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OPTIMAL CONTROL CONCEPTS IN DESIGN SENSITIVITY ANALYSIS

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ABSTRACT

In this paper, a close link is established between open loop optimal control theory and optimal design by noting certain similarities in the gradient calculations. The resulting benefits include a unified approach, together with physical insights in design sensitivity analysis, and an efficient approach for simultaneous optimal control and design. Both matrix displacement and matrix force methods are considered, and results are presented for dynamic systems, structures, and elasticity problems.

1. INTRODUCTION

Considerable interest is being shown in recent years on control of flexible systems such as robots and space structures. In control theory and optimal control in particular, the geometry (dimensions and shape) is given, and the task is to develop a control law so as to ensure proper operation of the system in an uncertain environment. In design, and optimal design in particular, the task is to determine the geometry. Evidently, at least in the preliminary design stages, there is interaction between optimal control and optimal design. There is a need for better understanding of this interaction. In this paper, a close link is established between these two disciplines. Specifically, the similarity of the sensitivity calculations and adjoint equations is examined. Practical benefits and new possibilities are discussed. Dynamic systems, structures, and continuum elasticity models are considered. Both displacement and matrix force methods of structural analysis are treated.

2. THE LAGRANGE MULTIPLIER RULE FOR CALCULATING SENSITIVITY COEFFICIENTS

The Lagrange multiplier rule is a well-known method for obtaining optimality conditions in the presence of constraints. The rule, however, serves equally well in obtaining expressions for sensitivity coefficients (or derivatives) of implicit functions, as shown below.

Consider the scalar valued function $f = f(\underset{\sim}{x},\underset{\sim}{b})$, where x is an (nx1) vector of 'state' variables and $\underset{\sim}{b}$ is a (kx1) vector of design variables. The function f is implicit in that for every vector $\underset{\sim}{b}$, x satisfies the state equation

$$g(\underset{\sim}{x}, \underset{\sim}{b}) = \underset{\sim}{0} \tag{1}$$

where $\underset{\sim}{g}$ is (nx1) vector function. It is desired to obtain the sensitivity vector $df/d\underset{\sim}{b}$. In design of structural and mechanical systems, f often represents the stress or displacement of a point and Eq. (1) is the equation of equilibrium. To illustrate the Lagrange multiplier rule for calculating $df/d\underset{\sim}{b}$, we first form the scalar valued function H as

$$H = f + \underset{\sim}{\lambda}^T g \tag{2}$$

where $\underset{\sim}{\lambda}$ is an (nx1) vector of Lagrange multipliers or 'adjoint' variables. Noting that f, g and H are functions of $\underset{\sim}{x}$ and $\underset{\sim}{b}$, we have, upon differentiating H with respect to $\underset{\sim}{b}$,

$$dH/d\underset{\sim}{b} = \partial H/\partial b + \partial H/\partial \underset{\sim}{x} \quad dx/d\underset{\sim}{b} \tag{3}$$

The idea behind the Lagrange multiplier rule is to require that $\underset{\sim}{\lambda}$ satisfy the equations

$$\partial H/\partial \underset{\sim}{x} = 0 \tag{4}$$

Assuming that $\partial g/\partial \underset{\sim}{x}$ is a nonsingular matrix -- which is necessary for $\underset{\sim}{x}$ to be a unique solution to Eq. (1) -- and using Eq. (4), we can obtain $\underset{\sim}{\lambda}$ from the following 'adjoint equations':

$$[\partial \underset{\sim}{g}/\partial \underset{\sim}{x}]^T \underset{\sim}{\lambda} = - \partial f/\partial \underset{\sim}{x}^T \tag{5}$$

Equation (3) now provides the result

$$df/d\underset{\sim}{b} = \partial H/d\underset{\sim}{b} \tag{6}$$

or,

$$df/d\underset{\sim}{b} = \partial f/\partial \underset{\sim}{b} + \underset{\sim}{\lambda}^T \partial g/\partial b. \tag{7}$$

In Eq. (7), the term $\underset{\sim}{g}^T \partial \underset{\sim}{\lambda}/\partial \underset{\sim}{b}$ does not appear because of Eq. (1).

