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MODAL SENSITIVITY FOR STRUCTURAL SYSTEMS WITH REPEATED FREQUENCIES

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ABSTRACT

Repeated or closely packed modal frequencies are common physical occurrences for vibrating structures which are complex or possess multi-planes of symmetry. The computation of the sensitivity to structural modifications for these frequencies and mode shapes is made difficult by the fact that the mode shapes are not unique, since any linear combination of eigenvectors corresponding to a repeated eigenvalue is also an eigenvector.

This paper extends the work of Chen and Pan [1], who used modal expansion techniques for accommodating the sensitivity analysis of structures with repeated eigenvalues. Starting with a discussion of the physical significance of sensitivity analysis for repeated frequency modes, the paper presents a derivation of the governing equations for the derivatives of a repeated eigenvalue. This is followed with a small example to illustrate the results. An efficient computation procedure, based upon an expansion of Nelson's ideas [2] for large banded systems, is then proposed for systems with repeated or closely spaced eigenvalues.

IMPORTANCE OF THE PROBLEM

The importance of obtaining gradients for eigenvalue problems stems from the fact that gradients, or derivatives with respect to system parameters, represent solution sensitivities. A knowledge of these sensitivities permits efficient design modifications, yields insight into the reasons for discrepancies between structural analyses and dynamic tests, and suggests model changes to improve correlations.

KNOWLEDGE OF GRADIENTS:

- YIELDS INSIGHT RE. PARAMETER SENSITIVITIES
- PERMITS EFFICIENT DESIGN MODIFICATIONS
- UNDERSTAND TEST/ANALYSIS DISCREPANCIES
- SUGGESTS MODEL CHANGES TO IMPROVE CORRELATION

WHEN DO REPEATED FREQUENCIES OCCUR?

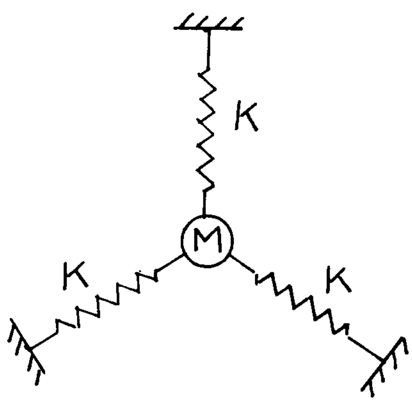
While a procedure for obtaining gradients efficiently was presented a decade ago by Nelson [2], the problems associated with repeated roots have not been adequately addressed.

The problem of repeated frequencies, or identical frequencies with different mode shapes, occurs in many physical situations. The most common circumstances under which multiple eigenvalues occur in engineering are cases where system symmetry exists, such as structures with two or more planes of reflective or cyclic symmetry (see Figure 1) or axis symmetry (see Figure 2).

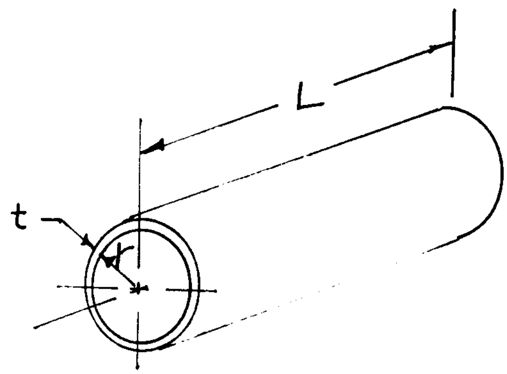
It is also possible for repeated or closely spaced eigenvalues to occur when physical symmetry is not present, such as with classical wing flutter when the first bending and twisting frequencies coalesce.

COINCIDENTAL PARAMETERS (E.G. WING TWIST/BENDING FLUTTER)

SYMMETRY: REFLECTIVE, CYCLIC, AXISYMMETRY



SYMMETRICALLY SUPPORTED MASS



RIGHT CIRCULAR CYLINDRICAL SHELL

TECHNICAL BACKGROUND

Assume [A] and [B] are symmetric n x n matrices and λ_i is a repeated eigenvalue with m+1 distinct orthogonal eigenvectors. Then $\{Z_i\}$ is also an eigenvector corresponding to λ_i where

$$\{Z_i\} = \sum_{j=0}^{m} \alpha_{i+j} \{X_{i+j}\} = [\mathrm{X}]\{\alpha\}$$

and

$$[\mathrm{X}] \equiv \begin{bmatrix} | & | & & | \\ X_i & X_{i+1} & \cdots\cdots & X_{i+m} \\ | & | & & | \end{bmatrix}$$

