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THE CASE FOR AERODYNAMIC SENSITIVITY ANALYSIS

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This paper is somewhat unusual since it does not offer any specific solutions, verified by applications, for its subject problem which is sensitivity analysis in Computational Fluid Dynamics (CFD). Instead, the paper makes a plea to the CFD community for extending their present capability to include sensitivity analysis. The plea is made from the viewpoint of an aeronautical engineer, not an expert in CFD methods, who needs the sensitivity information when working at the junction of aerodynamics, structures, active controls, and other disciplines whose inputs need to be integrated in aircraft design. The principal message of the paper is displayed on figure 1.

THE MESSAGE

- Computational fluid mechanics is advancing rapidly its capability to calculate aerodynamic forces on wing-body-nacelle-empennage configurations
- Next logical step: capability to compute sensitivity of these forces to configuration geometry, i.e., sensitivity derivatives
- Example: ∂(lift)/∂(wing sweep angle)
- Urgent need:
    - Intradisciplinary: aerodynamic shape optimization
    - Interdisciplinary: integrating aerodynamics with other disciplines

Fig. 1

The intradisciplinary applications of the postulated sensitivity analysis are obvious enough. It has now become quite common to optimize aerodynamic shapes (illustrated at the bottom of figure 2 by the inset showing an airfoil and an aircraft planform) by formal algorithms that iteratively change geometrical variables shown in the inset. Figure 2 depicts one such procedure composed of an OPTIMIZER which determines the increment of each geometrical variable (design variable, x), TERMINATOR containing a logic for stopping the iteration, and ANALYZER (a CFD program) whose task is to calculate the aerodynamic objective function (F) and constraints (g) for the geometry modified by the optimizer. Since most of the OPTIMIZER algorithms commonly in use require derivatives of F and g with respect to the design variables (x), it would be advantageous for the efficiency and accuracy of the aerodynamic optimization, if these derivatives were available in the ANALYZER's output. Thus, the need for a finite difference approximation to the derivatives, and the associated, costly, repetitive analysis would be eliminated.