The fact that the Lagrange multiplier rule offers a unified approach to design sensitivity analysis has been discussed in Ref. 1. Further, the adjoint method of design sensitivity analysis given in Ref. 2 results in the exact same equations as obtained using the Lagrange multiplier rule. In this paper, the use of this rule to obtain expressions for sensitivity coefficients helps to establish a close link between optimal design and optimal control, as discussed in the next section.

3. OPTIMAL CONTROL AND OPTIMAL DESIGN

Optimal Control

To present the basic concepts, consider a dynamic system described by the following nonlinear differential equations

$$\dot{\underset{\sim}{x}} = \underset{\sim}{q}\ (\underset{\sim}{x}(t),\ \underset{\sim}{u}(t),\ t);\ x(t_o)\ \text{given},\ t_o \leqq t \leqq t_f \tag{8}$$

where the 'state' $\underset{\sim}{x}(t)$, an (nx1) vector function, is determined by the 'control' $\underset{\sim}{u}(t)$, a (kx1) vector function. Consider a performance index given by the scalar functional

$$F = \int_{t_o}^{t_f} f\ (\underset{\sim}{x}(t),\ \underset{\sim}{u}(t),\ t)\ dt \tag{9}$$

The optimal control problem is to find $\underset{\sim}{u}(t)$ that minimizes (or maximizes) F [3]. The Lagrange multiplier rule as discussed in the previous section, is used to do this. Adjoin the system in Eq. (8) to F with multiplier functions (or adjoint variables) $\underset{\sim}{\lambda}(t)$:

$$\bar{F} = \int_{t_o}^{t_f} [f + \underset{\sim}{\lambda}^T\ (\underset{\sim}{q} - \dot{\underset{\sim}{x}})]\ dt \tag{10}$$

If we define the scalar function H, the Hamiltonian, as

$$H = f + \underset{\sim}{\lambda}^T\ \underset{\sim}{q} \tag{11}$$

and integrate the last term on the right side of Eq. (10) by parts, we obtain

$$\bar{F} = -\ \underset{\sim}{\lambda}^T\ (t_f)\ \underset{\sim}{x}\ (t_f) + \underset{\sim}{\lambda}^T\ (t_o)\ \underset{\sim}{x}\ (t_o) + \int_{t_o}^{t_f} [H + \dot{\underset{\sim}{\lambda}}^T\ \underset{\sim}{x}]\ dt \tag{12}$$

Now, consider the variation in F due to variation in the control vector $\underset{\sim}{u}(t)$ for fixed times t_o and t_f and fixed initial conditions,

$$\delta\ \bar{F} = -\ \underset{\sim}{\lambda}^T\ \delta \underset{\sim}{x} \Big|_{t\,=\,t_f} + \underset{\sim}{\lambda}^T\ \delta\ \underset{\sim}{x} \Big|_{t\,=\,t_o}^{0} + \int_{t_o}^{t_f} (\partial H/\partial \underset{\sim}{x} + \dot{\underset{\sim}{\lambda}}^T)\ \delta \underset{\sim}{x}\ dt + \int_{t_o}^{t_f} \frac{\partial H}{\partial \underset{\sim}{u}}\ \delta \underset{\sim}{u}\ dt \tag{13}$$

Since it is tedious to determine the variations $\delta \underset{\sim}{x}(t)$ produced by a given $\delta \underset{\sim}{u}(t)$, we choose the multiplier functions $\underset{\sim}{\lambda}(t)$ to satisfy

$$\dot{\lambda}^T = -\frac{\partial H}{\partial \underset{\sim}{x}} = -\frac{\partial f}{\partial \underset{\sim}{x}} - \underset{\sim}{\lambda}^T \frac{\partial \underset{\sim}{q}}{\partial \underset{\sim}{x}} \tag{14}$$

with boundary conditions

$$\lambda^T (t_f) = \underset{\sim}{0} \tag{15}$$

In view of the adjoint equations in (14) and (15), Eq. (13) yields

$$\delta F = \int_{t_o}^{t_f} \partial H/\partial \underset{\sim}{u} \;\; \delta \underset{\sim}{u} \; dt \tag{16}$$

