi-\Pi^1_{N'}\varphi)\Pi^1_{N'}\psi] \quad ,$$

so that

$$\begin{aligned}\|(1-\mathcal{J}_N)(\varphi\psi)\|_{0,\omega} \leqslant{}& \|(\varphi-\Pi^1_{N'}\varphi)(\psi-\Pi^1_{N'}\psi)\|_{0,\omega} + \|\Pi^1_{N'}\varphi(\psi-\Pi^1_{N'}\psi)\|_{0,\omega} \\ &+ \|(\varphi-\Pi^1_{N'}\varphi)\Pi^1_{N'}\psi\|_{0,\omega} + \|\mathcal{J}_N[(\varphi-\Pi^1_{N'}\varphi)(\psi-\Pi^1_{N'}\psi)]\|_{0,\omega} \\ &+ \|\mathcal{J}_N[\Pi^1_{N'}\varphi(\psi-\Pi^1_{N'}\psi)]\|_{0,\omega} + \|\mathcal{J}_N[(\varphi-\Pi^1_{N'}\varphi)\Pi^1_{N'}\psi]\|_{0,\omega} \quad .\end{aligned}$$

From (V.10) and (V.11) we derive

$$\begin{aligned}\|(1-\mathcal{J}_N)(\varphi\psi)\|_{0,\omega} \leqslant{}& \|(\varphi-\Pi^1_{N'}\varphi)(\psi-\Pi^1_{N'}\psi)\|_{0,\omega} + \|\Pi^1_{N'}\varphi(\psi-\Pi^1_{N'}\psi)\|_{0,\omega} \\ &+ \|(\varphi-\Pi^1_{N'}\varphi)\Pi^1_{N'}\psi\|_{0,\omega} + \|(\varphi-\Pi^1_{N'}\varphi)(\psi-\Pi^1_{N'}\psi)]\|_{\omega,N} \\ &+ \|\Pi^1_{N'}\varphi(\psi-\Pi^1_{N'}\psi)\|_{\omega,N} + \|(\varphi-\Pi^1_{N'}\varphi)\Pi^1_{N'}\psi\|_{\omega,N} \\ \leqslant{}& c\,[\|\varphi-\Pi^1_{N'}\varphi\|_{L^\infty(\Omega)}\, \|\psi-\Pi^1_{N'}\psi\|_{0,\omega} \\ &+ \|\Pi^1_{N'}\varphi\|_{L^\infty(\Omega)}\, \|\psi-\Pi^1_{N'}\psi\|_{0,\omega} + \|\varphi-\Pi^1_{N'}\varphi\|_{0,\omega}\, \|\Pi^1_{N'}\psi\|_{L^\infty(\Omega)}] \\ \leqslant{}& c\,[\|\varphi\|_{L^\infty(\Omega)} + \|\Pi^1_{N'}\varphi\|_{L^\infty(\Omega)}]\, \|\psi-\Pi^1_{N'}\psi\|_{0,\omega} \\ &+ \|\varphi-\Pi^1_{N'}\varphi\|_{0,\omega}\, \|\Pi^1_{N'}\psi\|_{L^\infty(\Omega)} \quad .\end{aligned}$$

Using now (IV.46) and the imbedding of $H^{1+\varepsilon/2}_\omega(\Omega)$ into $L^\infty(\Omega)$ for any $\varepsilon > 0$, we obtain for any $r \geqslant 0$

(VI.24) $\quad \|(1-\mathcal{J}_N)(\varphi\psi)\|_{0,\omega} \leqslant c\, N^{-r}\, [(\|\varphi\|_{1+\varepsilon/2} + \|\Pi^1_{N'}\varphi\|_{1+\varepsilon/2})\, \|\psi\|_{r,\omega} + \|\varphi\|_{r,\omega}\, \|\Pi^1_{N'}\psi\|_{1+\varepsilon/2}]$.

We derive the lemma as an easy consequence of the inverse inequality (IV.29) and of (IV.46).

We can now state the following result.