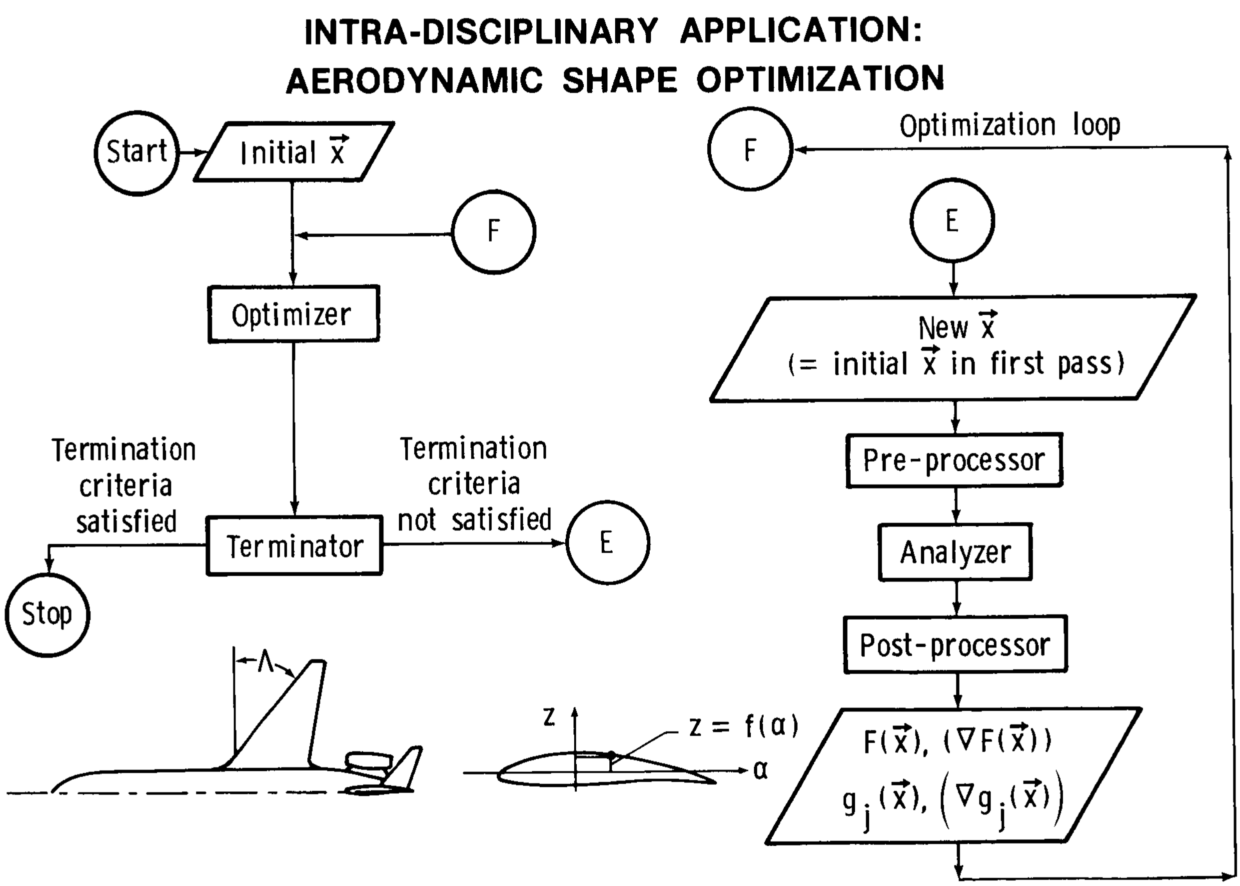


Fig. 2

Going beyond the confines of the discipline of aerodynamics, the aerodynamic sensitivity information is needed to quantify the effect of the changes in aerodynamic shape on other disciplines coupled to aerodynamics in the design process. Figure 3 shows aerodynamics at a central position in the process, its interactons with other disciplines depicted by two-headed arrows. The meaning of the arrows may be illustrated by an example of a coupling between the aerodynamics and structures. A change of the aerodynamic shape causes a change in the structural response, directly through the geometry and, indirectly, through the aerodynamic loads. In the opposite direction, the change in structural response will, of course, influence the aerodynamic loads through the change of deformation pattern.

To stay within a limited scope, the remainder of this discussion will concentrate on the interaction among only three disciplines: aircraft performance, aerodynamics, and structures, to show how the sensitivity information, including the aerodynamic sensitivity, could be used toward improving aircraft performance.

AERODYNAMICS INTERACTION WITH OTHER DISCIPLINES IN AIRCRAFT DESIGN

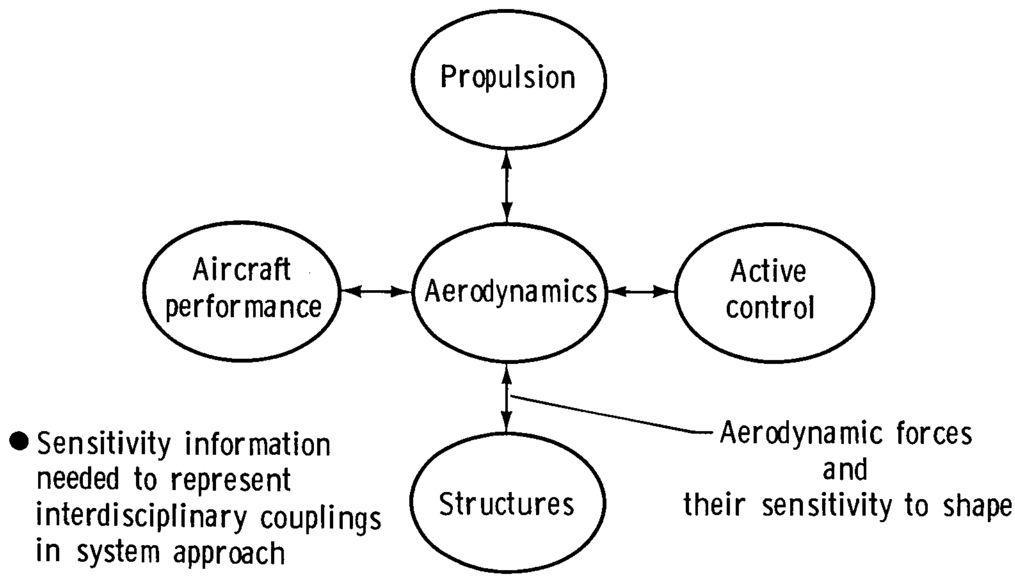


Fig. 3

To demonstrate that, figure 4 and the next two figures show what may happen when a problem encountered in one subsystem, or engineering discipline, is fixed by the means local to that subsystem or discipline. An example of a particular stage in the process of aircraft design will illustrate the point. Suppose that at that stage, the configuration designers had already set the value of the aspect ratio (AR), a typical configuration design variable, so as to maximize the aircraft range (R) under the constraint on the take-off gross weight (TOGW or T). In that decision, they accounted for the influence of the aerodynamic drag, represented by c_D, fuel weight W_f, and structural weight, W_s on R and TOGW. Of course, many more variables are involved in the real problem, but simplification of the example will help to make the point.

