

OPTIMIZATION OF SHALLOW ARCHES  
AGAINST INSTABILITY USING SENSITIVITY DERIVATIVES<sup>+</sup>

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SUMMARY

In this paper the author discusses the problem of optimization of shallow frame structures which involve a coupling of axial and bending responses. A shallow arch of a given shape and of given weight is optimized such that its limit point load is maximized. The cross-sectional area,  $A(x)$  and the moment of inertia,  $I(x)$  of the arch obey the relationship  $I(x) = \rho [A(x)]^n$ ,  $n = 1, 2, 3$  and  $\rho$  is a specified constant. Analysis of the arch for its limit point calculation involves a geometric nonlinear analysis which is performed using a corotational formulation.

The optimization is carried out using a second-order projected Lagrangian algorithm and the sensitivity derivatives of the critical load parameter with respect to the areas of the finite elements of the arch are calculated using implicit differentiation. Results are presented for an arch of a specified rise to span ratio under two different loadings and the limitations of the approach for the intermediate rise arches are addressed.

INTRODUCTION

With the advent of highly flexible large space structures the nonlinearity of response of such structures plays a dominant role in the control of such structures. Naturally, optimization of structures in nonlinear response is gaining prominence. This paper addresses the issue of optimizing shallow frame structures in nonlinear response involving a coupling of axial and bending actions. The objective is to optimize a shallow arch of a given shape and given weight such that its limit point load is maximized. Besides having to perform a nonlinear analysis in calculating the limit point load an issue of even greater concern is that of calculating the sensitivity derivatives of the critical load parameter with respect to the design variables, namely the cross-sectional areas of the elements of the discretized model of the arch. Two approaches are available for the calculation of sensitivity derivatives: the direct and the adjoint approach [1]. In general, the adjoint approach is preferred for problems involving nonlinear response [2] - [4]. The popularity of the adjoint approach stems from the fact that the differential equations governing the adjoint variable are linear even though the corresponding equilibrium equations in terms of the true displacement variables are nonlinear. But to date the author is unaware of the use of the adjoint approach for problems involving limit point instability. The present work outlines a direct approach

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similar to that used by the author and his co-worker in the case of shallow space trusses [5]. In this approach the sensitivity derivatives of the critical load parameter are obtained through an implicit differentiation of the nonlinear equilibrium equations as explained below. The present discussion is restricted to finite-element models of shallow arches whose cross sections obey the relationship

$$I(x) = \rho [A(x)]^n, \quad n = 1, 2, 3, \quad \rho = \text{specified constant} \quad (1)$$

#### SENSITIVITY DERIVATIVES OF THE CRITICAL LOAD PARAMETER

Consider a shallow arch under a given distribution of loading. Assume that  $\lambda_{cr}$  is the smallest value of the parameter by which the given distribution of loading must be scaled in order to produce instability of the arch. The parameter  $\lambda$  is then defined by the solution of the following system of equations of a finite-element model of the arch.

$$\frac{\partial \pi}{\partial q_j} = 0 \quad (2)$$

$$\left| \frac{\partial^2 \pi}{\partial q_i \partial q_j} \right| = 0 \quad (3)$$

where  $\pi$  denotes the total potential energy of the model undergoing finite displacements and  $q_j$ ,  $j = 1, 2 \dots N$  denote the generalized nodal displacements of the model. The load parameter  $\lambda$  occurs implicitly in Eqs. (2) and (3). Assume that  $A_k$  for  $k = 1, 2 \dots m$  are the  $m$  design variables, which for the arch are the cross-sectional areas of the finite elements. To obtain  $\frac{\partial \lambda}{\partial A_k}$  we proceed as follows:

