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THE ALGEBRAIC CRITERIA FOR THE STABILITY OF CONTROL SYSTEMS

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The present paper critically examines the standard algebraic criteria for the stability of linear control systems and their proofs, reveals important, previously unnoticed connections, and presents new representations. Algebraic stability criteria have also recently acquired significance for stability studies of nonlinear differential equation systems by the method of Krylov-Bogoljubov-Magnus, and allow realizability conditions to be determined for classes of broken rational functions as frequency characteristics of electrical networks.

From the easily obtained connection between the solutions of E.J. Routh and J. Schur for the stability problem, a characterization is derived for a class of polynomials with roots lying symmetric to the origin, which include the frequency formula of L. Cremer as a special case, possess significance for the construction of root and phase angle loci of control systems, and explain an error of K. Th. Vahlen (§ 1).

Then a parametric representation of the coefficients of real Hurwitz polynomials is presented which can be interpreted as a solution of a nonlinear system of equations. By setting suitable parameters equal to zero, this yields coefficient representations of the Hurwitz polynomials reduced by the method of J. Schur and H. Bückner, as well as a characteristic determinant notation for the first polynomials mentioned. These results make it possible to give the stability boundary hypersurface and quadratic control surface for linear control systems of any order, as a function of Routhian sample functions (§ 2).

The algebraic criteria for certain root distributions in

*Numbers in the margin indicate pagination in the foreign text.

areas of the complex plane, given by Ch. Hermite, J. Schur, A. Cohn and M. Fujiwara, are summarized, a general formation method for these conditions is derived, and it is shown that assuming real coefficients the Hermitian stability criteria lead to Hurwitz determinants that are already separated into main and secondary sequences. This yields the possibility of calculating the two sequences of determinants independently. This entails an advantage with positive coefficients, for a known result /329 shows that assuming positive coefficients, the positive nature of the determinant of one of these sequences is already characteristic of stability (§ 3).

A previously surprisingly unnoticed possibility is revealed for effortlessly deriving criteria for a certain zero distribution of polynomials with complex coefficients on the basis of the paper of A. Hurwitz published in this periodical. Thus, for example, K. Th. Vahlen considers his results to have a special advantage over those of A. Hurwitz in that they allow not only a stability study, but also a root enumeration, and J. Schur emphasizes that his criteria, unlike that of A. Hurwitz, are not limited to real coefficients. In this connection one also gets a considerable shortening of Hurwitz's proof (§ 4).

It is demonstrated that these generalized Hurwitz conditions, which are also valid for complex coefficients, are equivalent to the criteria derived by H. Bilharz from results of J. Schur. (§ 5).

Then the connection between the generalized Hurwitz criterion and Hermitian conditions is clarified. In two papers, M. Fujiwara proved the correspondence between the conditions of Hermite and Hurwitz for real coefficients, using the criterion of Liénard-Chipart as an intermediate term. This demonstration is performed directly here and extended to complex coefficients (§ 6).

The connection between the criteria of Ch. Hermite and H. Bilharz is then demonstrated directly without interposing the generalized criterion of A. Hurwitz (§ 7).

Finally, it is shown that from the vanishing of certain Hurwitz determinants one can directly conclude the number of zeros of a polynomial lying symmetric to the origin, and a coupling law for the vanishing of the Hurwitz determinants of the main and secondary sequence is presented (§ 8).

§ 1

It is known that systems of homogeneous linear differential equations

$$\sum_{k=1}^{n} L_{jk}(D)x_k(t) = 0 \qquad (j = 1, 2, \ldots, n), \tag{1}$$

where

$$L_{jk}(D) = \sum_{l=0}^{m_{jk}} d_{jkl} D^l,$$

$$D^l = \frac{d^l \ldots}{dt^l \ldots}$$

mean linear differential operators, can be solved by the formulation

$$x_k(t) = A_k e^{zt} \qquad (k = 1, 2, \ldots, n) \tag{2}$$

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This formulation leads to a system of homogeneous linear equations for the amplitudes A_k

$$\sum_{k=1}^{n} L_{jk}(z) A_k = 0 \qquad (j = 1, 2, \ldots, n),$$

which has non-trivial solutions if and only if the determinant

$$\Delta(z) = \begin{vmatrix} L_{11}(z) & L_{12}(z) & \ldots & L_{1n}(z) \\ L_{21}(z) & L_{22}(z) & \ldots & L_{2n}(z) \\ \vdots & \vdots & & \vdots \\ L_{n1}(z) & L_{n2}(z) & \ldots & L_{nn}(z) \end{vmatrix}$$

vanishes. $\Delta(z) = 0$ is called the "characteristic equation of the system." If $z = -\delta_\nu + i\omega_\nu$ is a root, then there exists a solution of the system in the form

$$x_k(t) = A_k e^{\lambda_\nu t} = A_k e^{-\delta_\nu t} e^{i\omega_\nu t},$$

i.e., with real coefficients one gets real solutions in the form

$$C_k e^{-\delta_\nu t} \cos(\omega_\nu t + \vartheta_\nu),$$

where C_k, δ_ν, ω_ν, θ_ν are real constants.

We call the system stable if all roots of the characteristic equation are in the left half-plane, i.e. if each solution of the system tends to 0 as t increases. If the differential equations describe a system capable of vibration this means that each free vibration of the system dies out.

Below, "stable" is always understood in this stricter sense. Solutions that do not tend to 0 as t increases are therefore not counted as stable -- in agreement with most authors, but in contrast to some who demand only boundedness of the solutions.

By generalizing the so-called harmonic balance method of N. Krylov and N. Bogoljubov [25], applied to a nonlinear differential equation system with the form

$$\dot{x}_j = \sum_{k=1}^{n} a_{jk} x_k + \varepsilon_j f_j(x_1, x_2, \ldots, x_n) \qquad (j = 1, 2, \ldots, n), \tag{3}$$

K. Magnus [27] arrives at a linear approximation system in form (1) with

$$L_{jk}(D) = \alpha_{jk} + (\alpha^*_{jk} - \delta_{jk}) D,$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases}$$

The preconditions given by K. Magnus [27] lead to coefficients α_{jk} and α^*_{jk} that can be represented as integral transforms of the nonlinear functions $f_j\ (x_1, x_2, \ldots, x_n)$. /331

Thus for example in the common case

$$f_j(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} f_{jk}(x_k):$$

one gets

$$\alpha_{jk} = a_{jk} + \frac{\varepsilon_j}{\pi A_k} \int_0^{2\pi} f_{jk}(A_k \sin \Phi_k) \sin \Phi_k \, d\Phi_k$$

$$\alpha^*_{jk} = \frac{\varepsilon_j}{\omega \pi A_k} \int_0^{2\pi} f_{jk}(A_k \sin \Phi_k) \cos \Phi_k \, d\Phi_k$$

with $\Phi_k = \omega t + \varphi_k$.

Here formulation (2) leads to a characteristic equation whose coefficients are still functions of a reference amplitude A [27, 17].

An "algebraic stability criterion" is a system of rules that makes it possible, from the given real or complex coefficients a_n and using only the four basic modes of computation, to determine whether all zeros (or how many zeros) of the polynomial

$$(4) \qquad f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu, \quad a_0 > 0,$$

lie in the left half-plane.