In the above set of quantities, c_D and W_f came from the analysis and experimentation carried out by the group of engineers working with the configuration and performance aerodynamics. In contrast, the value of structural weight was available to that group only as a rough estimate. Now, suppose that the process moves on into the phase of more detailed structural analysis and design.

A CONVENTIONAL APPROACH: LOCAL PROBLEM — LOCAL FIX

Aircraft: Range $R = f_1\left(W_s, C_D, \ldots\right)$ (1)

R is objective function, $R \rightarrow R_{max}$

W_s — structural weight

Constraint:

TOGW: $T = f_2\left(W_s, W_f, \ldots\right) \leqslant T_0$ (2)

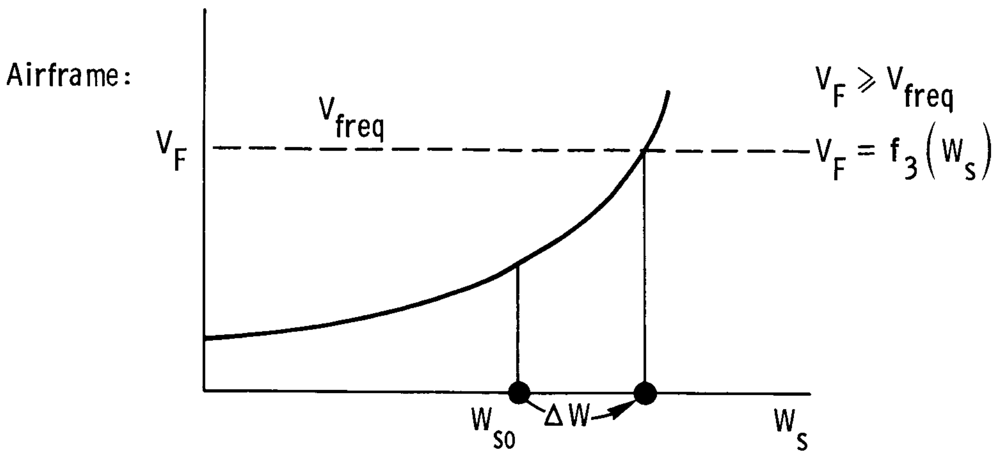
W_f — fuel weight

f_1, f_2 — computable functions, may be analytical expressions or computer programs

Fig. 4

At that stage, illustrated by figure 5 below, structural design has advanced to the point where a flutter analysis was carried out. Let us assume that it showed the flutter speed V_F falling short of the required value V_{freq}. With the wing geometry (AR) having been already set and frozen, the structural group fixed the flutter problem by stiffening the wing at the weight penalty made as small as possible, ΔW_{min}.

LOCAL PROBLEM, FLUTTER, FIXED BY A LOCAL MEANS A STRUCTURAL STIFFENING



Aeroelastic optimization: $\Delta W \rightarrow \Delta W_{min}$

ΔW — weight penalty

Fig. 5

The flutter weight penalty was sent back to the aircraft performance group who added it to the initial estimate of W_s and had to compensate for it by reducing the fuel weight W_f to keep TOGW within constraint (assuming constant payload). The result is a change in performance (R) estimated by eqs. 1 and 2 in figure 6.

GLOBAL (SYSTEM) CONSEQUENCES OF LOCAL FIX

Aircraft: $W_s \rightarrow W_s + \Delta W_{min}$ requires reduced

fuel $W_f \rightarrow W_f - \Delta W_f$ because of

constrained TOGW,

$T = f_2\left(W_s, W_f, \ldots\right) = T_0$, hence

Range reduction: $R = R_0 + \frac{\partial R}{\partial W_f} \bullet \Delta W_f$ to the first order approximation (1)

Since $\Delta W_f = -\Delta W_{min}$

$$R = R_0 + \frac{\partial R}{\partial W_s} \bullet \Delta W_{min} \qquad (2)$$

Fig. 6

Examination of the example unfolded thus far leads to the two observations, shown in figure 7, that summarize what may happen when a local problem is fixed by local means, but has an impact on the system performance.

TWO OBSERVATIONS

1. $\partial R/\partial W_s < 0$ (of course), hence $R \rightarrow R - \Delta R$. $(-\Delta R)$ is the system performance penalty for a subsystem modification.

