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THE THERMAL-VORTEX EQUATIONS

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ABSTRACT

The Boussinesq approximation is extended so as to explicitly account for the transfer of fluid energy through viscous action into thermal energy. Ideal and dissipative integral invariants are discussed, in addition to the general equations for thermal-fluid motion.

INTRODUCTION

In this paper the equations of motion for a thermo-gravitationally stratified fluid are developed. These equations are pertinent to atmospheric and oceanic flows in which temperature variations affect fluid flow through boundary forces and in which fluid motion affects temperature variations through viscous dissipation (in addition to convection). The equations developed here are an extension of the Boussinesq approximation in that they explicitly account for the transfer of fluid energy through viscous action into thermal energy. These equations thus conserve physical energy, while the Boussinesq equations do not.

This is not to say there are no conserved quantities in the Boussinesq approximation. In 2-D flows there are two: the first, sometimes called the energy, involves the velocity squared and the temperature (or density) perturbation squared. This is not strictly the physical energy, which is linear in the temperature perturbation. The second conserved quantity in 2-D flow, apparently unrecognized, is essentially the spatial correlation of the vorticity and temperature perturbation.

Here the general equations for thermal-fluid motion will be discussed first. Then the two-dimensional version of these will be detailed. Next, the standard Boussinesq approximation will be discussed; in particular, the two-dimensional form. Finally a summary and conclusion will be given.

GENERAL EQUATIONS[3]

In this work, the fluid under consideration will be treated as incompressible, i.e., density change with respect to pressure is negligible; however, density will change with temperature. The fluid density can then be written as

$$\rho = \rho_0 + \rho' \tag{1}$$

where ρ_0 is a constant and ρ' is the density variation. Similarly, the temperature is given by

$$T = T_0 + T' \tag{2}$$

where T_0 is a constant and T' is the temperature variation. The relation between ρ' and T' is linear and is given by

$$\rho' = -\rho_0 \beta T' \tag{3}$$

where β is the thermal-expansion coefficient of the fluid (at T_0). The pressure, in turn, is given by

$$p = p_0 + p' \tag{4}$$

where p' is the pressure variation and

$$p_0 = \rho_0 \vec{g} \cdot \vec{r} \tag{5}$$

Here $\vec{g}$ is the acceleration due to gravity and $\vec{r}$ is a position vector. In the present case, the Navier-Stokes equation must include the effects of gravity:

$$\frac{\partial \vec{v}}{\partial t} + v \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \vec{g} \tag{6}$$

(here ν is the kinematic viscosity). If we now place (1), (3), (4) and (5) into (6) and neglect second and higher order quantities, we arrive at

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla(p'/\rho_0) + \nu \nabla^2 \vec{v} - \beta T' \vec{g} \tag{7}$$

This, along with the incompressibility condition

$$\nabla \cdot \vec{v} = 0 \tag{8}$$

form the velocity equations of the Boussinesq approximation[1].

Now let us consider the temperature (and by eq. 3, the density) evolution. (It is here that we differ from the standard Boussinesq approximation and use a more general form). The general equation for temperature evolution in an incompressible fluid is

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \chi \nabla^2 T + \frac{\nu}{2C_p} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 \tag{9}$$

where $\chi = \frac{\kappa}{\rho C_p}$ is the thermometric conductivity (cm^2/sec), κ is the thermal conductivity, and C_p is the specific heat (at constant pressure) of the fluid. The last term in (9) is

$$\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 = \frac{\partial v_i}{\partial x_k}\frac{\partial v_i}{\partial x_k} + \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i} \tag{10}$$

where the summation convention is in effect (e.g., $v_i w_i \equiv v_1 w_1 + v_2 w_2 + v_3 w_3$).

At this point, let us write the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{v} = 0 \tag{11}$$

Expanding the divergence term yields

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla\rho = -\rho \nabla \cdot \vec{v} \tag{12}$$

Using (1) and keeping only the lowest order terms (remember that $\nabla \cdot v \approx 0$) gives

$$\frac{\partial \rho'}{\partial t} + \vec{v} \cdot \nabla\rho' = -\rho_0 \nabla \cdot \vec{v} \tag{13}$$

Now, because of (2), equation (9) may be written

$$\frac{\partial T'}{\partial t} + \vec{v} \cdot \nabla T' = \chi \nabla^2 T' + \frac{\nu}{2C_p}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 \tag{14}$$

Using (3), a comparison of (13) and (14) yields

$$\nabla \cdot \vec{v} = \beta\left[\chi \nabla^2 T' + \frac{\nu}{2C_p}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2\right] \tag{15}$$

The term within brackets in (15) and on the right hand side of (14) is small compared to the convective term ($\vec{v} \cdot \nabla T'$); also, the thermal expansion coefficient $\beta < 10^{-2}$. The product of β and the bracketed term in (15) is

therefore of a higher order than any terms being retained in this theoretical development. It is for this reason that we accept the incompressibility condition ($\nabla \cdot \vec{v} = 0$), i.e., any possible contribution of $\nabla \cdot \vec{v}$ to equation (7) is negligible. These assertions will be made quantitative in the next section.