Rewrite Eqs. (2) and (3) as

$$f_l(q_j(A_k), \lambda(A_k), A_k) = 0 \quad (4)$$

$$g(q_j(A_k), \lambda(A_k), A_k) = 0 \quad (5)$$

$$j, l = 1, 2 \dots N$$

$$k = 1, 2 \dots m$$

An implicit differentiation of Eqs. (4) and (5) with respect to  $A_k$  leads to

$$\begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \dots & \frac{\partial f_1}{\partial q_N} & \frac{\partial f_1}{\partial \lambda} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \dots & \frac{\partial f_2}{\partial q_N} & \frac{\partial f_2}{\partial \lambda} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial f_N}{\partial q_1} & \frac{\partial f_N}{\partial q_2} & \dots & \frac{\partial f_N}{\partial q_N} & \frac{\partial f_N}{\partial \lambda} \\ \frac{\partial g}{\partial q_1} & \frac{\partial g}{\partial q_2} & \dots & \frac{\partial g}{\partial q_N} & \frac{\partial g}{\partial \lambda} \end{bmatrix} \begin{pmatrix} \frac{\partial q_1}{\partial A_k} \\ \frac{\partial q_2}{\partial A_k} \\ \vdots \\ \frac{\partial q_N}{\partial A_k} \\ \frac{\partial \lambda}{\partial A_k} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_1}{\partial A_k} \\ -\frac{\partial f_2}{\partial A_k} \\ \vdots \\ -\frac{\partial f_N}{\partial A_k} \\ -\frac{\partial g}{\partial A_k} \end{pmatrix} \quad (6)$$

for every  $k = 1, 2 \dots m$ . These equations may be written symbolically as

$$\begin{bmatrix} \underline{H} & \underline{F} \\ \underline{G} & \underline{0} \end{bmatrix} \begin{pmatrix} \frac{\partial \underline{q}}{\partial A_k} \\ \frac{\partial \lambda}{\partial A_k} \end{pmatrix} = \begin{pmatrix} -\frac{\partial \underline{f}}{\partial A_k} \\ -\frac{\partial g}{\partial A_k} \end{pmatrix} \quad (7)$$

where  $\underline{H}$  is the Hessian matrix of the potential energy of the finite-element model of the arch,  $\underline{F}$  is the given vector of nodal forces, and  $\underline{G}$  is the row matrix of derivatives of the determinant  $g$  of the Hessian matrix with respect to nodal displacements. Equations (7) assume that  $\frac{\partial g}{\partial \lambda}$  is equal to zero since for constant directional loading parameter  $\lambda$  does not occur explicitly in the stability criterion. The elements of  $\underline{G}$  can be evaluated by using the formula

$$\frac{\partial g}{\partial q_j} = \text{trace} \left\{ (\text{adj } (\underline{H})) \left[ \frac{\partial \underline{H}}{\partial q_j} \right] \right\} \quad (8)$$

where  $\left[ \frac{\partial \underline{H}}{\partial q_j} \right]$  is the matrix obtained by differentiating each element of the determinant of  $\underline{H}$  with respect to a typical component  $q_j$ . With this, the sensitivity derivative  $\frac{\partial \lambda}{\partial A_k}$  can be calculated by the solution of Eqs. (7) at a given  $(\underline{q}, \lambda)$ .

Incidentally, Eqs. (7) apply everywhere along the loading path including the limit point. It is only at a bifurcation point that the determinant of  $\underline{H}$  is not differentiable. For very shallow arches instability typically occurs through snap-through and hence Eqs. (7) clearly apply.

# SPECIALIZATION TO A FRAME ELEMENT MODEL OF THE ARCH

We illustrate the derivation of the sensitivity derivatives for a finite element model with 3-D frame elements. For its kinematic description the frame element uses the co-rotational formulation as outlined in great detail in reference [6]. According to this formulation, which permits large rigid body motion of the element, the total motion is decomposed into a rigid body component and a strain-producing component. For an element p-q of length L, the displacements of the end q relative to the end p in the body fixed axes can be shown to be

$$\begin{Bmatrix} \delta_u \\ \delta_v \\ \delta_w \end{Bmatrix} = [T]_p \begin{Bmatrix} x_q - x_p \\ y_q - y_p \\ z_q - z_p \end{Bmatrix} - \begin{Bmatrix} L \\ 0 \\ 0 \end{Bmatrix} + [T]_p \begin{Bmatrix} U_q - U_p \\ V_q - V_p \\ W_q - W_p \end{Bmatrix} \quad (9)$$