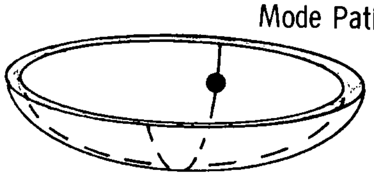
SYMMETRIC EIGENVALUE PROBLEM	$([A] - \lambda_i[B])\{x_i\} = \{0\}$
ORTHONORMALIZATION	$\{x_i\}^T [B] \{x_j\} = \delta_{ij}$
MULTIPLE EIGENVALUE	λ_i REPEATS M + 1 TIMES
CORRESPONDING EIGENVECTORS	$\{x_i\}, \{x_{i+1}\}, \ldots, \{x_{i+M}\}$
NONUNIQUENESS OF EIGENVECTORS	$\{Z_i\} = \sum_{j=0}^{M} \alpha_{i+j}\{x_{i+j}\} = [\mathrm{X}]\{\alpha\}$

$$[\mathrm{X}] \equiv \begin{bmatrix} | & | & & | \\ X_i & X_{i+1} & \cdots\cdots & X_{i+M} \\ | & | & & | \end{bmatrix}$$

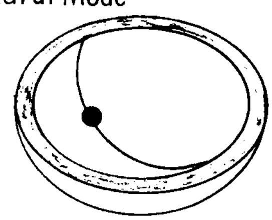
FRICTIONLESS PARTICLE IN A SHALLOW ELLIPTIC DISH
PHYSICAL INTERPRETATION

A simple physical interpretation of repeated eigenvalues was presented by Crandall [3] in which he considered a shallow elliptical bowl in which a frictionless mass particle is allowed to slide in the bottom of the bowl (left figure). The eigenvalue problem for this system consists in determining the paths and frequencies of back-and-forth motion in which each motion is repeated on the same path. The natural mode solution is obviously along the major and minor axes of the ellipse.

Next imagine that the elliptical bowl is gradually transformed into a spherical bowl (right figure). The eigenvalues will approach one another and any straight path, through the bottom of the bowl, is equally a natural mode. Thus, when m+1 eigenvalues coalesce, there is an infinity of mode shapes composed of a linear combination of the m+1 dependent, but somewhat arbitrary, basis modes.



Elliptic Dish - Unique Modes



Spherical Dish - Nonunique Modes

Natural Frequencies Coalesce As Dish Becomes Spherical

Distinct Mode Shapes Vanish As Ellipticity Disappears

Crandall (ref. 3)

MODAL GRADIENT EQUATIONS PRESENT AN ENIGMA

The modal sensitivity equations for a small change in a typical parameter, R, upon which certain matrix elements of [A] and [B] depend, are well known and summarized below. The problems are that they cannot be easily interpreted for the repeated eigenvalue problem since $\{x_i\}$ is not unique, and matrix ([A] - λ_i [B]) is not of order n-1 but lower (i.e., n-(m+1)) depending on the multiplicity, m, of eigenvalue λ_i .

$$(\quad)' = \frac{\partial(\quad)}{\partial R}$$

$$\lambda_i' = \{x_i\}^T ([A]' - \lambda_i [B]') \{x_i\}$$

$$([A] - \lambda_i [B]) \{x_i'\} = \{F_i\}$$

$$\{x_i\}^T [B] \{x_i'\} = b$$

$$\{F_i\} = - ([A] - \lambda_i [B])' \{x_i\}$$

$$b = -\tfrac{1}{2} \{x_i\}^T [B]' \{x_i\}$$

INTERPRETATION PROBLEMS

There are ambiguities associated with the gradient equations since $\{X_i\}$ is not unique and λ_i' depends upon which $\{X_i\}$ is chosen. In addition, the rank of $([A] - \lambda_i [B])$ is not n-1, but lower. Therefore, inclusion of the derivative of the normalization condition alone is not sufficient to uniquely determine $\{X_i'\}$.

WHICH $\{X_i\}$ SHOULD BE USED? IS $\{X_i\}$ DIFFERENTIABLE IN R?

USE OF DIFFERENT $\{X_i\}$ WILL YIELD DIFFERENT RESULTS.

RANK OF $([A] - \lambda_i[B])$ IS TOO LOW TO UNIQUELY DETERMINE $\{X_i'\}$.

