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EDDY CURRENTS IN A CONDUCTING SPHERE*

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Abstract

This report analyzes the eddy currents induced in a solid conducting sphere by a sinusoidal current in a circular loop. Analytical expressions for the eddy currents are derived as a power series in the vectorial displacement of the center of the sphere from the axis of the loop. These are used for first order calculations of the power dissipated in the sphere and the force and torque exerted on the sphere by the electromagnetic field of the loop.

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Table of Notation

$\mathbf{E}$ → Electric Field

$\mathbf{B}$ → Magnetic Induction

$\mathbf{A}$ → Vector Potential

$\mathbf{J}$ → Current Density

$\mathbf{I}$ → Current

ω → Frequency

μ_o → Permeability of free space

δ → Skin depth

σ → Conductivity

$\hat{\mathbf{r}}$ → Unit Radial Vector

$\hat{\boldsymbol{\theta}}$ → Unit Latitudinal Vector

$\hat{\boldsymbol{\phi}}$ → Unit Azimuthal Vector

e_x, e_y, e_z → Unit Cartesian Vectors

$\mathbf{s}=(s,\Theta_s,\Phi_s)$, $\mathbf{c}=(c,\Theta_c,\Phi_c)$, $\mathbf{r}=(r,\Theta,\Phi)$ → Postion vectors

$\mathbf{d}$ → Displacement of sphere center from axis of loop

a → Radius of sphere

b → Radius of loop

$\mathbf{n}$ → Normal unit vector to loop

P_n, P_n^m → Legendre functions

J_n, K_n, I_n, Y_n → Bessel functions

$C_n, A_n, C_{nm}, A_{nm}, B_{nm}, D_{nm}$ → constants

δ_{ij} → Kronecker delta

i → Unit imaginary

X_{nm} → Vector Spherical Harmonics

Introduction

This report is an extension of research by David Sonnabend [1,2] on the control of suspended objects by eddy current forces, and it was carried out at his suggestion. The results are potentially useful in the design of a quasi drag free gradiometer to be used in artificial space satellites.

In this study we are concerned with effects on a conducting sphere of the electromagnetic field produced by a steady state alternating current in a circular coil. The oscillation field induces eddy currents in the conductor which dissipate power and interact with the field to produce a force and torque on the sphere. Our objective is to derive analytical expressions describing these effects. The central task is to solve the steady state electromagnetic boundary value problem for a conducting sphere in the oscillating field of the current loop. The problem was solved some time ago for the special case where the center of the sphere lies on the axis of the loop [3,4]. Tegopoulos and Kriezis [4] give a valuable survey of analytical work on problems of this type. Hannakam [5] has found an elegant closed solution to our general problem in terms of an integral over the loop. His result is summarized in Appendix M of this report. However, after studying the difficulties in evaluating his integral, we decided that it would be more practical to solve the problem ab initio by a different method. We studied a variety of promising approaches. Expansions in terms of Lame' polynomials [6] seemed most promising because these functions possess both the symmetries and asymmetries of our problem. Unfortunately, the mathematical theory of Lame' polynomials is not sufficiently well developed to make all the necessary calculations easy. We finally settled on expansions in terms of spherical harmonics in large measure because many theorems about these functions are available to facilitate calculations. This enabled us to find a practical (if not optimal) solution to our problem.

To make our boundary value problem well defined and analytically tractable, we adopted the following idealizations of the physical situation:

1) the coil is replaced by a single, circular current loop. If the coil needs to be more accurately modelled, this can be accomplished by considering it as a series of current loops and supercomposing the individual solutions.

2) The current in the loop is taken to vary sinusoidally in time. That is

$$I = I_o \cos \omega t = \text{Re}\,\{I_o e^{i\omega t}\}$$

I_o, ω are real constants.

3) The propagation of the fields is regarded as instantaneous, that is, the fields change slowly with respect to their propagation time to points within the domain of this problem. Thus, all points in the conductor "see" a field of the same phase at the same instant in time, and the displacement current can be neglected.

4) The conductor is isotropic, homogeneous, and non-magnetic with a relative permeability of one. Hence, within it Ohm's law ($J = \sigma E$) applies and hysteresis can be ignored.

5) The conductor is modelled as a solid sphere. This model is valid for a spherical shell as long as the skin depth is small relative to the thickness of the shell.

6) The displacement of the center of the sphere from the axis of the loop is considered small compared to the radius of the sphere. This allows the use of an expansion about a position on the axis of the loop.

7) Transcient effects are ignored and only steady state solutions are considered.

I. The electro-magnetic vector potential

The first thing we do is introduce the idealizations of our problem into Maxwell's equations to derive a boundary value problem for the vector potential **A**. Neglecting the displacement current, and considering materials with relative permeability of one, Maxwell's equations are:

$$(1)\quad \nabla\cdot\mathbf{E} = \rho/\varepsilon_o$$

$$(2)\quad \nabla\mathrm{x}\mathbf{B} = \mu_o\ \mathbf{J}$$

$$(3)\quad \nabla\mathrm{x}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$(4)\quad \nabla\cdot\mathbf{B} = 0$$

From $\nabla\cdot\mathbf{B} = 0$, it follows that **B** can be written as the curl of the vector potential **A**.

$$(5)\quad \mathbf{B} = \nabla\mathrm{x}\mathbf{A}$$

The condition $\nabla\cdot\mathbf{A} = 0$ can also be imposed to determine **A** uniquely. Substituting equation (5) into equation (3) yields

$$(6)\quad \nabla\mathrm{x}\mathbf{E} = -\frac{\partial}{\partial t}(\nabla\mathrm{x}\mathbf{A}) = \nabla\ \mathrm{x}\ \left(-\frac{\partial A}{\partial A}\right)$$

so we can take

$$(7)\quad \mathbf{E} = -\frac{\partial A}{\partial t}\ .$$

Assuming Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$ and equation (7) yields

$$(8)\quad \mathbf{J} = -\sigma\frac{\partial A}{\partial t}\ .$$

We assume that the sources of the field are stationary and the field propagates instantaneously so it must have the same time dependence as the current loop. Accordingly, the vector potential can be written

$$\mathbf{A}\,(r,t) = \mathbf{A}\,(r)\; e^{i\omega t}$$

where only the real part is of physical interest. The eddy current density then is proportional to the vector potential

(9) $\mathbf{J} = -i\omega\sigma\mathbf{A}$.

Substituting equations (5) and (9) into equation (2) we obtain

(10) $\nabla\times(\nabla\times\mathbf{A}) = i\mu_o\omega\sigma\mathbf{A} = k^2\mathbf{A}$

where $k \equiv i\mu_o\omega\sigma$. By virtue of the vector identity

$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

and the relation $\nabla\cdot\mathbf{A}=0$, equation (10) reduces to Helmholz's equation

(11a) $\nabla^2\mathbf{A} = k^2\mathbf{A}$

inside the sphere, and Laplace's equation

(11b) $\nabla^2\mathbf{A} = 0$

immediately outside the sphere where the conductivity is zero. The solutions to equations (11a) and (11b) must be matched at the boundary of the sphere. According to Smythe [3, p. 305] the following boundary conditions must be satisfied.

(1) $\mathbf{A}$ is continuous across the boundary

(2) The normal component of $\mathbf{B}$ is continuous across the boundary

(3) The tangential component of $\mathbf{H} = \mu\mathbf{B}$ is continuous across any boundary. For this case $\mu = \mu_o$ inside the sphere as well as outside of it so the tangential componet of $\mathbf{B}$ is also continuous across the boundary.

