Rigorous Approaches to Tether Dynamics in Deployment and Retrieval

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ABSTRACT

Dynamics of Tethers in a linearized analysis can be
considered as the superposition of propagating waves.
This approach permits to have a new way for the analysis
of a Tether behaviour during deployment and retrival,
were a Tether can be considered composed by a part at
rest and a part sujected to propagation fenomena, being
the separating section depending on time.
The dependence on time of the separating section
requires the analysis of the reflection of the waves
travelling toward the part at rest. Such a reflection
generates a reflected wave, whose characteristics are
determined.
The propagation fenomena of major intest in a Tether are
transverse waves and longitudinal waves, all
mathematically modelled by the "vibrating chord"
equations, if the tension is considered constant along
the Tether itself. An interesting problem also
considered is concerned with the dependence of the
Tether tension from the longitudinal position, due to
microgravity, and the influence of this dependence on
propagation waves.

INTRODUCTION

Dynamics of Tethers('), as well as of any structure, in a linearized analysis can be considered as the superposition of propagating waves. This requires the study of dynamic propagation along the Tether with the appropriate boundary conditions. During deployment and retrieval, with reference to a lagrangian reference system("), Tether can be considered as composed of two parts - one at the rest and one subjected to the propagation phenomena. These two parts are separated by a section that changes with time, i.e. the Tether section that bounds the part constrained to rest is changing with time (the other part being free to move and vibrate and having the opposite end section subjected to the boundary conditions imposed by the satellite).
The propagation phenomena of major interest from a practical point of view are the following.

1) Transverse waves, mainly a "vibrating chord" behaviour, where inertial forces and the tension in the Tether - in combination with its local curvature - are the most important elements of the dynamic equilibrium.
2) Longitudinal waves, mathematically modelled by the "vibrating chord" equation, where inertial forces and longitudinal internal forces, due to elastic deformations, are the most important elements of the dynamic equilibrium.

The dependence on time of the section which bounds the part at rest requires the analysis of the reflection of the wave travelling toward the part at rest. Such a reflection generates a wave travelling outward, whose characteristics are to be determined.
The A. had previously considered from a theoretical point of view such problem in particular in order to analyze the behaviour of deployable booms subjected to longitudinal and flexural dynamic phenomena (see (1), (2), (3) and (4)). Also in the case of the problems concerning a structure like the Tether, the theoretical analysis gives rigorous solutions and permit an insight into experimentally observed effects,

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(') It can be suggested Ref. (8) for a general presentation of Tether concept and its (dynamic) problems.
(") A reference system which introduces a bi-univocal correspondence between a longitudinal coordinate and each Tether section.

that by some authors were erroneously thought to be "continuous" changes of frequencies and amplitudes of the proper modes.

This paper belongs to a series of works having the scope of opening a new way in the approach by means of mathematical models of several mechanical problems of deployable systems of telescopic and Tether type.

Recently several attempts have been made to solve the problem of the telescopic structures behaviour. Such attempts are mainly based in changes of the coordinates in order to take into account the changes during the time of the space where the problem is defined.

As a matter of fact these attempts don't seem obtain good results. They don't take into account energetic balances.

On the contrary this work introduces and developes to some extent the basic idea of considering each dynamic motion in a structure as the results of wave propagation, taking also into account energetic exchanges at the ends.

In the case of vibrating chord the problem of the time dependence of the definition space can't be resolved by means of the Cauchy, Goursat and Darboux results, (5). These results deal with the problems of time depending location of the sections where are imposed the boundary conditions. In the part that is external to such sections dynamic phenomena take place that are coherent with phenomena acting in the internal part and contribute to supply or spillover energy in it.

This work deals with the request of having external parts at the rest (not only the boundary sections). Therefore boundary sections have the behaviour of surface where internal dynamic phenomena "reflect".

Obviously reference is made to a constant section unaxial structure, as Tether can be considered. Longitudinal tension loads due to microgravity permits to consider additive small tension or compression loads without critical phenomena.

FUNDAMENTAL EQUATIONS

Let us have first a brief recall of concepts, with reference only to the case of longitudinal waves making use of Ref. (1). The reader can easily do the extension to transverse waves. The problem can be analyzed by means of an equilibrium and a continuity equations.