2. The system configuration was not touched. The constraint (flutter) was satisfied by purely local, subsystem, means. Since $\Delta W = \Delta W_{min}$, the system performance penalty is the smallest achievable by the local means. To reduce it further, one needs modification at system level.

Fig. 7

This, and the next three figures, will show the potential for improving the system performance by correcting the subsystem problem by design modifications at both the local and system levels - a system approach. In our example, that means unfreezing the configuration geometry (AR) and using it together with added structural material ΔW_{min} to meet the flutter constraint, while reducing the penalty in the system performance (R) subjected to the constraint on TOGW.

The upper box in figure 8 symbolizes the performance and configuration aerodynamics group who sends the data on geometry (AR) and on the aerodynamic loads magnitude and distribution $c_p(\alpha,\beta)$ to the structures group depicted by the lower box. The former group's objective is to maximize R under the constraint on TOGW by means of changing the configuration geometry AR. The latter group manipulates the structural cross-section dimensions to meet the flutter constraint at the minimum weight penalty. That penalty is a computable function of the geometry, (AR), and aerodynamic loads, c_p (the next to the last line on the figure). To the structures group these quantities are constants, but the configuration group can control them by means of AR, thus influencing the ΔW_{min}. That influence can be quantified by the chain differentiation shown on the bottom line on the figure.

In that line, the derivatives of f_4 are derivatives of the optimum design with respect to the constant parameters of the optimization - a type of constrained derivative. Algorithms exist (refs. 1 and 2) for computing such derivatives quasi-analytically, without engaging in repeated optimization of perturbed geometry. The derivative of c_p is a CFD sensitivity derivative postulated in this presentation.

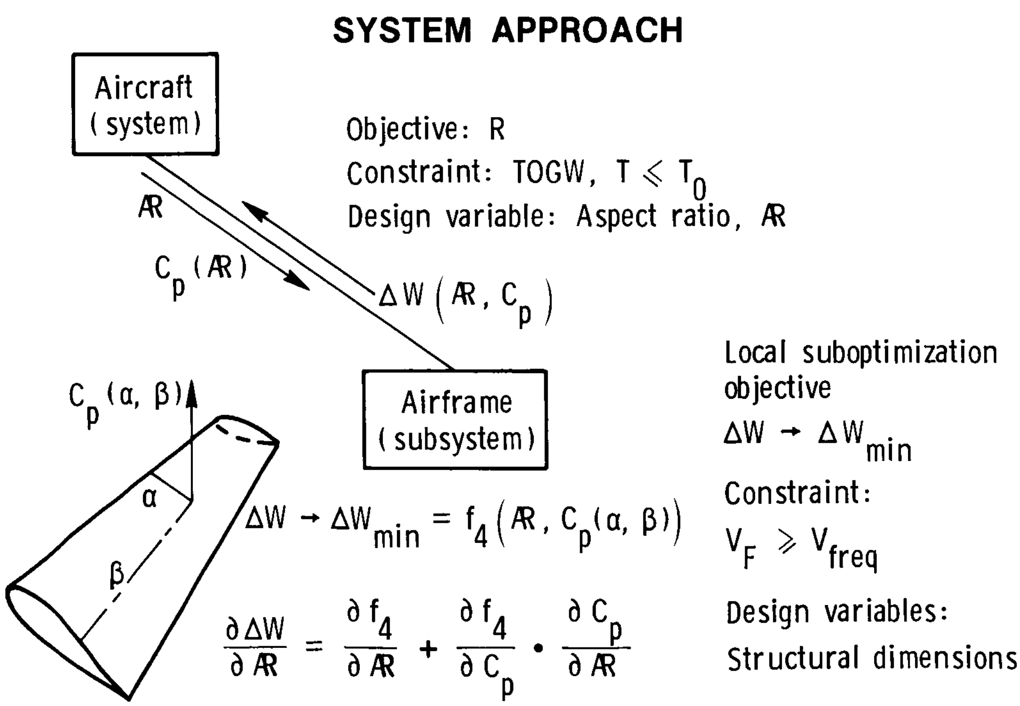


Fig. 8

Here, we return to the system level with the information generated in the discipline of structures. Under the system approach, the information has been enhanced by the sensitivity of the flutter weight penalty to geometry, quantified by the derivative of ΔW_{min} with respect to AR. The information now available to the performance and configuration group, and originating in that group's own work, is shown on line 1, figure 9 (subscript/superscript "0" refers to the design that has been accomplished and is now to be modified). The first two derivatives are computable from the performance analysis, and the third one was discussed at the end of the preceding figure. The chain differentiation relates the range to geometry.

