IN-39
61223

NASA CONTRACTOR
REPORT

NASA CR-178964

THE EFFECT OF A SMALL INITIAL CURVATURE ON THE FREE VIBRATION OF CLAMPED, RECTANGULAR PLATES

By A. A. Adeniji-Fashola
Resident Research Associate
National Research Council
Science and Engineering Directorate
Systems Dynamics Laboratory

and

A. A. Oyediran
Department of Mechanical Engineering
Lagos State University
Ojo, Lagos, Nigeria

Interim Report

October 1986

Prepared for
NASA-Marshall Space Flight Center
Marshall Space Flight Center, Alabama 35812

(NASA-CR-178964) THE EFFECT OF A SMALL INITIAL CURVATURE ON THE FREE VIBRATION OF CLAMPED, RECTANGULAR PLATES Interim Report (National Academy of Sciences - National Research) 24 p CSCL 20K G3/39

N87-18864

Unclas
43798

TECHNICAL REPORT STANDARD TITLE PAGE

1. REPORT NO. NASA CR-178964	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
4. TITLE AND SUBTITLE The Effect of a Small Initial Curvature on the Free Vibration of Clamped, Rectangular Plates		5. REPORT DATE October 1986
		6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) A. A. Adeniji-Fashola* and A. A. Oyediran**		8. PERFORMING ORGANIZATION REPORT #
9. PERFORMING ORGANIZATION NAME AND ADDRESS George C. Marshall Space Flight Center Marshall Space Flight Center, Alabama 35812		10. WORK UNIT NO.
		11. CONTRACT OR GRANT NO. NASW-3458
		13. TYPE OF REPORT & PERIOD COVERED Contractor Report
12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D.C. 20546		
		14. SPONSORING AGENCY CODE

15. SUPPLEMENTARY NOTES

Prepared by Systems Dynamics Laboratory, Science and Engineering Directorate.

*National Research Council.

16. ABSTRACT

An analytical method of obtaining the natural frequencies and mode shapes of clamped, rectangular plates having a small initial curvature is presented. Specifically, the singular perturbation technique is used to reduce the fourth-order plate vibration problem to the simpler membrane problem with modified boundary conditions that account for the bending effects. The eigenfrequencies for plates with inverse aspect ratios varying between 0.1 and 1.0 and for the dimensionless normal prestress between 0.1 and 1.0 have been presented for values of ε, the normalized bending rigidity, ranging between 0.0010 and 0.2500.

It is established that a small initial curvature has no effect on the frequency of vibration of the plate. However, its effect is manifested in the eigenmodes.

** Department of Mechanical Engineering, Lagos State University, Ojo, Lagos, Nigeria

17. KEY WORDS Small initial curvature; clamped, rectangular plates; normal prestress; singular perturbation technique; eigenfrequencies and eigenmodes	18. DISTRIBUTION STATEMENT Unclassified-Unlimited

19. SECURITY CLASSIF. (of this report)	20. SECURITY CLASSIF. (of this page)	21. NO. OF PAGES	22. PRICE
Unclassified	Unclassified	23	NTIS

MSFC - Form 3292 (May 1969)

For sale by National Technical Information Service, Springfield, Virginia 22151

TABLE OF CONTENTS

CONTRACTOR REPORT

THE EFFECT OF A SMALL INITIAL CURVATURE ON THE FREE VIBRATION OF CLAMPED, RECTANGULAR PLATES

SUMMARY

An analytical method of obtaining the natural frequencies and mode shapes of clamped, rectangular plates having a small initial curvature is presented. Specifically, the perturbation technique is used to reduce the fourth-order plate vibration problem to the simpler membrane problem with modified boundary conditions that account for the bending effects. Excellent results are obtained when the bending rigidities are small compared with the in-plane forces.

I. INTRODUCTION

The problem of a plate having a small initial curvature has long been recognized [1]. From small deflection theory, the total deflection is the sum of the initial deflection and the deflection resulting from the applied load on a flat plate. However, few particular solutions are available in the literature. Timoshenko and Woinowski-Krieger [1] presented an exact analysis for the static deflection of simply-supported plates having a small initial curvature. The solutions to the vibration problems of such plates are, however, not available in the open literature.

On the other hand, much work has been done on cylindrically-curved panels. The work of Sewall [2] revealed that natural frequencies of vibration of cylindrically-curved panels are obtainable in closed form for only two boundary conditions. Interestingly, it was found that the panel natural frequency deviates from the flat plate frequency only by a curvature term. This curvature term vanishes for large radii of curvature. However, Sewall's formulation, which is complex and involves integrals, does not lend itself to easy application. Blevins [3] presented a simplified analysis based entirely on Sewall's work. The Donnel equations for a cylindrical shell were used and mode shapes of a single-span beam satisfying pertinent boundary conditions were obtained. The Rayleigh-Ritz method was used to generalize Sewall's results to other boundary conditions.

The problem presented here is different from those of Sewall and Blevins. We consider the free vibrations of a clamped, normally-prestressed plate with a small initial curvature using the method of singular perturbations. The governing equations are those for curved plates. There are no known analytical expressions for the free vibration of clamped, initially-flat plates even in the absence of normal prestress forces. Explicit expressions for eigenvalues and mode shapes for the free vibration of initially-curved, normally-prestressed plates have, however, been obtained in the present study, in the limit when the normalized bending rigidity is small.

In the absence of shearing prestress, the asymptotic eigenvalue results for initially-curved, normally-prestressed plates are found to correspond to those of initially-flat, normally-prestressed plates. However, the eigenmodes are highly modified. It is shown here that the eigenmodes can be decomposed into the following three components: (i) the initial deflection, (ii) the flat plate deflection and (iii) the deflection resulting from the static equilibrium condition.

II. PROBLEM FORMULATION

We consider a thin rectangular plate having an initial deflection $w_o'(x',y')$. If the plate is further deflected by an amount $w_1'(x',y',\tau)$, then the total deflection, $w'(x',y',\tau)$, is

$$w'(x',y',\tau) = w_o'(x',y') + w_1'(x',y',\tau) \quad . \tag{2.1}$$

The equilibrium equation given by Timoshenko and Woinowski-Krieger [1] is

$$D\nabla^4 w_1' - N_x \frac{\partial^2 w_1'}{\partial x'^2} - N_y \frac{\partial^2 w_1'}{\partial y'^2} + \mu \frac{\partial^2 w_1'}{\partial \tau^2} = N_x \frac{\partial^2 w_o'}{\partial x'^2} + N_y \frac{\partial^2 w_o'}{\partial y'^2} \tag{2.2}$$

for the case in which there is no shearing prestress. In non-dimensional form, equation (2.2) becomes

$$\varepsilon^2 \nabla^4 w_1 - \beta_1^2 \frac{\partial^2 w_1}{\partial x^2} - \beta_2^2 \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial t^2} = \beta_1^2 \frac{\partial^2 w_o}{\partial x^2} + \beta_2^2 \frac{\partial^2 w_o}{\partial y^2} \tag{2.3}$$

where

$$\varepsilon^2 = D/N_o L^2 << 1 \tag{2.4}$$

N_o is the characteristic normal prestress value while L is the characteristic length used in normalizing the deflection w' and the co-ordinates x' and y'. Time is normalized using the characteristic frequency, ω, such that

$$t = \omega\tau \qquad \text{and} \qquad N_o/\mu\omega^2 L^2 = 1 \tag{2.5}$$

where μ is the mass per unit area of the thin plate. For a fully-clamped, rectangular plate, we require that $w = \partial w/\partial n = 0$ on all edges.

If we let the dimensionless initial deflection $w_o(x,y)$ be given by a double Fourier series of the form

$$w_o(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \operatorname{Sin}(n\pi x) \operatorname{Sin}(m\pi y/b) \tag{2.6}$$

where b is the dimensionless length of the rectangular plate in the y direction, the non-dimensional governing differential equation (2.3) becomes

$$\varepsilon^2 \nabla^4 w_1 - \beta_1^2 \frac{\partial^2 w_1}{\partial x^2} - \beta_2^2 \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial t^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \operatorname{Sin}(n\pi x) \operatorname{Sin}(m\pi y/b) \tag{2.7}$$

where

$$q_{mn} = -(\pi^2 P_{mn}/b^2)\ (b^2\beta_1^2 + \beta_2^2) \quad . \tag{2.8}$$

It should be noted that the plate length in the x direction has been taken as the characteristic length, L.