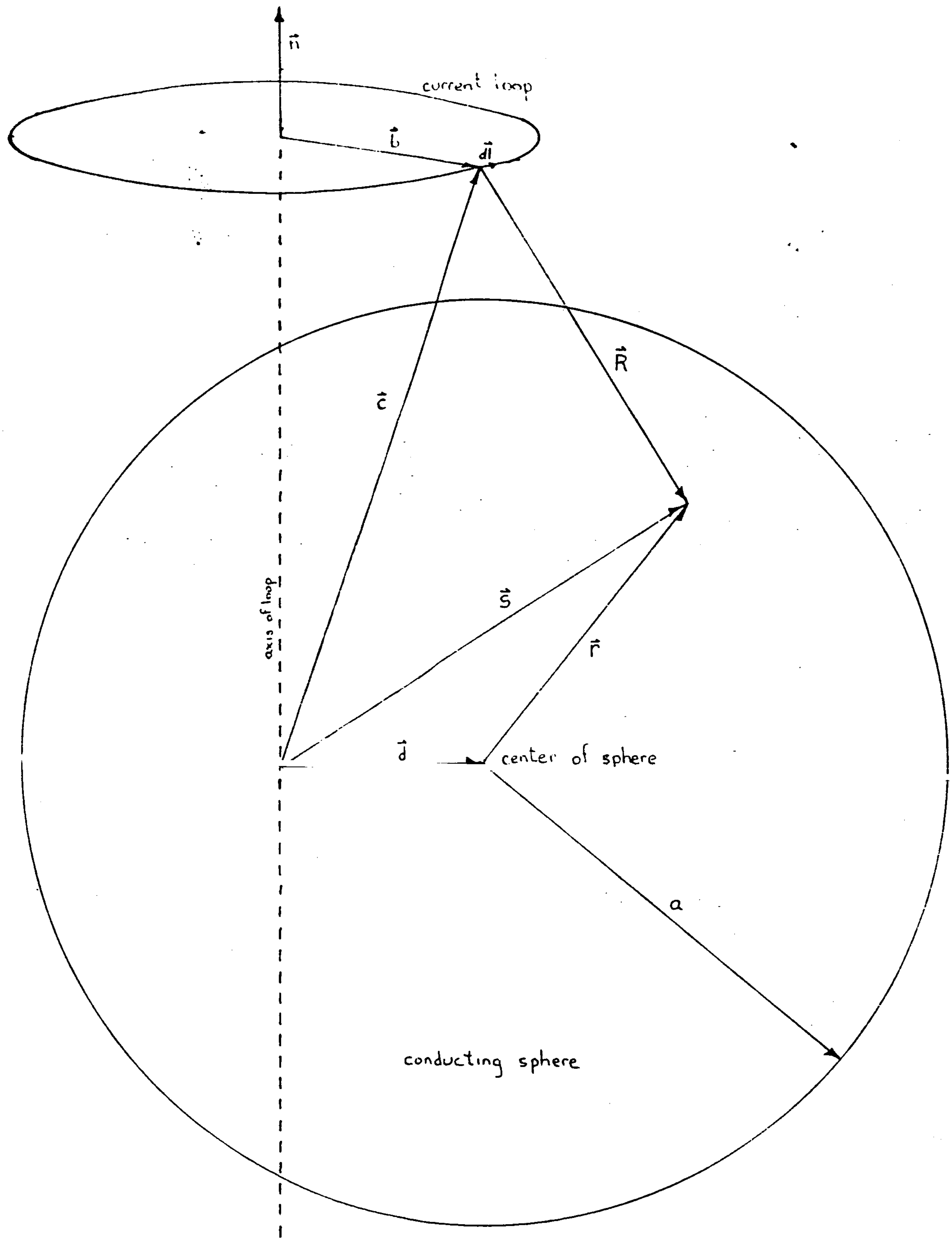
Thus,

(12a) $\mathbf{A}_o\big|_{r=a} = \mathbf{A}_i\big|_{r=a}$

(12b) $\nabla\times\mathbf{A}_o\big|_{r=a} = \nabla\times\mathbf{A}_i\big|_{r=a}$,

where $\mathbf{A}_o$ is the total vector potential outside the sphere and $\mathbf{A}_i$ is the total vector potential inside the sphere.

Figure 1



$\vec{n}$
current loop
$\vec{b}$
$\vec{dl}$
$\vec{R}$
$\vec{c}$
axis of loop
$\vec{s}$
$\vec{r}$
$\vec{d}$
center of sphere
a
conducting sphere

II. Calculating the vector potential of a current loop with respect to an origin off axis to the loop.

To calculate the vector potential of a current loop, it is convenient to choose an origin on the symmetry axis of the loop. However, in calculating the total vector potential in the sphere, the symmetry of the sphere is most easily exploited by using an origin at its center. To benefit from both choices of origin, the vector potential of the loop is first calculated with respect to an origin on the axis of the loop and then is expanded about the center of the sphere. This method would be inappropriate for large displacements of the center of the sphere from the axis of the loop. The geometrical parameters in our problem are shown in figure 1.

The vector potential of a line current is found from

$$(13)\quad \mathbf{A}_1 = \frac{\mu_o}{4\pi}\int \frac{I\,\mathbf{dl}}{|\mathbf{R}|} ,$$

where I is the current and can depend on time,

$\mathbf{dl} = dl\,\hat{\imath}$ is the line element directed along the current,

$|\mathbf{R}|$ is the distance from dl to the field point.

Expanding $|\mathbf{R}|^{-1}$ in Legendre polynomials and integrating over the current loop we find (Appendix A),

$$(14)\quad \mathbf{A}_1 = \frac{\mu_o I b}{2}\sum_{n=1}^{\infty}\frac{s^n}{c^{n+1}n(n+1)}\,P_n^1(\mathbf{n}\cdot\hat{\mathbf{c}})\,P_n^1(\mathbf{n}\cdot\hat{\mathbf{s}})\,\hat{\boldsymbol{\phi}}_s$$

$$|\mathbf{s}|<|\mathbf{c}|$$

b is the radius of the loop

s is the field point

c is a vector to the loop

$P_n^1(\mathbf{n}\cdot\hat{\mathbf{c}})$, $P_n^1(\mathbf{n}\cdot\hat{\mathbf{s}})$ are Associated Legendre functions

$\hat{\boldsymbol{\phi}}_s$ is the azimuthal unit vector

Using $s^n P_n^1(n\cdot\hat{s})\Phi s = \nabla_s x[P_n(n\cdot\hat{s})s^n s]$, (Appendix B)

$$(15)\quad A_1 = \frac{\mu_o Ib}{2} \sum_{n=1}^{\infty} \frac{P_n^1(n\cdot\hat{c})}{c^{n+1}n(n+1)} \nabla_s x[P_n(n\cdot\hat{s})s^n s].$$

The Taylor expansion for a vector function can be written

$$(16)\quad f(d+r) = \sum_{n=o}^{\infty} \frac{(d\cdot\nabla)^n}{n!} f(r).$$

A_1 is a function of $s = d+r$, where r is the vector of the field point with respect to an origin at the center of the sphere and d is the displacement of the center of the sphere from the axis of the loop. Employing the Taylor expansion,

$$(17)\quad A_1 = \sum_{k=o}^{\infty} \frac{(d\cdot\nabla)^k}{k!} \{ \frac{\mu_o Ib}{2} \sum_{n=1}^{\infty} \frac{P_n^1(n\cdot\hat{c})}{c^{n+1}n(n+1)} \nabla x P_n(n\cdot\hat{r})r^n r\}.$$

The displacement vector d has been chosen so that it is perpendicular to the axis of the loop ($d\cdot n = 0$). By the choice of the Legendre polymomial the axis of the loop is designated along the z-axis. It is natural then to designate d as being along the x axis so that

$$(d\cdot\nabla)^k = (d\frac{\partial}{\partial x})^k.$$

In this case, a useful identity is [8, p. 361]

$$(18)\quad \frac{\partial}{\partial x} P_n^m r^n e^{im\phi} = \frac{1}{2} \left[(m+n)(m+n-1)P_{n-1}^{m-1}e^{i(m-1)\phi} - P_{n-1}^{m+1}e^{i(m+1)\phi}\right]r^{n-1}.$$

Also true is,

$$\frac{\partial}{\partial x}\hat{r} = e_x, \text{ which implies that } d\frac{\partial}{\partial x} r = d, \text{ so}$$

$$(19)\quad (d\cdot\nabla)P_n^m r^n e^{im\phi} r = \frac{1}{2} r^{n-1} r\left[(m+n)(m+n-1)P_{n-1}^{m-1}e^{i(m-1)\phi} - P_{n-1}^{m+1}e^{i(m+1)\phi}\right] + P_n^m e^{im\phi} r^n d$$

The important aspect here is that even after differentiation the basic form of each term is preserved. Each term is either of the form

$$P_1^k e^{ik\phi} r^1 \mathbf{r} \quad \text{or} \quad P_1^k e^{ik\phi} r^1 \mathbf{d}.$$

Since $(\mathbf{d}\cdot\nabla)\mathbf{d} = 0$, repeated application of $\mathbf{d}\cdot\nabla$ yields only more terms of similar form. This makes it possible to find a closed form for any power of $\mathbf{d}\cdot\nabla$ operating on $P_n r^n \mathbf{r}$. The derivation is given in Appendix C. The result is

$$(20)\quad (\mathbf{d}\cdot\nabla)^k P_n r^n \mathbf{r} = \frac{d^k}{2^k}\sum_{m=0}^{k}\binom{k}{m}\frac{n!}{(n-2m)!}(-1)^{k-m}P_{n-k}^{k-2m}e^{i(k-2m)\phi}r^{n-k}\mathbf{r}$$

$$+\frac{Kd^{k-1}}{2^{k-1}}\sum_{m=0}^{k-1}\binom{k-1}{m}\frac{n!}{(n-2m)!}(-1)^{k-m-1}P_{n-k+1}^{k-2m-1}r^{n-k+1}\mathbf{d}.$$

Using this result in equation (17) and the fact that $\mathbf{d}\cdot\nabla(\nabla\times\mathbf{A}) = \nabla\times(\mathbf{d}\cdot\nabla\mathbf{A})$ we obtain

$$(21)\quad A_1 = \nabla\times\left[\frac{\mu_0 Ib}{2}\sum_{n=1}^{\infty}\frac{P_n^1(\mathbf{n}\cdot\hat{c})}{c^{n+1}n(n+1)}\sum_{k=0}^{n}\left(\frac{d}{2}\right)^k\frac{1}{k!}\sum_{m=0}^{kn}(-1)^{k-m}\frac{n!}{(n-2m)!}\binom{k}{m}\cdot\right.$$

$$\left.P_{n-k}^{k-2m}(\mathbf{n}\cdot\hat{r})e^{i(k-2m)\phi}r^{n-k}(\mathbf{r}+\mathbf{d})\right].$$

for $|r|>c$.

III. Calculating the total vector potential inside the sphere

The differential equations for the vector potential are linear so solutions can be superimposed. Therefore linearly independent terms in the harmonic expansion for the inducing vector potential can be separately matched to boundary conditions to determine corresponding terms for the vector potential inside the sphere. The terms can be recombined to give the total vector potential inside the sphere.