The extrapolation in eq. 2 using the optimum sensitivity derivative for ΔW_{min} with respect to AR establishes an approximation to the flutter weight penalty as a function of geometry. Substitutions shown by arrows into the linear extrapolation for R in eq. 3 lead to the approximation of R as a function of geometry in eq. 4. The first two terms represent the result obtained previously under the rule of frozen AR. The square parentheses term reflects the cumulative, first order effect of geometry on performance, exerted through a multitude of interdisciplinary effects, each quantified by a particular term in the parentheses.

SYSTEM SENSITIVITY AND OPTIMIZATION: OBJECTIVE

Aircraft (system)

ΔW^0_{min}

$\frac{\partial \Delta W_{min}}{\partial \text{AR}}$

$$R_0,\ \frac{\partial R}{\partial \Delta W},\ \frac{\partial R}{\partial C_D},\ \frac{\partial C_D}{\partial \text{AR}};\ \frac{\partial R}{\partial \text{AR}} = \frac{\partial R}{\partial C_D}\cdot\frac{\partial C_D}{\partial \text{AR}} \qquad (1)$$

Approximate:

$$\Delta W_{min} = \Delta W^0_{min} + \frac{\partial \Delta W_{min}}{\partial \text{AR}}\cdot \Delta \text{AR} \qquad (2)$$

$$R = R_0 + \frac{\partial R}{\partial(\Delta W)}\cdot \Delta W_{min} + \frac{\partial R}{\partial \text{AR}}\,\Delta \text{AR} \qquad (3)$$

$$R = R_0 + \frac{\partial R}{\partial(\Delta W)}\,\Delta W^0_{min} + \left[\frac{\partial R}{\partial(\Delta W)}\cdot\frac{\partial(\Delta W_{min})}{\partial \text{AR}} + \right.$$
$$\left. + \frac{\partial R}{\partial(\Delta W_{min})}\,\frac{\partial(\Delta W_{min})}{\partial C_p}\cdot\frac{\partial C_p}{\partial \text{AR}} + \frac{\partial R}{\partial C_D}\cdot\frac{\partial C_D}{\partial \text{AR}}\right]\cdot\Delta \text{AR} \qquad (4)$$

Fig. 9

A similar development is shown in figure 10 for the system level constraint on TOGW leading to a linear approximation in of the constraint as a function of geometry in eq. 4. Again, the terms in eq. 4 quantify the several, interdisciplinary influences involved.

$$\text{TOGW},\ T \leqslant T_0;\quad \frac{\partial T}{\partial(\Delta W)},\ \frac{\partial T}{\partial\left(W_f\right)},\ \frac{\partial W_f}{\partial C_D},\ \frac{\partial C_D}{\partial \text{AR}}; \tag{1}$$

Approximate:

$$T = T_0 + \frac{\partial T}{\partial(\Delta W)} \bullet \Delta W_{min} + \frac{\partial T}{\partial W_f} \bullet \frac{\partial W_f}{\partial C_D} \bullet \frac{\partial C_D}{\partial \text{AR}} \bullet \Delta \text{AR} \tag{2}$$

$$\Delta W_{min} = \Delta W^0_{min} + \frac{\partial \Delta W_{min}}{\partial \text{AR}} \bullet \Delta \text{AR} \tag{3}$$

From optimum sensitivity analysis

$$T = T_0 + \frac{\partial T \bullet \Delta W^0_{min}}{\partial\left(\Delta W_{min}\right)} + \left[\frac{\partial T}{\partial\left(\Delta W_{min}\right)} \bullet \left(\frac{\partial\left(\Delta W_{min}\right)}{\partial \text{AR}} + \frac{\partial\left(\Delta W_{min}\right)}{\partial C_p} \bullet \frac{\partial C_p}{\partial \text{AR}} \right) + \frac{\partial T}{\partial W_f} \bullet \frac{\partial W_f}{\partial C_D} \bullet \frac{\partial C_D}{\partial \text{AR}} \right] \bullet \Delta \text{AR} \leqslant T_0 \tag{4}$$

Fig. 10

Derivation of R and T as approximate functions of geometry (bottom line equations in fig. 9 and 10) enables the configuration group to modify the geometry (AR) toward better performance (R). When modifying AR, the group is assured that the flutter constraint will be kept satisfied to the first order of accuracy, because the flutter weight penalty will follow the change of AR in a way prescribed by eq. 2, figure 9. The change of AR may be obtained formally by solving an optimization problem defined by eqs.1 and 2, figure 11. The resulting performance improvement over the previous case of the frozen AR is shown by the last term in eq.4. The improvement comes about because we traded structural weight and aerodynamic drag for each other while modifying the geometry (a typical design trade-off), and we did it in a measured way on the basis of the sensitivity derivatives.