If we now make the substitution

$$w_1(x,y,t) = u(x,y,t) + \psi(x,y) \tag{2.9}$$

in equation (2.7), we obtain the following dynamic equation:

$$\varepsilon^2 \nabla^4 u - \beta_1^2 \frac{\partial^2 u}{\partial x^2} - \beta_2^2 \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial t^2} = 0 \tag{2.10}$$

with boundary conditions

$$\left.\begin{aligned}
& u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,b,t) = 0 \\
& \frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(1,y,t) = \frac{\partial u}{\partial y}(x,0,t) = \frac{\partial u}{\partial y}(x,b,t) = 0 \\
& \text{and initial conditions} \\
& u(x,y,0) = f(x,y) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,y,0) = g(x,y)
\end{aligned}\right\} \tag{2.11}$$

and the static equilibrium equation:

$$\varepsilon^2\nabla^4\psi - \beta_1^2 \frac{\partial^2\psi}{\partial x^2} - \beta_2^2 \frac{\partial^2\psi}{\partial y^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \operatorname{Sin}(n\pi x) \operatorname{Sin}(m\pi y/b) \tag{2.12}$$

with boundary conditions

$$\left.\begin{aligned} &\psi(0,y) = \psi(1,y) = \psi(x,0) = \psi(x,b) = 0 \\ &\frac{\partial\psi}{\partial x}(0,y) = \frac{\partial\psi}{\partial x}(1,y) = \frac{\partial\psi}{\partial y}(x,0) = \frac{\partial\psi}{\partial y}(x,b) = 0 \end{aligned}\right\} . \tag{2.13}$$

Equation (2.10) is the governing differential equation for the free vibration of an initially-flat, anisotropically-prestressed rectangular plate, while equation (2.12) corresponds to the static problem of a similar plate subjected to some generalized loading. Thus, the initial curvature effect is equivalent to that of some fictitious loading.

III. METHOD OF SOLUTION

The governing differential equations (2.10) and (2.12), subject to the condition equations (2.11) and (2.13), respectively, are solved by the method of singular perturbations, in the limit when $\varepsilon \to 0^+$. The procedure is to seek an approximate solution valid in the core region of the domain. Thus, we write

$$\left.\begin{aligned} u^0 &= u_o^0 + \varepsilon u_1^0 + \varepsilon^2 u_2^0 + O(\varepsilon^3) \\ \lambda^2 &= \lambda_o^2 + \varepsilon \lambda_1^2 + \varepsilon^2 \lambda_2^2 + O(\varepsilon^3) \\ \psi &= \psi_0^0 + \varepsilon \psi_1^0 + \varepsilon^2 \psi_2^0 + O(\varepsilon^3) \end{aligned}\right\} . \tag{3.1}$$

The resulting membrane-type governing differential equations, valid in the core region, are

$$\left.\begin{aligned} &\beta_1^2 u^0_{\nu_{xx}} + \beta_2^2 u^0_{\nu_{yy}} + \lambda_0^2 u_\nu^0 = \nabla^4 u^0_{\nu-2} - \sum_{\sigma=1}^{\nu} \lambda_\sigma^2 u^0_{\nu-\sigma} \\ &\text{and} \\ &\beta_1^2 \psi^0_{\nu_{xx}} + \beta_2^2 \psi^0_{\nu_{yy}} = \nabla^4 \psi^0_{\nu-2} - \left(\beta_1^2 \frac{\partial^2 w_o}{\partial x^2} + \beta_2^2 \frac{\partial^2 w_o}{\partial y^2}\right) , \end{aligned}\right\} \tag{3.2}$$

$$\nu = 0, 1, 2, \ldots$$

Here, the subscript ν denotes the order in ε, while the superscript o denotes outer solution. Terms in which the dependent variable has a negative subscript are discarded. We note that the equations (3.2) for u_ν^o and ψ_ν^o are of second order as opposed to equations (2.10) and (2.12) which are fourth order.

It is easy to see that the expansions given in equation (3.1) cannot satisfy all the boundary conditions. Thus, the resulting solutions for u_ν^o and ψ_ν^o, which are equivalent to states of membrane deformation, will not be valid near the plate edges where bending deformations exist. Across these bending regions (boundary layers), the deformations change rapidly to satisfy the boundary conditions while the fourth order governing differential equations (2.10) and (2.12) are preserved. The solutions of these equations in the bending regions will then serve as modified boundary conditions for the membrane-type equations (3.2).

For example, near x = 0, a stretching transformation takes the form

$$X = x/\varepsilon \quad . \tag{3.3}$$

In this case, we seek expansions of the form

$$\left.\begin{aligned} u &= u_o^i + \varepsilon u_1^i + \varepsilon^2 u_2^i + O(\varepsilon^3) \\ \psi &= \psi_o^i + \varepsilon \psi_1^i + \varepsilon^2 \psi_2^i + O(\varepsilon^3) \end{aligned}\right\} \quad . \tag{3.4}$$

The governing differential equations become

$$u^i_{\nu_{XXXX}} - \beta_1^2 u^i_{\nu_{XX}} = -2 u^i_{\nu-2_{XXyy}} + \beta_2^2 u^i_{\nu-2_{yy}} - u^i_{\nu-4_{yyyy}} - \sum_{\sigma=0}^{\nu-2} (\lambda_\sigma^2 u^i_{\nu-\sigma-2})$$

and

$$\psi^i_{\nu_{XXXX}} - \beta_1^2 \psi^i_{\nu_{XX}} = -2 \psi^i_{\nu-2_{XXyy}} + \beta_2^2 \psi^i_{\nu-2_{yy}} - \psi^i_{\nu-4_{yyyy}}$$

$$- \left(\beta_1^2 \frac{\partial^2}{\partial(\varepsilon X)^2} + \beta_2^2 \frac{\partial^2}{\partial y^2}\right) w_o(\varepsilon X, y) \tag{3.5}$$

Equations similar to (3.5) can be written for the regions near x = 1, y = 0, and y = b. The dynamic and static problems will now be considered for arbitrary initial curvature.

3.1 The Dynamic Problem

The dynamic problem for a fully-clamped, rectangular plate undergoing sinusoidal vibration can be described by the equation

$$\varepsilon^2 \nabla^4 u - \beta_1^2 \frac{\partial^2 u}{\partial x^2} - \beta_2^2 \frac{\partial^2 u}{\partial y^2} - \lambda^2 u = 0 \tag{3.6}$$

subject to

$$\left.\begin{array}{ll} u(0,y) = \dfrac{\partial u}{\partial x}(0,y) = 0 \; ; & \\ u(1,y) = \dfrac{\partial u}{\partial x}(1,y) = 0 \; ; & \\ & 0 < b \le 1 \\ u(x,0) = \dfrac{\partial u}{\partial y}(x,0) = 0 \; ; & \\ u(x,b) = \dfrac{\partial u}{\partial y}(x,b) = 0 \; . & \end{array}\right\} \tag{3.7}$$

It is easily verified from the literature [4,5,6] or otherwise that an approximate solution to equation (3.6), subject to the boundary conditions (3.7), can be written as

$$\begin{aligned} u(x,y) = {} & A_o \sin(n\pi x) \sin(m\pi y/b) + \varepsilon A_o \{(n\pi/\beta_1)(2x-1)\cos(n\pi x)\sin(m\pi y/b) \\ & + (m\pi/\beta_2 b^2)(2y-b)\sin(n\pi x)\cos(m\pi y/b)\} + \varepsilon^2\, 2A_o \{(n^2\pi^2/\beta_1^2)x(1-x) \\ & \cos(n\pi x)\cos(m\pi y/b) + (n\pi/2\beta_1^2\beta_2)(2\beta_2 b-\beta_1)(2x-1)\cos(n\pi x)\sin(m\pi y/b) \\ & + (m\pi/2\beta_1\beta_2^2 b^3)(2\beta_1-\beta_2 b)(2y-b)\sin(n\pi x)\cos(m\pi y/b) \\ & + (m^2\pi^2/\beta_2^2 b^4)\, y(b-y)\sin(n\pi x)\sin(m\pi y/b)\} + \varepsilon\{(z_1/\beta_1)\exp(-\beta_1 x/\varepsilon) \\ & + (z_2/\beta_1)\exp(-\beta_1(1-x)/\varepsilon) + (z_3/\beta_2)\exp(-\beta_2 y/\varepsilon) \\ & + (z_4/\beta_2)\exp(-\beta_2(b-y)/\varepsilon)\} + \varepsilon^2\{(z_5/\beta_1)\exp(-\beta_1 x/\varepsilon) \\ & + (z_6/\beta_1)\exp(-\beta_1(1-x)/\varepsilon) + (z_7/\beta_2)\exp(-\beta_2 y/\varepsilon) \\ & + (z_8/\beta_2)\exp(-\beta_2(b-y)/\varepsilon)\} + O(\varepsilon^3) \end{aligned} \tag{3.8}$$