$U_i, V_i,$  and  $W_i$  ( $i = p$  or  $q$ ) denote the global displacements of the nodes and matrix  $[T]_p$  is [6].

$$[T]_p = [T_1(\phi_x, \phi_y, \phi_z)][T_1(\theta_{xp}, \theta_{yp}, \theta_{zp})] \quad (10)$$

with

$$[T_1(\alpha_x, \alpha_y, \alpha_z)] = \begin{bmatrix} c_y c_z & c_y s_z & -s_y \\ -c_x s_z + s_x s_y c_z & c_x c_z + s_x s_y s_z & s_x c_y \\ s_x s_z + c_x s_y c_z & -s_x c_z + c_x s_y s_z & c_x c_y \end{bmatrix} \quad (11)$$

$c_i = \cos \alpha_i$ ,  $s_i = \sin \alpha_i$  for  $i = x, y$  and  $z$ . Angles  $\phi_x, \phi_y$  and  $\phi_z$  are the initial orientation angles of the frame element and the angles  $\theta_{xp}, \theta_{yp}$  and  $\theta_{zp}$  are the rigid-body rotations of the end p. In deriving Eqs. (10) Euler angle transformation is implied with the order of the rotations being  $\alpha_z, \alpha_y$  and  $\alpha_x$ .

Similarly, with the restriction of small relative rotation within the element, the rotations  $\psi_x, \psi_y$  and  $\psi_z$  of the end q relative to the end p are

$$\begin{Bmatrix} \psi_x \\ \psi_y \\ \psi_z \end{Bmatrix} = [T]_p \begin{Bmatrix} \theta_{xq} - \theta_{xp} \\ \theta_{yq} - \theta_{yp} \\ \theta_{zq} - \theta_{zp} \end{Bmatrix} \quad (12)$$

Assuming the relative axial and transverse displacements to be linear and cubic, respectively, the strain energy of the (p-q)th or the e-th element,  $e = 1, 2, \dots, m$  can be shown to be [6].

$$U_e = U_{(p-q)} = \frac{E}{2L_e} \left\{ A_e (\delta_u)^2 + \frac{12}{L_e^2} \rho_z A_e^n [(\delta_v)^2 + \frac{L_e^2}{3} \psi_z^2 - L_e (\delta_v)(\psi_z)] \right. \\ \left. + \frac{12}{L_e^2} \rho_y A_e^n [(\delta_w)^2 + \frac{L_e^2}{3} \psi_y^2 + L_e (\delta_w)(\psi_y)] \right\} \quad (13)$$

Hence

$$\pi = \sum_{e=1}^m U_e - \underline{F}^T \underline{q} \quad (14)$$

where  $\underline{q}^T = (U_p, V_p, W_p, \theta_{xp}, \theta_{yp}, \theta_{zp}, U_q, V_q, W_q, \theta_{xq}, \theta_{yq}, \theta_{zq})$ . All the expressions for the evaluation of matrices in Eqs. (7) are now available and, in principle, can be evaluated even though the algebra may be rather tedious. The above expressions, especially the  $[T]_p$  matrix, can be simplified using the assumption of small rigid body motions within a load step.

Indeed, Updated Lagrangian formulation for the kinematic description may have simplified matters quite a bit especially if the expressions are linearized within a load step but the above expressions using the co-rotational formulation permit truly large displacements and with an highly efficient algorithm for the solution of nonlinear equations like for instance the BFGS algorithm [7], it can permit relatively large load steps resulting in a fewer number of load steps to attain a given load level.