WHICH ADDITIONAL EQUATIONS SHOULD BE USED?

REPEATED FREQUENCY SENSITIVITY EQUATION

To determine how the eigenvectors are perturbed by the infinitesimal change in R, we postulate an aribtrary vector $\{Z_i\}$ which is linearly composed of all the $\{x_l\}$ ($l = i, i+1, \ldots, i+m$) and premultiply the eigenvector gradient equations by the transpose of all the eigenvectors corresponding to λ_i.

This yields an auxiliary matrix eigenvalue equation in λ_i', which is of order m+1, the solution of which defines the specific eigenvectors, through the eigenvectors $\{\alpha(i)\}$, affected by the change in parameter R.

$$\text{LET} \quad \{Z_i\} = \sum_{j=0}^{M} \alpha_{i+j} \{X_{i+j}\} = [X]\{\alpha\}$$

$$([A] - \lambda_i [B]) \{Z_i'\} = \{F_i\} = -([A] - \lambda_i [B])' \{Z_i\}$$

$$[X]^T ([A] - \lambda_i [B]) = [0]$$

$$[X]^T \{F_i(Z)\} = \{0\} \Leftrightarrow [D]\{\alpha\} = \lambda_i' \{\alpha\}$$

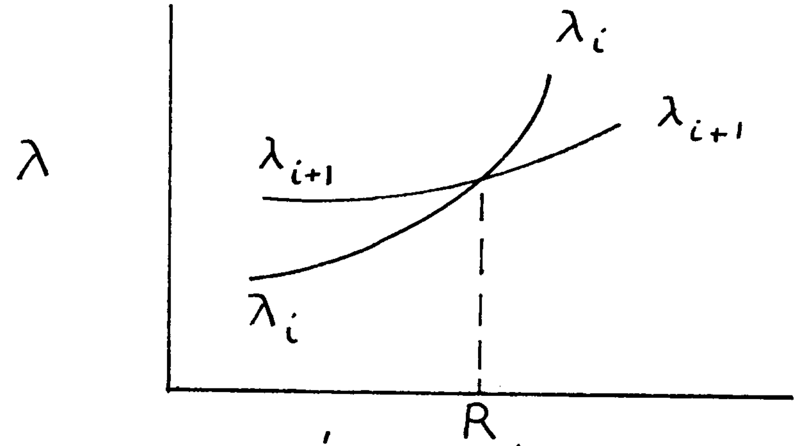
$$\underset{(M+1)\times(M+1)}{[D]} \equiv [X]^T ([A]' - \lambda_i [B]') [X]$$

PHYSICAL INTERPRETATION

Solution of the (m+1) eigenvalues and eigenvectors of the [D] matrix will yield the m+1 gradients of λ_i as well as the eigenvectors $\{Z_i\}$ to which they correspond. The figure below displays how the eigenvalues coalesce for a particular value of the parameter R and also shows how they correspond to different gradients. In general, there will be as many derivatives as there are curves intersecting at a particular parameter value R.

M + 1
SOLUTIONS

$$[D]\ \{\alpha\} = \lambda_i'\ \{\alpha\}$$



NOTE: THERE ARE TWO λ_i' WHERE λ_i COALESCES WITH λ_{i+1}

DETERMINATION OF λ_i' AND CORRESPONDING $\{\alpha\}$

UNIQUELY DETERMINES MODE FOR GRADIENT SOUGHT

PROPOSED SOLUTION PROCEDURE: OVERVIEW

The solution procedure proposed is an extended version of Nelson's method for non-repeated roots. The method maintains the original matrix bandwidth while eliminating m+1 equation redundancies in the original eigenvalue system.

The equations to be eliminated are determined by examining each eigenvector which corresponds to the eigenvalue whose gradients are desired, and establishing which elements are the maximum for each vector. These then correspond to which m+1 rows of $([A] - \lambda_i [B])$ should be considered as redundant. If the maximum elements of any two eigenvectors correspond to the same row, then it is necessary to go to the next smaller element until a set of m+1 equations for removal is obtained.

Rather than eliminate these rows and upset the system bandedness, we propose to extend Nelson's ideas by zeroing out the corresponding element of $\{F_j\}$ and then solve for $\{V_j\}$.