where

$$z_1 = A_o\ n\pi \sin(m\pi y/b)$$

$$z_2 = (-1)^{n+1}\ z_1$$

$$z_3 = (A_o\ m\pi/b) \sin(n\pi x)$$

$$z_4 = (-1)^{m+1}\ z_3$$

$$z_5 = (A_o\ n\pi/b^2\beta_1\beta_2)\ \{2b^2\beta_2 \sin(m\pi y/b) + (m\pi\beta_1)(2y-b) \cos(m\pi y/b)\} \tag{3.9}$$

$$z_6 = (-1)^{n+1}\ z_5$$

$$z_7 = (A_o\ m\pi/b^2\beta_1\beta_2)\ \{n\pi b\beta_2(2x-1) \cos(n\pi x) + 2\beta_1 \sin(n\pi x)\}$$

$$z_8 = (-1)^{m+1}\ z_7$$

while the eigenvalues λ^2 are given as

$$\lambda^2 = \beta_1^2\ n^2\ \pi^2 + \beta_2^2\ m^2\ \pi^2/b^2 + \varepsilon\ 4\ \{\beta_1^2\ n^2\ \pi^2 + \beta_2^2\ m^2\ \pi^2/b^2\}$$
$$+ \varepsilon^2\ \{(\pi^4/b^4)(n^2b^2 + m^2)^2 + (12\ \pi^2/b^4)\ (n^2b^4 + m^2)\} + O\ (\varepsilon^3)\ . \tag{3.10}$$

The eigenvalues for plates with inverse aspect ratios, b, varying between 0.1 and 1.0 and for the dimensionless normal prestress values β_1, β_2 between 0.1 and 1.0 are presented in Figures 3.1 through 3.4 for values of ε ranging between 0.0010 and 0.2500.

It is seen from Figure 3.1(a) that the zeroth and second order solutions for b = 1 (square plate) are essentially indistinguishable for values of ε of 0.0025 and 0.0010. However, as b decreases, a considerable difference develops between these two curves, indicating the solution to be more sensitive to the value of the normalized bending rigidity, ε. A similar trend, albeit more pronounced, is observed for decreasing values of the normalized prestress β_1 and β_2 as shown in Figure 3.1(a)iii and Figure 3.1(b). It is observable from Figure 3.1(b)iii that the difference between the second order and zeroth order eigenfrequency (λ^2) can be as much as four orders of magnitude at mode numbers m = n = 15 for values of $\varepsilon = 0.0250$ and higher.

Figure 3.2 shows the second order eigenfrequencies for $\varepsilon = 0.0010$ plotted against the mode numbers with the inverse aspect ratio b as parameter. The observed trend of decreasing values of the eigenfrequency with increasing b is easily inferred from the eigenfrequency expression equation (3.10).

The same trend of decreasing eigenfrequency with increasing b is also evident in Figures 3.3(a) and (b) in which the additional effects of combinations of m and n, the mode numbers in the x and y directions respectively, Figure 3.3(a), and combinations of β_1 and β_2, the normalized prestress in the x and y directions respectively, Figure 3.3(b), have been highlighted.

Finally, Figures 3.4(a) and (b) show the eigenfrequencies plotted against the normalized prestress for various combinations of values of m, n, and b. At higher values of m and n, Figure 3.4(b), the eigenfrequency is essentially independent of β_1 and β_2 at the higher ε values.

Next, we consider the static problem for an arbitrary initial curvature.

3.2 The Static Problem

The governing differential equation for a fully-clamped, rectangular plate takes the form

$$\varepsilon^2 \nabla^4 \psi - \beta_1^2 \frac{\partial^2 \psi}{\partial x^2} - \beta_2^2 \frac{\partial^2 \psi}{\partial y^2} = \beta_1^2 \frac{\partial^2 w_o}{\partial x^2} + \beta_2^2 \frac{\partial^2 w_o}{\partial y^2} \tag{3.11}$$

subject to

$$\left.\begin{aligned}
\psi(0,y) &= \frac{\partial \psi}{\partial x}(0,y) = 0 \quad ; \\
\psi(1,y) &= \frac{\partial \psi}{\partial x}(1,y) = 0 \quad ; \\
\psi(x,0) &= \frac{\partial \psi}{\partial y}(x,0) = 0 \quad ; \qquad 0 < b \leq 1 \\
\psi(x,b) &= \frac{\partial \psi}{\partial y}(x,b) = 0 \quad .
\end{aligned}\right\} \tag{3.12}$$

Using the particular form of $w_o(x,y)$ given in equation (2.6), and by comparison with equation (2.12), the right hand side of equation (3.11) is seen to take the form

$$q_{mn} \operatorname{Sin}(n\pi x) \operatorname{Sin}(m\pi y/b)$$

where q_{mn} was defined in equation (2.8) in terms of p_{mn}, the Fourier coefficients obtained by expressing any function f(x,y), describing the initial curvature, as a double Fourier series. The procedure is to solve equation (3.2b) for $\nu = 0, 1, 2, \ldots$ subject to appropriate boundary conditions. It is easy to see that the boundary value problem for ψ_o takes the form

$$\beta_1^2 \frac{\partial^2 \psi_o}{\partial x^2} + \beta_2^2 \frac{\partial^2 \psi_o}{\partial y^2} = - q_{mn} \operatorname{Sin}(n\pi x) \operatorname{Sin}(m\pi y/b) \tag{3.13}$$

subject to

$$\psi_o(0,y) = \psi_o(1,y) = \psi_o(x,0) = \psi_o(x,b) = 0 \quad . \tag{3.14}$$

Clearly, the solution to equation (3.13) satisfying all the conditions of equation (3.14) is

$$\psi_o = \gamma_{mn} \operatorname{Sin}(n\pi x) \operatorname{Sin}(m\pi y/b) \tag{3.15}$$

with

$$\gamma_{mn} = q_{mn}/(\beta_1^2 n^2 \pi^2 + \beta_2^2 m^2 \pi^2/b^2) \quad . \tag{3.16}$$

To the next order of approximation, ψ_1 must satisfy the differential equation

$$\beta_1^2 \frac{\partial^2 \psi_1}{\partial x^2} + \beta_2^2 \frac{\partial^2 \psi_1}{\partial y^2} = 0 \tag{3.17}$$

subject to

$$\left.\begin{aligned} \psi_1(0,y) &= - \gamma_{mn}\, n\pi \operatorname{Sin}(m\pi y/b) \\ \psi_1(1,y) &= (-1)^n\, \psi_1(0,y) \\ \psi_1(x,0) &= - \gamma_{mn}\, (m\pi/b) \operatorname{Sin}(n\pi x) \\ \psi_1(x,b) &= (-1)^m\, \psi_1(x,0) \end{aligned}\right\} \tag{3.18}$$

In previous studies [7,8], equation (3.17) subject to equations (3.18) was solved numerically. In general, any of the standard numerical methods of solving parabolic partial differential equations such as the Gauss-Seidel, Jacobi and the under- and over-relaxation (SOR) methods can be used. However, using the singular perturbation analysis technique, analytical expressions can be obtained to leading and higher orders with increasing accuracy as $\varepsilon \to 0^+$. It can be shown that

$$\psi_1 = \{A_1 \cosh(\alpha_1 y) + A_2 \sinh(\alpha, y)\} \operatorname{Sin}(n\pi x)$$

$$+ \{A_3 \cosh(\alpha_2 x) + A_4 \sinh(\alpha_2 x)\} \sin(m\pi y/b) \tag{3.19}$$

where

$$\alpha_1 = n\pi\beta_1/\beta_2 \qquad \text{and} \qquad \alpha_2 = m\pi\beta_2/\beta_1 b \tag{3.20}$$

and the constants $A_1, \ldots, A_4$ are observed to take the form

$$\left.\begin{aligned} A_1 &= -\, m\pi\, \gamma_{mn}/\beta_2 b \\ A_2 &= -\, A_1\{(-1)^m + \cosh(\alpha_1 b)\}/\sinh(\alpha_1 b) \\ A_3 &= -\, n\pi\, \gamma_{mn}/\beta_1 \\ A_4 &= -\, A_3\{(-1)^n + \cosh(\alpha_2)\}/\sinh(\alpha_2) \end{aligned}\right\} \tag{3.21}$$