## CONSTRAINED OPTIMIZATION

The optimization problem consists of maximizing the critical value of the load parameter  $\lambda$  subject to the constraint on total volume of the structure and side constraints on member sizes. Although it is perfectly permissible to pose the problem as

$$\min (f(\underline{A}) = -\lambda) \quad (15)$$

Subject to

$$\frac{\partial \pi}{\partial q_j} = 0 \quad (16)$$

$$\left| \frac{\partial^2 \pi}{\partial q_i \partial q_j} \right| = 0 \quad (17)$$

$$\sum_{i=1}^m A_i L_i - V_0 = 0 \quad (18)$$

$$A_i - A_{\min} \geq 0 \quad (19)$$

experience suggests the following well-posed problem

$$\min (-\lambda_{cr}) \quad (20)$$

subject to

$$\sum_{i=1}^m A_i L_i - V_0 = 0 \quad (21)$$

$$A_i - A_{\min} \geq 0 \quad (22)$$

where  $\lambda_{cr}$  is located by incrementing the load parameter and locating its level at which the determinant of the Hessian vanishes. This can be done by monitoring either the determinant or the inertia of the eigenvalues of the Hessian matrix  $H$ . Once an interval is located where the critical point is supposed to lie its exact location is determined by a root-finding technique. With Eqs. (15)-(19) there is no guarantee that the lowest value of  $\lambda$  that satisfies Eqs. (17) will always be found.

The problem as posed by Eqs. (20)-(22) is solved by using Powell's variable metric algorithm for constrained optimization (VMCON) [8]. The required gradient of the Lagrangian function corresponding to Eqs. (20)-(22) involves the gradient of the load parameter which is calculated using the expressions derived in the previous section.

## DISCUSSION OF RESULTS AND CONCLUSIONS

The first step was to validate the accuracy of the sensitivity derivatives. This validation was performed by comparing the analytically calculated derivatives using the expressions (6)-(8) with those calculated using central differences. Since no previous studies exist that address the problem being discussed herein, it was essential to generate a basis for comparison. Such a basis was provided by designs that corresponded to maximum potential energy of the nonlinear deformations. Even though previous studies on shallow trusses [5] have confirmed the non-optimality of such designs they are relatively easy to generate and provide a basis for comparison with truly optimum designs.

It can be easily verified that designs which correspond to maximum potential energy satisfy the condition

$$S_e = \frac{(U_e^s + n U_e^b)}{V_e} = C = \text{constant}; \quad e = 1, 2, \dots, m \quad (23)$$

where  $U_e^s$ ,  $U_e^b$  and  $V_e$  are the strain energy due to stretching, the strain energy due to bending, and the volume of the eth element, respectively.

Relations (23) can be easily met by a recurrence procedure that evolves design for the  $(r+1)$ st iteration from that of the  $r$ th iteration according to

$$A_e^{r+1} = \alpha A_e^r \left( \frac{S_e}{S_{avg}} \right)^p \quad (24)$$

where

$$S_{avg} = \left( \sum_{e=1}^m U_e^s + n U_e^b \right) / V_0 \quad (25)$$

$\alpha$  is a constant such that

$$\sum_{e=1}^m A_e^{r+1} L_e = V_0$$

and  $p$  is a suitable exponent usually chosen to be equal to  $1/2$ . Several designs for a concentrated load at the crown and a uniformly distributed vertical loading were generated for  $n = 1, 2, 3$  using the mathematical programming procedure, VMCON and the recurrence relations (23)-(24). Table 1 provides a comparison of these designs. Differences between the two designs are indeed drastic especially for  $n = 3$ . A curious phenomenon was observed during the recurrence procedure namely, that several non-converged intermediate designs had higher critical (limit) loads than the final fully covered designs with a uniform specific energy density distribution. This is to be expected since the fully converged designs are non-optimal. Table 2 provides the material distributions in terms of the non-dimensional areas of the five frame elements used to model half the arch.

An attempt to optimize a five element arch model with  $y(x) = 5 \sin \frac{\pi x}{100}$  failed for  $n = 2, 3$  because no limit point load could be determined. This is not surprising since for very low rise to span ratios the arch is likely to behave more like a flexible nonlinear beam with no susceptibility to snap-through. Likewise for arches with high rise to span ratios instability occurs by bifurcation at load levels far below their limit points and hence the problem belongs to the class of linear eigenvalue problems. For arches with intermediate rise to span ratios the type of instability can change from the initial limit point to a bifurcation type at convergence. In fact the two points may coincide during optimization at which point the critical load parameter is no longer differentiable with respect to the design variables. Recourse must be then made to techniques of nondifferentiable optimization [8].