If $\sigma = \sigma(x,t)$ is the stress at point x and time t and $u = u(x,t)$ is the velocity of the motion, the equilibrium linearized equation is

1) $$\frac{\partial \sigma}{\partial t} = -\rho \frac{\partial u}{\partial t} \quad .$$

The continuity linearized equation on the other hand is

2) $$\frac{\partial u}{\partial x} = -\frac{1}{E} \frac{\partial \sigma}{\partial t} \quad .$$

If we put $c^2 = E/\rho$, operating we obtain

1') $$\frac{\partial^2 \sigma}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \sigma}{\partial t^2} \quad ,$$

2') $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad .$$

The general solutions of eq.s 1') and 2') are

3) $$\sigma = p_1 \left(t - \frac{x}{c} \right) + p_2 \left(t + \frac{x}{c} \right) ,$$

4) $$u = u_1 \left(t - \frac{x}{c} \right) + u_2 \left(t + \frac{x}{c} \right) .$$

Eqs. 3) and 4) indicate that the motion of the bar is composed of two waves: one travelling in the increasing x direction and another travelling in the decreasing x direction.

On the base of these considerations it is possible to obtain a relation between p and u of each travelling wave.

If $t_1 = t - \frac{x}{c}$ and $t_2 = t + \frac{x}{c}$, from 3) and 4) one obtains

5) $$\frac{\partial \sigma}{\partial x} = \frac{1}{c}\left(\frac{\partial p_2}{\partial t_2} - \frac{\partial p_1}{\partial t_1}\right) ,$$

6) $$\frac{\partial u}{\partial t} = \frac{\partial u_1}{\partial t_1} + \frac{\partial u_2}{\partial t_2} .$$

Taking into account eq. 1) from 5) and 6) we have the relation

7) $$\frac{1}{c}\left(\frac{dp_1}{dt_1} - \frac{dp_2}{dt_2}\right) = \rho\left(\frac{du_1}{dx_1} + \frac{du_2}{dt_2}\right) .$$

If $p_2 = 0$, $u_2 = 0$, eq. 7), gives the relation
$$\frac{dp_1}{dx_1} = \rho c \frac{du_1}{dt_1} = \frac{E}{c}\frac{du_1}{dt_1}$$
On the other hand if $p_1 = 0, u_1 = 0$, we have the relation
$$\frac{dp_2}{dt_2} = -\rho c \frac{du_2}{dt_2} = -\frac{E}{c}\frac{du_2}{dt_2} .$$

With the initial conditions $u_1(t_0) = 0$, $p_1(t_0) = 0$ or (separately) $p_2(t_0) = 0, u_2(t_0) = 0$ the following relations between u and p hold for each travelling wave

8) $$u_1\left(t - \frac{x}{c}\right) = \frac{c}{E} p_1\left(t - \frac{x}{c}\right),$$

9) $$u_2\left(t + \frac{x}{c}\right) = -\frac{c}{E} p_2\left(t + \frac{x}{c}\right).$$

Such relations enable us to determine one of the two values u_1 or p_1 when the other is known (eq. 8) and also to determine one of the values u_2 or p_2 when the other is know (eq. 9).
They are connected respectively with the separate behaviour of the two travelling waves.

REFLECTION ON THE TIME DEPENDENT SECTION

Also in this paragraph longitudinal waves are considered as a sample problem, still making use of Ref (1). The extension to transverse wave is easy. In practical applications the boundary conditions that are usually considered for uniaxial extensional bars are free edge ($\sigma = 0$) and fixed edge ($u = 0$). At the edge where deployment is done, it is $(u(x_c)=0$, where $x_c = x_c(t)$ is the time dependent section that can be considered as fixed and $dx_c/dt = v$ is the velocity of displacement of the constraint. Here x_c is an abscissa on the indeformed bar.

As a first analysis of the behaviour at a time dependent fixed section we can consider the problem of an extensional bar having a free edge at $x = 0$ and constrained with $u(x_c) = 0$ at $x_c = l_0 + c v$, where l, v are constant. The bar is subjected in $x = 0$ to an external extensional specific force p_1 indipendent of the time t.

Such force produces a wave travelling in the direction of the increasing x when such wave reaches the moving constraint section x_c a reflected wave of specific force p_2 is generated that runs in the direction of the decreasing x.