The method of solution is further extended to ψ_2. The boundary value problem to be solved for ψ_2 is

$$\beta_1^{\,2} \frac{\partial^2 \psi_2}{\partial x^2} + \beta_2^{\,2} \frac{\partial^2 \psi_2}{\partial y^2} = \nabla^4 \psi_o \tag{3.22}$$

subject to

$$\psi_2(0,y) = -\,(A_1 \cosh(\alpha_1 y) + A_2 \sinh(\alpha_1 y)) n\pi/\beta_1 - (A_4\, \alpha_2/\beta_1) \sin(m\pi y/b)$$

$$\psi_2(1,y) = (A_1 \cosh(\alpha_1 y) + A_2 \sinh(\alpha_1 y))\, (-1)^n\, n\pi/\beta_1$$

$$+ (\alpha_2/\beta_1)\, (A_3 \sinh(\alpha_2) + A_4 \cosh(\alpha_2)) \sin(m\pi y/b)$$

$$\psi_2(x,0) = -(A_2\alpha_2/\beta_2)\sin(n\pi x) - (m\pi/b\beta_2)(A_3\cosh(\alpha_2 x) + A_4\sinh(\alpha_2 x))$$

$$\psi_2(x,b) = (A_3\cosh(\alpha_2 x) + A_4\sin(\alpha_2 x))(-1)^m\, m\pi/\beta_2 b$$

$$+ (\alpha_1/\beta_2)(A_1\sinh(\alpha_1 b) + A_2\cosh(\alpha_1 b))\sin(n\pi x) \quad . \qquad (3.23)$$

We seek the solution to equation (3.22), subject to equations (3.23), in the form

$$\psi_2 = \psi_{2a} + \psi_{2b} \qquad (3.24)$$

where

$$\psi_{2a} = [g_1\cosh(\alpha_1 y) + g_2\sinh(\alpha_1 y)]\sin(n\pi x)$$

$$+ [g_3\cosh(\alpha_2 x) + g_4\sinh(\alpha_2 x)]\sin(m\pi y/b)$$

$$+ \tilde{\gamma}\sin(n\pi x)\sin(m\pi y/b) \qquad (3.25)$$

with

$$\left.\begin{array}{l}
\tilde{\gamma} = -(n^2\pi^2 + m^2\pi^2/b^2)/(\beta_1^2 n^2\pi^2 + \beta_2^2 m^2\pi^2/b^2) \\
\text{and} \\
g_1 = -A_2\alpha_1/\beta_2 \\
g_2 = \alpha_1[A_1\sinh(\alpha_1 b) + 2A_2\cosh(\alpha_1 b)]/\beta_2\sinh(\alpha_1 b) \\
g_3 = -A_4\alpha_2/\beta_1 \\
g_4 = \alpha_2[A_3\sinh(\alpha_2) + 2A_4\cosh(\alpha_2)]/\beta_1\sinh(\alpha_2)
\end{array}\right\} \qquad (3.26)$$

Also, the expression for ψ_{2b} takes the form

$$\psi_{2b} = [E_1 \sinh(\alpha_1 y) + E_2 \cosh(\alpha_1 y) + E_3 y \sinh(\alpha_1 y) + E_4 y \cosh(\alpha_1 y)] \sin(n\pi x)$$

$$+ [E_5 \sinh(\alpha_2 x) + E_6 \cosh(\alpha_2 x) + E_7 x \sinh(\alpha_2 x) + E_8 x \cosh(\alpha_2 x)] \sin(m\pi y/b) \tag{3.27}$$

where

$$\left.\begin{aligned}
E_1 &= -E_2[(-1)^m + \cosh(\alpha_1 b)]/\sinh(\alpha_1 b) - E_3 b - E_4 b \coth(\alpha_1 b) \\
E_2 &= -m\pi [A_3 K_1 + A_4 K_2]/\beta_2 b \\
E_3 &= A_1 \beta_1 n^2 \pi^2/\beta_2^2 \alpha_1 \\
E_4 &= E_3 A_2/A_1 \\
E_5 &= -E_6 [(-1)^n + \cosh(\alpha_2)]/\sinh(\alpha_2) - E_7 - E_8 \coth(\alpha_2) \\
E_6 &= -n\pi [A_1 K_3 + A_2 K_4]/\beta_1 \\
E_7 &= A_3 \beta_2 m^2 \pi^2/\beta_1^2 \alpha_2 \\
E_8 &= E_7 A_4/A_3
\end{aligned}\right\} \tag{3.28}$$

while

$$\left.\begin{aligned}
K_1 &= n\pi[1 + (-1)^{n+1} \cosh(\alpha_2)]/(\alpha_2^2 + n^2 \pi^2) \\
K_2 &= (-1)^{n+1} n\pi \sinh(\alpha_2)/(\alpha_2^2 + n^2 \pi^2) \\
K_3 &= m\pi [1 + (-1)^{m+1} \cosh(\alpha_1 b)]/b(\alpha_1^2 + m^2 \pi^2/b^2) \\
K_4 &= (-1)^{m+1} m\pi \sinh(\alpha_1 b)/b(\alpha_1^2 + m^2 \pi^2/b^2)
\end{aligned}\right\} \tag{3.29}$$

This completes the solution of the problem to $O(\varepsilon^2)$.

IV. CONCLUSION

The small initial curvature effect on the free vibration of clamped, rectangular plates has been presented. The eigenfrequencies for plates with inverse aspect ratios varying between 0.1 and 1.0 and for the dimensionless normal prestress between 0.1 and 1.0 have been presented for values of ε ranging between 0.0010 and 0.2500 in Figures 3.1 through 3.4.

It is established here that a small initial curvature has no effect on the frequency of vibration of the plate. However, its effect is manifested in the eigenmodes. The eigenmodes obtained are of increasing accuracy as $\varepsilon \to 0^+$, similar to the trend for the eigenfrequencies presented in Figures 3.1 through 3.4.

The solutions obtained from above provide a guide for researchers that implement the finite difference, finite element or other numerical schemes on the computer for the solution of curved plate problems. This utility of the singular perturbation technique in providing simplified expressions which yield solutions valid in the asymptotic limit and which can serve as useful guides for the pure numerical analysis approach has been attested to by Nayfeh and Kamat [7] and Ramkumar et al. [8].

REFERENCES

1. Timoshenko, S. P. and Woinowski-Krieger, S.: Theory of Plates and Shells. 2nd Edition, McGraw-Hill Kogakusha, 1959, pp. 393-395.

2. Sewall, J. L.: Vibration Analysis of Cylindrically-Curved Panels with Simply-Supported or Clamped Edges and Comparison with Some Experiments. NASA-TN-D-3791, 1967.

3. Blevins, R. D.: Natural Frequencies of Shallow, Cylindrically-Curved Panels. J. Sound and Vibration, Vol. 75, 1981, pp. 145-149.