The functions $\partial H/\partial \underset{\sim}{u}$ can be interpreted as impulse response functions since each component of $\partial H/\partial \underset{\sim}{u}$ represents the variation in F due to a unit impulse in the corresponding component of $\underset{\sim}{u}$ at time t [3]. Furthermore, $\partial H/\partial \underset{\sim}{u}$ can be interpreted as the function-space gradient of F with respect to $\underset{\sim}{u}$. This last interpretation is useful when using gradient methods to extremize F. For example, choosing $\underset{\sim}{u}(t) = -\alpha \; \partial H/\partial \underset{\sim}{u}$ corresponds to a steepest descent step to minimize F.

Finally, it should be noted that setting $\partial H/\partial \underset{\sim}{u} = 0$ yields the optimality conditions. In the special case when F is quadratic in $\underset{\sim}{x}$ and $\underset{\sim}{u}$ and Eqs. (8) are linear, the optimality conditions together with the state equations (8) and adjoint equations (14) and (15) can be solved in closed form, leading to the Ricatti equations, which are very attractive in closed loop control since the feedback law is independent of the state vector $\underset{\sim}{x}$ and can be computed 'off-line'.

Optimal Design

In optimal design of mechanical systems, it is required to obtain the sensitivity vector $dF/d\underset{\sim}{b}$ where F is a cost or constraint functional of the form

$$F = \int_{t_o}^{t_f} f(\underset{\sim}{x}(t), \underset{\sim}{b}, t) \; dt \tag{17}$$

with $\underset{\sim}{b}$ a (kx1) vector of design variables. For example, F represents a time-averaged performance measure of a vehicle traversing a rough terrain. Most gradient-based nonlinear programming codes require input of the vector

$dF/d\underset{\sim}{b}$. In Eq. (17), for a given $\underset{\sim}{b}$, $\underset{\sim}{x}$ should satisfy the equations of motion given by

$$\dot{\underset{\sim}{x}} = \underset{\sim}{g}\ (\underset{\sim}{x}(t),\ \underset{\sim}{b},\ t);\ x(t_o)\ \text{given},\ t_o \leqq t \leqq t_f \tag{18}$$

As before, the use of the Lagrange multiplier rule requires the function

$$\bar{F} = \int_{t_o}^{t_f} [f + \underset{\sim}{\lambda}^T\ (\underset{\sim}{g} - \dot{\underset{\sim}{x}})]\ dt \tag{19}$$

Integrating the last term on the right side of Eq. (19) by parts yields

$$\begin{aligned}\bar{F} = &- \underset{\sim}{\lambda}^T(t_f)\ \underset{\sim}{x}(t_f) + \underset{\sim}{\lambda}^T(t_o)\ \underset{\sim}{x}(t_o) \\ &+ \int_{t_o}^{t_f} [H + \dot{\underset{\sim}{\lambda}}^T\ \underset{\sim}{x}]\ dt\end{aligned} \tag{20}$$

where H is defined in Eq. (11). Now, consider the variation in F due to variations (or differentials) in $\underset{\sim}{b}$ for fixed times t_o and t_f, and fixed initial conditions:

$$\begin{aligned}\delta\ \bar{F} = &- \underset{\sim}{\lambda}^T\ \delta\underset{\sim}{x}\Big|_{t_f} + \underset{\sim}{\lambda}^T\ \delta\underset{\sim}{x}\Big|_{t_o}^{0} + \int_{t_o}^{t_f} (\partial H/\partial\underset{\sim}{x} + \dot{\underset{\sim}{\lambda}}^T)\ \delta\underset{\sim}{x}\ dt \\ &+ \int_{t_o}^{t_f} \partial H/\partial\underset{\sim}{b}\ \delta\underset{\sim}{b}\ dt\end{aligned} \tag{21}$$