## REFERENCES

1. Haftka, R. T. and Kamat, M. P., Elements of Structural Optimization, Martinus Nijhoff Publishers, 1984.
2. Mroz, Z., Kamat, M. P., and Plaut, R. H., "Sensitivity Analysis and Optimal Design of Nonlinear Beams and Plates", Journal of Structural Mechanics, Vol. 13, No. 3, 1985, pp. 245-266.
3. Mroz, Z. "Sensitivity Analysis and Optimal Design with Account for Varying Shape and Support Conditions", Computer Aided Optimal Design, NATO Advanced Study Institute, Portugal, 1986, Vol. 2, pp. 109-144.
4. Arora, J. S. and Wu, C. C., "Design Sensitivity Analysis of Nonlinear Structures, Computer Aided Optimal Design, NATO Advanced Study Institute, Portugal, 1986, Vol. 2, pp. 228-246.
5. Kamat, M. P. and Ruangsilasingha, P., "Optimization of Space Trusses Against Instability Using Design Sensitivity Derivatives", Engineering Optimization Vol. 8, 1985, pp. 177-188.
6. Kamat, M. P., "Nonlinear Transient Analysis by Energy Minimization,- Theoretical Basis for the ACTION Computer Code, NASA-CR-3287, July 1980.
7. Avriel, M., "Nonlinear Programming", Prentice Hall, Englewoods Cliff, N. J. 1979.
8. Wardi, Y. Y., and Polak, E., "A Nondifferentiable Optimization Algorithm for Structural Problems with Eigenvalue Inequality Constraints", Journal of Structural Mechanics, Vol. 11, No. 4, 1984, pp. 561-577.



Table 1. Comparison of Designs for Different Loadings on an Arch

$$y(x) = a \sin \frac{\pi x}{L}; \quad a = 10, \quad L = 100$$

| Type<br>of Design  | $(\lambda_{cr})_{OPT} / (\lambda_{cr})_{unif.}$ |       |       |   |       |       |
|--|---|-------|-------|---|-------|-------|
|  | Concentrated Load at the<br>Crown               |       |       | Uniformly Distributed Vertical<br>Loading |       |       |
|  | n = 1   | n = 2 | n = 3 | n = 1                                     | n = 2 | n = 3 |
| VMCON<br>with Sensitivity<br>Derivatives                 | 1.033   | 1.305 | 2.15  | 1.0013                                    | 1.207 | 1.932 |
| Max. Potential<br>Energy<br>with Recurrence<br>Procedure | 1.047   | 1.064 | 1.092 | 1.005                                     | 1.024 | 1.048 |

Table 2. Material Distributions for the Optimal Arch Designs of Table 1 Using VMCON

| Type of Loading                              | n | $(A_e)_{OPT} / (A_e)_{unif.}$ |        |        |        |        |
|--|---|-------------------------------|--------|--------|--------|--------|
|  |   | e = 1                         | e = 2  | e = 3  | e = 4  | e = 5  |
| Concentrated<br>Load at<br>the Crown         | 1 | 0.8774                        | 0.8662 | 0.8839 | 1.0452 | 1.340  |
|  | 2 | 0.7036                        | 0.8370 | 0.9963 | 1.1577 | 1.3153 |
|  | 3 | 0.5780                        | 0.8860 | 1.0700 | 1.1953 | 1.2907 |
| Uniformly<br>Distributed<br>Vertical Loading | 1 | 0.9471                        | 1.0240 | 1.0455 | 1.0080 | 0.9760 |
|  | 2 | 0.7662                        | 0.9526 | 1.0663 | 1.1080 | 1.1168 |
|  | 3 | 0.6122                        | 0.9285 | 1.0927 | 1.1738 | 1.2095 |