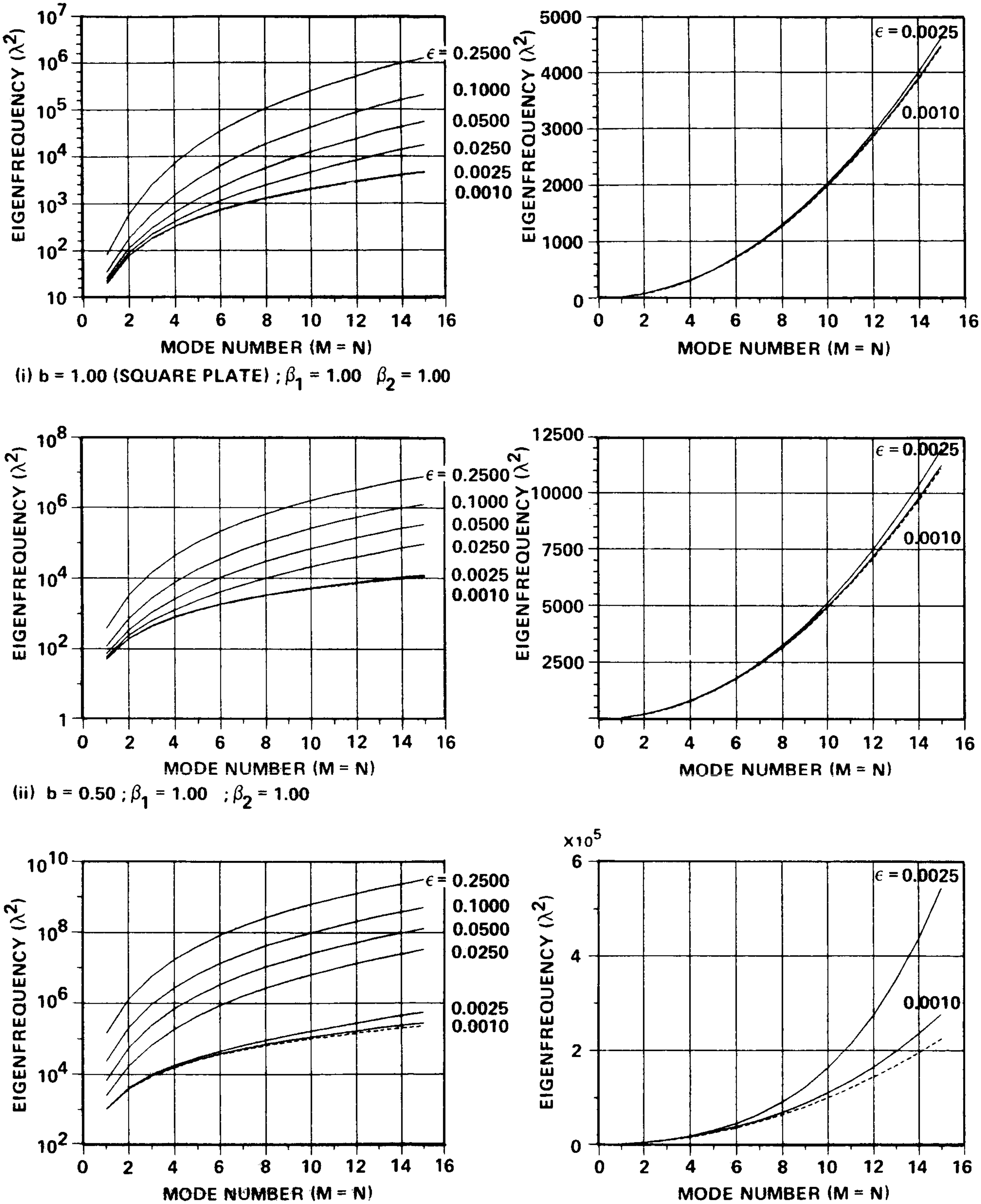
4. Schneider, W.: Einfluß Einer Kleinen Biegesteifigkeit auf die Querschwingungen Einer Eingespannten Rechteckigen Membran. Acta Mechanica, Vol. 13, 1972, pp. 293-302.

5. Hutter, K., Olunloyo, V. O. S.: Vibration of an Anisotropically Prestressed Thick Rectangular Membrane with Small Bending Rigidity. Acta Mechanica, Vol. 20, 1974, pp. 1-22.

6. Oyediran, A. A. and Gbadeyan, J. A.: Vibration of a Prestressed Orthotropic Rectangular Thin Plate Via Singular Perturbation Technique. Acta Mechanica, accepted for publication, 1986.

7. Nayfeh, A. H. and Kamat, M. P.: A Numerical Perturbation Technique for the Transverse Vibrations of Highly Prestressed Plates. J. Acoust. Soc. Am., Vol. 61(1), 1977, pp. 95-100.

8. Ramkumar, R. L., Kamat, M. P., and Nayfeh, A. H.: Vibrations of Highly Prestressed Anisotropic Plates via a Numerical-Perturbation Technique. Int. J. Solids Structures, Vol. 13, 1977, pp. 1037-1044.