Let us now examine the energy conservation implicit in (7), (8) and (14). To begin, we use the vector identity

$$\vec{v} \times (\nabla \times \vec{v}) = \frac{1}{2} \nabla v^2 - \vec{v} \cdot \nabla \vec{v} \tag{16}$$

to change (7) into

$$\frac{\partial \vec{v}}{\partial t} = \vec{v} \times \vec{\omega} - \nabla(p'/\rho_0 + v^2/2) + \nu \nabla^2 \vec{v} - \beta T' \vec{g} \tag{17}$$

where the vorticity is $\vec{\omega} \equiv \nabla \times \vec{v}$. Now, (17) is multiplied by $\vec{v}$. , (14) by C_p and the resulting equations are added, resulting in

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2} v^2 + C_p T'\right) = -\nabla \cdot \left[\vec{v}\left(\frac{p'}{\rho_0} + \frac{v^2}{2} + C_p T'\right) - C_p \chi \nabla T'\right] \\ + \nu[\vec{v} \cdot \nabla^2 \vec{v} + (\partial_k v_i)(\partial_k v_i) + (\partial_k v_i)(\partial_i v_k)] \\ - [\beta T' \vec{v} \cdot \vec{g}] \end{aligned} \tag{18}$$

(Here $\partial/\partial x_k \equiv \partial_k$ for convenience). At this point we must treat the last two bracketed terms on the right hand side of (18) in more detail.

First, consider the term whose coefficient is ν :

$$\begin{aligned} \vec{v} \cdot \nabla^2 \vec{v} + (\partial_k v_i)(\partial_k v_i) + (\partial_k v_i)(\partial_i v_k) \\ = v_i \partial_k \partial_k v_i \\ + \partial_k(v_i \partial_k v_i) - v_i \partial_k \partial_k v_i \\ + \partial_k(v_i \partial_i v_k) - v_i \partial_k \partial_i v_k \\ = \partial_k(v_i \partial_k v_i + v_i \partial_i v_k) \\ = \partial_k\left(\frac{1}{2} \partial_k v^2 + v_i \partial_i v_k\right) \end{aligned} \tag{19}$$

Here $\nabla \cdot \vec{v} = \partial_k v_k = 0$ is the incompressibility condition. Using $\vec{g} = -\nabla\phi$, the last term on the right hand side of (18) is

$$-\beta T' \vec{v} \cdot \vec{g} = \frac{\rho'}{\rho_0} \vec{v} \cdot \vec{g}$$

$$= \frac{\rho'}{\rho_0^2} \vec{v} \cdot \nabla(\rho_0 \vec{g} \cdot \vec{r})$$

$$= \frac{\rho'}{\rho_0^2} \vec{v} \cdot \nabla p_0$$

$$= \nabla \cdot \left(\frac{\rho' p_0}{\rho_0^2} \vec{v}\right) - \frac{p_0}{\rho_0^2} \nabla \cdot (\rho' \vec{v}) \qquad (20)$$

Using the continuity condition, (20) becomes

$$-\beta T' \vec{v} \cdot \vec{g} = \nabla \cdot \left(\frac{\rho' p_0}{\rho_0^2} \vec{v}\right) + \frac{p_0}{\rho_0^2} \frac{\partial \rho'}{\partial t}$$

$$= \frac{\partial}{\partial t} \left(\frac{p_0}{\rho_0^2} \rho'\right) + \nabla \cdot \left(\frac{\rho' p_0}{\rho_0^2} \vec{v}\right)$$

$$= -\frac{\beta}{\rho_0} \left[\frac{\partial}{\partial t} (p_0 T') + \nabla \cdot (p_0 T' \vec{v})\right] \qquad (21)$$