SYSTEM SENSITIVITY AND OPTIMIZATION

Conclusion

Find ΔAR, such that

$$R = R_0 + \frac{\partial R}{\partial(\Delta W)} \cdot \Delta W^0_{min} + \frac{\partial R}{\partial AR} \cdot \Delta AR \rightarrow \max \qquad (1)$$

Subject to

$$T = T_0 + \frac{\partial T}{\partial\left(\Delta W_{min}\right)} \cdot \Delta W^0_{min} + \frac{\partial T}{\partial AR} \cdot \Delta AR \leqslant T_0 \qquad (2)$$

Obtain $(\Delta AR)_{opt}$ from 1 and 2, to get R_{max}:

$$R_{max} = R_0 + \frac{\partial R}{\partial(\Delta W)} \cdot \Delta W^0_{min} + \frac{\partial R}{\partial AR} \cdot \Delta AR \qquad (3)$$

$$R_{max} = \underbrace{R_0 - \Delta R}_{\text{Obtained previously}} + \underbrace{\frac{\partial R}{\partial AR} \cdot \Delta AR}_{\text{Additional term}} \qquad (4)$$

Fig. 11

The sensitivity of R to geometry represented by the derivative in the last term on the preceding figure is the key piece of information necessary to reduce the system performance penalty paid for the fix of the subsystem problem (flutter). The expression for the derivative is reproduced in figure 12 (see eq. 4, figure 9), with the source of each partial identified by a letter code inscribed beneath.

DISCUSSION OF THE OPTIMAL SOLUTION

Total chain-derivative expression for $\partial R / \partial AR$ is:

$$\frac{\partial R}{\partial AR} = \underset{P}{\frac{\partial R}{\partial (\Delta W)}} \cdot \underset{S}{\frac{\partial \left(\Delta W_{min}\right)}{\partial AR}} + \underset{P}{\frac{\partial R}{\partial \left(\Delta W_{min}\right)}} \cdot \underset{ASF}{\frac{\partial \left(\Delta W_{min}\right)}{\partial C_p}} \cdot \underset{\substack{A \\ U}}{\frac{\partial C_p}{\partial AR}} + \underset{P}{\frac{\partial R}{\partial C_D}} \cdot \underset{\substack{A \\ ST}}{\frac{\partial C_D}{\partial AR}}$$

- Existence of the additional term in equation for R_{max} allows to recover a part of the performance penalty —

- Sources of derivatives: P - performance, S - structures,
  ASF - aeroelasticity and flutter,
  A - aerodynamics, ST - steady,
  U - unsteady

Fig. 12

Before we take a closer look at availablity of the derivatives at the appropriate sources, let us devote one figure (fig. 13) to adress the obvious question that arises at this point: "Why not to get whatever derivatives are needed by a straightforward finite difference technique ?". To supplement the figure, let us assure the reader that we do not dogmatically favor the quasi-analytical way over the finite difference way of computing the derivatives. If someone overcomes the computational cost impediment in a finite difference technique built on top of a CFD analysis - the resulting tool will certainly be eagerly accepted. However, the point is that a quasi-analytical alternative to finite difference techniques exists, and due to experience garnered in other disciplines it deserves a serious consideration. We will come back to this point again, soon after we examine, briefly, the derivative availability under the state of the art.

SENSITIVITY DERIVATIVES BY FINITE DIFFERENCE?

- For N variables, the simplest finite difference technique requires, at least, N + 1 repetitions of analysis
- In real world of engineering design, that erects a time n and cost barrier
- Experience from other engineering disciplines suggests an alternative: quasi-analytical algorithms
- Only one paper in this symposium program refers to aerodynamic sensitivity analysis — that fact is symptomatic for the state of the art in CFD

Fig. 13

Although it is quite clear where each derivative should originate, the availability is distributed very unevenly, as shown in figure 14. Most of the pertinent capability exists in structures for derivatives with respect to cross-sectional dimensions and overall shape (see survey in ref. 3). Some of that capability became available in production level codes. In aeroelasticity, algorithms exist for computation of the flutter velocity derivatives with respect to the cross-sectional dimensions (ref. 4), but not with respect to the overall shape variables. Unfortunately, to the best of available information, sensitivity analysis in CFD is currently limited to the capability described in ref. 5 that applies only to linear subsonic aerodynamics.