BASED UPON MAXIMUM ELEMENTS OF $\{Z_i\}, \{Z_{i+1}\}, \ldots, \{Z_{i+M}\}$

ZERO-OUT M+1 ROWS AND COLUMNS OF $([A] - \lambda_i [B])$

ZERO-OUT M+1 ELEMENTS FROM $\{F_j\}$, $j = i, i+1, \ldots, i+M$

SOLVE: $\left(\overline{[A] - \lambda_i [B]}\right) \{V_j\} = \{\bar{F}_j\}$

NOTE: $\left(\overline{[A] - \lambda_i [B]}\right)$ HAS SAME BANDWIDTH AS $([A] - \lambda_i [B])$

SOLUTION PROCEDURE: AUGMENTATION OF EQUATION

The process described on the previous page yields a solution vector $\{V_j\}$ with m+1 zeroes. To this we append the m+1 eigenvectors $\{Z_\ell\}$ with appropriate constants $C_{j\ell}$. This combination is then substituted into the derivatives of the m+1 orthonormalization equations and the (m) additional optional equations to uniquely determine the m+1 constants $C_{j\ell}$.

INTRODUCE M+1 ADDITIONAL EQUATIONS:

$$\left(\{Z_j\}^T [B] \{Z_i\} - \delta_{ij}\right)' = 0$$

$$i, j = i, i+1, i+2, \ldots, i+m$$

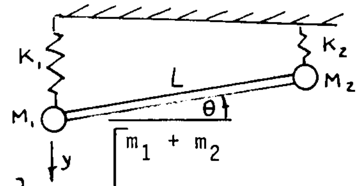
LET $\{Z_j'\} = \{V_j\} + \sum_{\ell=i}^{i+m} C_{j\ell} \{Z_\ell\}$

AND $\{Z_j\}^T [B] \{Z_K'\} = \{Z_K\}^T [B] \{Z_j'\}$ $K \neq j$ (OPTIONAL)

THEN $C_{Kj} = -\frac{1}{2} \{Z_K\}^T [B] \{Z_j\} - \{Z_j\}^T [B] \{V_K\}$

SIMPLE EXAMPLE: BASIC DESCRIPTION

As a very simple application of this procedure consider a weightless straight bar of length L with end masses supported by linear springs. As the spring stiffnesses and masses approach one another, so do the two system frequencies. Thus, depending upon the method by which the normal modes are obtained, the mode shapes may vary. For the mode shapes presented below, either both masses vibrate simultaneously up and down together $\{x_1\}$, or in opposition, $\{x_2\}$.

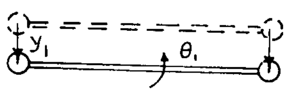


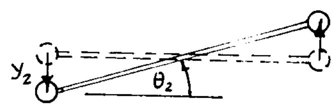
$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} y \\ \theta L \end{Bmatrix} = \lambda \begin{bmatrix} m_1 + m_2 & -m_2 \\ -m_2 & m_2 \end{bmatrix} \begin{Bmatrix} y \\ \theta L \end{Bmatrix}$$

FOR $k_1 = k_2 \equiv k$, $m_1 = m_2 \equiv m$

$$\lambda_1 = \lambda_2 = \frac{k}{m}$$

$$\{x_1\} = \begin{Bmatrix} y_1 \\ L\theta_1 \end{Bmatrix} = \frac{1}{\sqrt{2m}} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad \{x_2\} = \begin{Bmatrix} y_2 \\ L\theta_2 \end{Bmatrix} = \frac{1}{\sqrt{2m}} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$





SIMPLE EXAMPLE: NORMAL MODES

If we follow the procedure outlined earlier, and compute the system frequency gradients with respect to changes in m_2, we obtain normal modes $\{Z_1\}$ and $\{Z_2\}$. These modes are associated with motions for which only m_1 moves, and motions for which only m_2 moves.

$$(\quad)' = \frac{\partial(\ )}{\partial M_2}, \qquad [D]\{\alpha\} = \lambda'\{\alpha\}$$

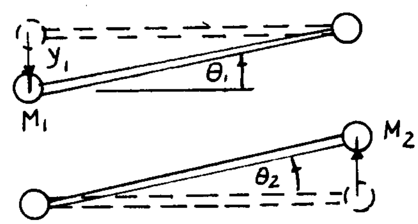
$$-\frac{K}{2M^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \lambda' \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\lambda' = 0 \quad \text{AND} \quad -\frac{k}{M^2} = \frac{\partial\left(\frac{K}{M}\right)}{\partial M}$$

$$\left\{\alpha^{(1)}\right\} = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \left\{\alpha^{(2)}\right\} = \frac{1}{\sqrt{2}}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\{Z_1\} = \alpha_1^{(1)}\{X_1\} + \alpha_2^{(1)}\{X_2\} = \frac{1}{\sqrt{M}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{Z_2\} = \alpha_1^{(2)}\{X_1\} + \alpha_2(1)\{X_2\} = \frac{1}{\sqrt{M}}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$