Proposition VI.3 : *For N large enough, the operator $\mathbf{DF}_N(\mathbf{u}^*_N) = \mathbf{1} + \mathbf{T}_N \mathbf{DG}_N(\mathbf{u}^*_N)$ is an isomorphism of X_N, and γ_N is bounded by a constant γ independent of N.*

Proof : We write $\mathbf{DF}_N(\mathbf{u}^*_N)$ in the form

(VI.25) $$\mathbf{DF}_N(\mathbf{u}^*_N) = [\mathbf{1} + \mathbf{T}\,\mathbf{DG}(\mathbf{u})] - (\mathbf{T}-\mathbf{T}_N)\mathbf{DG}(\mathbf{u}) - \mathbf{T}_N(\mathbf{DG}(\mathbf{u})-\mathbf{DG}(\mathbf{u}^*_N)) - \mathbf{T}_N(\mathbf{DG}-\mathbf{DG}_N)(\mathbf{u}^*_N) \quad .$$

Let $\mathbf{w}_N$ be any element of X_N. Since $\mathbf{1} + \mathbf{T}\,\mathbf{DG}(\mathbf{u})$ is an isomorphism of X, there exists a constant c_0 independent of N such that

(VI.26) $\quad \|[\mathbf{1} + \mathbf{T}\mathbf{DG}(\mathbf{u})].\mathbf{w}_N\|_{1,\omega} \geqslant c_0 \|\mathbf{w}_N\|_{1,\omega} \quad .$

It remains to bound the three other terms in (VI.25).

1) It follows from (VI.18) and from the compactness of the operator $\mathbf{DG}(\mathbf{u})$ (see Lemma VI.1) that

$$\lim_{N \to \infty} \|(\mathbf{T}-\mathbf{T}_N)\mathbf{DG}(\mathbf{u})\|_{\mathcal{L}(X,X)} = 0 \quad .$$

Hence, for N large enough, one has

(VI.27) $\quad \|(\mathbf{T}-\mathbf{T}_N)\mathbf{DG}(\mathbf{u}).\mathbf{w}_N\|_{1,\omega} \leqslant (c_0/4)\,\|\mathbf{w}_N\|_{1,\omega} \quad .$

2) It follows from (VI.17) and from the continuity of the operator $\mathbf{DG}$ (see Lemma VI.1) that

$$\|\mathbf{T}_N(\mathbf{DG}(\mathbf{u})-\mathbf{DG}(\mathbf{u}_N^*))\|_{\mathcal{L}(X,X)} \leqslant c\,\|\mathbf{DG}(\mathbf{u})-\mathbf{DG}(\mathbf{u}_N^*)\|_{\mathcal{L}(X,X')} \leqslant c'\,\|\mathbf{u}-\mathbf{u}_N^*\|_{1,\omega} \quad .$$

From the convergence of $\mathbf{u}_N^*$ to $\mathbf{u}$ (see (VI.22)), we deduce that, for N large enough,

(VI.28) $\quad \|\mathbf{T}_N(\mathbf{DG}(\mathbf{u})-\mathbf{DG}(\mathbf{u}_N^*)).\mathbf{w}_N\|_{1,\omega} \leqslant (c_0/4)\,\|\mathbf{w}_N\|_{1,\omega} \quad .$

3) By (VI.4), (VI.14) and (VI.17), we know that

$$\|\mathbf{T}_N(\mathbf{DG}-\mathbf{DG}_N)(\mathbf{u}_N^*).\mathbf{w}_N\|_{1,\omega} = \sum_{1 \leqslant i \leqslant 2} [\|(1-\mathcal{J}_N)(w_{Ni}\mathbf{u}_N^*)\|_{0,\omega} + \|(1-\mathcal{J}_N)(u_{Ni}^*\mathbf{w}_N)\|_{0,\omega}] \quad .$$

From (VI.23), we derive for any $\varepsilon > 0$

$$\|\mathbf{T}_N(\mathbf{DG}-\mathbf{DG}_N)(\mathbf{u}_N^*).\mathbf{w}_N\|_{1,\omega} \leqslant c\,N^{\varepsilon-1}\,\|\mathbf{w}_N\|_{1,\omega}\,\|\mathbf{u}_N^*\|_{1,\omega} \quad ,$$

whence

(VI.29) $\quad \|\mathbf{T}_N(\mathbf{DG}-\mathbf{DG}_N)(\mathbf{u}_N^*).\mathbf{w}_N\|_{1,\omega} \leqslant c\,N^{\varepsilon-1}\,\|\mathbf{u}\|_{s,\omega}\,\|\mathbf{w}_N\|_{1,\omega} \quad .$

Finally, we conclude from (VI.25) to (VI.29) that, for N large enough,

$$\forall\, \mathbf{w}_N \in X_N\,, \quad \|\mathbf{DF}_N(\mathbf{u}_N^*).\mathbf{w}_N\|_{1,\omega} \geqslant (c_0/4)\,\|\mathbf{w}_N\|_{1,\omega} \quad ,$$

which proves the proposition.