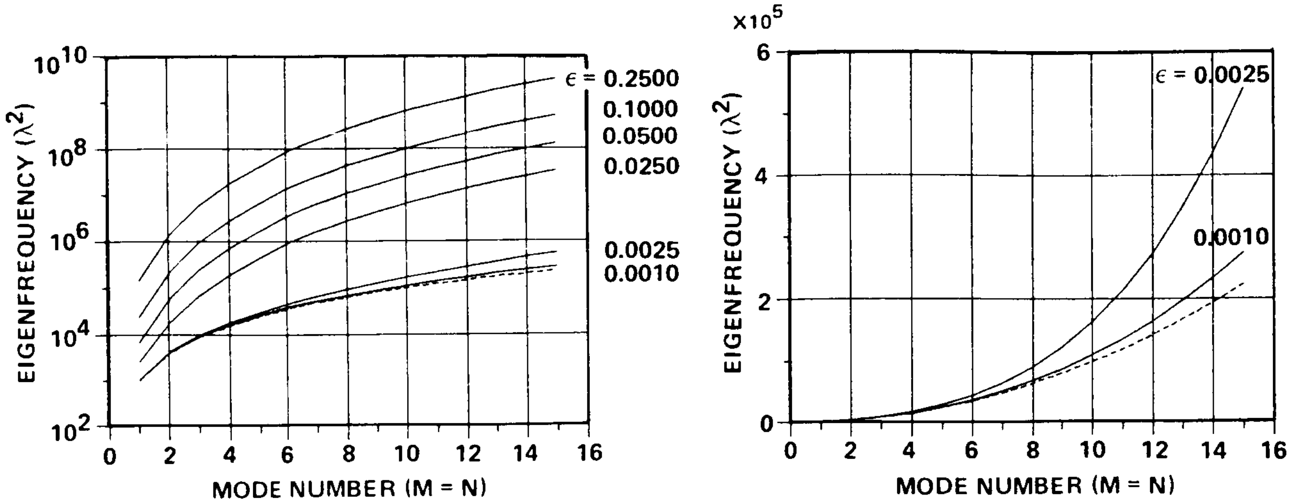


(i) b = 1.00 (SQUARE PLATE) ; β_1 = 1.00 β_2 = 1.00

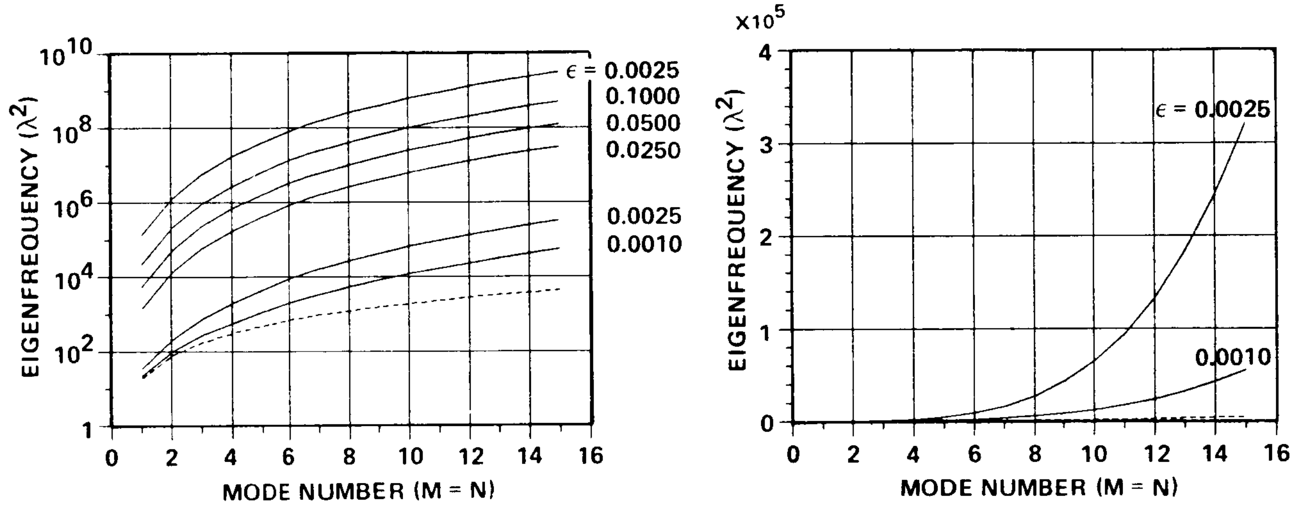
(ii) b = 0.50 ; β_1 = 1.00 ; β_2 = 1.00

(iii) b = 0.10 ; β_1 = 1.00 ; β_2 = 1.00

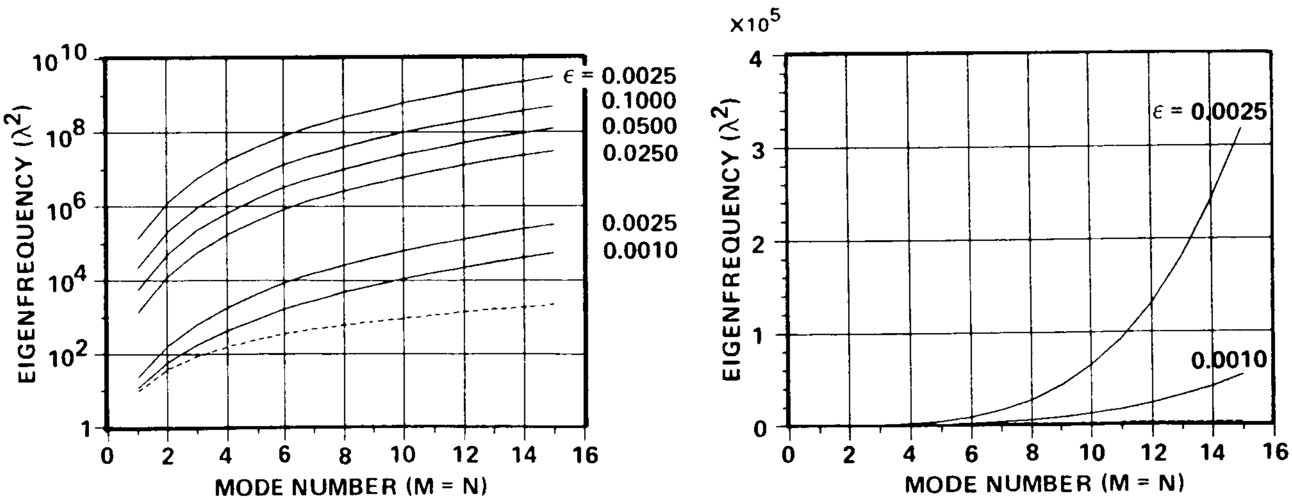
FIGURE 3.1(a) SECOND ORDER EIGEN FREQUENCIES SHOWING THE EFFECT OF b. ZEROTH ORDER VALUES ARE SHOWN DOTTED



(i) b = 0.10 ; β_1 = 0.10 ; β_2 = 1.00

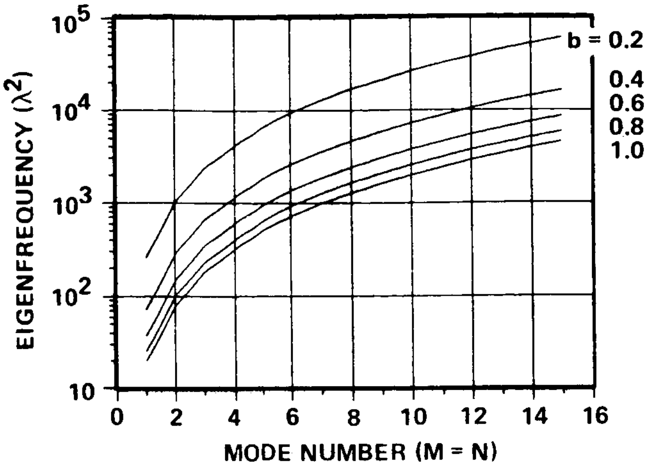


(ii) b = 0.10 ; β_1 = 1.00 ; β_2 = 0.10

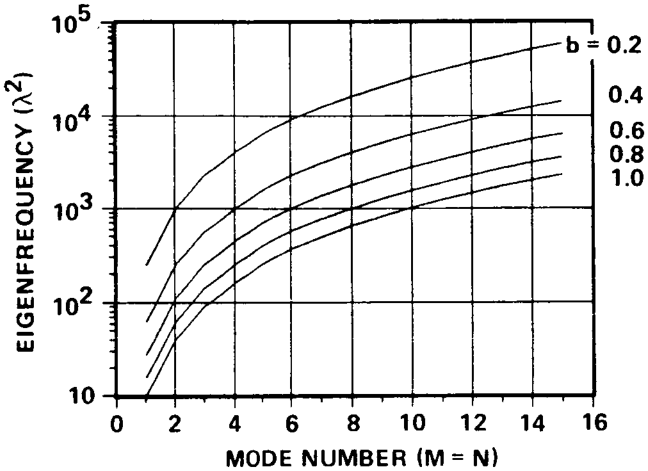


(iii) b = 0.10 ; β_1 = 0.10 ; β_2 = 0.10

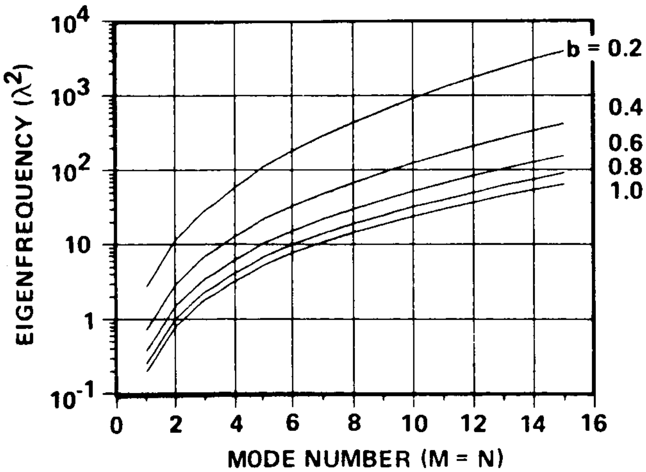
FIGURE 3.1(b) SECOND ORDER EIGEN FREQUENCIES SHOWING THE EFFECT OF β_1 AND β_2 ZEROTH ORDER VALUES ARE SHOWN DOTTED.