If we choose $\lambda(t)$ to satisfy the same adjoint equations as in the optimal control problem in (14) and (15), we obtain

$$\delta F = \int_{t_o}^{t_f} \partial H/\partial\underset{\sim}{b}\ \ \delta\underset{\sim}{b}\ dt \tag{22}$$

Since $\underset{\sim}{b}$ is not a function of time, Eq. (22) yields the sensitivity coefficient vector

$$dF/d\underset{\sim}{b} = \int_{t_o}^{t_f} \partial H/\partial \underset{\sim}{b} \, dt \qquad (23)$$

Expressions as in Eq. (23) can then be fed into nonlinear programming codes to obtain improved design vectors $\underset{\sim}{b}$. The main emphasis here, however, is to show that the calculation of design sensitivity vectors is simply a special case of open loop optimal control. That is, treating the control variables $u(t)$ as design variables enables us to obtain expressions for the sensitivity vectors.

The following advantages result from this observation:

(1) A general approach to design sensitivity analysis is established.

(2) Physical significance of the adjoint variables is established. In particular, in the above discussion, the functions $\partial H/\partial u$ are interpreted to be influence functions. The importance of such a physical interpretation in structural design is discussed subsequently.

(3) The fact that the adjoint equations are the same in the optimal control and optimal design problems motivates an efficient gradient approach for simultaneous handling of control variables and design parameters. That is, functionals of the form

$$F = \int_{t_o}^{t_f} f\,(x(t), \underset{\sim}{u}(t), \underset{\sim}{b}, t) \qquad (24)$$

where both control variables $u(t)$ and design parameters $\underset{\sim}{b}$ have to be optimally chosen, can be treated efficiently. Such problems may arise, for example, when designing both a control law as well as determining the dimensions and shape for a robot or for a flexible space structure.

4. STRUCTURES

Matrix Displacement Method

The general results discussed in the preceding section lead to special insights when applied to structural systems. Consider a scalar function

$$f \equiv f\,(\underset{\sim}{x}, \underset{\sim}{b}) \qquad (25)$$

where f typically represents the stress or displacement at some point in the structure, $\underset{\sim}{b}$ is a (kx1) vector of design variables, and $\underset{\sim}{x}$ is the nodal displacement vector. If a finite element model of the structure exists, then the (nx1) vector $\underset{\sim}{x}$ is obtained from the matrix displacement (finite element) equations

$$\underset{\sim}{K}(\underset{\sim}{b})\underset{\sim}{x} = \underset{\sim}{P}(\underset{\sim}{b}) \tag{26}$$

where $\underset{\sim}{K}$ is an (nxn) structural stiffness matrix and $\underset{\sim}{P}$ is a vector of applied nodal loads.

The importance of applying optimal control concepts to structural systems described by (25) and (26) will now be shown. The sensitivity vector $df/d\underset{\sim}{b}$ will be obtained by using the Lagrange multiplier rule. Define the function H as

$$H = f + \underset{\sim}{\lambda}^T (\underset{\sim}{P} - \underset{\sim}{K}\ \underset{\sim}{x}) \tag{27}$$

where $\underset{\sim}{\lambda}$ is the (nx1) adjoint vector. The variation of H due to a variation in $\underset{\sim}{b}$ is given by

$$\delta H = \partial H/\partial \underset{\sim}{b}\ \delta \underset{\sim}{b} + \partial H/\partial \underset{\sim}{x}\ \delta \underset{\sim}{x} \tag{28}$$