Since the real part of $1/z$ has the same sign as the real part of z, a stability criterion for the polynomial (4) also applies unchanged for the polynomial

$$(5) \qquad F(z) = z^n f\left(\frac{1}{z}\right) = \sum_{\nu=0}^{n} a_\nu z^{n-\nu}.$$

We will repeatedly employ this fact tacitly in the proofs below.

The problem of setting up algebraic stability criteria was solved in principle by Cauchy as early as 1837. He reduced the determination of the number of roots within a given area to the determination of the "index" of a rational function. For the case where the area is a half-plane, this index can be determined especially easily by the "Sturm method," published shortly before (1829).

This solved the problem in principle. Of course, Cauchy set up no explicit criterion.

Ch. Hermite [23] in 1856 reduced the question to whether certain assigned forms (the famous "Hermitian forms") are positively definite or not.

Not until twenty years later was E.J. Routh [32, 33] led to the question by a technical problem, and in 1877 in the "Adams Price [sic] Essay" gave a very elegant and practically handy solution for the problem when assuming real coefficients. /332
According to Routh's criterion, the coefficients of polynomial (5) are arranged in two rows as follows:

$$\begin{array}{lllll} a_0 & a_2 & a_4 & a_6 & \cdots \\ a_1 & a_3 & a_5 & a_7 & \cdots \end{array}$$

and by cross-multiplication the new row is formed:

$$\frac{a_1 a_2 - a_0 a_3}{a_1} \quad \frac{a_1 a_4 - a_0 a_5}{a_1} \quad \frac{a_1 a_6 - a_0 a_7}{a_1} \quad \cdots$$

and likewise each additional row is formed by a corresponding cross-multiplication of the elements of the two previous rows. If all zeros of polynomial (4) lie in the left half-plane, the process can only be continued to the $(n + 1)^{th}$ row, which then contains only one element. The coefficient functions of the first column of this "Routh scheme" are called "Routhian sample functions." If they are all positive for the normalization $a_0 > 0$, then all zeros of polynomial (4) have negative real parts, and vice versa.

Again almost twenty years later, A. Hurwitz [24] was led to the problem by an inquiry from the turbine builder Stodola. Hurwitz did not know Routh's work, but knew that of Cauchy, Sturm and Hermite. Shortly before, Frobenius had published his paper on the "Law of inertia of quadratic forms" in the Proceedings of the Royal Prussian Academy. Hurwitz based his proof on this paper, and using the theory of quadratic forms and functional theoretical means with a relatively elaborate mathematical effort, which must be understood historically, he arrived at his criterion:

The characteristic for whether polynomial (4) with real coefficients has only zeros with negative real parts is that the determinants H_1, H_2, ..., H_n are all positive, with

$$H_\beta = \begin{vmatrix} a_1 & a_3 & a_5 & \dots a_{2\beta-1} \\ a_0 & a_2 & a_4 & \dots a_{2\beta-2} \\ 0 & a_1 & a_3 & \dots a_{2\beta-3} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{2-\beta} & a_{4-\beta} & a_{6-\beta} & \dots a_\beta \end{vmatrix} \tag{6}$$

and $a_k = 0$, if $k > n$ or $k < 0$. Thus H_n is a determinant of the n^{th} degree, whose general element is given by

$$a_{\mu\nu} = a_{2\nu-\mu} \qquad (\mu, \nu = 1, 2, \dots, n)$$

The determinants (6) are the principal minors of H_n.

This Hurwitz criterion is attractively elegant and became famous, although it contains nothing materially new beyond Routh's scheme. Polynomials whose zeros all lie in the left half-plane are called "Hurwitz polynomials." Accordingly, equations $f(z) = 0$ in which $f(z)$ stands for a Hurwitz polynomial are called "Hurwitz equations." /333

As H. Bilharz [3] first showed, the connection between the Routhian sample functions and the Hurwitz determinants is very simple. If R is the Routhian sample function of the ν^{th} row, then

$$R_\nu = \frac{H_\nu}{H_{\nu-1}} \qquad H_0 = 1, \; H_{-1} = a_0^{-1} \; (\nu = 0, 1, \dots, n). \tag{7}$$

The complete formation of the Routhian scheme is thus linked to the prerequisite $H_{-1} \neq 0$. With Hurwitz polynomials this is satisfied. In fact all elements of the Routhian scheme can be represented as quotients of minors of the highest Hurwitz determinant [6].

According to J. Schur [36], the following proposition applies:

The polynomial (4) is a Hurwitz polynomial of degree n if and only if $a_1 > 0$ and the equation of degree $n - 1$

$$K(z) = z^{-2}[(a_0 z + a_0 \zeta)\, h(z) - a_1 \zeta\, z\, g(z)] = 0 \tag{8}$$

with

$$g(z) = \frac{1}{2}[f(z) + f(-z)]$$

$$h(z) = \frac{1}{2}[f(z) - f(-z)]$$

for any ζ with a negative real part represents a Hurwitz equation.

If one proceeds from the polynomial

$$f_\beta(z) = \sum_{\nu=0}^{n-\beta} a_{\nu,n-\beta} z^\nu \quad (\beta = 0, 1, \ldots, n - 1)$$

and specializes in (8) the parameter as
then assuming real coefficients, the above proposition can be stated as follows:

For $a_{0,n-\beta} > 0$, $_{\beta}(z)$ is a Hurwitz polynomial of the degree $(n - \beta)$ if and only if $a_{1,n-\beta} > 0$ and the polynomial of the degree $n - (\beta + 1)$

$$f_{\beta+1}(z) = \frac{f_\beta(z)}{z} - \frac{a_{0,n-\beta}}{a_{1,n-\beta}} \cdot \frac{h_\beta(z)}{z^2} = \sum_{\nu=0}^{n-(\beta+1)} a_{\nu,n-(\beta+1)} z^\nu \tag{9}$$

with

$$h_\beta(z) = \frac{1}{2}\left[f_\beta(z) - f_\mu(-z)\right]$$

represents a Hurwitz polynomial.

If the coefficients of the Hurwitz polynomial $f(z) = f_0(z)$ and of the reduced polynomials $f_\mu(z)$ resulting from algorithm (9) ($\mu = 1, 2, \ldots, n$) are written in rows under each other, one arrives at the Routhian scheme from this "Schur scheme" simply by striking out every second term in each row, i.e., by striking out from each row $a_{\nu,n-\mu}$ ($\nu = 0, 1, \ldots, n - \mu$) the elements $a_{\nu,n-\mu}$ with an uneven index ν. Conversely, one gets the μ^{th} row of the Schur scheme by combining the μ^{th} and the $(\mu + 1)^{th}$ row of the Routhian scheme. Here the μ^{th} row yields the terms $a_{\nu,n-\mu}$ with an even index ν, and the $(\mu + 1)^{th}$ row yields those with an uneven index ν.

Thus the Routhian scheme stands in a simple relation to the Hurwitz polynomials reduced after J. Schur. /334

If we multiply the μ^{th} row of the Routhian scheme by H_μ ($\mu = 2, \ldots, n$), according to [6] the following representation applies to this row:

$$H^{1,2,\ldots,\mu-1,\mu}_{1,2,\ldots,\mu-1,\mu},\ H^{1,2,\ldots,\mu-1,\mu}_{1,2,\ldots,\mu-1,\mu+1},\ H^{1,2,\ldots,\mu-1,\mu}_{1,2,\ldots,\mu-1,\mu+2},\ \ldots$$

Here

$$H^{m_1, m_2, \ldots, m_k}_{n_1, n_2, \ldots, n_k}$$

means that minor of the k^{th} degree of the highest Hurwitz determinant H_n which contains the rows $m_1, m_2, \ldots, m_k$ and the columns $n_1, n_2, \ldots, n_k$.