<u>Lemma VI.4</u> : *The constant* $L_N(\eta)$ *satisfies*

(VI.30) $\quad L_N(\eta) \leqslant c\,\eta \quad .$

Proof : Let $\mathbf{w}_N$ be any element in X_N. From the linearity of $\mathbf{DG}_N$ and $\mathbf{DG}$, we write

$$\|\mathbf{DF}_N(\mathbf{w}_N)-\mathbf{DF}_N(\mathbf{u}_N^*)\|_{\mathcal{L}(X_N,X_N)} = \|\mathbf{T}_N(\mathbf{DG}_N(\mathbf{w}_N-\mathbf{u}_N^*))\|_{\mathcal{L}(X_N,X_N)}$$
$$\leqslant \|\mathbf{T}_N(\mathbf{DG}(\mathbf{w}_N-\mathbf{u}_N^*))\|_{\mathcal{L}(X_N,X_N)} + \|\mathbf{T}_N(\mathbf{DG}-\mathbf{DG}_N)(\mathbf{w}_N-\mathbf{u}_N^*)\|_{\mathcal{L}(X_N,X_N)} \quad .$$

Using (VI.17) and the continuity of the operator $\mathbf{DG}$ (see Lemma VI.1) yields that

$$\|\mathbf{T}_N(\mathbf{DG}(\mathbf{w}_N-\mathbf{u}_N^*))\|_{\mathcal{L}(X_N,X_N)} \leqslant c\,\|\mathbf{DG}(\mathbf{w}_N-\mathbf{u}_N^*)\|_{\mathcal{L}(X_N,X')} \leqslant c'\,\|\mathbf{w}_N-\mathbf{u}_N^*\|_{1,\omega} \quad .$$

On the other hand, as in the proof of Proposition VI.3, we know that, for any $\mathbf{z}_N$ in X_N,

$$\| T_N(DG-DG_N)(w_N-u_N^*).z_N \|_{1,\omega}$$
$$= \sum_{1\leqslant i\leqslant 2} [\|(1-\mathcal{J}_N)[(w_{Ni}-u_{Ni}^*)z_N]\|_{0,\omega} + \|(1-\mathcal{J}_N)[z_{Ni}(w_N-u_N^*)]\|_{0,\omega}] \quad ,$$

so that, by (VI.23) and for any $\varepsilon > 0$,

$$\|T_N(DG-DG_N)(w_N-u_N^*)\|_{\mathcal{L}(X_N,X_N)} \leqslant c\, N^{\varepsilon-1} \|w_N-u_N^*\|_{1,\omega} \quad .$$

These two inequalities, together with the definition (VI.20) of $L_N(\eta)$, imply (VI.30).

<u>Lemma VI.5</u> : *The constant* ε_N *satisfies*

(VI.31) $\quad \varepsilon_N \leqslant c(\mathbf{u})\, N^{1-s} + c(\mathbf{f})\, N^{-\sigma} \quad .$

Proof : From (VI.10) we write $\mathbf{F}_N(\mathbf{u}_N^*)$ in the form

$$\mathbf{F}_N(\mathbf{u}_N^*) = \mathbf{u}_N^* + \mathbf{T}_N\mathbf{G}_N(\mathbf{u}_N^*) - \mathbf{u} - \mathbf{T}\mathbf{G}(\mathbf{u})$$
$$= (\mathbf{u}_N^*-\mathbf{u}) + (\mathbf{T}_N-\mathbf{T})\mathbf{G}(\mathbf{u}) + \mathbf{T}_N(\mathbf{G}(\mathbf{u}_N^*)-\mathbf{G}(\mathbf{u})) + \mathbf{T}_N(\mathbf{G}_N(\mathbf{u}_N^*)-\mathbf{G}(\mathbf{u}_N^*)) \quad ,$$

which gives

(VI.32) $\quad \varepsilon_N \leqslant \|\mathbf{u}-\mathbf{u}_N^*\|_{1,\omega} + \|(\mathbf{T}-\mathbf{T}_N)\mathbf{G}(\mathbf{u})\|_{1,\omega} + \|\mathbf{T}_N(\mathbf{G}(\mathbf{u})-\mathbf{G}(\mathbf{u}_N^*))\|_{1,\omega}$
$\qquad + \|\mathbf{T}_N(\mathbf{G}(\mathbf{u}_N^*)-\mathbf{G}_N(\mathbf{u}_N^*))\|_{1,\omega} \quad .$

It remains to estimate these four terms.