(i) $\beta_1 = 1.00$; $\beta_2 = 1.00$

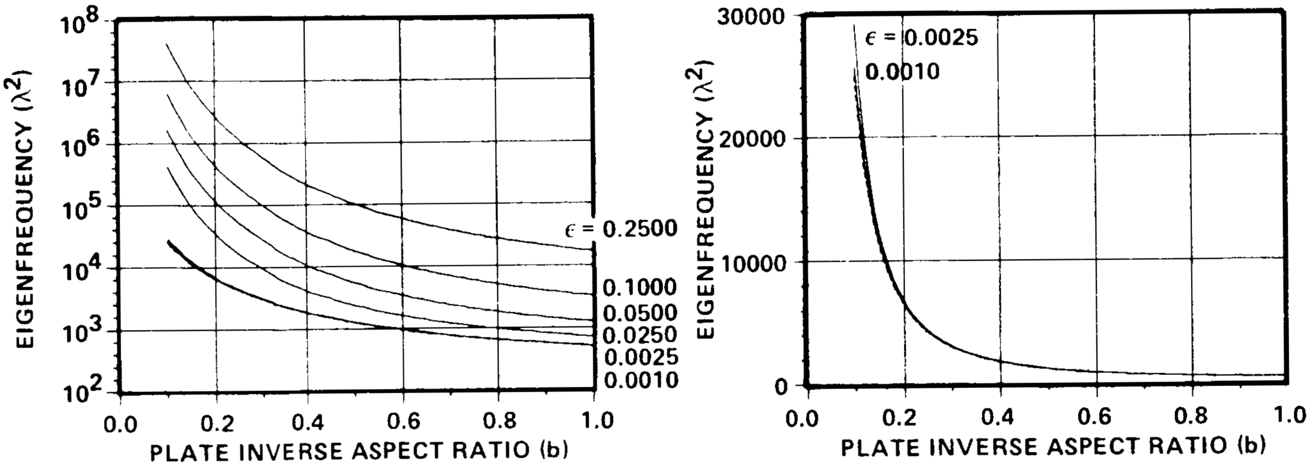


(ii) $\beta_1 = 0.10$; $\beta_2 = 1.00$

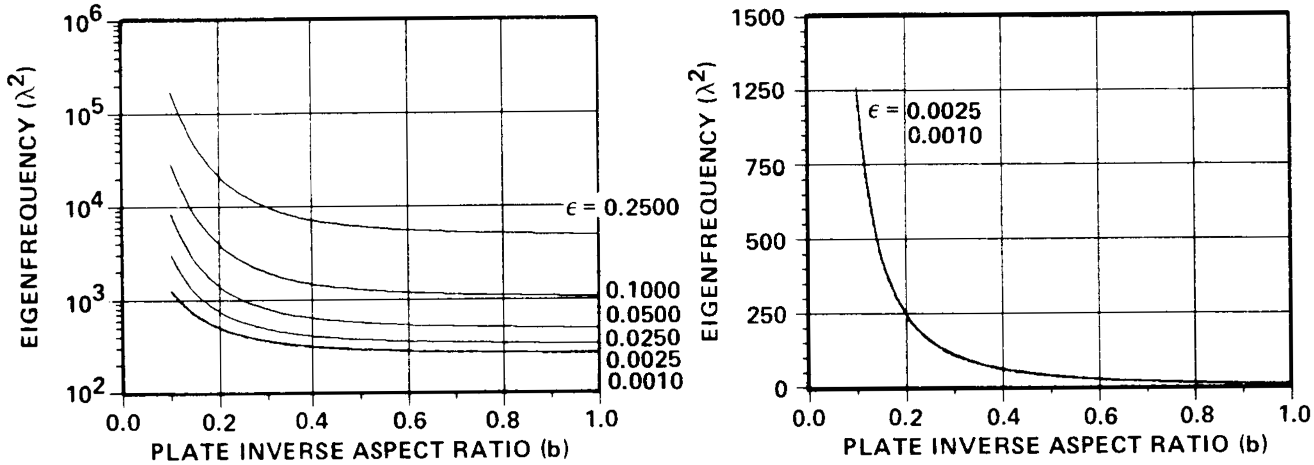


(iii) $\beta_1 = 0.10$; $\beta_2 = 0.10$

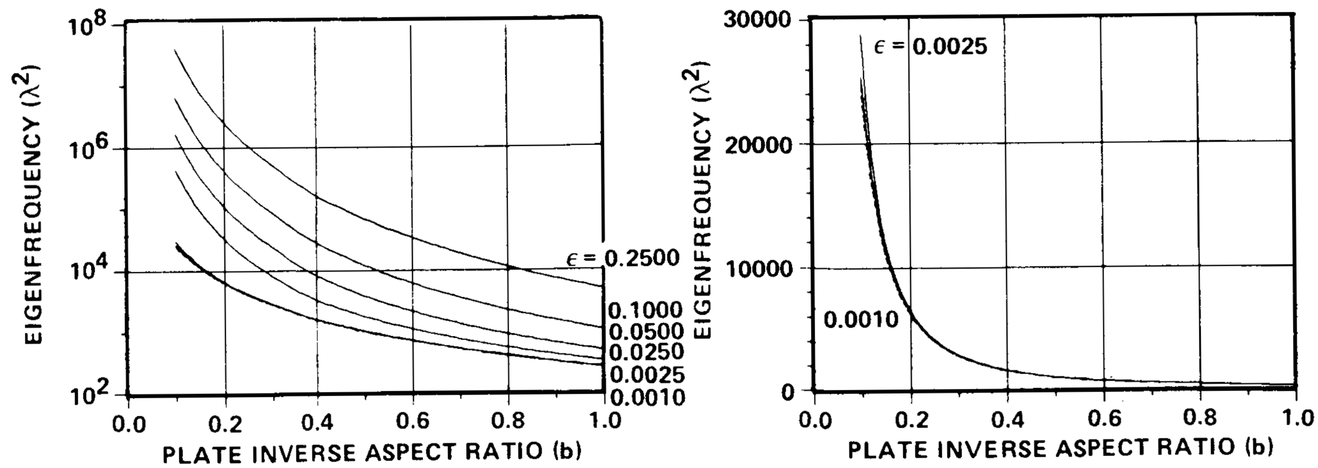
FIGURE 3.2 SECOND ORDER VALUES ($\epsilon = 0.0010$) OF THE EIGEN FREQUENCY FOR COMBINATION OF VALUES OF β_1 AND β_2



(i) M = 1 ; N = 1 ; β_1 = 1.00 ; β_2 = 1.00

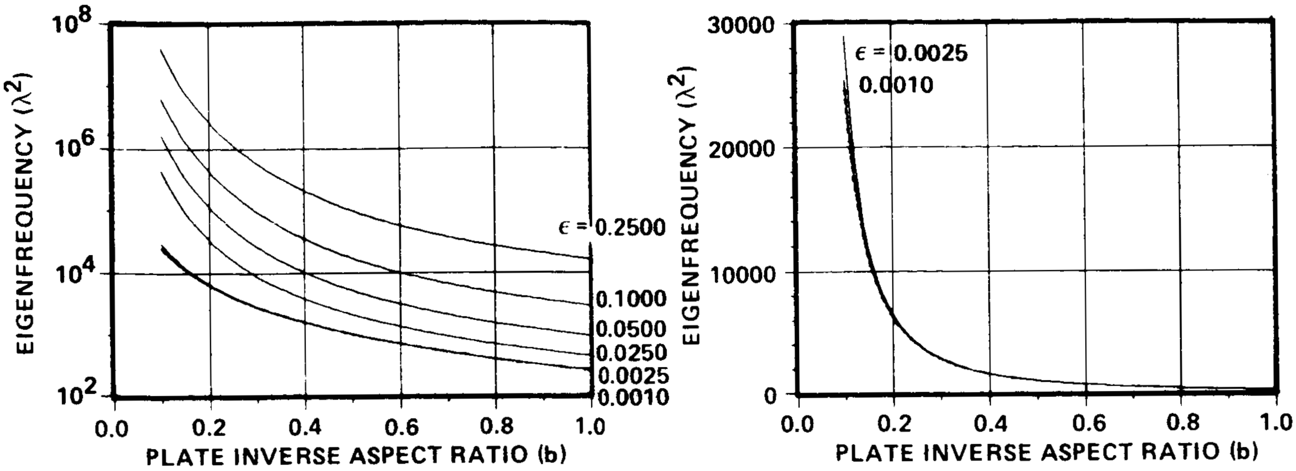


(ii) M = 1 ; N = 5 ; β_1 = 1.00 ; β_2 = 1.00

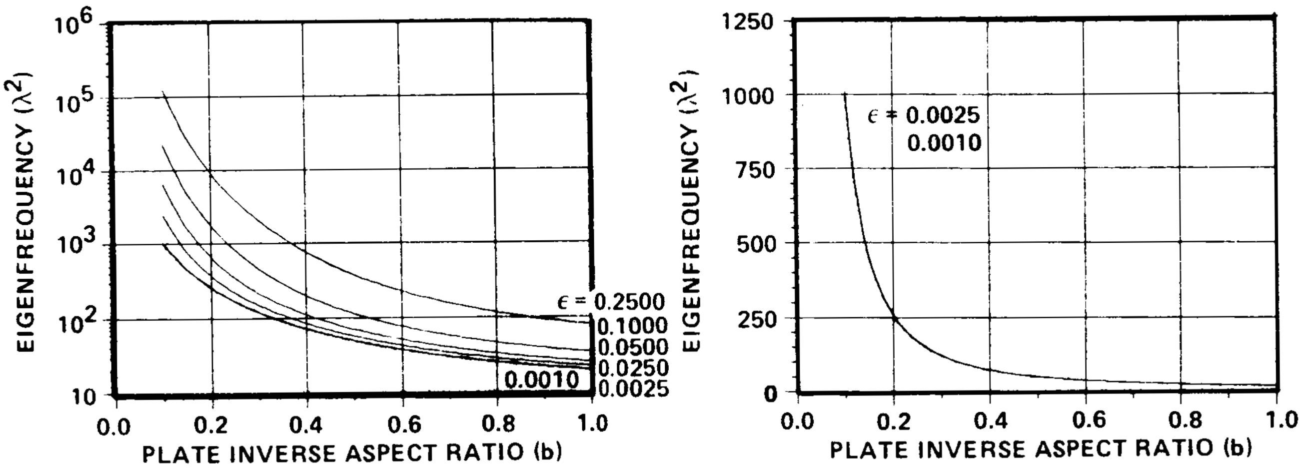


(iii) M = 5 ; N = 1 ; β_1 = 1.00 ; β_2 = 1.00

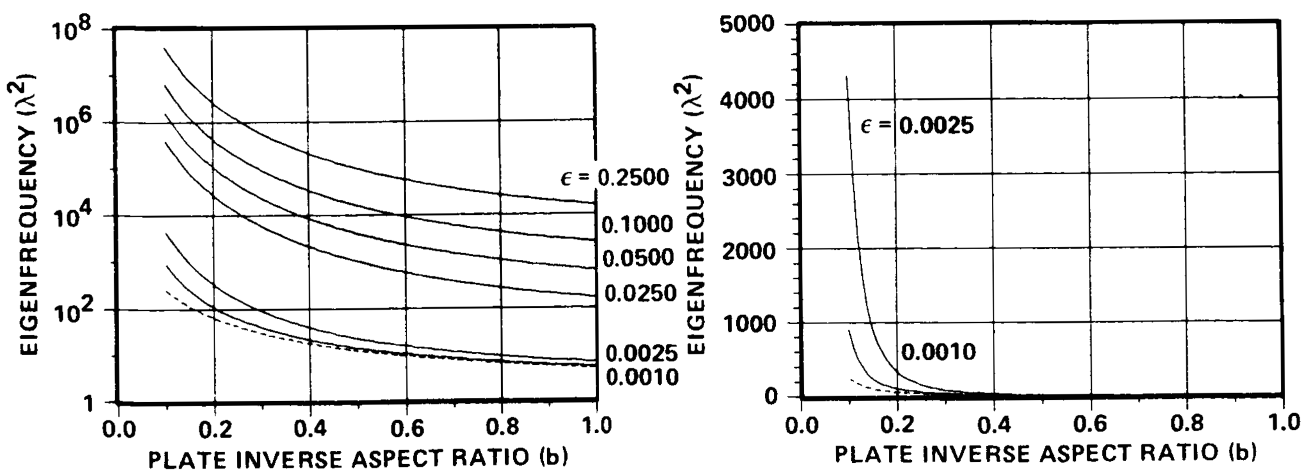
FIGURE 3.3(a) SECOND ORDER EIGEN FREQUENCIES SHOWING THE EFFECT OF M AND N ZEROTH ORDER VALUES ARE SHOWN DOTTED