Choosing $\underset{\sim}{\lambda}$ to satisfy

$$\partial H/\partial \underset{\sim}{x} = \underset{\sim}{0} \tag{29}$$

which can also be written as $\underline{K}\ \underset{\sim}{\lambda} = \partial f/\partial \underset{\sim}{x}^T$, we have from Eq. (28),

$$\delta f = \partial H/\partial \underset{\sim}{b}\ \ \delta \underset{\sim}{b} \tag{30}$$

which yields

$$df/d\underset{\sim}{b} = \partial H/\partial \underset{\sim}{b} \tag{31}$$

Now, in the foregoing analysis, let us consider the variation in H due to a variation in $\underset{\sim}{P}$. That is, we consider variations in the 'control' vector $\underset{\sim}{P}$ instead of the design vector $\underset{\sim}{b}$. We have

$$\delta H = \partial H/\partial \underset{\sim}{P}\ \ \delta \underset{\sim}{P} + \partial H/\partial \underset{\sim}{x}\ \ \delta \underset{\sim}{x} \tag{32}$$

Choosing $\underset{\sim}{\lambda}$ to satisfy the adjoint equation in (29), and noting that f and K do not depend explicitly on $\underset{\sim}{P}$ for linear structures, we get

$$\delta f = \underset{\sim}{\lambda}^T\ \delta \underset{\sim}{P} \tag{33}$$

or,

$$\lambda^T = df/d\underset{\sim}{P} \tag{34}$$

Since the adjoint equations in (29) are the same regardless of whether the fundamental variation is in $\underset{\sim}{b}$ or $\underset{\sim}{P}$, Eq. (34) shows that the adjoint vector

λ used in structural design sensitivity analysis represents the sensitivity of the function f to variations in the applied loads $\underset{\sim}{P}$. Further, if f is linear in P, then λ_i = value of f due to $P_i = 1$. In civil engineering, λ is the influence coefficient vector associated with the function f, as discussed in Ref. 4. Further, since Eqs. (29) can be written as

$$\underset{\sim}{K}\,\underset{\sim}{\lambda} = \partial f/\partial \underset{\sim}{x}^T \tag{35}$$

we can think of $\underset{\sim}{\lambda}$ as a displacement vector associated with the load vector $\partial f/\partial \underset{\sim}{x}$. This motivates the use of element shape functions to obtain expressions for λ within the elements from knowing the nodal values. Thus, we can write

$$\lambda(\xi) = \sum_i \lambda_i \, N_i(\xi) \tag{36}$$

where λ_i are the nodal values obtained by solving Eq. (35) and N_i are the familiar shape functions used in finite element analysis.

The beam in Fig. 1(a) is solved to illustrate this. A finite element model of the beam is shown in Fig. 1(b). The function f is taken to be the moment at support b. The adjoint vector $\underset{\sim}{\lambda}$, representing the values of f due to unit loads along each degree of freedom, is obtained by solving Eq. (35). Equation (36) is used to obtain expressions for λ along the beam, which is used to draw the influence line as shown in Fig. 1(c). The results are in agreement with those in Ref. 5, and show that the adjoint method is a new and powerful approach for determining influence lines.

Some other interesting aspects relating to Eqs. (34) and (35) may now be noted. If we let f be the strain energy function U given by

$$U = \frac{1}{2}\, \underset{\sim}{x}^T \underset{\sim}{K}\, \underset{\sim}{x} \tag{37}$$

then Eq. (35) yields $\underset{\sim}{K}\,\underset{\sim}{\lambda} = \underset{\sim}{K}\,\underset{\sim}{x}$, from which $\underset{\sim}{\lambda} = \underset{\sim}{x}$. Equation (34) then yields

$$\underset{\sim}{x}^T = dU/d\underset{\sim}{P} \tag{38}$$

which is a discrete version of Castiglaiano's theorem for linear structures. Also, letting $f = \frac{1}{2}\, x^T K x - x^T F$ = potential energy, results in $\underset{\sim}{\lambda} = \underset{\sim}{0}$ and $d\pi/d\underset{\sim}{P} = 0$, which is a statement of the minimum potential energy theorem.