AVAILABILITY OF DERIVATIVES

- Performance: Finite difference is inexpensive
- Structures: Analytical derivatives available in production codes (e.g., NASTRAN)
- Aeroelasticity and flutter: Analytical derivatives of V_F available
- Aerodynamics: A beginning made in steady, subsonic, NASA CR 3713, 1983 (Bristow, MCDAC)

  Nothing in transonic } Steady
  Nothing in supersonic } Steady

  Nothing in unsteady

  Nothing in production level codes

Fig. 14

C-2

Let us contrast, in figure 15, the finite difference technique with a quasi-analytical manner of computing the derivatives. Both techniques apply to a set of equations that, in general, govern a physical problem (this is a generic discussion, not limited to aerodynamics). The set of equations appears as the topmost equation on the figure, with y denoting the vector of solution variables (behavior variables), and x standing for a vector of design variables that are constant in the process of solving the equations F, but may vary in the associated design (optimization) problem.

The computational cost of the finite difference approach (line 1) was noted before. That cost may be avoided by means of a quasi-analytical approach described by line 2. It begins with setting to zero the first variation of F with respect to perturbation of an element of the vector x, and leads to a universal sensitivity equation (eq. 2). That equation can be directly solved to obtain the vector of derivatives which, in effect, relate change of the output (y) of the solution of the governing equations (F(y,x) = 0) to the input (x). Three comments on the nature of the sensitivity equation (eq.2) are noted at the bottom of the figure. Appendix A provides a self-contained elaboration on the generic quasi-analytical approach, and Appendix B illustrates that approach in linear static structural analysis.

ANALYTICAL DERIVATIVES VERSUS FINITE DIFFERENCES

$F(y,x) = 0, \rightarrow y; \; y = y(x)$ implicitly

e.g., $y = C_p$ (location), $x = AR$, F( ) — an algorithm

1. Finite difference: $x \rightarrow x + \Delta x \rightarrow F(y,x) \rightarrow y + \Delta y; \; \frac{\partial y}{\partial x} \cong \frac{\Delta y}{\Delta x}$ (1)

   N + 1 times for N x's

2. Analytical: $\frac{\partial}{\partial x}(F(y,x)) = 0 \rightarrow \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = -\frac{\partial F}{\partial x}$ (2)

- Eq. 2 is linear with respect to $\partial y/\partial x$, even though F(y,x) may be nonlinear
- Eq. 2 is noniterative, even if F(y,x) = 0 is iterative
- In eq. 2, $\partial F/\partial y$ and $\partial F/\partial x$ obtainable either analytically or by finite difference, then F(y,x) is evaluated, rather than solved F(y,x) = 0

Fig. 15

The conclusion we are now arriving at is that demonstratable improvements in aircraft performance are achievable by including interdisciplinary interactions in the configuration shaping decisions. Much of the potential for these improvements remains either unused, or its exploitation is being achieved at an excessive computational cost because of the lack of sensitivity analysis capability in CFD. The postulated remedy is development of a capability for computation of derivatives with respect to shape as a routinely available option in the CFD codes. Hence, the challenge to the CFD community posed in figure 16 closes this paper.

A CHALLENGE FOR COMPUTATIONAL AERODYNAMICS COMMUNITY

- Derivatives of: $C_p(x,y)$, C_D, C_L, C_M
- With respect to: Configuration variables,
  e.g., Aspect ratio
  Sweep angle
  Taper
  Airfoil shape
  Camber . . .
  Twist, etc. . . .
- For sub-, tran-, supersonic, steady, unsteady wing + full configuration
- Basic formulation + production codes

Fig. 16

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APPENDIX A

GENERAL EQUATION FOR SENSITIVITY

This Appendix is a self-contained tutorial on sensitivity analysis arising in a generic problem whose governing equations are given. Let

$$F(y,x) = 0 \quad (1)$$

represent governing equations of a problem in which y is a vector of unknowns to be obtained by solving eq. 1, and x is a vector of given constants. The quantities y and x may be vectors, and F may be a vector of functions. If y is a vector, eq. 1 implies a set of equations whose number is equal to the length of vector y; however, the x vector may be shorter than y. Existence of the solution of eq. 1 makes, implicitly, $y = f(x)$. The functions F may be anything computable : linear algebraical equations, PD equations, integral equations, or integral-differential equations, transcendental functions, etc. It may be nonlinear, and may require an iterative method for solution of eq. 1.