Now, using (19) and (21), equation (18) becomes

$$\frac{\partial}{\partial t} \left[\frac{1}{2} v^2 + \left(C_p + \frac{\beta}{\rho_0} p_0\right) T'\right] =$$

$$-\nabla \cdot \left[\vec{v} \left(\frac{p'}{\rho_0} + \frac{v^2}{2} + \left\{C_p + \frac{\beta}{\rho_0} p_0\right\} T'\right) - \frac{\kappa}{\rho_0} \nabla T' - \nu \left(\frac{1}{2} \nabla v^2 + \vec{v} \cdot \nabla \vec{v}\right)\right] \qquad (22)$$

This energy conservation equation is accurate up to first order in terms such as p' and T'. Multiplying both sides of (22) by ρ_0 and integrating this expression over the volume of interest, V, we arrive at the following

$$\frac{d}{dt} E = -\iint_S \vec{F} \cdot d\vec{S} \qquad (23)$$

where the total energy in volume V is

$$E = \int_V \left[\frac{1}{2} \rho_0 v^2 + (\rho_0 C_p + \beta p_0) T'\right] dv \qquad (24)$$

and the energy flux is

$$\vec{F} = \vec{v}\,[\rho' + \rho_0 V^2/2 + (C_p + \beta p_0)\,T'] - \kappa \nabla T' - \eta\,(\frac{1}{2}\nabla v^2 + \vec{v} \cdot \nabla \vec{v}) \qquad (25)$$

where the viscosity is $\eta = \rho_0 \nu$. Note that, in (24),

$\frac{1}{2}\rho_0 v^2$: Kinetic energy density

$\rho_0 C_p T'$: Thermal energy density

$\beta \rho_0 T' = -\rho' g z$: Gravitational potential energy density (26)

Several points should be noted concerning the energy conservation formulas (23), (24) and (25). First, the reference point for the total energy is arbitrary; we can add a constant E_0 to E in (24) without changing (23). Second, if the volume V is large enough so that there is no energy flow through the bounding surface (or if the flow is periodic or balanced across the boundaries) then the surface integral in (23) is zero and dE/dt = 0.

In the next section, the equations will be written in non-dimensional form. This will allow for the relative size of various terms to be determined and establish criteria for deciding on the similarity of flow in different physical situations. Application of these results to flows within the NASA/LaRC Vortex Research Facility will provide a quantitative example.

NONDIMENSIONAL EQUATIONS

At this point we will introduce the presence of a temperature gradient. The fluid, at rest, will have a uniform vertical temperature gradient γ (°C/cm); the temperature variation will thus be composed of two parts:

$$T' = \gamma z + \tau \qquad (27)$$

Here τ is the temperature anomaly field. Placing (27) into (14) gives

$$\frac{\partial \tau}{\partial t} + \vec{v} \cdot \nabla \tau = -\vec{v} \cdot (\gamma \hat{z}) + \chi \nabla^2 \tau + \frac{\nu}{2C_p}(\partial_k v_i + \partial_i v_k)^2 \qquad (28)$$

This, along with the equation for fluid motion (7) and incompressibility condition (8) are the equations of motion for our thermal-fluid system; these are the equations we wish to put in nondimensional form.

In order to produce our nondimensional equations, we will use the following characteristic values:

Table I Characteristic Values

L: length

V: velocity

γ: temperature gradient

β: coefficient of thermal compressibility

g: gravitational acceleration

ν: kinematic viscosity

χ: thermometric conductivty

c_p: specific heat at constant pressure

$N = \sqrt{\beta g \gamma}$: Brunt-Vaisala frequency

The quantities in Table I can be combined into nondimensional groups[4]; these groups have values for specific flows which allow them to be compared to other flows. The pertinent groups are given in Table II.

Table II Dimensionless Groups

Reynolds number:	$R = \frac{VL}{\nu}$
Prandtl number:	$P = \frac{\nu}{\kappa}$
Strouhal number:	$S = \frac{V}{NL} = \frac{1}{N^*}$
Froude number:	$F = \frac{V^2}{Lg}$
Grashof number:	$G = (RN^*)^2$
DuLong number:	$Du = \frac{V^2}{\gamma L\, C_p}$

We may now take equations (7) and (14) and transform them into dimensionless equations. In these equations, $\vec{v}$ will be measured in terms of V, T' in terms of γL, (p'/ρ_0) in terms of gL, and time t in terms of N^{-1}. Remembering this, the equations are

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{N^*}\vec{v} \cdot \nabla\vec{v} = -\frac{1}{FN^*}\nabla\left(\frac{p'}{\rho_0}\right) + \frac{1}{RN^*}\nabla^2\vec{v} + N^*\, T'\,\hat{z} \tag{29}$$

$$\frac{\partial T}{\partial t} + \frac{1}{N^*}\vec{v} \cdot \nabla T' = \frac{1}{N^*RP}\nabla^2 T' + \frac{Du}{2N^*R}\left(\partial_k V_i + \partial_i V_k\right)^2 \tag{30}$$

In order to estimate the values of the coefficients in (29) and (30) for a practical situation, we have drawn the characteristic values from a typical experiment at the Vortex Research Facility at NASA/LaRC[5]. These are presented in Table III, along with the associated dimensionless numbers.