This notation now allows us also to present directly the μ^{th} reduced Hurwitz polynomial $f_\mu(z)$ of the degree $(n - \mu)$:

$$f_1(z) = \sum_{\nu=0}^{\left[\frac{n-1}{2}\right]} a_{2\nu+1} z^{2\nu} + \sum_{\nu=0}^{\left[\frac{n-2}{2}\right]} H_{1,2+\nu}^{1,2} z^{2\nu+1}$$

$$f_\mu(z) = \frac{1}{H_{\mu-1}} \sum_{\nu=0}^{\left[\frac{n-\mu}{2}\right]} H_{1,2,\ldots,\mu-1,\mu+\nu}^{1,2,\ldots,\mu-1,\mu} z^{2\nu} + \tag{10}$$

$$+ \frac{1}{H_\mu} \sum_{\nu=0}^{\left[\frac{n-\mu-1}{2}\right]} H_{1,2,\ldots,\mu,\mu+1+\nu}^{1,2,\ldots,\mu,\mu+1} z^{2\nu+1}$$

$$(\mu = 2, 3, \ldots, n).$$

Here $n \geq 2$, and a sum with a negative upper summation limit must be omitted. The symbol $[a]$, as usual, means the largest integer m with $m \leq a$.

If s zeros $z_1, z_2, \ldots, z_4$, whose position is characterized by the fact that they either vanish or can be combined in mutually exclusive pairs whose elements differ only in their sign, are designated as s zeros lying symmetric to the origin, then from the representation (10) we easily get the following proposition:

"The polynomial (5) with real coefficients has s ($1 \leq s \leq n - 1$) zeros $z_1, z_2, \ldots, z_s$ lying symmetric to the origin, as well as $r = n - s$ zeros with a negative real part, if and only if the Hurwitz determinants H_β ($\beta = 1, 2, \ldots, r$) are positive and the determinants

$$H_{1,2,\ldots,n-s,n-s+1+t}^{1,2,\ldots,n-s,n-s+1} \qquad \left(t = 0, 1, \ldots, \left[\frac{s-1}{2}\right]\right) \tag{11}$$

all vanish. The s symmetric zeros z_i solve the equation

$$\sum_{t=0}^{\left[\frac{s}{2}\right]} H_{1,2,\ldots,n-s-1,n-s+t}^{1,2,\ldots,n-s-1,n-s} z^{s-2t} = 0 \tag{12a}$$

for $s \leq n - 2$, or

$$\sum_{t=0}^{\left[\frac{n-1}{2}\right]} a_{2t+1} z^{n-1-2t} = 0 \tag{12b}$$

for $s = n - 1$." /335

This proposition includes the frequency formula of L. Cremer [7] as a special case, and can also be used to construct root and phase angle loci of control systems [13].

To prove the theorem, we write the polynomial $F(z) = F_0(z)$ in the form $F_0(z) = \Phi_0(z)\ \Psi_0(z)$, where $\Phi_0(z)$ possesses only the s zeros lying symmetric to the origin, and $\Psi_0(z)$ is a Hurwitz polynomial. If one applies algorithm (13):

$$(13) \qquad F_{\beta+1}(z) = F_\beta(z) - \frac{a_{0,n-\beta}}{2a_{1,n-\beta}}\, z\,[F_\beta(z) - (-1)^{n-\beta} F_\beta(-z)]$$

obtained from (9) by

$$z^{n-k} f_k\left(\frac{1}{z}\right) = F_k(z) = a_{0,n-k}\, z^{n-k} + \cdots + a_{n-k,n-k} \qquad (k = \beta, \beta+1)\,,$$

to both $\Psi_0(z)$ and $F_0(z)$, then between the reduced polynomials $F_i(z)$ and $\Psi_i(z)$ there exists the relationship $F_i(z) = \Phi_0(z)\ \Psi_i(z)$ $(i = 1, 2, \ldots, r)$. It follows that the $(r + 1)$ first Routhian sample functions for $F_0(z)$ are positive and thus so are the Hurwitz determinants H_β $(\beta = 1, 2, \ldots, r)$. The representation for $F_r(z)$ to be obtained from (10) shows the vanishing of the determinants (11) and the validity of (12a) or (12b), since $F_r(z)$ only contains the s symmetric zeros. If, conversely, the determinants (11) vanish for $F(z)$, then the equation (12a) or (12b), which we now write in the form $F_r(z) = 0$, describes the r^{th} $(r = n - s)$ reduced equation of $F(z)$, which obviously has exactly s symmetric roots. If we apply the converse of algorithm (13)

$$(14) \qquad F_{\beta-1}(z) = F_\beta(z) + \frac{a_{0,n-\beta+1}}{2a_{1,n-\beta+1}}\, z\,[F_\beta(z) + (-1)^{n-\beta} F_\beta(-z)]$$

to $F_\beta(z) = F_r(z)\ M_\beta(z)$ with $a_{0,n-(\beta-1)} > 0$ and $a_{1,n-(\beta-1)} = a_{0,n-\beta}$, with $M_\beta(z)$ being a polynomial of the $(r - \beta)^{th}$ degree and $M_r(z) = 1$, then $F_{\beta-1}(z) = F_r(z)\ M_{\beta-1}(z)$ $(\beta = r, r - 1, \ldots, 1)$. Here

$$M_{\beta-1}(z) = \sum_{\nu=0}^{r-(\beta-1)} b_{\nu,r-(\beta-1)}\, z^{r-(\beta-1)-\nu}$$

is the Hurwitz polynomial of the $(r = \beta + 1)^{th}$ degree, which is obtained from $M_\beta(z)$ by applying algorithm (14) after selecting $b_{0,r-(\beta-1)} > 0$ with $b_{1,r-(\beta-1)} = b_{0,r-\beta}$. It follows from this that the Routhian sample functions for $F_j(z)$ $(j = r - 1, r - 2, \ldots, 0)$ are all positive. The same then also applies for the corresponding Hurwitz determinants H_β $(\beta = r, r - 1, \ldots, 1)$.

The theorem proved above contradicts a result of K. Th. Vahlen [38], which can easily be demonstrated as an oversight. If we use the terms of the present paper, he erroneously concludes that a polynomial reduced according to (13) has only purely imaginary zeros when the coefficients of the even or uneven powers vanish. Moreover, he overlooks that the expansion into continued fractions introduced there includes a third possibility beyond his defined regular and irregular cases. This extra case arises if in a step of his expansion into continued fractions, the degree of the resulting function decreases by more than 1, but the function does not vanish identically. Hence /336 the work of Vahlen supplies the claimed root enumeration in a strip parallel to the imaginary axis only if the aforementioned third case does not occur in the expansion into continued fractions.