We will now determine the characteristic of the reflected wave, before it reaches the section $x = 0$.

In order to determine the reflected wave by means of an energy balance is necessary to dispose of an evaluation of the energy exchanged at the constrained end. A discussion on this subject is performed in ref. (1). The conclusion is that the constraint has and energy exchange different from zero e that the reflected wave can be determined by means of a behavioural analysis like the following.

During the time we have the already introduced displacement of the section where the constraint is imposed. Such displacement corresponds to the internal deformation of the rod.

The interval $dx = v dt$ during dt, withstands a length change $dl = -(u_1 + u_2) dt$ with a strain $\epsilon = dl/v dt$.

On the other hand the final stress in $v dt$ must be $\sigma = p_1 + p_2 =$ $= (u_1 - u_2) \frac{E}{c}$, that means a strain $\epsilon = (u_1 - u_2) 1/c$.

Equating the two expressions of the strain gives

$$10) \qquad u_2 = - u_1 \; \frac{c - v}{c + v} \quad .$$

This result 10) coincides with that of the application of the Goursat Darboux and Cauchy problem solution, (5), with the condition $u = o$ at $x = l_o + vt$. In spite of the observed coincidence with the well known results of the vibrating chord analysis, the proposed model presents the advantage of the applicability to more complex problems as dispersive systems, (see for instance (6) and (3)).

EXTENSION TO TIME DEPENDENT AMPLITUDE (OF THE TRAVELLING WAVE) AND SPEED (OF THE BOUNDARY CONDITION)

Let us consider in a non dispersive system the wave u_1 travelling inward the time variable restraint and the u_2 travelling outward expanded as follows, (see also (2)),

$$u_1 = \alpha_o + \alpha_1 (x - ct) + \alpha_2 (x - ct)^2 + \cdots \quad ,$$

$$u_2 = \beta_o + \beta_1 (x + ct) + \beta_2 (x + ct)^2 + \cdots \quad .$$

If at a time t the restraint condition is

$$u(x,t) = 0 \qquad (x \geqslant x_1)$$

and at the time $t + \Delta t$ where Δt will tends to zero,

$$u(x, t + \Delta t) = 0 \qquad (x \geqslant x_1 + c \Delta t),$$

without loss in generality we can put $x_1 = 0$ and introduce a dummy variable τ such that $t < \tau < t + \Delta t$.

During Δt at the section x_1 the displacement

12)
$$\Delta_1 = \int_0^{\Delta t} \left[u_1 (0, \tau) + u_2 (0, \tau) \right] d\tau$$

take place. During the same Δt in the region $x > 0$ the internal deformation generates a change in length

13)
$$\Delta_1 = \int_0^{\Delta t} \left[u_1 (v\tau, \Delta t) - u_2 (v\tau, \Delta t) \right] \frac{v}{c} d\tau.$$

The equation between such two Δ_1, substituting 14) and taking into account only the terms of the lower order for respect to the principal Δt, operating gives,

$$\left(1 + \frac{v}{c} \right) \beta_o = \alpha_o \left(1 - \frac{v}{c} \right)$$

or

14)
$$\beta_o \cong - \alpha_o \left(1 - 2 \frac{v}{c} \right).$$

Relation 14) means that, beside the higher order terms,

$$u_2 \cong - u_1 \left(1 - 2 \frac{v}{c} \right),$$

like in the case of u_1 and u_2 of constant value.

EXTENSION TO PROPAGATION SPEED DEPENDING ON THE LONGITUDINAL POSITION

The present paragraph concerns the extension of the analysis to the case where eq. 1') and 2') become as follows

$$\frac{\partial^2 \sigma}{\partial x^2} = \frac{1}{c^2(x)} \frac{\partial^2 \sigma}{\partial t^2} \quad , \qquad\qquad 1'')$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2(x)} \frac{\partial^2 u}{\partial t^2} \quad . \qquad\qquad 2'')$$

This case includes the dynamics of transverse waves in Tethers where longitudinal load is depending on the position due to microgravity.

Because $c(x)$ is not constant but a function of x, $c = c(x)$, expressions 3) and 4) are not still valid, and it is necessary to find an appropriate way of solution.