At this point we can estimate the various sizes of the terms in (29) and (30). The right hand sides of (29) and (30) are important in relation to the size of the $\vec{v} \cdot \nabla$ terms on the left sides, which have coefficients of $1/N^*$.

Table III Typical Values

$V = 50$ cm/sec

$L = 10$ cm

$\nu = .145$ cm^2/sec

$\chi = .202$ cm^2/sec

$\gamma = .01$°C/cm

$g = 980$ cm/sec^2

$\beta = 3.48 \times 10^{-3}$/°C

$C_p = 1.012 \times 10^7$ cm^2/(sec^2-°C)

$N = 0.185$ sec^{-1}

$R = 3450$

$P = 0.72$

$N^* = 0.037$

$Du = 0.0025$

$F = 0.255$

$G = 16300$

Thus, the relative sizes of the various terms to these "convective" or inertial terms is given by their coefficients with the $1/N^*$ factor divided out. These relative coefficients are given in Table IV.

Table IV Relative Coefficients

$\vec{v}$ equation	T equation
$\frac{1}{F} = 3.92$	$\frac{1}{RP} = 4.0 \times 10^{-4}$
$\frac{1}{R} = 2.9 \times 10^{-4}$	
$(N^*)^2 = 1.4 \times 10^{-3}$	$\frac{Du}{R} = 7.2 \times 10^{-7}$

At this point, let us estimate the validity of assuming incompressibility. If we nondimensionalize (15) we have

$$\nabla \cdot \vec{v} = \left(\frac{\beta\chi\gamma}{V}\right) \nabla^2 T' + \left(\frac{\nu\beta V}{C_p L}\right) \frac{1}{2} \left(\partial_k v_i + \partial_i v_k\right)^2$$

$$= \left(1.4 \times 10^{-7}\right) \nabla^2 T' + \left(2.5 \times 10^{-8}\right) \frac{1}{2} \left(\partial_k v_i + \partial_i v_k\right)^2$$

Thus, in a dimensionless representation where all variables and their spatial derivatives are nominally of unit magnitude, the error in assuming incompressibility is about one part in ten million.

TWO DIMENSIONAL FLOW

It is often useful (and computationally expedient) to approximate certain flows as being two dimensional. In particular, the evolution of vortex wakes trailing aircraft and submarines can be studied by examining the dynamics of the pertinent fluid in a cross-sectional plane transverse to the direction of motion. In this section we will reduce the general equations of motion (8), (17) and (28) to those appropriate to two-dimensional thermally-stratified flow.

As is well known, the incompressibility condition (8) can be automatically satisfied in 2-D flow by expressing the velocity in terms of a stream function $\psi(y,z)$:

$$\begin{aligned} \vec{v} &= \nabla \times (\hat{x}\psi) \\ &= -\hat{x} \times \nabla \psi \\ &= \hat{y} \frac{\partial \psi}{\partial z} - \hat{z} \frac{\partial \psi}{\partial y} \end{aligned} \tag{31}$$

The vorticity, in turn, can also be expressed in terms of the stream function:

$$\vec{\omega} = \nabla \times \vec{v} = \nabla \times (\nabla \times \hat{x}\,\psi)$$

$$= \nabla(\hat{x} \cdot \nabla\psi) - \hat{x}\nabla^2\psi$$

$$= -\hat{x}\,\nabla^2\,\psi \qquad (32)$$

The scalar vorticity is thus $\omega = -\nabla^2\,\psi$ where the operator $\nabla^2 \equiv \partial^2/\partial y^2 + \partial^2/\partial z^2$ because there is no x dependence in the 2-D approximation.