§ 2

The following proposition answers the question of the ranges of variability of the coefficients of real Hurwitz polynomials. It reads:

"The polynomial (5) with real coefficients and the normalization $a_0 > 0$ is a Hurwitz polynomial if and only if the following representation applies for the coefficients:

$$a_0 = R_0$$
$$a_1 = R_1 \tag{15}$$
$$a_k = R_s \sum_{m_1 = s+2}^{n-k+s+2} \frac{R_{m_1}}{R_{m_1-1}} \sum_{m_2 = m_1+2}^{n-k+s+4} \frac{R_{m_2}}{R_{m_2-1}} \cdots \sum_{m_l = m_{l-1}+2}^{n} \frac{R_{m_l}}{R_{m_l-1}}$$

with

$$2s = 1 - (-1)^k \ (k = 2, 3, \ldots, n); \ m_0 = s$$

and

(16) $$R_\nu > 0 \qquad (\nu = 0, 1, \ldots, n).$$

"The parameters R_ν prove to be the Routhian sample functions of the polynomial $F(z)$."

As proof, consider that the application of algorithm (14) -- after selecting $a_{0,n-(\beta-1)} = R_{\beta-1} > 0$ -- to the Hurwitz polynomial $F_\beta(z)$ of the $(n - \beta)^{th}$ degree with the Routhian sample function $a_{0,n-\beta} = R_\beta$ yields a Hurwitz polynomial $F_{\beta-1}(z)$ of the degree $n - \beta + 1$. Furthermore, $a_{1,n-(\beta-1)} = a_{0,n-\beta}$. By applying (14) $(n - 2)$ times with any $a_{0,n-(\beta-1)} = R_{\beta-1} > 0$ $(\beta = n - 2, n - 1, \ldots, 1)$ to

$$F_{n-2}(z) = R_{n-2}z^2 + R_{n-1}z + R_n,$$

which is a Hurwitz polynomial with $R_\nu > 0$ $(\nu = n - 2, n - 1, n)$, the coefficient representation (15) of the Hurwitz polynomial $F_0(z) = F(z)$ follows after complete induction. Conversely, if a polynomial $F_0(z)$ is given for which (15) and (16) apply, it follows from the converse (13) of (14) that $F_0(z)$ is a Hurwitz polynomial.

Representation (15) can be interpreted as the solving of the nonlinear equation system $R_\nu = \phi_\nu(a_0, \ldots, a_n)$ $(\nu = 0, 1, \ldots, n)$ for the coefficients, which is always uniquely possible under the assumption $R_k > 0$ $(k = 1, 2, \ldots, n - 2)$.

If instead of (16) all R_ν $(\nu = 0, 1, \ldots, n)$ is other than zero and exactly r sign changes occur in the sequence $R_0, R_1, \ldots, R_n$, then the parametric representation (15) yields the coefficients of a polynomial $F(z)$ having r zeros with a positive real part and n - r zeros with a negative real part.

On the other hand, if instead of (16), $R_{n-1} = 0$ and $R_\nu \neq 0$ for $\nu = 0, 1, \ldots, n - 2, n$, and if exactly r sign changes occur in the sequence $R_0, R_1, \ldots, R_{n-2}, R_n$, then $F(z)$ has a pair of zeros on the imaginary axis, and r zeros lie in the right /337 half-plane and n - r - 2 zeros lie in the left one. This remark leads to a complete solution of the problem of giving a parametric representation of the stability boundary hypersurface of a linear control system of any order [17, 11].

The parametric representation (15) has the following noteworthy property: If in (15) one successively sets the parameters R_0, R_1, ..., $R_{\beta-1}$ equal to zero, one gets a parametric representation of the coefficients of the Hurwitz polynomial $F_\beta(z)$ of the $(n - \beta)^{\text{th}}$ degree reduced after (13) (β = 1, 2, ..., n).

If F(z) is a Hurwitz polynomial and if

$$A(z) = \frac{1}{2}[F(z) + (-1)^n F(-z)]$$
$$B(z) = \frac{1}{2}[F(z) - (-1)^n F(-z)],$$

then from the expansion into continued fractions presented by W. Bader [1]

$$\frac{A(z)}{B(z)} = \frac{R_0}{R_1} z + \frac{1|}{\left|\frac{R_1}{R_2} z\right.} + \cdots + \frac{1|}{\left|\frac{R_{n-1}}{R_n} z\right.}$$

by a known continued fraction theorem [31], in addition to (10) one gets the further explicit representation for the reduced Hurwitz polynomials defined by (13), with $F_0(z) = F(z)$:

$$(10a) \quad F_\beta(z) = R_n \begin{vmatrix} 1 + \frac{R_\beta}{R_{\beta+1}} z & -1 & 0 & \cdots & 0 \\ 1 & \frac{R_{\beta+1}}{R_{\beta+2}} z & -1 & \cdots & 0 \\ 0 & 1 & \frac{R_{\beta+2}}{R_{\beta+3}} z & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \frac{R_{n-1}}{R_n} z \end{vmatrix}$$
$$(\beta = 0, 1, \ldots, n-2)$$

H. Bückner [4] gave another reduction algorithm for Hurwitz polynomials $F_\beta^{(B)}(z)$ of the degree $(n - \beta)$, which can be represented as follows:

$$F_\beta^{(\beta)}(z) = R_{n-\beta} \begin{vmatrix} 1 + \frac{R_0}{R_1} z & -1 & 0 & \dots & 0 \\ 1 & \frac{R_1}{R_2} z & -1 & \dots & 0 \\ 0 & 1 & \frac{R_2}{R_3} z & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & \frac{R_{n-\beta-1}}{R_{n-\beta}} z \end{vmatrix}$$

$$(\beta = 0, 1, \dots, n-2).$$

The coefficients of the β^{th} reduced polynomial $F_\beta{}^{(B)}(z)$ /338 can be obtained by successively setting R_n, R_{n-1}, ..., $R_{n-\beta+1}$ in representation (15) equal to zero.

This yields the possibility of giving the quadratic control surface of a linear control system as a function of the initial values and the Routhian sample functions without calculating determinants [14].

Both reduction methods for Hurwitz polynomials can be interpreted simply by using electrical networks [9, 10, 12].

§ 3

As early as 1854, in a letter to Borchardt, Ch. Hermite [23] posed and solved the problem of giving characteristic conditions for the case that the roots of an equation with complex coefficients all have positive imaginary parts.

The algebraic criteria for a given root distribution inside and outside the unit circle, or in the left and right half-planes, or upper and lower half-planes, can be derived by his method and the results of Ch. Hermite [23], J. Schur [36], A. Cohn [5] can be summarized after M. Fujiwara [21] as follows:

If (4) is the polynomial under consideration, in each case one forms the expression:

$$k(f) = \frac{f(x) f^*(y) - f(y) f^*(x)}{x - y} = \sum_{i,k=0}^{n-1} A_{ik} x^i y^k. \tag{17}$$

Here $f^*(z)$ means the polynomial $z^n\bar{f}(z^{-1})$ or $\bar{f}(-z)$ or $-i\bar{f}(z)$, where $\bar{f}(z)$ is the polynomial with conjugately complex coefficients relative to $f(z)$. Thus equations $f^*(z) = 0$ are considered whose roots are mirror images of the corresponding roots of the original equation $f(z) = 0$, with regard to the unit circle, the imaginary axis or the real axis, as the case may be. Now in the three cases we consider the forms:

I. $$\sum_{i,k=0}^{n-1}{}' A_{i,n-1-k} U_i \bar{U}_k$$

II. $$\sum_{i,k=0}^{n-1}{}' (-1)^k A_{ik} U_i \bar{U}_k$$

III. $$\sum_{i,k=0}^{n-1}{}' A_{ik} U_i \bar{U}_k .$$

The following proposition then applies:

A characteristic of whether the roots of the equation $f(z) = 0$ with any real or complex coefficients all lie

I. inside the unit circle

II. in the left half-plane

III. in the upper half-plane

is that the associated Hermitian form is positively definite. If the associated form has the rank n and if π or ν respectively gives the number of positive or negative squares in the normal representation of the form, then the equation has /339

I. π roots inside and ν roots outside the unit circle

II. π roots in the left and ν roots in the right half-plane

III. π roots in the upper and ν roots in the lower half-plane.

If the rank of the associated form is $r < n$ and if π and ν mean the same as above, then

I. $\pi + \lambda$ roots lie inside and $\nu + \lambda$ roots lie outside the unit circle,

II. $\pi + \lambda$ roots lie in the left and $\nu + \lambda$ roots lie in the right half-plane,

III. $\pi + \lambda$ roots lie in the upper and $\nu + \lambda$ roots lie in the lower half-plane,
where

$$0 \leq \lambda \leq \frac{n-r}{2}.$$

If the presented equation has no pair of roots α, β with the properties

$$\begin{aligned} &\text{I.} \quad \alpha\bar{\beta} = 1 \\ &\text{II.} \quad \alpha + \bar{\beta} = 0, a_0 \neq 0 \\ &\text{III.} \quad \alpha - \bar{\beta} = 0, \end{aligned}$$

then $\lambda = 0$ and $n - r$ gives the number of roots

I. on the unit circle

II. on the imaginary axis

III. on the real axis.

This statement can also be formulated thus: If $f(z)$ and $f^*(z)$ have a shared zero, then $\lambda = 0$ and $n - r$ gives the number of roots on the unit circle, imaginary axis or real axis.

Compared to those of Routh and Hurwitz, the conditions formed by Hermite's method have the drawback of not being stated explicitly and in compact form, like the Hurwitz determinants in particular. This too may have moved Hurwitz to derive a new criterion, despite his knowledge of Hermite's work.

Below, developing an idea of W. Schmeidler [35], we explicitly present a general method for forming Hermitian conditions. It is sufficient to explain the formation method for Case II.

First one forms

$$f(x)f^*(y) - f(y)f^*(x) = (x^0 \cdots x^n)\begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & & \vdots \\ a_{n0} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} y^0 \\ \vdots \\ y^n \end{pmatrix} = \sum_{i,k=0}^{n}{}' a_{ik} x^i y^k .$$

Here,

$$a_{ik} = (-1)^k a_i \bar{a}_k + a_k \bar{a}_i (-1)^{i+1}, \tag{18}$$

and thus in particular

$$a_{ik} = -a_{ki}. \tag{18a}$$

It furthermore follows from (17) that:

$$A_{ik} = a_{i+1,k} + a_{i+2,k-1} + a_{i+3,k-2} + \cdots \tag{19a}$$

or

$$A_{ik} = a_{k+1,i} + a_{k+2,i-1} + a_{k+3,i-2} + \cdots. \tag{19b}$$

The end term in both cases is

$$\begin{aligned} &\text{a) } a_{n,i+k+1-n}, \quad \text{if} \quad i+k+1 \geqq n \\ &\text{b) } a_{i+k+1,0}, \quad \text{if} \quad i+k+1 \leqq n. \end{aligned} \tag{20}$$

The series must therefore be continued as begun until one of the indices becomes zero or n.

Formulas (19) can be interpreted simply. One must form the sum of all elements appearing in the matrix of a_{ik}, which we call $\mathfrak{a}$, in the diagonal leading down to the left from $a_{i+1,k}$ or $a_{k+1,i}$. Thus the zeroth row and the nth column of $\mathfrak{a}$ do not need to be calculated.

The matrix of A_{ik}, which we call $\mathfrak{A}$, is symmetric. It is sufficient to calculate A_{ik} with $i \geq k$, which forms the triangular matrix $\mathfrak{A}'$, or A_{ik} with $i \leq k$, which forms the triangular matrix $\mathfrak{A}''$. One gets the fewest terms in (19a) in the former case and in (19b) in the latter case. If, on the other hand, one uses (19b) e.g. in the first case, one gets terms still symmetric to the principal diagonal, which rise away because of antisymmetry. The second case works analogously.

We limit ourselves to forming $\mathfrak{A}'$ from (19a). For this it is sufficient to include from matrix $\mathfrak{a}$ the elements of the triangular matrix $\mathfrak{a}'$ with the corner terms a_{10}, a_{n0} and $a_{n,n-1}$.

In enumerating the terms a_{st} appearing for A_{ik} it is sufficient to limit oneself to A_{ik} with $i \geq k$. However, it follows from (19a) that the term A_{ik} and the term $A_{n-1-k,n-1-i}$ appearing symmetrically to the secondary diagonal of $\mathfrak{A}$ will yield the same number of terms. With $i + k + 1 \leq n$, according to (20b), A_{ik} yields $(k + 1)$ terms. According to (20a), $A_{n-1-k,n-1-i}$ also yields $(k + 1)$ terms. If an element A_{st} is in either the $(k + 1)^{th}$ column or the $(n - k)^{th}$ row of $\mathfrak{A}'$, one gets $(k + 1)$ terms. If an element A_{st} is in the $(k + 1)^{th}$ row or the $(n - k)^{th}$ column of $\mathfrak{A}''$, then because of symmetry one likewise gets $(k + 1)$ terms. This generalizes the fact, noted and proved in detail for equations up to the sixth degree by W. Schmeidler [35], that the element A_{ik} in the "r^{th} outer frame" of the matrix $\mathfrak{A}$ consists of r summands $(r = 1, 2, \ldots)$.

With Schmeidler, we say that an element lies in the ν^{th} outer frame of a matrix $\mathfrak{A}$, if it lies in the ν^{th} or $n - (\nu - 1)^{th}$ row or column and in the rth column or row $(r = \nu, \nu + 1, \ldots, n - (\nu - 1)$, with the row and column count beginning with 1. /341

For the element A_{ik} $(i \geq k)$ in the r^{th} outer frame of matrix $\mathfrak{A}$, according to (19) and (18) the following applies:

$$
\begin{aligned}
&A_{ik} = \sum_{\nu=0}^{r-1} a_{i+1+\nu,k-\nu} \\
(21)\qquad &A_{ik}(-1)^k = \sum_{\nu=0}^{r-1} [a_{i+1+\nu}\bar{a}_{k-\nu}(-1)^\nu + \bar{a}_{i+1+\nu}a_{k-\nu}(-1)^{i+k+\nu}],
\end{aligned}
$$

where $\begin{cases} r = k+1 \\ r = n-i \end{cases}$ if $\begin{cases} i+k+1 \leqq n \\ i+k+1 \geqq n. \end{cases}$

Thus the Hermitian conditions can easily be formed for complex coefficients. We will now explain these conditions for the case of real coefficients.

For real coefficients and $i \geq k$, it follows from (21) that:

(22a) $i + k$ uneven: $A_{ik} = 0$

(22b) $i + k$ even:

$$(-1)^k A_{ik} = 2\sum_{\nu=0}^{r-1}(-1)^\nu a_{i+1+\nu}a_{k-\nu}$$

with $\begin{cases} r = k+1 \\ r = n-i \end{cases}$ if $\begin{cases} i+k+1 \leqq n \\ i+k+1 \geqq n \end{cases}$.