1) Using (VI.22) yields

(VI.33) $\quad \|\mathbf{u}-\mathbf{u}_N^*\|_{1,\omega} \leqslant c\, N^{1-s} \|\mathbf{u}\|_{s,\omega} \quad .$

2) It follows from (VI.19) that

$$\|(\mathbf{T}-\mathbf{T}_N)\mathbf{G}(\mathbf{u})\|_{1,\omega} \leqslant c\, N^{1-s} \|\mathbf{T}\mathbf{G}(\mathbf{u})\|_{s,\omega} \quad ,$$

whence, thanks to (VI.10),

(VI.34) $\quad \|(\mathbf{T}-\mathbf{T}_N)\mathbf{G}(\mathbf{u})\|_{1,\omega} \leqslant c\, N^{1-s} \|\mathbf{u}\|_{s,\omega} \quad .$

3) Due to (VI.17) and to the continuity of $\mathbf{G}$ (see Lemma VI.1), we have

$$\|\mathbf{T}_N(\mathbf{G}(\mathbf{u})-\mathbf{G}(\mathbf{u}_N^*))\|_{1,\omega} \leqslant c\, \|\mathbf{u}-\mathbf{u}_N^*\|_{1,\omega} \quad ,$$

so that

(VI.35) $\quad \|\mathbf{T}_N(\mathbf{G}(\mathbf{u})-\mathbf{G}(\mathbf{u}_N^*))\|_{1,\omega} \leqslant c\, N^{1-s} \|\mathbf{u}\|_{s,\omega} \quad .$

4) From (VI.17), we derive

$$\|\mathbf{T}_N(\mathbf{G}(\mathbf{u}_N^*)-\mathbf{G}_N(\mathbf{u}_N^*))\|_{1,\omega} \leqslant c \sup_{\mathbf{v}_N \in X_N} \frac{((\mathbf{G}(\mathbf{u}_N^*)-\mathbf{G}_N(\mathbf{u}_N^*),\mathbf{v}_N)_\omega}{\|\mathbf{v}_N\|_{1,\omega}} \quad .$$

Using the definitions (VI.4) and (VI.14) of $\mathbf{G}$ and $\mathbf{G}_N$, we have for any $\mathbf{v}_N$ in X_N

$$(\mathbf{G}(\mathbf{u}_N^*)-\mathbf{G}_N(\mathbf{u}_N^*),\mathbf{v}_N)_\omega$$
$$= \sum_{1\leqslant i\leqslant 2} [(u_{Ni}^*\mathbf{u}_N^*,\, \omega^{-1}\, \partial(\omega\mathbf{v}_N)/\partial x_i)_{\omega,N} - (u_{Ni}^*\mathbf{u}_N^*,\, \omega^{-1}\, \partial(\omega\mathbf{v}_N)/\partial x_i)_\omega]$$
$$- (\mathbf{f},\mathbf{v}_N)_\omega + (\mathbf{f},\mathbf{v}_N)_{\omega,N} \quad .$$

But, since $u_{Ni} * \mathbf{u}_N^*$ belongs to $[P_{N-1}(\Omega)]^2$ and $\omega^{-1}\, \partial(\omega \mathbf{v}_N)/\partial x_i$ belongs to $[P_N(\Omega)]^2$, the exact and discrete scalar products coincide. Hence, we obtain from (V.48), (IV.46) and (V.49)

$$|(\mathbf{G}(\mathbf{u}_N^*)-\mathbf{G}_N(\mathbf{u}_N^*),\mathbf{v}_N)_\omega| = |(\mathbf{f},\mathbf{v}_N)_\omega - (\mathbf{f},\mathbf{v}_N)_{\omega,N}| \leqslant c\, N^{-\sigma}\, \|\mathbf{f}\|_{\sigma,\omega}\, \|\mathbf{v}_N\|_{0,\omega} \quad ,$$

so that

$$\text{(VI.36)} \quad \|\mathbf{T}_N(\mathbf{G}(\mathbf{u}_N^*)-\mathbf{G}_N(\mathbf{u}_N^*))\|_{1,\omega} \leqslant c\, N^{-\sigma}\, \|\mathbf{f}\|_{\sigma,\omega} \quad .$$

Finally, we derive the desired estimate for ε_N from inequalities (VI.32) to (VI.36).