If eq. 1 governs a physical system being designed, then the designer wants to know not only the y for a given x, but also the sensitivity of b to those x-quantities that he controls as design variables. For instance, $F(y,x)$ might be the Euler equations from which to compute y - the pressure distribution on a body in airflow, and x might be the body geometry variables. The designer of the body shape needs to know $\partial y/\partial x$.

One way to obtain $\partial y/\partial x$ is by finite differences. This requires solving eq. 1 for given x to obtain y. Then assume, for one element of x,a perturbation x $x + \Delta x$, and repeat solution of eq. 1 to get $y + \Delta y$. Approximation to $\partial y/\partial x$ is

$$\partial y/\partial x = \Delta y/\Delta x; \quad (2)$$

This operation must be repeated for all x-quantities of interest and may be prohibitively computer-intensive, if eq. 1 is expensive to solve. In addition, the accuracy of $\partial y/\partial x$ will depend on the proper choice of Δx.

An alternative is a quasi-analytical approach. It is called "quasi-" because the y(x) is known only numerically. However, we know that for Δx, we must have

$$F(y+\Delta y,\ x+\Delta x) = 0; \quad (3)$$

in other words, increase of x must be compensated for by change in y to preserve the zero value of F. Hence, recognizing that the total derivative (TD) of F with respect to x is according to the textbook rules of differentiation for implicit functions

$$dF/dx = \partial F/\partial x + \partial F/\partial y\ \partial y/\partial x; \quad (4)$$

eq. 3 will be satisfied if

$$dF/dx\ \Delta x = 0 \quad (5)$$

Substituting eq. 4 into 5, and rearranging, yield

$$\partial F/\partial y \; \partial y/\partial x = - \partial F/\partial x \quad (6)$$

Eq. 6 is a general sensitivity equation in which the desired sensitivity appears directly as the unknown $\partial y/\partial x$. For vector y of length n, the term $\partial F/\partial y$ is a matrix n * n whose each column is a vector of gradients with respect to y (a Jacobian matrix), the term $\partial y/\partial x$ is a vector of unknown derivatives of y with respect to one particular x variable, and the term $\partial F/\partial x$ is a vector of derivatives with respect to the same particular variable x. Computation of the derivatives of y with respect to several variables x requires solutions of eq. 6 with many right hand sides - one per each variable x. Since the Jacobian matrix remains the same for all variables x, a solution algorithm arranged so as to factor the matrix only once will be preferred for computational economy.

It is important that eq. 6 is simply a set of linear, algebraical equations even though eq.1 may be far more complicated than that. The terms $\partial F/\partial y$ and $\partial F/\partial x$ may still not be obtainable analytically. If so, they can be computed by finite difference, i.e., assuming perturbation $x=x+\Delta x$ and $y=y+\Delta y$ for each element of x and each element of y separately, and substituting into eq. 1, one obtains the respective ΔF values (upon substitution of $x+\Delta x$, or $y+\Delta y$, F in eq. 1 is no longer equal zero, it becomes ΔF) from which the terms $\partial F/\partial y$ and $\partial F/\partial x$ can be computed as in eq. 2.

Computation of the terms $\partial F/\partial y$ and $\partial F/\partial x$ by finite difference is accomplished by repetitve evaluations of F(y,x) for known y and x, as opposed to repetitive solutions of F(y,x) = 0 (eq.1) for unknown y required by eq.2. Hence, the quasi-analytical approach is inherently less computer intensive than the finite difference procedure based on eq. 2.

APPENDIX B

Application of the generic, quasi-analytical algorithm for sensitivity derivatives is illustrated with one example from linear, static, structural analysis. The governing equations - the counterpart of $F(y,x) = 0$ - are the load-deflection equations involving a stiffness matrix K, unknown displacements y, and the cross-sectional dimensions x as design variables. The structural sensitivity equation recursively connects to the load-deflection equations through the solution vector y. Since the matrix K has to be factored (decomposed) in the process of solving for y, significant computational economy may be realized by saving the factored matrix and reusing it in the solution of the sensitivity equation.

ANALYTICAL DERIVATIVES IN LINEAR STATIC STRUCTURAL ANALYSIS

Generic	Structural
$F(y,x) = 0;\ y = y(x)$ $\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x} = -\frac{\partial f}{\partial x}$	$K(x) \cdot y = P(x);\ y = y(x)$ $K \cdot \frac{\partial y}{\partial x} = -\frac{\partial K}{\partial x} \cdot y + \frac{\partial P}{\partial x}$ y — displacement x — cross-section dimension $\frac{\partial K}{\partial x}, \frac{\partial P}{\partial x}$ Analytically or by finite differences