As has been mentioned, all of the terms of A_1 are either of the form

$$A_{nm}\ \nabla x[P_n^m(n\cdot\hat{r})e^{im\phi}r^n r] = A_{nm}r^n r x\nabla[P_n^m(n\cdot\hat{r})e^{im\phi}]$$

or

$$-A_{nm}\nabla x[r^n P_n^m(n\cdot\hat{r})e^{im\phi}d].$$

The first are recognized as vector spherical harmonics. The general solution to Laplace's equation (using only terms defined for all $n\cdot\hat{r}$) is

$$(22)\ A_o = \sum_{n=o}^{\infty}\sum_{m=-n}^{n} A_{nm}[r^n+B_{nm}r^{-n-1}][rx\nabla(P_n^m(n\cdot\hat{r})e^{im\phi})].$$

The potential outside the sphere is a combination of that due to the current loop and that due to the eddy currents in the sphere. The potential due to the eddy currents must vanish as r approaches infinity so A_{nm} must be completely determined by the current loop. There are no r^{-n-1} terms in the vector potential of the current loop so B_{nm} is completely determined by the eddy currents. Immediately A_{nm} is known.

Likewise, the general solution to Helmholtz's equation can be writtin

$$A_i = \sum_{n=o}^{\infty}\sum_{m=-n}^{n}\{C_{nm}I_{n+1/2}(kr) + D_{nm}K_{n+1/2}(kr)\}rx\nabla P_n^m(n\cdot\hat{r})e^{im\phi}$$

where

$I_n(kr) = i^{-n}J_n(ikr)$

$K_n(kr) = 1/2\pi i^{n+1}[J_n(ikr)+iY_n(ikr)]$

$J_n(ikr)$ is a Bessel function of the first kind

$$Y_n(ikr) = \frac{J_n(ikr)\cos(ikr\pi) - J_{-n}(ikr)}{\sin(ikr\pi)}$$

The function $K_{n+1/2}$ is singular at the origin, which is within the sphere, so $D_{nm}=0$ for all n,m. So

$$(23)\quad A_i = \sum_{n=o}^{\infty} r^{-1/2}C_{nm}I_{n+1/2}(kr)rx\nabla(P_n^m(n\cdot\hat{r})e^{im\phi}).$$

The boundary conditions are met if (Appendix D)

$$(24)\quad C_{nm} = \frac{(2n+1)a^{n-1/2}A_{nm}}{kI_{n-1/2}(ka)}$$

A similar process is used for the terms of the form $A_{nm}\ \nabla x(P_n^m e^{im\phi} r^n e_x)$. For these terms, the curl is taken first and is broken up into its cartesian components. From Appendix E

$$A_{nm}\nabla x(P_n^m(n\cdot\hat{r})e^{im\phi}r^n e_x) = A_{nm}\{(m+n)P_{n-1}^m(n\cdot\hat{r})e^{im\phi}r^{n-1}e_y -$$

$$\frac{i}{2}\ [(m+n)(m+n-1)P_{n-1}^{m-1}(n\cdot\hat{r})e^{i(n-1)\phi}\ r^{n-1}+P_{n-1}^{m+1}(n\cdot\hat{r})e^{i(m+1)\phi}r^{n-1}]e_z\}.$$

In cartesian coordinates $\nabla^2\mathbf{A}=\nabla^2A_x e_x+\nabla^2A_y e_y+\nabla^2A_z e_z$. Each component of Helmboltz's vector equation is an identical Helmholtz scalar equation. From [p. 375] and the arguments stated for the vector spherical harmonics, the general solutions for $\mathbf{A}_i$ and $\mathbf{A}_o$ are

$$A_i^j = \sum_{n=o}^{\infty} A_{nm}^{\ j}(r^n+B_{nm}r^{-n-1})P_n^m(n\cdot\hat{r})e^{im\phi}\ ,\ j=x,y,z$$

$$A_o^j = \sum_{n=o}^{\infty} C_{nm}^{\ j}\ r^{-1/2}I_{n+1/2}(kr)P_n^m(n\cdot\hat{r})e^{im\phi}$$

The similarities between these equations and equations (22) and (23) are obvious and thus

$$C_{nm}^j = \frac{(2n+1)a^{n-1/2}A_{nm}^j}{kI_{n+1/2}(ka)}$$

The complete expression for the vector potential inside the sphere is

$$(25)\ A_1 = \frac{\mu_o Ib}{2} \sum_{n=1}^{\infty} A_n \sum_{k=o}^{\infty} \frac{d^k}{2^k k!} \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} \frac{n!}{(n-2m)!} r^{-1/2} \{ I_{n-k+1/2}(kr) C_{n-k}$$

$$(\nabla P_{n-k}^{k-2m}(n\cdot\hat{r}) e^{i(k-2m)\phi}) x r + d I_{n-k-1/2}(kr) C_{n-k-1} \Big[(n-2m) P_{n-k-1}^{k-2m}(n\cdot\hat{r}) e^{i(k-2m)\phi} e_y$$

$$- \frac{i}{2} ((n-2m)(n-2m-1) P_{n-k-1}^{k-2m-1}(n\cdot\hat{r}) e^{i(k-2m-1)\phi} + P_{n-k-1}^{k-2m+1}(n\cdot\hat{r}) e^{i(k-2m+1)\phi} e_z) \Big] \}$$

where

$$A_n = \frac{P_n^1(n\cdot\hat{c})}{c^{n+1} n(n+1)}, \quad C_n = \frac{(2n+1)a^{n-1/2}}{k I_{n-1/2}(ka)}$$

IV. Calculating the average power dissipated in the sphere

For a sinousoidally time dependent current density, the average power dissipated is

$$P = \frac{1}{2\sigma} \int J^* \cdot J dV. \qquad [3\ p.\ 369]$$

σ is the conductivity.

The factor of one half arises from the time averaging over one period.

Using equation (9)

$$J = -i\sigma\omega A,$$

the average power becomes

$$P = \frac{\omega^2 \sigma}{2} \int A^* \cdot A dV.$$

The displacement d of the center of the sphere from the axis of the loop is small, so only terms to first order in d will be considered. This can be justified by examining equation (25). Each term of A_1 is propotional to

$$\frac{d^k a^{n-k-1/2}}{c^{n+1}} = \left(\frac{d}{c}\right)^k \frac{a^{n-k-1/2}}{c^{n-k+1}} .$$

The loop must be outside the sphere so a<c and d is assumed small so (d/c)is small. From equation (25), to first order in d

$$(27) \quad A_1^{1,0} = \frac{\mu_o Ib}{2} \sum_{n=1}^{\infty} A_n r^{-1/2}\{I_{n+1/2}(kr)\, C_n \nabla P_n(n\cdot\hat{r})xr$$

$$-I_{n-1/2}(kr)C_{n-1}d[\nabla(P^1_{n-1}(n\cdot\hat{r})\cos\phi)xr - nP_{n-1}(n\cdot\hat{r})e_y - P^1_{n-1}(n\cdot\hat{r})\sin\phi e_z]\}$$

Using $\quad -n(n-1)P^{-1}_{n-1} = P^1_{n-1} \qquad$ (7, p. 560)

and $\quad \cos\phi = \dfrac{e^{i\phi}+e^{-i\phi}}{2}$; $\sin\phi = \dfrac{e^{i\phi}-e^{i\phi}}{2i}$.

Study of equation (27) reveals that the angular part of the integral (equation (26)) to first order in d has terms of the form

$$(28a) \quad \oint (\nabla P_n(n\cdot\hat{r}) \times r) \cdot (\nabla P_1(n\cdot\hat{r}) \times r)d\Omega$$

$$(28b) \quad \oint (\nabla P_n(n\cdot\hat{r}) \times r) \cdot (\nabla[P^1_{n-1}(n\cdot\hat{r})\cos\phi] \times r)d\Omega$$

$$(28c) \quad \oint (\nabla P_n(n\cdot\hat{r}) \times r) \cdot (1P_{1-1}(n\cdot\hat{r})e_y + P^1_{1-1}(n\cdot\hat{r})\sin\phi\, e_z)d\Omega$$

These are evaluated in Appendix F with the results

$$\oint (\nabla P_n xr) \cdot (\nabla[P^1_{1-1}\cos\phi] \times r)d\Omega = 0$$

$$\oint (\nabla P_n xr) \cdot (1\, P_{1-1}e_y + P^1_{1-1}\sin\phi\, e_z)d\Omega = 0, \text{ and}$$

$$\oint (\nabla P_n xr) \cdot (\nabla P_1 xr)d\Omega = \frac{4\pi n(n+1)}{2n+1}\delta_{n,1}.$$

Thus, all of the first order terms vanish and only the zeroth order contributes to the power dissipation. The expression for the power is now

$$P = \frac{\sigma\omega^2\mu_o I_o{}^2b^2}{8} \sum_{n=1}^{\infty} A_n{}^2 \frac{4\pi n(n+1)}{2n+1} \int_o^a |C_n|^2 r^{-1}|I_{n+1/2}(kr)|^2 r^2 dr .$$

Following the logic of Sonnabend (see Appendix I)

$$I_{n+1/2}(x) \sim \frac{e_x}{(2\pi x)^{1/2}} \quad ; \ r\sim a.$$

so $$\int_0^a |Cn|^2 |I_{n+1/2}(kr)|^2 r dr \sim \frac{\delta^3 a^{2n}(2n+1)^2}{2}$$

where $\delta = \left(\frac{2}{\mu_o \omega\sigma}\right)^{1/2}$ is the skin depth .