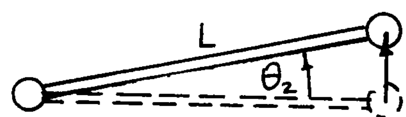
SIMPLE EXAMPLE: EIGENVECTOR GRADIENTS

Following the computation procedure outlined earlier, the eigenvector gradients, $\{Z'_1\}$ and $\{Z'_2\}$, for changes in m_2 are the null vector and the value of $\partial \{Z_2\}/\partial m_2$ shown below.

$$\{Z_1'\} = C_{11}\{Z_1\} + C_{12}\{Z_2\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$ I.E. $\{Z_1\}$ NOT AFFECTED BY M_2 CHANGE

$$\{Z_2'\} = C_{21}\{Z_1\} + C_{22}\{Z_2\} = \begin{Bmatrix} 0 \\ -\dfrac{1}{2M^{3/2}} \end{Bmatrix}$$

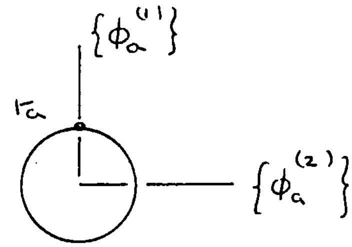
I.E. $\{Z_2\} = \dfrac{1}{\sqrt{M}} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

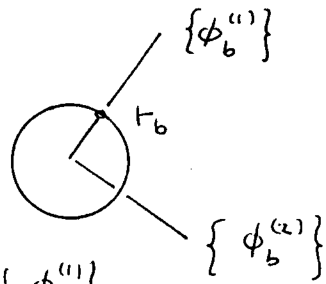


$$\frac{\partial\{Z_2\}}{\partial M_2} = \begin{Bmatrix} 0 \\ -\frac{1}{2} M^{-3/2} \end{Bmatrix}$$

USE CAUTION WHEN WORKING WITH THE TOTAL DIFFERENTIAL

$$d\{\phi_a^{(1)}\} = \frac{\partial\{\phi_a^{(1)}\}}{\partial r_a}\, dr_a \qquad\qquad d\{\phi_b^{(1)}\} = \frac{\partial\{\phi_b^{(1)}\}}{\partial r_b}\, dr_b$$





$$d\{\phi^{(1)}\} \neq \frac{\partial\{\phi_a^{(1)}\}}{\partial r_a}\, dr_a + \frac{\partial\{\phi_b^{(1)}\}}{\partial r_b}\, dr_b$$

THE TOTAL DIFFERENTIAL MAY NOT EXIST EVEN THOUGH THE PARTIAL DERIVATIVES DO.

CONCLUSIONS

The coordinate system and mode shapes initially selected for this example gave little physical insight regarding how the initial system would decompose due to a change in m_2. Yet, this example yields a simple demonstration of the insight to be gained by following the proposed procedure. Thus, it is seen that the proposed mathematical procedure automatically yields m+1 distinct gradients for repeated frequencies and m+1 distinct modes, without requiring user dependence.

The computational efficiencies suggested by Nelson[3] have been expanded. These include: maintaining system bandwidth and consideration of only the m+1 repeated root frequencies.

EIGENVALUE GRADIENTS FOR REPEATED FREQUENCIES
GENERALLY YIELD MULTIPLE DISTINCT VALUES

EFFICIENT COMPUTATION OF EIGENVECTOR GRADIENTS
FOR REPEATED FREQUENCIES IS POSSIBLE

I.E. BANDWIDTH MAY BE MAINTAINED
MODAL EXPANSION IS NOT NECESSARY

BUT, MUST INTRODUCE MODAL ORTHOGONALITY CONDITIONS IN ADDITION TO NORMALIZATION CONDITION

EXERCISE CAUTION WHEN USING A TOTAL DIFFERENTIAL WHICH IS A COMBINATION OF PARTIAL DERIVATIVES FOR REPEATED FREQUENCIES.

REFERENCES

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3. Crandall, S. H., "Engineering Analysis", McGraw-Hill, New York, 1956.