VI.4. Existence result and convergence estimates.

We can now prove the main result of this section.

Theorem VI.3 : *Assume that there exists a solution* $(\mathbf{u},p)$ *of the Navier-Stokes equations* (VI.1)(VI.2) *such that the operator* $\mathbf{1} + \mathbf{TDG}(\mathbf{u})$ *is an isomorphism of* X *; assume moreover that it belongs to* $[H^s(\Omega)]^2 \times H^{s-1}(\Omega)$ *for a real number* $s > 1$ *and that the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma > 1$. *For* N *large enough, problem* (VI.12) *admits a solution* $(\mathbf{u}^N,p^N)$ *in* $X_N \times M_{1N}$ *where* M_{1N} *is defined by* $(V.30)_1$ *(resp. in* $X_N \times \tilde{M}_{1N}$ *, where* $\tilde{M}_{1N}$ *satisfies the hypothesis* (V.35)*). Moreover, the approximate velocity* $\mathbf{u}^N$ *satisfies the convergence estimate*

$$\text{(VI.37)} \quad \|\mathbf{u}-\mathbf{u}^N\|_{1,\omega} \leqslant c(\mathbf{u})\, N^{1-s} + c(\mathbf{f})\, N^{-\sigma}$$

for constants $c(\mathbf{u})$ *and* $c(\mathbf{f})$ *independent of* N.

Proof : Using Proposition VI.3 and Lemmas VI.4 and VI.5, we notice that

$$2\gamma_N\, L_N(2\gamma_N\varepsilon_N) \leqslant 2\gamma\, .\, (\, c(\mathbf{u})\, N^{1-s} + c(\mathbf{f})\, N^{-\sigma}\,) \quad ,$$

so that we have :

$$\lim_{N\to\infty}\; 2\gamma_N\, L_N(2\gamma_N\varepsilon_N) = 0 \quad ,$$

and the assumptions of Theorem VI.2 are satisfied for N large enough. Hence, for each η such that

$$2\gamma\, L_N(\eta) < 1 \quad ,$$

there exists a unique solution $\mathbf{u}^N$ of (VI.16) in the ball $B_N = \{\, \mathbf{w}_N \in X_N \;;\; \|\mathbf{w}_N-\mathbf{u}_N^*\|_{1,\omega} \leqslant \eta \,\}$. Next, from (VI.21), we derive the estimate

$$\|\mathbf{u}^N-\mathbf{u}_N^*\|_{1,\omega} \leqslant c\, \|\mathbf{F}_N(\mathbf{u}_N^*)\|_{1,\omega} \quad ,$$

which, together with (VI.22) and Lemma VI.5, yields (VI.37).

Next, by Proposition V.3 (resp. Proposition V.5), there exists a unique p^N in M_{1N}, where M_{1N} is defined by $(V.30)_1$ (resp. in $\tilde{M}_{1N}$, where $\tilde{M}_{1N}$ satisfies the hypothesis (V.35)) such that

$$\forall\, \mathbf{v} \in X_N \,, \quad b_{1N}(\mathbf{v},p^N) = -\, a_N(\mathbf{u}^N,\mathbf{v}) - (\mathbf{G}_N(\mathbf{u}^N),\mathbf{v})_\omega$$

and the pair $(\mathbf{u}^N,p^N)$ is a solution of problem (VI.12).

Remark VI.1 : The error bound we obtain is the same as for the Stokes problem ; it is still optimal with respect to the regularity of the solution and of the data.

In order to state an error bound for the pressure, we need a lemma.