Thus,

$$P = \frac{\sigma\omega^2 \mu_o \pi\delta^3 I_o b^2}{4} \sum_{n=1}^{\infty} A_n{}^2 (2n+1)n(n+1)a^{2n}$$

$$= \frac{\pi I_o{}^2 b^2}{\sigma\delta} \sum_{n=1}^{\infty} \frac{|P_n^1(n\cdot\hat{c})|^2 (2n+1)a^{2n}}{n(n+1)C^{2n+2}} .$$

V. Calculating the force on the sphere

The force on the sphere due to the external magnetic field is

$$F = \int JxBdV = \text{Re}\ \{-i\omega\sigma \int A_i \times (\nabla x A_1)dV\}$$

where A_i is the total vector potential inside the sphere and A_1 is the vector potential due to the current loop alone. The interaction between the eddy currents and the field of the eddy currents is of no interest. The sphere cannot move itself and thus that force must be negated by internal stresses about which little is known.

Evaluating $\nabla x A_1$ to first order in d, there are terms of the form (Appendix H):

zeroth order

(29a) $\nabla x(\nabla r^n P_n x r) = n(n+1)r^{n-1}P_n \hat{r} - (n+1)r^{n-1}\sin\theta P_n{}'\hat{\theta}$

where $P_n{}'(n\cdot\hat{r}) = \frac{dP_n(n\cdot\hat{r})}{d(n\cdot\hat{r})}$

first order

(29b) $\nabla x[r^{n-1}\nabla(P^1_{n-1}\cos\phi)xr] = n(n-1)r^{n-2}P^1_{n-1}\cos\phi\hat{r}$

$$-nr^{n-2}\sin\theta P^{1\,\prime}_{n-1}\cos\phi\;\hat{\theta} - n\,\frac{r^{n-2}\sin\phi}{\sin\theta}P^1_{n-1}\hat{\phi}$$

(29c) $\nabla x[\nabla(r^nP_n)xe_x] = r^{n-2}\;[-(n-1)P^1_{n-1}\cos\phi\;\hat{r} + \sin\theta P_n'\hat{\theta} + \frac{\sin\phi}{\sin\theta}P_n\hat{\phi}].$

The terms to first order is d of A_i are found from equation (27). These are of the form:

zeroth order

(30a) $r^{-1/2}C_nI_{n+1/2}(kr)\nabla P_nxr = r^{-1/2}C_nI_{n+1/2}(kr)\sin\theta P^1_n\hat{\phi}$

first order

(30b) $-r^{-1/2}C_{n-1}I_{n-1/2}(kr)\nabla(P^1_n\cos\phi)xr =$

$$-r^{-1/2}(C_{n-1}I_{n-1/2}(kr)[\sin\theta P'_{n-1}\cos\phi\;\hat{\phi} - \frac{\sin\phi}{\sin\theta}P^1_{n-1}\hat{\theta}]$$

(30c) $r^{-1/2}C_{n-1}I_{n-1/2}(kr)n\;P_{n-1}e_y$

(30d) $r^{-1/2}C_{n-1}I_{n-1/2}(kr)P^1_{n-1}\sin\phi e_z$

From the cross product of the zeroth order terms of A_i with the zeroth order terms of ∇xA_i the zeroth order terms of the force are obtained. From results in Appendix K the force to zeroth order, averaged over one period of the current is

$$F^o = \frac{-\mu_o I_o{}^2b^2\pi}{2}\sum_{n=1}^{\infty}\frac{P^1_n(n\cdot\hat{c})P^1_{n+1}(n\cdot\hat{c})a^{2n+1}}{2^{2n+3}(n+1)}\;e_z\,.$$

This is identical to the force, found by Sonnabend [p. 54], on a sphere on the axis of the loop.

The first order term for the force is found by taking the cross product between the zeroth order terms of A_i with the first order terms of A_i, and the cross product between the first order terms of A_i and the zeroth order terms of A_i. The calculations are in Appendix J. The results is

$$F^1 = \frac{-\pi\mu_o I_o{}^2 b^2 d}{4} \sum_{n=1}^{\infty} \left\{ \frac{|P_n^1(n\cdot\hat{c})|^2 a^{2n-1}(n^2+2n-1)}{c^{2n+2}n(n+1)} \right.$$

$$\left. + \frac{P_n^1(n\cdot\hat{c})P_{n+2}^1(n\cdot\hat{c})a^{2n+1}(n+1)}{c^{2n+4}(n+3)} \right\}$$

The zeroth order force is one strictly along the axis of the loop, and the first order force is one strictly in the direction of the displacement from the loop. At first glance the first order force appears to return the sphere to the axis of the loop, however evaluation of the summation should be done to be sure. The first term in the summation is definitely restoring because

$$|P_n^1(n\cdot\hat{c})|^2 \geq 0$$

and it should dominate because of its $\frac{a^{2n-1}}{c^{2n+1}}$ dependence versus the other terms $\frac{a^{2n+2}}{c^{2n+4}}$ dependence.

VI Calculating the torque about the center of the sphere

The torque is found from

$$\Gamma = \int \mathbf{r} \times (\mathbf{J} \times \mathbf{B})dV.$$

J x **B** was calculated to first order in d when calculating the force. Finding the terms of **r** x (**J** x **B**) is then trivial. Calculating the torque then becomes simply a matter of trudging through the integrals as was done in calculating the force. This is done in Appendix L. The results are that there is no torque to zero order, and to first order the torque is

$$\Gamma^1 = -e_y \frac{\pi\mu_o I^2 b^2 d}{4} \sum_{n=1}^{\infty} \left[\frac{P_n^1(n\cdot\hat{c})a^{2n-1}}{c^{2n+1}} + \frac{P_n^1(n\cdot\hat{c})P_{n+1}^1(n\cdot\hat{c})a^{2n+1}(2n^3+3n^2+3n-2)}{c^{2n+3}(n+1)(n+2)(2n+1)} \right]$$

Thus the axis of rotation is perpendicular to the displacement off axis. For a displacement in the opposite direction ($d \rightarrow -d$) the torque changes sign. This means that as long as the sphere stays near the axis of the loop, the angular acceleration will average to zero.

APPENDIX A

From equation (13) the vector potential of the loop is

$$A_1 = \frac{\mu_o}{4\pi} \oint \frac{dl}{|R|}$$

Expand $\frac{1}{|R|}$ in Legendre functions.

$$\frac{1}{|R|} = \sum_{n=o}^{\infty} \frac{s^n}{c^{n+1}} P_n(\hat{s}\cdot\hat{c}), \qquad [7,p. 539]$$

c is the vector to the loop, s to the field point. Functions of c can be separated from functions of s by using the addition theorem for Legendre functions in terms of spherical harmonics. Let n be the normal unit vector to the loop define the z-axis.

$$P_n(\hat{s}\cdot\hat{c}) = P_n(n\cdot\hat{s})P_n(n\cdot\hat{c})+2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(n\cdot\hat{s})P_n^m(n\cdot\hat{c})\cos m(\phi_c-\phi_s)$$

where $P_n^m(x) = (1-x^2)^{1/2} \frac{d^m}{dx^m} P_n(x)$. [7, p.582].

For the purpose of simplicty, the coordinate system can be defined where ϕ_s=o. Since c=|c| is constant over the loop as in n·c, the line integral amounts to an integration over φ_c.

$$\oint dl = \int_o^{2\pi} bd\phi(-\sin\phi e_x+\cos\phi e_y)$$

Using the following orthogonality relations,

$$\int_o^{2\pi} \sin\phi \cos m\phi \, d\phi = o \qquad [7, p. 435]$$

$$\int_o^{2\pi} \cos\phi \cos m\phi \, d\phi = \pi\delta_{m,1}, \qquad \delta_{m,1} = \begin{cases} o & m\neq 1 \\ 1 & m=1 \end{cases}$$

$$A_1 = \frac{\mu_o Ib}{2} \sum_{n=1}^{\infty} \frac{P_n^1(n\cdot\hat{c})}{c^{n+1}n(n+1)} S^n P_n^1(n\cdot\hat{s})e_y .$$

The coordinates were fixed relative to s when ϕ_s was set to zero. This can be "unfixed" by letting $e_y = \hat{\phi}_s$, and the coordinates are again independent of s. Thus

$$A_1 = \frac{\mu_o Ib}{2} \sum_{n=1}^{\infty} \frac{s^n}{c^{n+1}n(n+1)} P_n^1(n\cdot\hat{c})P_n^1(n\cdot\hat{s})\hat{\phi}_s .$$

APPENDIX B

To see that $s^n P_n^1(n\cdot\hat{s})\hat{\phi}_s = \nabla_s[P_n(n\cdot\hat{s})s^n s]$, first note that

$$\nabla(s^n P_n(n\cdot\hat{s})) = ns^{n-1}P_n(n\cdot\hat{s})\hat{s} + s^{n-1}\frac{\partial}{\partial\theta_s}P_n(n\cdot\hat{s})\hat{\theta}_s .$$

From the chain rule,

$$\frac{\partial}{\partial\theta_s}P_n(n\cdot\hat{s}) = \frac{\partial}{\partial(n\cdot\hat{s})}P_n(n\cdot\hat{s})\frac{\partial(n\cdot\hat{s})}{\partial\theta_s}$$

$$= -P_n'(n\cdot\hat{s})\sin\theta_s = -P_n^1(n\cdot\hat{s}).$$

$s x s = o$ and $s x\hat{\theta}_s = s\hat{\phi}_s$, so

$$\nabla_s(s^n P_n(n\cdot\hat{s})) \times s = s^n P_n^1(n\cdot\hat{s})\hat{\phi}_s$$

$$\nabla_s x\{P_n(n\cdot\hat{s})s^n s] = \nabla_s[P_n(n\cdot\hat{s})s^n]xs + P_n(n\cdot\hat{s})s^n\nabla_s xs$$

and $\nabla_s xs = 0$. So,

$$\nabla_s \times [P_n(n\cdot\hat{s})s^n s] = \nabla_s[P_n(n\cdot\hat{s})s^n]xs = s^n P_n^1(n\cdot\hat{s})\hat{\phi}_s .$$