Matrix Force Method

The systematic use of the Lagrange multiplier rule or adjoint method for design sensitivity analysis and physical significance of adjoint variables, which was discussed in the context of displacement finite element analysis, will now be extended to structures analyzed by the the matrix force method.

For indeterminate structures, the equilibrium equations in the matrix force method take the form [6]

$$n_F \underset{\sim}{F} + n_x \underset{\sim}{X} = \underset{\sim}{P} \tag{39}$$

where $\underset{\sim}{x}$ is a vector of 'redundant' or independent forces (and reactions), $\underset{\sim}{F}$ is a vector of dependent forces, and $\underset{\sim}{P}$ is the vector of externally applied forces. The redundants $\underset{\sim}{X}$ are obtained from compatibility conditions of the form

$$\underset{\sim}{X} = C(\underset{\sim}{b})\ \underset{\sim}{P} + \underset{\sim}{d}(\underset{\sim}{b}) \tag{40}$$

Consider now a function f $(\underset{\sim}{b}, \underset{\sim}{F}, \underset{\sim}{X})$. Note that matrices C and d also depend on the design vector $\underset{\sim}{b}$. Form the function H as

$$\begin{aligned} H = f\ (\underset{\sim}{b}, \underset{\sim}{F}, \underset{\sim}{X}) + \underset{\sim}{\lambda}^T\ (\underset{\sim}{P} - \underset{\sim}{n}_F\ \underset{\sim}{F} - \underset{\sim}{n}_x\ \underset{\sim}{X}) \\ + \underset{\sim}{\mu}^T\ (\underset{\sim}{X} - \underset{\sim}{C}\ \underset{\sim}{P} - \underset{\sim}{d}) \end{aligned} \tag{41}$$

Consider the variation in H due to a variation in $\underset{\sim}{b}$:

$$\begin{aligned} \delta H = \partial f/\partial \underset{\sim}{b}\ \delta \underset{\sim}{b} + (\partial f/\partial \underset{\sim}{F} - \underset{\sim}{\lambda}^T\ n_F)\ \delta \underset{\sim}{F} \\ + (\partial f/\partial \underset{\sim}{X} - \underset{\sim}{\lambda}^T\ n_x + \mu^T)\ \delta \underset{\sim}{X} - \mu^T\ (\delta C \cdot \underset{\sim}{P} + \delta \underset{\sim}{d}) \end{aligned} \tag{42}$$

Choosing $\underset{\sim}{\lambda}$ to satisfy

$$n_F^T\ \underset{\sim}{\lambda} = \partial f/\partial \underset{\sim}{F}^T \tag{43}$$

and letting

$$\mu^T = \underset{\sim}{\lambda}^T\ n_X - \partial f/\partial \underset{\sim}{X} \tag{44}$$

we have

$$\delta f = \partial f/\partial \underset{\sim}{b}\ \delta \underset{\sim}{b} - \mu^T\ (\delta C \cdot \underset{\sim}{P} + \delta \underset{\sim}{d}) \tag{45}$$

from which the sensitivity vector is obtained as

$$df/d\underset{\sim}{b} = \partial f/\partial \underset{\sim}{b} - \underset{\sim}{\mu}^T\ \ \partial/\partial \underset{\sim}{b}\ (\underset{\sim}{C}\ \underset{\sim}{P} + \underset{\sim}{d}) \tag{46}$$

The physical significance of the adjoint variables $\underset{\sim}{\lambda}$ and $\underset{\sim}{\mu}$ is obtained in the usual manner by considering the variation of H in Eq. (41), due to a variation $\delta\underset{\sim}{P}$ is $\underset{\sim}{P}$. Upon defining $\underset{\sim}{\lambda}$ as in Eq. (43) and $\underset{\sim}{\mu}$ as in Eq. (44), we get

$$\delta f = (\underset{\sim}{\lambda}^T - \underset{\sim}{\mu}^T C)\ \delta \underset{\sim}{P} \tag{47}$$

Thus, $(\underset{\sim}{\lambda} - C^T \underset{\sim}{\mu})$ is now the influence coefficient vector.