Lemma VI.6 : *The approximate velocity* $\mathbf{u}^N$, *as defined in Theorem* VI.3, *satisfies*

$$(VI.38) \quad \sup_{\mathbf{v}_N \in X_N} \frac{(\mathbf{G}(\mathbf{u})-\mathbf{G}_N(\mathbf{u}^N),\mathbf{v}_N)_\omega}{\|\mathbf{v}_N\|_{1,\omega}} \leqslant c(\mathbf{u})\, N^{1-s} + c(\mathbf{f})\, N^{-\sigma} \quad .$$

Proof : Let $\mathbf{v}_N$ be any element in X_N . We compute

$$(\mathbf{G}(\mathbf{u})-\mathbf{G}_N(\mathbf{u}^N),\mathbf{v}_N)_\omega = (\mathbf{G}(\mathbf{u})-\mathbf{G}(\mathbf{u}^N),\mathbf{v}_N)_\omega + (\mathbf{G}(\mathbf{u}^N)-\mathbf{G}_N(\mathbf{u}^N),\mathbf{v}_N)_\omega \quad .$$

Lemma VI.1 and (VI.37) give at once

$$|\,(\mathbf{G}(\mathbf{u})-\mathbf{G}(\mathbf{u}^N),\mathbf{v}_N)_\omega\,| \leqslant (\, c(\mathbf{u})\, N^{1-s} + c(\mathbf{f})\, N^{-\sigma}\,)\, \|\mathbf{v}_N\|_{1,\omega} \quad .$$

From the definitions (VI.4) and (VI.14) of $\mathbf{G}$ and $\mathbf{G}_N$, we obtain

$$|\,(\mathbf{G}(\mathbf{u}^N)-\mathbf{G}_N(\mathbf{u}^N),\mathbf{v}_N)_\omega\,| \leqslant [\textstyle\sum_{1\leqslant i\leqslant 2} \|\,(1-\mathcal{J}_N)(u_i^N\mathbf{u}^N)\|_{0,\omega}]\, \|\mathbf{v}_N\|_{1,\omega} + |\,(\mathbf{f},\mathbf{v}_N)_\omega - (\mathbf{f},\mathbf{v}_N)_{\omega,N}\,| \quad ,$$

and using (VI.24), (IV.29), (V.48), (IV.46) and (V.49) yields for $\varepsilon > 0$

$$|\,(\mathbf{G}(\mathbf{u}^N)-\mathbf{G}_N(\mathbf{u}^N),\mathbf{v}_N)_\omega\,| \leqslant (\, c(\mathbf{u})\, N^{\varepsilon-s} + c\, N^{-\sigma}\, \|\mathbf{f}\|_{\sigma,\omega}\,)\, \|\mathbf{v}_N\|_{1,\omega} \quad .$$

Finally, these two bounds imply (VI.38).

Theorem VI.4 : *Assume that hypotheses* (V.34) *and* (V.35) *hold and that there exists a solution* $(\mathbf{u},p)$ *of the Navier-Stokes equations* (VI.1)(VI.2) *such that the operator* $\mathbf{1} + \mathbf{TDG}(\mathbf{u})$ *is an isomorphism of* X *; assume moreover that it belongs to* $[H^s(\Omega)]^2 \times H^{s-1}(\Omega)$ *for a real number* $s > 1$ *and that the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma > 1$. *Then, the approximate pressure* p^N *in* $\tilde{M}_{1N}$, *as defined in Theorem* VI.3, *satisfies the convergence estimate*

$$(VI.39) \quad \|p-p^N\|_{0,\omega} \leqslant c(\mathbf{u},p)\, N^{3-s} + c(\mathbf{f})\, N^{2-\sigma}$$

for constants $c(\mathbf{u},p)$ *and* $c(\mathbf{f})$ *independent of* N.

Proof : Let us introduce the solution $(\tilde{\mathbf{u}}^N,\tilde{p}^N)$ in $X_N \times M_N$ of the following problem :

$$(VI.40) \quad \left|\begin{array}{l} \forall\, \mathbf{v} \in X_N\,, \quad a_N(\tilde{\mathbf{u}}^N,\mathbf{v}) + b_1(\mathbf{v},\tilde{p}^N) + (\mathbf{G}(\mathbf{u}),\mathbf{v})_\omega = 0 \quad , \\ \forall\, q \in P_N(\Omega), \quad b_2(\tilde{\mathbf{u}}^N,q) = 0 \quad . \end{array}\right.$$