APPENDIX C

Using equation (19) with m=o,

$$(d\cdot\nabla)\ (P_n r^n \mathbf{r}) = \frac{d^2}{2}\ r^{n-1}[n(n-1)P_{n-1}^{-1}e^{-i\phi} - P_{n-1}^{1}e^{i\phi}]\mathbf{r} + P_n r^n \mathbf{d}.$$

Repeating this yields

$$(d\cdot\nabla)^2(P_n r^n \mathbf{r}) = \frac{d^2}{4} r^{n-2}[n(n-1)(n-2)(n-3)P_{n-2}^{-2}e^{-2i\phi} - 2n(n-1)P_{n-2}^{2} + P_{n-2}e^{2i\phi}]\mathbf{r}$$

$$+ \frac{d}{2}\ r^{n-1}[n(n-1)P_{n-1}^{-1}e^{-i\phi} - P_{n-1}^{1}e^{i\phi}]\mathbf{d}$$

$$= (\frac{d}{2})^2 r^{n-2} \sum_{m=o}^{2} \binom{2}{m} \frac{n!}{(n-2m)!}\ (-1)^{2-m}P_{n-2}^{2-2m}e^{i(2-2m)\phi}$$

$$+ 2\ (\frac{d}{2}) \sum_{m=o}^{1} \binom{1}{m} \frac{n!}{(n-2m)!}\ (-1)^{2-m-1}P_{n-2+1}^{2-2m+1}\ r^{n-2+1}\mathbf{d}.$$

The pattern holds for further differentiations with the result

$$(d\cdot\nabla)^k(P_n r^n \mathbf{r}) = \frac{d^2}{4} \sum_{m=o}^{k} \binom{k}{m} \frac{n!}{(n-2m)!}\ (-1)^{k-m}P_{n-k}^{k-2m}e^{i(k-2m)\phi}r^{n-k}\mathbf{r}$$

$$+ \frac{kd^{k-1}}{2^{k-1}} \sum_{m=o}^{k} \binom{k-1}{m} \frac{n!}{(n-2m)!}\ (-1)^{k-m-1}\ P_{n-k+1}^{k-2m-1}\ r^{n-k+1}\ \mathbf{d}.$$

APPENDIX D

Taking each term independently, $A_i|_{r=a} = A_o|_{r=a}$ implies that

(D1) $[A_{nm}(r^n+B_{nm}r^{-n-1})r\times\nabla P_n^m e^{im\phi}|_{r=a} = [C_{nm}r^{-1/2}I_{n+1/2}(kr)r\times\nabla P_n^m e^{im\phi}|_{r=a}$.

$r\times\nabla P_n^m e^{im\phi}$ can be cancelled so,

(D2) $A_{nm}(a^n+B_{nm}a^{-n-1}) = C_{nm}a^{-1/2}I_{n+1/2}(ka)$.

The next boundary condition is that $\nabla\times A_i|_{r=a} = \nabla\times A_o|r=a$. Taking the curl of equation (D1), cancelling the angular dependence and using equation (F2) yields

$$\nabla\times[A_{nm}(r^n+B_{nm}r^{-n-1})|_{r=a} = \nabla\times[C_{nm}r^{-1/2}I_{n+1/2}(kr)|_{r=a}.$$

Carrying out this differentiation in spherical coordinates and equating components yields.

(D3) $A_{nm}(na^n-(n+1)B_{nm}a^{-n-1}) = C_{nm}\{-1/2a^{-3/2}I_{n+1/2}(ka)+a^{-1/2}kI'_{n+1/2}(ka)\}$.

From [3, p. 98]

(D4) $$I'_{n+1/2}(ka) = I_{n-1/2}(ka) - \frac{n+1/2}{ka} I_{n+1/2}(ka).$$

Substituting this into (D3), solving for B_{nm} in (D2) and then solving (D3) for C_{nm} in terms of A_{nm} yields

$$C_{nm} = \frac{A_{nm}a^{n-1/2}(2n+1)}{kI_{n-1/2}(ka)}.$$

Appendix E

$$\nabla \mathrm{x}(r^n P_n^m e^{im\phi} e_x) = \frac{\partial}{\partial z}(r^n P_n^m e^{im\phi})e_y - \frac{\partial}{\partial y}(r^n P_n^m e^{im\phi})e_z$$

From [8, p. 361]

$$\frac{\partial}{\partial y} r^n P_n^m e^{im\phi} = \frac{i}{2}[(m+n)(m+n-1)P_{n-1}^{m-1} r^{n-1} e^{i(m-1)\phi} + P_{n-1}^{m+1} r^{n-1} e^{i(m+1)\phi}].$$

$$\frac{\partial}{\partial z} r^n P_n^m e^{im\phi} = n\cos\theta r^{n-1} P_n^m e^{im\phi} + \sin^2\theta P_n^{m\prime} r^{n-1} e^{im\phi}$$

$$= (m\mathrm{x}n)P_{n-1}^m r^{n-1} e^{im\phi}. \quad \text{(using equation (L17))}$$

Thus,

$$\nabla \mathrm{x}(r^n P_n^m e^{im\phi} e_x) = (m+n)P_{n-1}^m r^{n-1} e^{im\phi} e_y - \frac{i}{2}[(m+n)(m+n-1)P_{n-1}^{m-1} r^{n-1} e^{i(m-1)\phi}$$

$$+ P_{n-1}^{m+1} r^{n-1} e^{i(m+1)\phi}]e_z.$$

Appendix F

To evaluate (28a) and (28b), refer to the theory of vector spherical harmonies which are defined in [9, p. 211-214] as

$$X_{nm} = \frac{-i}{[n(n+1)]^{1/2}} \; r x \nabla\{(-1)^m [\frac{(2n+1)(n-m)!}{4\pi(n+m)!}]^{1/2} P_n^m e^{im\phi}\} \; .$$

Thus,

$$\nabla P_n x r = -i \; [\frac{4\pi n(n+1)}{2n+1}]^{1/2} \; X_{no} \, ,$$

and

$$\nabla P_{l-1}^{\ 1} \cos\phi \; xr = 1/2 \nabla [P_{l-1}^{\ 1} e^{i\phi} - l(l+1) P_{l-1}^{1} e^{-i\phi}] \; xr$$

$$= \frac{-i}{2} \; (\frac{4\pi}{2l-1})^{1/2} l(l-1)(X_{l,1} - X_{l,-1}) .$$

The vector spherical Harmonies satisfy the orthogonality relation

$$\int X_{lm}^{*} \cdot X_{l'm'} d\Omega = \delta_{ll'} \delta_{mm'} , \qquad [9, \text{ p. } 211]$$

It can be immediately seen that (28b) vanishes. To evaluate (28a),

$$\int (\nabla P_n xr) \cdot (\nabla P_l xr) d\Omega = -4\pi [\frac{n(n+1)}{2n+1}]^{1/2} [\frac{l(l+1)}{2l+1}]^{1/2} \int X_{no}^{*} \cdot X_{lo} d\Omega$$

$$= 4\pi [\frac{n(n+1)}{2n+1}]^{1/2} [\frac{l(l+1)}{2l+1}]^{1/2} \int X_{no}^{*} \cdot X_{lo} d\Omega$$

$$= \frac{4\pi n(n+1)}{2n+1} \; \delta_{nl}$$

To evaluate (28c), the following relations are useful:

i) $\nabla P_n xr = - \frac{\partial P_n}{\partial \theta} \hat{\phi}$

ii) $\hat{\phi} \cdot e_y = \cos\phi$

iii) $\hat{\phi} \cdot e_z = 0$

iv) $\int_0^{2\pi} \cos\phi d\phi = 0$

Using i), ii) and iii), (28c) becomes

$$\int (\nabla P_n x r)\cdot[e_y l P_{l-1} + e_z P^1_{l-1}\sin\phi]d\Omega = \int -\frac{\partial p_n}{\partial\theta} lP_{l-1}\cos\phi d\Omega$$

which, using iv) vanishes.