Because of (22a), the determinants $C_\nu = |A_{ik}(-1)^g|_{i,k=0}^{\nu-1}$ can be decomposed into the product of two determinants as follows:

$$\text{(23a)} \quad C_{2m-1} = |A_{ik}(-1)^k|_{i,k=0}^{2m-2} = |A_{2i,2k}|_{i,k=0}^{m-1} \cdot |-A_{2i+1,2k+1}|_{i,k=0}^{m-2} \quad \left(m = 1, 2, \ldots, \left[\frac{n+1}{2}\right]\right).$$

(here we set $|-A_{2i+1,2k+1}|_{i,k=0}^{-1} = 1$.)

$$\text{(24a)} \quad C_{2m} = |A_{ik}(-1)^k|_{i,k=0}^{2m-1} = |A_{2i,2k}|_{i,k=0}^{m-1} \cdot |-A_{2i+1,2k+1}|_{i,k=0}^{m-1} \quad \left(m = 1, 2, \ldots, \left[\frac{n}{2}\right]\right)$$

For brevity's sake we write the decomposition equations (23a) and (24a) in the form:

$$\text{(23b)} \quad C_{2m-1} = C_{0,m} \cdot C_{1,m-1}\,; \; (C_{1,0} \equiv 1)\,; \; m = 1, 2, \ldots, \left[\frac{n+1}{2}\right]$$

$$\text{(24b)} \quad C_{2m} = C_{0,m} \cdot C_{1,m}\,; \; m = 1, 2, \ldots, \left[\frac{n}{2}\right].$$

According to (23b) and (24b) the inequalities:

$$\text{(25)} \quad \begin{aligned} C_{0,m} &> 0\,; \; m = 1, 2, \ldots, \left[\frac{n+1}{2}\right] \\ C_{1,m} &> 0\,; \; m = 1, 2, \ldots, \left[\frac{n}{2}\right]. \end{aligned}$$

are equivalent to the conditions $C_\nu > 0$; $\nu = 1, 2, \ldots, n$.

The following connection exists between the Hermitian and Hurwitz determinants:

$$C_\nu = |A_{ik}(-1)^k|_{i,k=0}^{\nu-1} = 2^\nu a_0 H_\nu H_{\nu-1}\,; \; (H_0 = 1)\,; \; \nu = 1, 2, \ldots, n.$$

Furthermore, /342

$$\text{(26)} \quad \begin{aligned} C_{0\nu} &= |A_{2i,2k}|_{i,k=0}^{\nu-1} = 2^\nu a_0 H_{2\nu-1}\,; \; \nu = 1, 2, \ldots, \left[\frac{n+1}{2}\right] \\ C_{1\nu} &= |-A_{2i+1,2k+1}|_{i,k=0}^{\nu-1} = 2^\nu H_{2\nu}\,; \; \nu = 1, 2, \ldots, \left[\frac{n}{2}\right]. \end{aligned}$$

Thus it is shown that aside from inessential factors, the

Hermitian conditions convert into the Hurwitz determinants when real coefficients are assumed.

Thus from (25) and (26) one gets a compact formation method for Hermitian stability criteria with the formation theorem that follows from (21):

$$A_{ik} = 2(-1)^k \sum_{\nu=0}^{r-1}{}' (-1)^\nu a_{i+1+\nu} a_{k-\nu} \tag{27}$$

$$\text{with } \begin{cases} r = k+1 \\ r = n-i \end{cases} \text{if } \begin{cases} i+k+1 \leq n \\ i+k+1 \geq n \end{cases} \text{for } i \geq k .$$

Because of the symmetry of the determinants $C_{0\nu}$ and $C_{1\nu}$, the restriction of (27) to $i \geq k$ suffices.

L. Cremer [8] studied the question of reducing the numbers of stability criteria assuming positive coefficients in the characteristic equation. Among other things, he reaches the result that with $a_\nu > 0$ ($\nu = 0, 1, 2, \ldots, n$), the Hurwitz determinants of the secondary sequence:

$$H_n, H_{n-2}, H_{n-4}, \ldots, H_2 | H_1 |$$

are always positive, if this is so for the main sequence:

$$H_{n-1}, H_{n-3}, H_{n-5}, \ldots, H_1 | H_2 |$$

and vice versa. A.T. Fuller [22] recently refined this result by showing that only certain coefficients need to be given as positive.

Then L. Cremer notes "that the way to the (n - 1)-series Hurwitz determinant, the most laborious to calculate, quite automatically leads past all Hurwitz determinants with lower numbers of places, whether one tries to work them out directly or takes the more practical route by way of Routh's algorithm."

The above discussion shows that it is now possible to calculate the Hurwitz determinants of both sequences independently of one another, and that the result of L. Cremer is of practical significance.

If the coefficients of the characteristic equation are positive, only the Hurwitz determinants of the main sequence need to be calculated, i.e.

$C_{0\nu}; \nu = 1, 2, \ldots, \frac{n}{2}$ for even n and

$C_{1\nu}; \nu = 1, 2, \ldots, \frac{n-1}{2}$ for uneven n.

A simple diagonal scheme for calculating these determinants was reported elsewhere [12, 15, 16]. /343

In the stability testing of an equation with a positive coefficient, now instead of one determinant each of the degree 1 to (n - 1), only one determinant each of the degree 1 to $\left[\frac{n}{2}\right]$ needs to be calculated. This yields the drawback that the elements of these determinants are coefficient functions; but they are easily formed and calculated.

§ 4

At the start of his paper, A. Hurwitz [24] presents a criterion for whether all roots of an algebraic equation with real coefficients lie in the left half-plane. In different ways, independently of Hurwitz's work, H. Bilharz [3] and E. Frank [18, 19] extended Hurwitz's theorem in that their results allow an enumeration of the roots lying in the left and right half-planes for equations with complex coefficients.

Iterative methods to decide whether all roots of an equation with complex coefficients lie in the left half-plane were presented by S. Sherman, J. di Paola and H.F. Frissel [37]. These methods seem especially useful when one is using computers.

K. Th. Vahlen [38] notes in his paper "Root enumeration in stability questions" that A. Hurwitz takes a considerable detour to arrive at his determinant criterion, and certainly finds a criterion for whether all roots lie in the left half-plane, "but does not, as one must hope, arrive at an enumeration of the roots in this half-plane." J. Schur [36], who presented two criteria that are also valid for complex coefficients, especially emphasizes this advantage of his criteria over that of A. Hurwitz. The response to these two remarks is that the work of A. Hurwitz offers all the prerequisites for obtaining the desired root enumeration even for the case of complex coefficients. Completing Hurwitz's train of thought for complex coefficients yields the following result:

Working from (5) with $a_\nu = a'_\nu + i a''_\nu \quad (\nu = 0, 1, \ldots, n)$ and

$$i^n F(-iz) = U(z) + i V(z)$$

with U(z) and V(z) being polynomials in z with real coefficients, we get

$$R(z) = \frac{V(z)}{U(z)} = \frac{\sum_{\nu=0}^{n} \beta_\nu z^{n-\nu}}{\sum_{\nu=0}^{n} \alpha_\nu z^{n-\nu}}$$

with

$$\begin{aligned} \alpha_\nu &= \frac{1}{2}\left[a_\nu i^\nu + \bar{a}_\nu i^{-\nu}\right] \\ \beta_\nu &= \frac{1}{2i}\left[a_\nu i^\nu - \bar{a}_\nu i^{-\nu}\right] \end{aligned} \qquad (\nu = 0, 1, \ldots, n). \tag{28}$$