At this point let us take the curl of eq (17):

$$\frac{\partial\vec{\omega}}{\partial t} = \nabla\times(\vec{v} \times \vec{\omega}) + \nu\nabla^2\omega - \beta\nabla T' \times \vec{g} \qquad (33)$$

Similarly, the divergence of (17) gives

$$\nabla^2(\frac{p'}{\rho_0} + v^2/2) = \nabla \cdot (\vec{v} \times \vec{\omega}) - \beta\,\nabla T' \cdot \vec{g} \qquad (34)$$

These equations, (33) and (34), are still applicable to 3-D incompressible flow; in particular, they show that the pressure p' is given, at each instant of time, by the instantaneous values of $\vec{v}$ and T', both of which obey non-linear coupled evolution equations. If we now put (27), (31) and (32) into (28), (33) and (34), we get a set of equations for studying the 2-D flow of a thermally-stratified fluid:

$$\frac{\partial\omega}{\partial t} = (\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\omega}{\partial y}) + \nu\nabla^2\omega + \beta g\frac{\partial\tau}{\partial y} \qquad (35)$$

$$\frac{\partial\tau}{\partial t} = (\frac{\partial\psi}{\partial y}\frac{\partial\tau}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\tau}{\partial y}) + \chi\nabla^2\tau + \gamma\frac{\partial\psi}{\partial y}$$

$$+ \frac{\nu}{c_p}\,[4(\frac{\partial^2\psi}{\partial x\,\partial y})^2 + (\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2})^2] \qquad (36)$$

$$\nabla^2(p'/\rho_0) = \omega^2 - \frac{1}{2}\nabla^2(v^2) + \beta g(\gamma + \frac{\partial\tau}{\partial z}) \qquad (37)$$

We also note that the quantity conserved by these equations, for a closed system, comes from (24); this is the energy of the system (leaving off time independent terms):

$$E = \int_v [\tfrac{1}{2}\rho_0 v^2 + (\rho_0 C_p + \beta p_0)\tau]dydz \qquad (38)$$

We will call this assembly of equations, (35), (36) and (37), the thermal-vortex equations.

Let us note here that the energy given by (38) is conserved in the presence of viscosity and thermal conductivity. As has already been pointed out, the energy as given by (38) is explicitly composed of a kinetic part, a thermal part, and a gravitational part. This conserved quantity (integral invariant) E can thus be considered to be the actual physical energy of the fluid system we are modelling.

In comparison, let us look at the system of equations which are obtained when we set $\nu = \chi = 0$: the inviscid equations. It is well known that in this case, (35) and (36) conserve a certain quantity which has been denoted the 'energy'[2]; this energy E′ is

$$E' = \int_v \rho_o (\tfrac{1}{2}v^2 + \frac{\beta g}{\gamma} \tau^2)dydz \qquad (39)$$

This, however, is not the physical energy (although, in some computer models, it will be numerically close). In addition, with $\nu = \chi = o$, it is easy to show that, for a closed system, there is also another conserved quantity:

$$C = \int_v \omega\tau \, dydz \qquad (40)$$

The presence of two conserved quantities fundamentally changes the evolutionary behavior of a dynamic system[6]. If the physical system which is being modelled does not actually have two (or more) integral invariants, although our model system does, then the predictions we make with our model system must surely be suspect.

These model systems arise in a number of applications, notably inviscid hydrodynamics and magnetohydrodynamics. These are interesting models; their numerical realizations indeed give rise to very interesting statistical systems whose fundamental behavior depends on the number of integral invariants[7]. However, viscosity (and thermal conductivity) are always present in real systems so that the accurate modelling of any real system requires the incorporation of the proper dissipative mechanisms (viscosity and conductivity).

The presence of integral invariants allows us a numerical check of our simulations: we set $\nu = \chi = 0$ and see that the integral invariants are conserved: The real system has $\nu \neq 0$, $\chi \neq 0$; in this case there is only one "integral invariant": the total energy.

CONCLUSION

In this paper, the equations of motion suitable for studying the incompressible motion of a thermally stratified fluid were developed. It was shown that the two quantities which satisfy time evolution equations were the temperature anomaly and the vorticity. The equations were therefore termed the thermal-vortex equations.

These expressions (for a closed system) contain one integral invariant, which is clearly the physical energy. In contrast, the quantity commonly called the energy of the non-dissipative equations ($\nu = \chi = 0$) is not, in fact, the physical energy. In addition, the non-dissipative equations contain a second integral invariant. The presence of two invariants, plus the fact that neither is the actual energy, divorces the non-dissipative model from physical reality.

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16. Abstract

The Boussinesq approximation is extended so as to explicitly account for the transfer of fluid energy through viscous action into thermal energy. Ideal and dissipative integral invariants are discussed, in addition to the general equations for thermal-fluid motion.

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