APPENDIX G

The following argument is from [1, p. 53].

$$I_{n+1/2}(x) = (2\pi x)^{-1/2} \sum_{l=o}^{\infty} \frac{(e^{x}-(-1)^{n}e^{-x}(n+l)!}{l!(n-l)!(-2x)^{l}}$$

It has already been assumed (assumption 5 of the introduction) that the skin depth δ is small compared to the radius of the sphere. This means that the current density will be negligible except when $r \sim a$. In the equation for the current density the argument

$$x = kr = (i\mu_o \omega\sigma)^{1/2} r = (2i)^{1/2} \frac{r}{\delta} = (1+i) \frac{r}{\delta}.$$

But for $r \sim a \rightarrow x \sim (1+i)\, a/\delta$ is a large number, so the summation e^{x} dominates over e^{-x}. $(-2x)^{-l}$ goes quickly to zero as l increases, so the $l=o$ term of the summation dominates. Thus

$$I_{n+1/2}(kr) \sim (2\pi kr)^{-1/2} e^{kr}.$$

Remembering that most of the current density is at $r \sim a$ and that over small variations in r, $I_{n+1/2}(kr)$ will fluctuate much more than small powers of r, it is reasonable to let all of the functional variation lie in the exponential and set $r=a$ in the radial integral for small powers of r. Thus

$$\int_o^a |c_n|^2 \; |I_{n+1/2}(kr)|^2 r\,dr \sim \int_o^a \frac{(2n+1)^2 a^{2n}}{k^2 e^{2ka}} e^{2kr} dr = \frac{\delta^3 a^{2n} (2n+1)^2}{2}.$$

APPENDIX H

Zeroth order terms of A_1 have the form $\nabla(r^n P_n) x r$, and first order terms are of the form $\nabla(r^n P_n) x\ e_x$ and $\nabla(r^{n-1} P_{n-1}^1 \cos\phi) x r$. So the terms of $\nabla x A_1$ have the form

$$\nabla x\ [\nabla(r^n P_n) x r],\ \nabla x[\nabla(r^{n-1} P_{n-1}^1 \cos\phi) x r]$$

and

$$\nabla x[\nabla(r^n P_n) x\ e_x].$$

For a function $f(r)\ Y\ (\theta,\phi)$ where $\nabla^2[f(r)Y(\theta,\phi)]=o$

$$\nabla x[\nabla(fY)xr] = (r\cdot\nabla)\nabla(fY) - [\nabla(fY)\cdot\nabla]r + (\nabla\cdot r)\nabla(fY) - [\nabla^2(fY)]r$$

$$= (r\cdot\nabla)\nabla(fY) + 2\nabla(fY).$$

For $f(r) = r^n$,

$$\nabla(fY) = nr^{n-1} Y\hat{r} + r^{n-1} \frac{\partial Y}{\partial\theta}\hat{\theta} + \frac{r^{n-1}}{\sin\theta}\frac{\partial Y}{\partial\phi}\hat{\phi}$$

and $(r\cdot\nabla)\nabla(fY) = n(n-1)r^{n-1}\ Y\hat{r} + (n-1)r^{n-1} \dfrac{\partial Y}{\partial\theta}\hat{\theta} + (n-1)\dfrac{r^{n-1}}{\sin\theta}\dfrac{\partial Y}{\partial\phi}\hat{\phi}.$

Using these results,

$$\nabla x[\nabla(r^n P_n)xr] = n(n+1)r^{n-1} P_n \hat{r} - (n+1)r^{n-1}\sin\theta\ P_n'\ \hat{\theta}$$

and

$$\nabla x[\nabla(r^{n-1} P_{-}^1{}_{,}\cos\phi)xr] = n(n-1)r^{n-2} P_{n-1}^1 \cos\phi\hat{r} - nr^{n-2}\sin\theta P_{n-1}^{1\,\prime}\cos\phi\hat{\theta}$$

$$-n\frac{r^{n-2}\sin\phi}{\sin\theta} P_{n-1}^1 \hat{\phi}.$$

similarly,

$$\nabla x[\nabla(fY)x\ e_x] = (e_x\cdot\nabla)\nabla(fY) - [\nabla(fY)\cdot\nabla]e_x + [\nabla\cdot e_x]\nabla fY - \nabla^2(fY)e_x$$

$$= \frac{\partial}{\partial x}\nabla(fY).$$

so,

$$\nabla x[\nabla(r^n P_n)xe_x] = \frac{\partial}{\partial x}\nabla(r^n P_n) = r^{n-2}[-(n-1)P_{n-1}^1\cos\phi\hat{r} + \sin\theta\ P'_n\ \hat{\theta} + \frac{\sin\phi}{\sin\theta} P_n\hat{\phi}].$$

APPENDIX I

Using (29a) and (30a), the zeroth order terms of $A_1 \times (\nabla \times A_1)$ have the form,

$$r^{-1/2} C_n I_{n+1/2}(kr) r^{l-1} [l(l+1)P_l \sin\theta P_n' \hat{\phi}\times\hat{r} - (l+1)\sin\theta P_l' \sin\theta P_n' \hat{\phi}\times\hat{\theta}].$$

$$\hat{\phi}\times\hat{r} = \hat{\theta} = -\sin\theta e_z + \cos\theta \cos\phi e_x + \cos\theta \sin\phi e_y.$$

$$-\hat{\phi}\times\hat{\theta} = \hat{r} = \cos\theta e_z + \sin\theta \cos\phi e_x + \sin\theta \sin\phi e_y.$$

Performing the angular part of the integral.

Use $\sin\theta \, P_n' = P_n^1$, $\sin\theta \, P_l = \dfrac{1}{2l+1} [P_{l+1}^{\,1} - P_{l-1}^{\,1}]$ (from Appendix N)

and that $\displaystyle\int_0^{2\pi} \sin\phi d\phi = \int_0^{2\pi} \cos\phi d\phi = 0$. Then

$$\text{(I1)} \int l(l+1)P_l \sin\theta P_n' \hat{\theta} d\Omega = \frac{-4\pi n(n+1)}{2n+1} \left[\frac{n(n-1)}{2n-1} \delta_{n,l+1} - \frac{(n+1)(n+2)}{2n+3} \delta_{n,l-1}\right] e_z.$$

Then use $\sin\theta P_l' = P_l^1$, $\sin\theta P_n' = P_n^1$, and

$$\cos\theta P_l^1 = \frac{1}{2l+1} [(l+1)P_{l-1}^{\,1} + l P_{l+1}^1] \text{ to find}$$

$$\text{(I2)} \int \sin\theta P_l' \sin\theta P_n' l(l+1) \hat{r} d\Omega = \frac{4\pi n(n+1)}{2n+1} \left[\frac{(n+2)^2}{2n+3} \delta_{n,l-1} + \frac{n(n-1)}{2n-1} \delta_{n,l+1}\right] e_z$$

Adding (I1) and (I2) yields the angular part of the integral

$$\int l(l+1)P_l \sin\theta P_n' \hat{\theta} + (l+1) \sin\theta P_l' \sin\theta P_n' \hat{r} = \frac{4\pi n(n+1)(n+2)}{2n+1} e_z \delta_{n,l+1}$$

Using the results from Appendix G, the radial part is

$$\int_0^a r^{-1/2} C_n I_{n+1/2}(kr) r^n r^2 dr \sim \frac{\delta(2n+1)a^{2n+1}}{(1+i)k}.$$

Putting in the necessary constants and averaging over one cycle yields,

$$F^o_{ave} = \frac{-\mu_o I_o{}^2 b^2 \pi}{2} \sum_{n=1}^{\infty} \frac{P_n^1(n\cdot\hat{c}) P_{n+1}^1(n\cdot\hat{c}) a^{2n+1}}{c^{2n+3}(n+1)} e_z.$$

APPENDIX J

The cross product between (29a) and (30b) yields

(J1) $r^{-1/2}C_{n-1}I_{n-1/2}(kr)r^{l-1}\{$

$$\text{a)}\ -\sin\theta P^{1\,\prime}_{n-1}\cos\phi\, l(l+1)P_l \begin{bmatrix}-\sin\theta\ e_z\\ \cos\theta\cos\phi\ e_x\\ \cos\theta\sin\phi\ e_y\end{bmatrix}.$$

$$\text{b)}\ -\frac{\sin\phi}{\sin\theta}P^1_{n-1}l(l+1)P_l \begin{bmatrix}-\sin\phi\ e_x\\ \\ \cos\phi\ e_y\end{bmatrix}$$

$$\text{c)}\ -\sin^2\theta P^{1\,\prime}_{n-1}\cos\phi(l+1)P_l{}' \begin{bmatrix}\cos\theta\ e_z\\ \sin\theta\cos\phi\ e_x\\ \sin\theta\sin\phi\ e_y\end{bmatrix}\}.$$

Similarly for (29a) with (30c):

(J2) $r^{-1/2}C_{n-1}I_{n-1/2}(kr)r^{l-1}\ \{$

$$\text{a)}\ nP_{n-1}l(l+1)P_l \begin{bmatrix}\cos\theta\ e_x\\ \\ -\sin\theta\cos\phi\ e_z\end{bmatrix}$$

$$\text{b)}\ -nP_{n-1}(l+1)\sin\theta P_l{}' \begin{bmatrix}-\sin\theta\ e_x\\ \\ -\cos\theta\cos\phi\ e_z\end{bmatrix}$$

For (29a) with (30d)

(J3) $r^{-1/2}C_{n-1}I_{n-1/2}(kr)r^{l-1}\ \{$

$$\text{a)}\ P^{\ 1}_{n-1}\sin\phi\, l(l+1)P_l \begin{bmatrix}\sin\theta\cos\phi\ e_y\\ \\ -\sin\theta\sin\phi\ e_x\end{bmatrix}.$$

$$\text{b)}-P^1_{n-1}\sin\theta(l+1)\sin\theta P_l{}' \begin{bmatrix}\cos\theta\cos\phi\ e_y\\ \\ -\cos\theta\sin\phi\ e_x\end{bmatrix}\}.$$

For (29b) and (29c) with (30a)

(J4) $r^{-1/2}C_nI_{n+1/2}(kr)r^{l-2}\ \{$

$$\text{a)}(l-1)^2P^{\ 1}_{l-1}\cos\phi\ \sin\theta P_n{}' \begin{bmatrix}-\sin\theta\ e_z\\ \cos\theta\cos\phi\ e_x\\ \cos\theta\sin\phi\ e_y\end{bmatrix}$$

$$\text{b)}\sin^2\theta P_l^1{}'(l-1)P_n{}'\cos\phi \begin{bmatrix}\cos\theta\ e_z\\ \sin\theta\ \cos\phi\ e_x\\ \cos\theta\ \sin\phi\ e_y\end{bmatrix}$$

From the ϕ dependence, it can readily be seen that only the x-components of these terms will survive integration.