If for $\nu = 1, 2, \ldots, n$ with (28) we form /344

$$\hat{R}_\nu = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{2\nu-1} \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{2\nu-1} \\ 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_{2\nu-2} \\ 0 & \beta_0 & \beta_1 & \cdots & \beta_{2\nu-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & \alpha_0 \cdots & \alpha_\nu \\ 0 & \cdots & 0 & \beta_0 \cdots & \beta_\nu \end{vmatrix} \tag{29}$$

and if none of these determinants vanishes, the following theorem applies:

In the sequence

$$\hat{R}_1, \frac{\hat{R}_2}{\hat{R}_1}, \ldots, \frac{\hat{R}_n}{\hat{R}_{n-1}} \tag{30}$$

the number of positive terms indicates the number of roots with a negative real part, and the number of negative terms indicates the number of roots with a positive real part.

The last part of the Hurwitz proof can be simplified and considerably shortened by the relationship

$$\breve{R}_\nu = a_0 H_{\nu-1} H_\nu \quad (\nu = 1, 2, \ldots, n) \qquad (H_0 = 1) \tag{29a}$$

which applies for real coefficients. Section 7 of Hurwitz's paper becomes entirely dispensable.

§ 5

H. Bilharz [3] devoted a work to extending the Hurwitz criterion to complex coefficients. He works not from Hurwitz's paper, but from Schur's second criterion [36]. It reads:

Let η be any fixed value with a negative real part. Then (4) is a Hurwitz polynomial if and only if with $a_0 \neq 0$ the inequality

$$Re\left(\frac{a_1}{a_0}\right) > 0$$

applies and the equation of the $(n - 1)^{th}$ degree

$$H(z) = z^{-2}[f(z)\,\varphi(z) - f(-z)\,\psi(z)] = 0$$

with

$$\varphi(z) = \bar{a}_0 z - \bar{a}_1 \eta z + \bar{a}_0 \eta$$

and

$$\psi(z) = a_0 z + a_1 \eta z + a_0 \eta$$

represents a Hurwitz equation.

As a necessary and sufficient condition for a Hurwitz equation with complex coefficients one gets n inequalities. H. Bilharz [3] has now shown that these inequalities can be written as quotients of determinants, with the determinants easily being formed from the coefficients of the equation. /345

According to H. Bilharz [3], with a suitable choice of η one gets the representation

$$\frac{D_\nu}{D_{\nu-1}} > 0, \quad \nu = 1, 2, \ldots, n \quad (D_0 = 1), \tag{31}$$

for the inequalities, with D_ν meaning the principal minors of degree 2ν of the following matrix:

$$\mathfrak{M} = (a_{ik}) \text{ with } \begin{array}{ll} a_{\mu,2\nu-1} = (-i)^{\mu-\nu} a_{\mu-\nu}, & \mu = 1, 2, \ldots, 2n \\ a_{\mu,2\nu} = i^{\mu-\nu+3}\, \bar{a}_{\mu-\nu}, & \nu = 1, 2, \ldots, n \end{array} \tag{32}$$

where $a_i = 0$ if $i > n$ or $i < 0$.

$D_n(-1)^{\binom{n}{2}}$ is the resultant of $f(-iz)$ and $-i\bar{f}(iz)$.

We now examine the connection between the Bilharz criterion and the Hurwitz criterion extended to complex coefficients. H. Bilharz has also shown that

$$D_\nu = 2^\nu \cdot d_\nu \tag{33}$$

where d_ν is the principal minor of degree 2ν of the matrix M:

$M = (a_{ik})$ with $a_k = a_k' + i a_k''$ and

$$\begin{array}{ll}
\left.\begin{array}{l} a_{\mu,2\nu-1} = a'_{\mu-\nu} \\ a_{\mu,2\nu} = -a''_{\mu-\nu} \end{array}\right\} & \text{for } \mu - \nu \equiv 0 \pmod 4 \\
\left.\begin{array}{l} a_{\mu,2\nu-1} = a''_{\mu-\nu} \\ a_{\mu,2\nu} = -a'_{\mu-\nu} \end{array}\right\} & \text{for } \mu - \nu \equiv 1 \pmod 4 \\
\left.\begin{array}{l} a_{\mu,2\nu-1} = -a'_{\mu-\nu} \\ a_{\mu,2\nu} = a''_{\mu-\nu} \end{array}\right\} & \text{for } \mu - \nu \equiv 2 \pmod 4 \\
\left.\begin{array}{l} a_{\mu,2\nu-1} = -a''_{\mu-\nu} \\ a_{\mu,2\nu} = -a'_{\mu-\nu} \end{array}\right\} & \text{for } \mu - \nu \equiv 3 \pmod 4
\end{array}$$

$$\mu = 1, 2, \ldots, 2n$$
$$\nu = 1, 2, \ldots, n.$$

If in (29) one multiplies the rows r with r ≡ 2,3 (mod 4) and the columns s with s ≡ 2,4 (mod 4) by -1, which does not change the value of (29), then one gets d_ν with the addition, as follows from (30), that a root enumeration is also proved. Thus the following relationship applies:

$$R_\nu = d_\nu \quad (\nu = 1, 2, \ldots, n).$$

§ 6

W. Schmeidler [35] called an investigation of the connection between the criteria of Bilharz and Hermite desirable. Now that the connection between the criterion of Bilharz and the extended criterion of Hurwitz has been shown, we can limit ourselves to explaining the connection between the extended criterion of Hurwitz and the Hermitian conditions. Hurwitz proceeds from polynomial (5). For comparison, it is also useful to state the Hermitian criterion for this notation of the polynomial. /346

After some computation one gets: With (5) and

$$F^*(z) = \sum_{\nu=0}^{n} (-1)^\nu \bar{a}_\nu z^{n-\nu}$$

one forms, corresponding to (17):

$$\frac{F(x)\,F^*(y) - F(y)\,F^*(x)}{x-y} = \sum_{i,k=1}^{n} B_{ik}\, x^{n-i} y^{n-k},$$

thus F(z) = 0 is a Hurwitz equation if and only if the Hermitian form

$$\sum_{i,k=1}^{n} B_{ik}(-1)^k U_i \overline{U}_k$$

is positively definite. If the rank of the Hermitian form is n and of one designates the ν^{th} primary minor as B_ν, then:

In the sequence

$$B_1, \frac{B_2}{B_1}, \ldots, \frac{B_n}{B_{n-1}}$$

the number of positive terms indicates the number of roots with a negative real part, and the number of negative terms indicates the number of roots with a positive real part. With (18), the following applies:

$$B_{ik} = a_{k-1,i} + a_{k-2,i+1} + \cdots$$

or

$$B_{ik} = a_{i-1,k} + a_{i-2,k+1} + \cdots.$$

In both cases the final term reads:

$$\begin{array}{ll} a_{i+k-1-n,n} & \text{if } i+k-1 \leq n \\ a_{0,i+k-1} & \text{if } i+k-1 \geq n. \end{array}$$

For Hermite, B_ν is of the order ν. For Hurwitz, however, R_ν is of the order 2ν. For $\hat{R}_\nu$ we will first form equivalent determinants R^*_ν of the ν^{th} order, and then show that $2^\nu R^*_\nu = B_\nu$.