(i) M = 5 ; N = 5 ; β_1 = 1.00 ; β_2 = 1.00

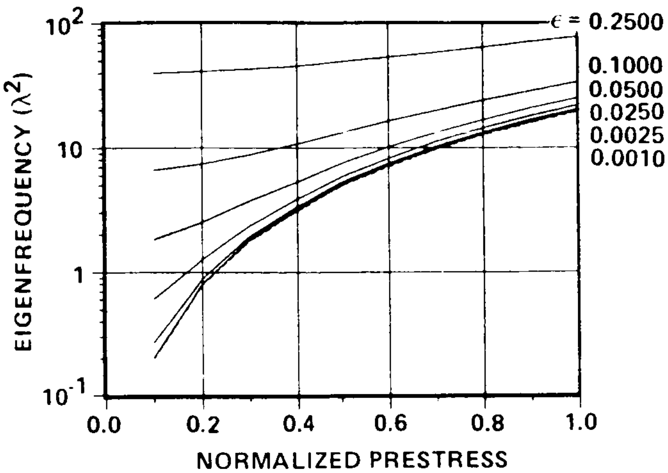


(ii) M = 5 ; N = 5 ; β_1 = 0.10 β_2 = 1.00

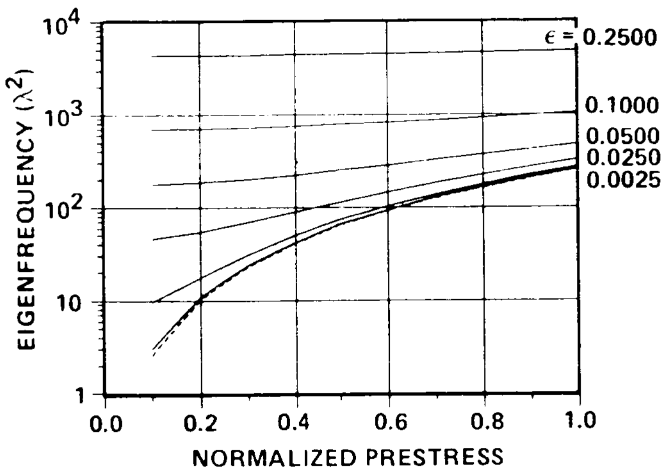


(iii) M = 5 ; N = 5 ; β_1 = 0.10 β_2 = 0.10

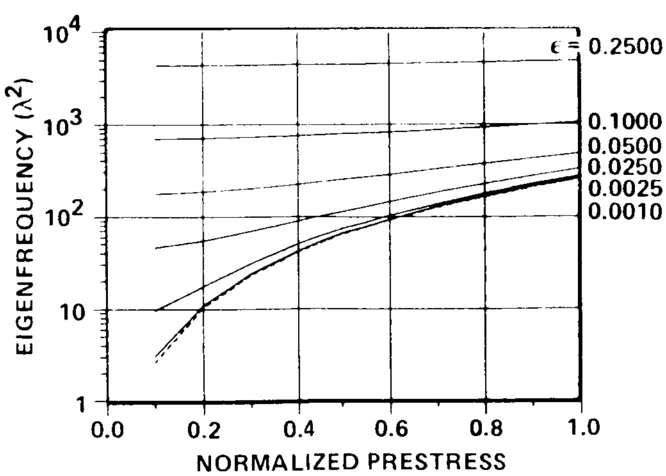
FIGURE 3.3(b) SECOND ORDER EIGEN FREQUENCIES SHOWING THE EFFECT OF β_1 AND β_2
ZEROTH ORDER VALUES ARE SHOWN DOTTED



(i) M = 1 ; N = 1 ; b = 1.00

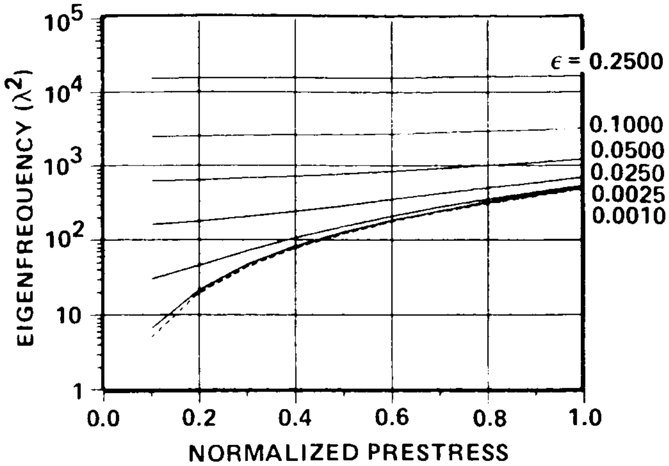


(ii) M = 1 ; N = 5 ; b = 1.00

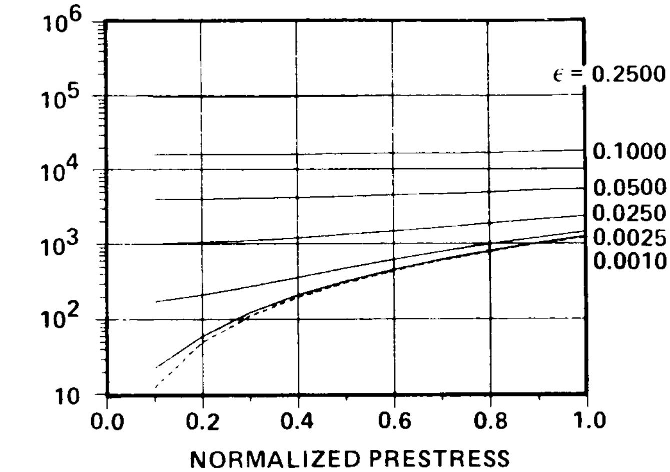


(iii) M = 5 ; N = 1 ; b = 1.00

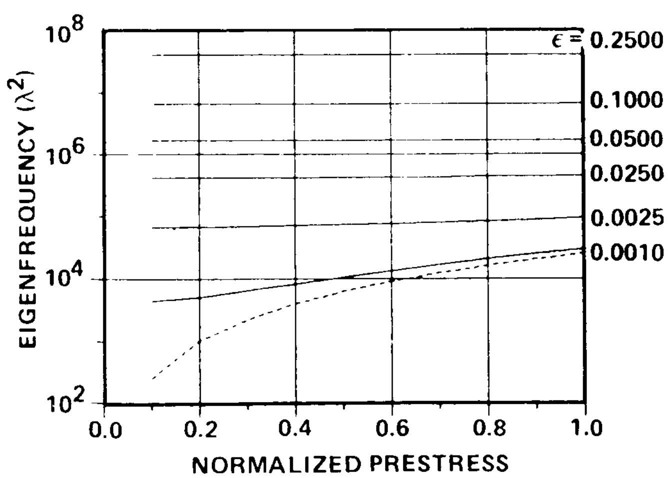
FIGURE 3.4(a) SECOND ORDER EIGEN FREQUENCIES SHOWING THE EFFECT OF M AND N (ZEROTH ORDER VALUES ARE SHOWN DOTTED)



(i) M = 5 ; N = 5 ; b = 1.00



(ii) M = 5 ; N = 5 ; b = 0.50



(iii) M = 5 ; N = 5 ; b = 0.10

FIGURE 3.4(b) SECOND ORDER EIGEN FREQUENCIES SHOWING THE EFFECT OF b. ZEROTH ORDER VALUES ARE SHOWN DOTTED

APPROVAL

THE EFFECT OF A SMALL INITIAL CURVATURE ON THE FREE VIBRATION OF CLAMPED, RECTANGULAR PLATES

By A. A. Adeniji-Fashola and A. A. Oyediran

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

G. F. McDONOUGH
Director, Systems Dynamics Laboratory

☆U.S. GOVERNMENT PRINTING OFFICE 1986—730-067/40036