Taking (J1) first and completing the angular integration, use the following identities:

i) $\int_0^{2\pi}\cos^2\phi d\phi = \int_0^{2\pi}\sin^2\phi d\phi = \pi$

ii) $-\sin\theta\cos\theta P_{n-1}^1{}' + \dfrac{1}{\sin\theta}P_{n-1}^1 = n(n-1)P_{n-2}-(n-1)\sin\theta P_{n-1}^1$

iii) $\sin^2\theta P_{n-1}^1{}' = \dfrac{1}{2n-1}[-(n-1)^2P_n^1 + n^2P_{n-2}^1]$

iv) $\sin\theta P_l{}' = P_l^1$.

Then,

$$\int[-\sin\theta P_{n-1}^1{}' l(l+1)\cos\theta\cos^2\phi P_l + \frac{\sin^2\phi}{\sin\theta}P_{n-1}^1 l(l+1)P_l - \sin^3\theta P_{n-1}^1{}'(l+1)P_l{}'\cos^2\phi]d\Omega$$

$$=\pi\int_{-1}^{1}[(n-1)nP_{n-2}l(l+1)P_l-(n-1)\sin\theta P_{n-1}^1 l(l+1)P_l - \frac{1}{2n-1}[-(n-1)^2P_n^1+n^2P_{n-2}^1](l+1)P_l^1]d\cos\theta$$

Use $\sin\theta\ P_l = \dfrac{1}{2l+1}[P_{l+1}^1 - P_{l-1}^1]$ and this equals

$$\pi\int_{-1}^{1}n(n-1)P_{n-2}l(l+1)P_l - (n-1)P_{n-1}^1\frac{l(l+1)}{2l+1}[P_{l+1}^1 - P_{l-1}^1]$$

$$-\frac{1}{2n-1}[-(n-1)^2n-1)^2P_n^1 + n^2\ P_{n-2}^1](l+1)P_l^1\ d\cos\theta.$$

Using the orthogonality relation for Legendre polynomials, this simply equals

$$\frac{2n\pi(n+1)(n-1)^2}{2n-1}\delta_{n,l}.$$

To do the same for (J2) use,

i) $\cos\theta\, P_l = \frac{1}{2l+1}\,[lP_{l-1} + (l+1)P_{l+1}]$

ii) $\sin^2\theta P_l{}' = \frac{1}{2l+1}\,[-l(l+1)P_{l+1} + l(l+1)P_{l-1}]$

and orthogonality to find that

$$\int (nP_{n-1}\, l(l+1)P_l \cos\theta + nP_{n-1}(l+1)\sin^2\theta P_l{}')d\Omega$$

$$= \frac{4\pi n^2(n+1)\delta_{n,l}}{2n-1}.$$

For (J3) use

i) $\sin\theta\, P_l = \frac{1}{2l+1}\,[P^1_{l+1} - P^1_{l-1}]$

ii) $\sin\theta\,\cos\theta P_l{}' = \frac{1}{2l+1}\,[(l+1)P^1_{l-1} + lP^1_{l+1}]$

to find that

$$\int (-P^1_{n-1}\ \sin\theta\, l(l+1)P_l \sin^2\phi + P^1_{n-1}\sin\theta\ \cos\theta(l+1)P_l{}'\ \sin^2\phi)d\Omega$$

$$= \frac{4\pi n(n-1)(n+1)\delta_{n,l}}{2n-1}.$$

for (J4) use

i) $\sin\theta\ \cos\theta\ P_n{}' = \frac{1}{2n+1}\,[(n+1)P^1_{n-1} + nP^1_{n+1}]$

ii) $\sin\theta P_n{}' = P^1_n$

iii) $\sin^2\theta P^1{}'_{l-1} = \frac{1}{2l-1}\,[-(l-1)^2 P^1_l + l^2 P^1_{l-2}]$

to find that

$$\int[(1-1)^2 P^1_{1-1}\cos\theta\sin\theta P_n'\cos^2\phi + \sin^3\phi P_1^{1\,\prime}(1-1)P_n{}'\ \cos^2\phi]d\Omega$$

$$= \frac{2\pi n(n+1)^2(n+2)}{2n+1}\ \delta_{n,1-2}.$$

(J1), (J2), (J3) all have the same radial terms and constants in the total force expression, thus their angular terms can be added together.

$$2\pi\left[\frac{n(n+1)(n-1)^2}{2n-1} + \frac{2n^2(n+1)}{2n-1} + \frac{2n(n-1)(n+1)}{2n-1}\right]\ \delta_{n,1}$$

$$= \frac{2\pi(n^2 + 2n-1)\ n\ (n+1)}{2n-1}\ \delta_{n,1}.$$

Because of the Kronecker delta, the radial part has the form

$$\int_0^a \frac{2n-1}{k}\ a^{2n-1}\ \exp\left[\frac{(1+i)(r-a)}{\delta}\right] dr = \frac{\delta^2(2n-1)a^{2n-1}}{2i}\quad .$$

The radial part of (J4) becomes

$$\int_{-}^{a} \frac{2n+1}{k}\ a^{2n+1}\exp\left[\frac{(1+i)(r-a)}{\delta}\right]\ dr = \frac{\delta^2(2n+1)a^{2n+1}}{2i}\quad .$$

Putting all of this together with the appropriate constants and averaging over one cycle yields the first order term of the force.

$$F^1_{ave} = \frac{-\pi\mu_o I_o{}^2 b^2 d}{4}\sum_{n=1}^{\infty}\left[\frac{|P_n{}'(n\cdot\hat{c})|^2 a^{2n-1}(n^2+2n-1)}{c^{2n+2}n(n+1)} + \frac{P_n^1(n\cdot\hat{c})P_{n+2}^{\,1}(n\cdot\hat{c})(n+1)}{c^{2n+4}(n+3)}\right]e_z.$$

APPENDIX K

For the zeroth order part of the torque, take the cross product of r with (J1) and (J2). For (J2), $r \times \hat{r} = o$, so that integral vanishes. For (J1), $\hat{r} \times \hat{\theta} = \phi$ so,

$$(K1)\quad r \times \int l(l+1)P_l \sin\theta P_n^1 \hat{\theta} d\Omega = r\int l(l+1)P_l \sin\theta P_n^1 \hat{\phi} d\Omega.$$

But, $\hat{\phi} = -\sin\phi\, e_x + \cos\phi\, e_y$

and $\int_o^{2\pi} \sin\phi d\phi = \int_o^{2\pi} \cos\phi d\phi = o.$

so (K1) vanishes also. Thus there is no torque to zeroth order in d.

The first order terms can be found by taking the cross product with r and equations (J1), (J2), (J3) and (J4). Noting that

$$r \times r = o$$

$$r \times \hat{\theta} = r\hat{\phi}$$

$$r \times \hat{\phi} = -r\hat{\theta}$$

$$\hat{r} \times e_x = -\sin\theta \sin\phi\, e_z + \cos\theta\, e_y$$

$$\hat{r} \times e_y = -\cos\theta\, e_x + \sin\theta \cos\phi\, e_z$$

$$\hat{r} \times e_z = -\sin\theta \cos\phi\, e_y + \sin\theta \sin\phi\, e_x$$

These terms are readily found. The cross product of r with equation (J1) yields

$$(K2)\quad r^{-1/2} C_{n-1} I_{n-1/2}(kr) r^l \{$$

$$\text{a)}\ -\sin\theta P_{n-1}^{1\prime} \cos\phi\, l(l+1)P_l \quad \begin{bmatrix} -\sin\phi\, e_z \\ \cos\phi\, e_y \end{bmatrix}$$

$$\text{b)}\ \frac{\sin\phi}{\sin\theta} P_{n-1}^1 l(l+1)P_l \quad \begin{bmatrix} -\sin\theta\, e_z \\ \cos\theta \cos\phi\, e_x \\ \cos\theta \sin\phi\, e_y \end{bmatrix}$$

The cross product with equation (J2) yields

(K3) $r^{-1/2}C_{n-1}I_{n-1/2}(kr)r^{l}\{$

$$\text{a) } nP_{n-1}\,l(l+1)P_l \begin{bmatrix} -\sin^2\theta\cos\phi\sin\phi\ e_z \\ \cos^2\theta + \sin^2\theta\cos^2\phi\ e_y \\ -\cos\theta\sin\theta\sin\phi\ e_z \end{bmatrix}$$

$$\text{b) } -nP_{n-1}(l+1)\sin\theta P_l' \begin{bmatrix} -\cos\theta\sin\theta\sin^2\phi\ e_y \\ \sin^2\theta\sin\phi\ e_z \\ \cos^2\theta\cos\phi\ e_x \end{bmatrix}$$

For (J3)

(K4) $r^{-1/2}C_{n-1}I_{n-1/2}(kr)r^{l}\{$

$$\text{a) } P^1_{n-1}\sin\phi\, l(l+1)P_l \begin{bmatrix} -\cos\theta\sin\theta\cos\phi\ e_x \\ -\cos\theta\sin\theta\sin\phi\ e_y \\ \sin^2\theta\ e_z \end{bmatrix}$$

$$\text{b) } -P^1_{n-1}\sin\phi(l+1)\sin\theta P_l' \begin{bmatrix} -\cos^2\theta\cos\phi\ e_x \\ -\cos^2\theta\sin\phi\ e_y \\ \sin\theta\cos\theta\ e_z \end{bmatrix}$$

Finally, for (J4)