For statically determinate structures, the terms n_x, $\underset{\sim}{X}$ and $\underset{\sim}{\mu}$ vanish, and Eq. (47) becomes

$$\delta f = \underset{\sim}{\lambda}^T\ \delta \underset{\sim}{P} \tag{48}$$

which is analogous to Eq. (33) that was obtained in the displacement method of analysis. In this case, λ_i = value of f due to $P_i = 1$, provided f is linear and homogeneous in $\underset{\sim}{P}$.

5. ELASTICITY

This section will focus on elasticity problems. Consider a functional

$$F = \int_\Omega f\ (\sigma_{ij})\ d\Omega \tag{49}$$

where σ_{ij} is the stress tensor, Ω is the domain of the elastic body, and the equilibrium equations in variational form are

$$\int_\Omega \sigma_{ij}\ (u)\ \varepsilon_{ij}\ (\phi)\ d\Omega = \int_\Gamma \phi_i\ T_i\ d\Gamma + \int_\Omega \phi_i\ B_i\ d\Omega \tag{50}$$

Equation (50) is satisfied for every displacement field ϕ satisfying $\phi_i = 0$ on Γ_1, u is the actual displacement field due to traction forces T_i and body forces B_i, and kinematic boundary conditions u = 0 is imposed on a portion Γ_1 of the total boundary. Equation (50) is simply the principle of virtual work, with $\varepsilon_{ij}\ (\phi)$ being the virtual strain due to a kinematically admissible virtual displacement ϕ. As before, form the functional H as

$$H = \int_\Omega [f(\sigma_{ij}) - \sigma_{ij}(u)\ \varepsilon_{ij}(\lambda)]\ d\Omega + \int_\Gamma \lambda_i\ T_i\ d\Gamma + \int_\Omega \lambda_i\ B_i\ d\Omega \tag{51}$$

where λ satisfies $\lambda = 0$ on Γ_1. Consider a variation δT_i in T_i, and δB_i in B_i, and let $\sigma_{ij}(v)$ and $f_{ij}(v)$ be the corresponding variations in stress and strain, respectively. The variation in H is given by

$$\delta H = \int_\Omega [\partial f/\partial \sigma_{ij}\ \sigma_{ij}\ (v) - \sigma_{ij}\ (v)\ \varepsilon_{ij}\ (\lambda)]\ d\Omega$$

$$+ \int_\Gamma \lambda_i\ \delta T_i\ d\Gamma + \int_\Omega \lambda_i\ \delta B_i\ d\Omega \tag{52}$$

Equation (52) holds true for all kinematically admissible λ, and consequently holds true for a λ determined from the following adjoint equations:

$$\int_\Omega \sigma_{ij}(\lambda)\, \varepsilon_{ij}(\Psi)\, d\Omega = \int_\Omega \partial f/\partial\sigma_{ij}\, \sigma_{ij}(\Psi)\, d\Omega \qquad (53)$$

which is satisfied for all Ψ, $\Psi = 0$ on Γ_1. Since

$$\int_\Omega \sigma_{ij}(\lambda)\, \varepsilon_{ij}(\Psi)\, d\Omega = \int_\Omega \sigma_{ij}(\Psi)\, \varepsilon_{ij}(\lambda)\, d\Omega$$

putting $\Psi = v$ in Eq. (53), Eq. (52) yields

$$\delta F = \int_\Gamma \lambda_i\, \delta T_i + \int_\Omega \lambda_i\, \delta B_i\, d\Omega \qquad (54)$$