The A. in a previous paper, (7) here largely recalled, proposed that the general solution of eq.s 1'') were composed by means of two waves in opposite directions travelling and having speed depending on x. Such waves reduce to eq.s 3) and 4) when $c(x)$ reduce to a constant C. If ϕ indicates σ or u as necessary, the following expression was adopted (')

$$\phi = \phi_1\left(t - \int \frac{dx}{k\,c(x)}\right) + \phi_2\left(t + \int \frac{dx}{k\,c(x)}\right) \quad , \qquad 15)$$

where k is an arbitrary constraint and $\pm\, k\, c(x)$ could be the propagation speed at x.

Obviously, not all the functions ϕ_1 and ϕ_2 are usefull to satisfy 1''). The problem is now reduced to the determination of the functions ϕ_1 and ϕ_2 if any, that can satisfy 1'').

The functions ϕ_1 and ϕ_2 can be examined separately.

If we put $\, \tau = t - \int \frac{dx}{k\,c(x)}\,$, we have

$$\frac{\partial \phi_1}{\partial x} = \frac{d\phi_1}{d\tau}\frac{\partial \tau}{\partial x} = \frac{d\phi_1}{d\tau}\left(-\frac{1}{k\,c(x)}\right)$$

$$\frac{\partial^2 \phi_1}{\partial x^2} = \frac{d^2\phi_1}{d\tau^2}\left(\frac{1}{k\,c(x)}\right)^2 + \frac{d\phi_1}{d\tau}\left(\frac{1}{k}\frac{c'(x)}{c^2(x)}\right) \quad , \quad \left(c'(x) = \frac{dc(x)}{dx}\right) ,$$

$$\frac{\partial \phi_1}{\partial t} = \frac{d\phi_1}{d\tau}\frac{\partial \tau}{\partial t} = \frac{d\phi_1}{d\tau} \quad , \qquad \frac{\partial^2 \phi_1}{\partial t^2} = \frac{d^2\phi_1}{d\tau^2} \quad .$$

(') When $c(x) = $const. and $k = 1$, the proposed expression reduces to 3) and 4).

Recalling now 1") we obtain

$$\frac{d^2\phi_1}{d_1^2}\frac{1}{k^2} + \frac{d\phi_1}{d_1}\frac{1}{k}c(x) - \frac{d^2\phi_1}{d_1^2} = 0. \qquad 16)$$

The function ϕ_1, if it there exists, must satisfy eq. 16). To the same conclusion we lead if we consider ϕ_2.

As an application we can now restrict to the case $c = c_0 x$
We have $c'(x) = c_0$ and

$$\left(1 - \frac{1}{k^2}\right)\frac{d^2\phi_1}{d_1^2} - \frac{c_0}{k}\frac{d\phi_1}{d_1} = 0 \qquad 16')$$

This is a linear ordinary second order equation.
Its solution is of the form $\phi_1 = e^{\lambda_1}$ and precisely

$$\phi_1 = exp + \frac{1}{1-k^2}\left(-kc_0 + l_n x\right) \qquad 17)$$

If we consider ϕ_2 we obtain also

$$\phi_2 = exp + \frac{1}{1-k^2}\left(kc_0 + l_n x\right) \qquad 17')$$

In the previous analysis k is an arbitrary constant. If we let k assume all the values $0' \longmapsto k \longmapsto \infty$ we obtain ϕ as ϕ_1 and ϕ_2 complete sets of functions, which allow us to expande by integral whatever function. Each dynamic fenomenum in, a structure where eq.s 1") are valid and $c = c_0 x$ can be analysed as, (see 17) and 17')),

$$\phi = \int_0^\infty \mathscr{C}_1(k) \, exp + \frac{1}{1-k^2}\left(-kc_0 + l_n x\right) dk +$$

$$+ \int_0^{\infty} \mathscr{C}_2(k) \, exp + \frac{1}{1-k^2}\left(kc_0 + l_n x\right) dk. \qquad 18)$$

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Obviously 18) is not the only way by which to expand by integral such a dynamic fenomenum, but this way allows us to consider component functions when the propagation speed at any x is well know. Such speed, as we know, is a fundamental datum in order to evaluate the speed of a reflected wave, in particular in the case of time depending restraint conditions.

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