(K5) $r^{-1/2}C_n I_{n+1/2}(kr)r^{l-1}\{$

$$(l-1)^2 P^{\,1}_{l-1}\cos\phi\,\sin\theta P_n' \begin{bmatrix} -\sin\phi\ e_x \\ \cos\phi\ e_y \end{bmatrix} \}.$$

Using

$$0 = \int_0^{2\pi}\sin\phi d\phi = \int_0^{2\pi}\cos\phi d\phi = \int_0^{2\pi}\cos\phi\sin\phi d\phi = \int_0^{2\pi}\cos^2\phi\sin\phi d\phi = \int_0^{2\pi}\sin^2\phi\cos\phi d\phi$$

only the y-components survive the integration over ϕ. To evaluate the angular part of the integral for terms of the form of equation (K2) use

$$\text{i) } \int_0^{2\pi}\cos^2\phi d\phi = \pi = \int_0^{2\pi}\sin^2\phi d\phi$$

$$\text{ii) } \sin\theta\, P^{\,1\,\prime}_{n-1} - \frac{\cos\theta}{\sin\theta}P^{\,1}_{n-1} = -\,n(n-1)P_{n-1}$$

$$\text{iii) } \int P_{n-1}P_l\, d\Omega = \frac{2}{2n-1}\,\delta_{n-1,l}.$$

Then integration of the angular part of (K2) yields

$$\frac{2\pi}{2n-1} n^2 (n-1)^2 \delta_{n-1,1}$$

To evaluate the angular part of (K3) use

$$\int_0^{2\pi} \sin^2\phi = \int_0^{2\pi} \cos^2\phi = \pi$$

to eliminate the ϕ dependence. The angular part being integrated is then

$2\pi n P_{n-1} l(l+1) P_l \cos^2\theta + \pi n P_{n-1} l(l+1) P_l \sin^2\theta + \pi n P_{n-1}(l+1) P_l' \sin^2\theta \cos\theta$.

Using

$\cos\theta P_l' = l P_l + P_{l-1}'$ this becomes

$2\pi n P_{n-1} l(l+1) P_l + \pi n P_{n-1}(l+1) P_{l-1}' \sin^2\theta$.

The first term can immediately be evaluted. It's integral is

$$\int_{-1}^{1} 2\pi n P_{n-1} l(l+1) P_l \, d\cos\theta = \frac{4\pi n^2 (n-1)}{2n-1} \delta_{n-1,1}$$

Using

$$\sin^2\theta P_{l-1}' = \frac{l(l-1)}{2l-1} [P_{l-2} - P_l],$$

the second term yields

$$\frac{2\pi n^2 (n+1)(n+2)}{(2n+1)(2n-1)} \delta_{n,l-1} - \frac{2\pi n^2 (n-1)(n-2)}{(2n-3)(2n-1)} \delta_{n,l+1} .$$

Doing the same with (K4). Use

i) $P_{l-1}' = -l P_l + \cos\theta P_l'$

ii) $\cos\theta P_{n-1}^1 = \frac{1}{2n-1} [(n-1) P_n^1 + n P_{n-2}^1]$

iii) $\sin\theta \, P_{l-1}' = P_{l-1}^1$.

Integrating, the angular part yields

$$\frac{2\pi n(n+2)(n-1)(n+1)}{(2n-1)(2n+1)} \delta_{n,l-1} + \frac{2\pi n^2 (n-1)(n-2)}{(2n-1)(2n-3)} \delta_{n,l+1} .$$

Equations (K2), (K3), (K4) all have the same radial dependence so their angular parts can be added. This yields

$$\frac{2\pi n(n+1)(n+2)}{2n+1}\,\delta_{n,l-1} + \frac{2\pi n^2(n-1)(n+1)}{2n-1}\,\delta_{n,l+1}\ .$$

For (M5) use

$$\sin\theta P_n{}' = P_n{}^1 .$$

The integral then yields

$$\frac{2\pi n^3(n+1)}{2n+1}\,\delta_{n,l-1}\ .$$

Integrating the radial part of these terms is similar to that integrations done in finding the force.

$$\int C_{n-1} I_{n-1/2}(kr) r^{l+2-1/2} dr = \int C_{n-1} I_{n-1/2}(kr) r^{n+3-1/2}\, dr \text{ for } \delta_{n,l-1} \text{ terms}$$

$$= \int C_{n-1} I_{n-1/2}(kr) r^{n+1-1/2}\, dr \text{ for } \delta_{n,l+1} \text{ terms.}$$

and $\int C_n I_{n+1/2}(kr) r^{l+1-1/2} dr = \int C_n I_{n+1/2}(kr) r^{n+2-1/2} dr$ for (M5) terms.

Again, using the same approximations as were used in finding the force and power,

$$\int_0^a C_{n-1} I_{n-1/2}(kr) r^{n+3-1/2} dr \sim \frac{\delta(2n-1)a^{2n+1}}{(1+i)k}\ ,$$

$$\int_0^a C_{n-1} I_{n-1/2}(kr) r^{n+1-1/2} \sim \frac{\delta(2n-1)a^{2n-1}}{(1+i)k}\ ,$$

$$\int_0^a C_n I_{n+1/2}(kr) r^{n+2-1/2} \sim \frac{\delta(2n+1)a^{2n+1}}{(1+i)k}\ .$$

So, the final expression for the first order term of the torque is

$$\Gamma' = \frac{-e_y \pi \mu_o I_o{}^2 b^2 d}{4} \sum_{n=1}^{\infty} \left[\frac{P_n^1(n\cdot c)P_{n-1}^1(n\cdot c)a^{2n-1}}{c^{2n+1}} + \frac{P_n^1(n\cdot c)P_{n+1}^1(n\cdot c)a^{2n+1}(2n^3+3n^2+3n-2)}{c^{2n+3}(n+1)(n+2)2n+1)} \right]$$

APPENDIX L

From [7, p. 541]

(L1) $P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x)$

(L2) $P_{n-1}'(x) = -nP_n(x) + xP_n'(x)$

(L3) $(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x)$

(L4) $(1-x^2)P_n'(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x)$

From [3,p. 162-3)

L5) $P_{n+1}^{m+1}(x) = (n+m+1)(1-x^2)^{1/2}P_n^m(x) + xP_n^{m+1}(x)$

(L6) $P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x) = (2n+1)(1-x^2)^{1/2}P_n^m(x)$

(L7) $P_{n-1}^{m+1}(x) = (m-n)(1-x^2)^{1/2}P_n^m(x) + xP_n^{m+1}(x)$

(L8) $P_n^{m+1}(x) - 2mx(1-x^2)^{-1/2}P_n^m(x) + (m+n)(n-m+1)P_n^{m-1}(x) = 0$

(L9) $(m-n-1)P_{n+1}^m(x) + (2n+1)xP_n^m(x) - (m+n)P_{n-1}^m(x) = 0$

(L10) $(1-x^2)^{1/2}P_n^{m\prime}(x) = -mx(1-x^2)^{-1/2}P_n^m(x) + P_n^{m+1}(x)$

(L11) $= -1/2(m+n)(n-m+1)P_n^{m-1}(x) - 1/2P_n^{m+1}(x)$

(L12) $= mx(1-x^2)^{-1/2}P_n^m(x) - (n+m)(n-m+1)P_n^{m-1}(x)$

(L13) $2m(1-x^2)^{-1/2}P_n^m(x) = P_{n-1}^{m+1}(x) + (m+n-1)(m+n)P_{n-1}^{m-1}(x)$

(L14) $(1-x^2)P_n^{m\prime}(x) = mxP_n^m(x) - (m+n)(n-m+1)(2n+1)^{-1}[P_{n+1}^m(x) - P_{n-1}^m(x)]$

(L15) $= (n+1)\ x\ P_n^m(x) - (n-m+1)P_{n+1}^m(x)$

(L16) $= (2n+1)^{-1}[(m-n-1)nP_{n+1}^m(x) + (n+1)(m+n)P_{n-1}^m(x)]$

(L17) $= -nxP_n^m(x) + (m+n)P_{n-1}^m(x)$

APPENDIX M

Another expression for the current density in the sphere

Ludwig Hannakam found an exact integral form for the eddy current density in conducting solid sphere in the presence of an alternating current loop of arbitrary shape or orientation to the sphere. He first constructed a surface by drawing lines through the center of the sphere and the current loop. He then showed that the vector potential due to the loop can be calculated by considering radial magnetic dipoles on the constructed surface. The problem of a conducting sphere in the presence of a magnetic dipole of oscillating dipole moment has a known solution. He partially completes the integral over the surface leaving only an integral over the current loop. His solution is

$$J = \frac{i\omega\mu\sigma I}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \frac{1}{\alpha_n} f_n(kr)\bar{P}_n(r)$$

$$\text{where } \alpha_n = (\frac{\mu}{\mu_o} -1)n\, f_n(ka) + ka f_{n-1}(ka)$$

$$f_n(kr) = (\frac{2}{kr})^{1/2} I_{n+1/2}(kr)$$

$$\bar{P}_n = \hat{r} \times \oint (\frac{a}{r_c})^n \{(\hat{r}_c \times \frac{dS_c}{r_c}) P_n'(\hat{r}\cdot\hat{r}_c) + \hat{r}_c[(\hat{r}_c \times \frac{dS_c}{r_c})\cdot\hat{r}] P_n''(\hat{r}\cdot\hat{r}_c)\}$$

a is the radius of the sphere

r_c is a vector to a point on the current loop (source point)

r is the field point in the sphere

dS_c is a line element on the current loop

$I = I_o e^{i\omega t}$ is the current in the loop

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