Since $\tilde{\mathbf{u}}^N$ is equal to $-\,\mathbf{T}_N\mathbf{G}(\mathbf{u})$, we deduce from (VI.19) that

$$(VI.41) \quad \|\mathbf{u}-\tilde{\mathbf{u}}^N\|_{1,\omega} \leqslant c(\mathbf{u})\, N^{1-s} \quad ,$$

moreover, we obtain (see Remark V.3)

$$(VI.42) \quad \|p-\tilde{p}^N\|_{0,\omega} \leqslant c(\mathbf{u},p)\, N^{3-s} \quad .$$

Next, due to (VI.40) and (VI.15), we notice that, for any $\mathbf{v}_N$ in X_N,

$$b_1(\mathbf{v}_N\,,\,p^N-\tilde{p}^N) = a_N(\tilde{\mathbf{u}}^N,\mathbf{v}_N) + (\mathbf{G}(\mathbf{u}),\mathbf{v}_N)_\omega - a_N(\mathbf{u}^N,\mathbf{v}_N) - (\mathbf{G}_N(\mathbf{u}^N),\mathbf{v}_N)_\omega \quad ,$$

so that, from Proposition V.5, we deduce

$$\text{(VI.43)} \qquad \|p^N-\tilde{p}^N\|_{0,\omega} \leqslant c\,N^2 \sup_{\mathbf{v}_N \in X_N} \frac{a_N(\mathbf{u}^N-\tilde{\mathbf{u}}^N,\mathbf{v}_N) + (\mathbf{G}(\mathbf{u})-\mathbf{G}_N(\mathbf{u}^N),\mathbf{v}_N)_\omega}{\|\mathbf{v}_N\|_{1,\omega}} \quad .$$

Let $\mathbf{v}_N$ be any element in X_N. By the uniform continuity of a_N, we have

$$a_N(\mathbf{u}^N-\tilde{\mathbf{u}}^N,\mathbf{v}_N) \leqslant c\,\|\mathbf{u}^N-\tilde{\mathbf{u}}^N\|_{1,\omega}\,\|\mathbf{v}_N\|_{1,\omega} \leqslant c\,(\,\|\mathbf{u}-\tilde{\mathbf{u}}^N\|_{1,\omega} + \|\mathbf{u}-\mathbf{u}^N\|_{1,\omega}\,)\;\|\mathbf{v}_N\|_{1,\omega} \quad ,$$

so that

$$\text{(VI.44)} \qquad a_N(\mathbf{u}^N-\tilde{\mathbf{u}}^N,\mathbf{v}_N) \leqslant (\,c(\mathbf{u})\,N^{1-s} + c(\mathbf{f})\,N^{-\sigma}\,)\,\|\mathbf{v}_N\|_{1,\omega} \quad .$$

Now using (VI.44) and (VI.38) in (VI.43) yields

$$\text{(VI.45)} \qquad \|p^N-\tilde{p}^N\|_{0,\omega} \leqslant c(\mathbf{u})\,N^{3-s} + c(\mathbf{f})\,N^{2-\sigma} \quad ,$$

which, together with (VI.42), gives (VI.39).

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Standard Bibliographic Page

1. Report No. NASA CR-178220 ICASE Report No. 86-61	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle GENERALIZED INF-SUP CONDITION FOR CHEBYSHEV APPROXIMATION OF THE NAVIER-STOKES EQUATIONS		5. Report Date December 1986
		6. Performing Organization Code
7. Author(s) Christine Bernardi, Claudio Canuto, Yvon Maday		8. Performing Organization Report No. 86-61
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No.
		11. Contract or Grant No. NAS1-18107
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code 505-90-21-01

15. Supplementary Notes

Langley Technical Monitor: J. C. South

Submitted to SIAM J. Numerical Analysis

Final Report

16. Abstract

We study an abstract mixed problem and its approximation; both are well-posed if and only if several inf-sup conditions are satisfied. These results are applied to a spectral Galerkin method for the Stokes problem in a square, when it is formulated in Chebyshev weighted Sobolev spaces. Finally, a collocation method for the Navier-Stokes equations at Chebyshev nodes is analyzed.

17. Key Words (Suggested by Authors(s)) Navier-Stokes equations, Chebyshev collocation, inf-sup conditions		18. Distribution Statement 64 - Numerical Analysis Unclassified - unlimited	
19. Security Classif.(of this report) Unclassified	20. Security Classif.(of this page) Unclassified	21. No. of Pages 67	22. Price A04