Equation (54) is essentially a variational principle. If we let the functional F be the complementary strain energy density, that is, we let

$$F = \int_\Omega \left[\int_0^{\sigma_{ij}} \varepsilon_{ij}\, d\sigma_{ij}\right] d\Omega \qquad (55)$$

then Eq. (54) yields (upon using Leibnitz's rule)

$$\int_\Omega \varepsilon_{ij}\, \delta\sigma_{ij}\, d\Omega = \int_\Gamma \lambda_i\, \delta T_i\, d\Gamma + \int_\Omega \lambda_i\, \delta B_i\, d\Omega \qquad (56)$$

which is the principle of complementary virtual work [7]. Finally, sensitivity expressions can be readily obtained if variation of H due to design variations is considered as done in previous sections. This approach holds true for a changing domain, as in shape optimal design [1]. From Eq. (54), we can see that λ_i at a point represents the value of F due to a unit load at that point. This fact can be written in terms of Green's function as

$$\lambda_i = \int_\Omega f\,(\sigma_{ij}\,(G_i))\, d\Omega \qquad (57)$$

where the Green's function G is the displacement field due to a unit load.

6. FUTURE WORK

In both optimal control and optimal design, it is shown that the Hamiltonian function and the Lagrange multiplier rule play a similar role. Optimal control theory helps to obtain a unified framework for design sensitivity analysis and physical understanding of adjoint variables. Some areas which may merit future investigation are noted below.

1. This work motivates an efficient gradient approach for optimal design of systems with control in mind. That is, both control variables and geometric design parameters can be considered simultaneously in the preliminary design stages.

2. In structures, the adjoint method provides a powerful method for constructing influence lines in the framework of finite element analysis. Also, the equation $\lambda^T = df/d\underset{\sim}{P}$ can be used to design structures which are insensitive to loads P_i by minimizing $\lambda^2{}_i$, or can be used to optimally locate the loads for maximum utilization of the structure, by maximizing $\underset{\sim}{\lambda}^T\underset{\sim}{P}$ subject to suitable constraints.

3. The stability analysis of the adjoint equations that has been carried out extensively in optimal control theory may turn out to be of importance to the design engineer.

7. REFERENCES

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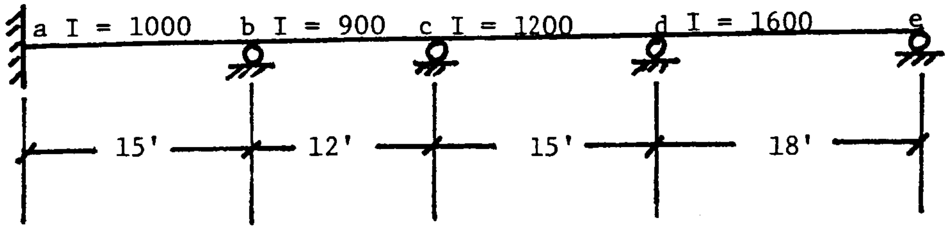
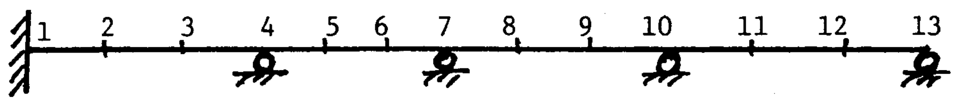


(a)



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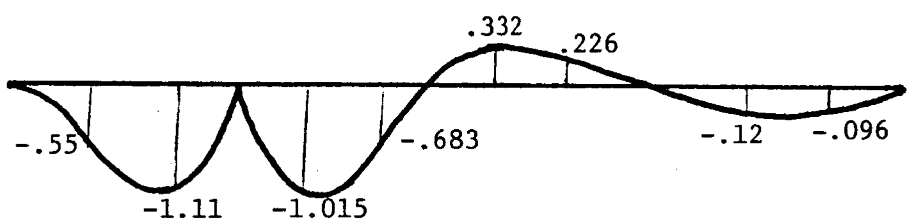

(c)

Figure 1. (a) Beam Problem,
(b) Finite Element Model of Beam,
(c) Influence Line for Moment at Support b of Beam