If with $i^n F(-iz) = U(z) + iV(z)$ one forms

$$\frac{U(x)\,V(y) - U(y)\,V(x)}{x - y} = \sum_{i,k=1}^{n}{}' R_{ik} x^{n-i} y^{n-k},$$

then for R_{ik} the representation

$$R_{ik} = r_{k-1,i} + r_{k-2,i+1} + \cdots$$

or

$$R_{ik} = r_{i-1,k} + r_{i-2,k+1} + \cdots,$$

applies, with the end term in both cases reading:

$$\begin{array}{ll} r_{i+k-1-n,n} & \text{if } i+k-1 \geqq n \\ r_{0,i+k-1} & \text{if } i+k-1 \leqq n\,. \end{array}$$

We have:

$$r_{ik} = \alpha_i \beta_k - \alpha_k \beta_i$$

and with

(34)
$$R^*_\nu = \begin{vmatrix} R_{11} \ldots R_{1\nu} \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\ R_{\nu 1} \ldots R_{\nu\nu} \end{vmatrix}$$

we find that /347

$$R^*_\nu = 2^{-\nu} B_\nu \qquad (\nu = 1, 2, \ldots, n)\,.$$

For according to a notation of E. Netto [28], $R^*_\nu = \hat{R}_\nu$, so that the connection of R^*_ν with B_ν remains to be examined. We find:

$$r_{jk} = \frac{1}{2} i^{j+k+1} a_{jk}$$

and

$$R_{jk} = \frac{1}{2} i^{j+k} B_{jk}\,.$$

If in (34) we multiply the k^{th} column ($k = 1, 2, \ldots, \nu$) by i^{k-1} and the j^{th} row by i^{1-j} ($j = 1, 2, \ldots, \nu$), R^*_ν remains unchanged and R_{jk} converts into $1/2\ B_{jk}(-1)^k$:

$$R_{jk} i^{k-1} i^{1-j} = \frac{1}{2} i^{j+k} i^{k-1} i^{1-j} B_{jk} = \frac{1}{2} B_{jk} i^{2k} = \frac{1}{2} B_{jk}(-1)^k\,.$$

Thus: $R^*_\nu = 2^{-\nu} B_\nu$. The Hurwitz and Hermitian conditions for equations with complex coefficients therefore correspond except for positive factors.

M. Fujiwara [21] has noted that the Hermitian conditions convert in the case of real coefficients into the conditions of Liénard-Chipart [26]. The equivalence of the criterion of Liénard-Chipart and that of Hurwitz for real coefficients has already been shown by M. Fujiwara [20]. His result therefore is included in the present one as a special case.

§ 7

Having shown the connection between the criteria of H. Bilharz and Ch. Hermite via the extended criterion of A. Hurwitz as an intermediate term, we now directly verify the connection between the two coefficient conditions.

We proceed from Bilharz's condition. In the primary minor D_r ($r = 1, 2, \ldots, n$) from (32) we multiply the ν^{th} row by $i^{\nu-1}$ ($\nu = 1, 2, \ldots, 2r$), the $(2s-1)^{th}$ column by i^{2-s} and the $2s^{th}$ column by i^{1-s} ($s = 1, 2, \ldots, r$). Here D_r converts to $\hat{D}_r$ and the following applies:

$$\hat{D}_r = i^{r(r+1)} D_r. \tag{35}$$

$\hat{D}_r$ are the 2r-serial principal minors of the following matrix:

$$\hat{\mathfrak{M}} = (a_{ik}) \text{ with } \begin{array}{ll} a_{\mu,2\nu-1} = a_{\mu-\nu} & (\nu = 1, 2, \ldots, n) \\ a_{\mu,2\nu} = (-1)^{\mu-\nu}\bar{a}_{\mu-\nu} & (\mu = 1, 2, \ldots, 2n). \end{array}$$

Here $a_i = 0$ if $i > n$ or $i < 0$. $\hat{D}_n (-1)^{\binom{n}{2}}$ is the resultant of $f(z)$ and $\bar{f}(-z)$.

The determinants $\hat{D}_r$ ($r = 1, 2, \ldots, n$) can also be formed like the Bézout form of the resultant of $f(z)$ and $\bar{f}(-z)$:

$$\frac{f(x)\bar{f}(-y) - f(y)\bar{f}(-x)}{x-y} = \sum_{i,k=0}^{n-1} A_{ik} x^i y^k.$$

Here $\hat{D}_r = (-1)^r |A_{ik}|_{i,k=0}^{r-1}$ and from /349
(35) it follows that:

$$D_r = i^{r(r+3)} |A_{ik}|_{i,k=0}^{r-1} \cdot$$

According to Hermite, the determinants

$$A_r = |A_{ik}(-1)^k|_{i,k=0}^{r-1} = |A_{ik}|_{i,k=0}^{r-1} \cdot i^{r(r+3)}$$

must be considered. This yields $A_r = D_r$ and the agreement of the conditions of Bilharz and Hermite.

§ 8

Finally we return to Section 4. Using a proposition of W. Scheibner [34] and M. Noether [29] we can easily prove the following theorem:

"The following conditions are necessary and sufficient in order for the equation F(z) = 0 to possess [exactly] ν roots lying symmetric to the imaginary axis:

$$\hat{R}_n = \hat{R}_{n-1} = \cdots = \hat{R}_{n-(\nu-1)} = 0;\ [\hat{R}_{n-\nu} \neq 0,\ \hat{R}_0 = 1].$$

Here roots are called symmetric to the imaginary axis if they either lie on the imaginary axis or can be combined in mutually exclusive pairs whose elements differ only in the sign of the real part."

If we now limit ourselves to real coefficients and take relationship (29a) into account, it follows that:

"It is necessary and sufficient in order for F(z) = 0 to have [exactly] ν roots symmetric to the origin if in each of the following rows

$$\begin{array}{ll} H_n & H_{n-1} \\ H_{n-1} & H_{n-2} \\ \cdots & \cdots \\ H_{n-(\nu-2)} & H_{n-(\nu-1)} \\ H_{n-(\nu-1)} & \end{array}$$

at least one of the Hurwitz determinants vanishes [and $H_{n-\nu}$ and $H_{n-(\nu+1)}$ are not equal to zero ($H_0 = 1$, $H_{-1} = a_0^{-1}$)]."

If we designate the roots of $F(z) = 0$ as $z_1, z_2, \ldots, z_n$, from the last proposition with any numeration of the roots one can derive the following representations for the Hurwitz determinants:

$$H_{n-(r-1)} = \sum_{k=1}^{\left[\frac{r+1}{2}\right]} z_{2k-1}\,\psi_{r,2k-1} + \sum_{k=1}^{\left[\frac{r}{2}\right]}{}' z_{2k}\,\psi_{r,2k-1}\,. \tag{36}$$

$\psi_{r,s}$ are functions of the roots, and $\psi_{r,2i-1} \neq 0$ for $z_{2k-1} = -z_{2k}$ with $k \neq i\left(i = 1, 2, \ldots, \left[\frac{r}{2}\right]\right)$. Furthermore, $\psi_{r,r}$ contains none of the sums $z_{2k-1} + z_{2k}$ for $k = 1, 2, \ldots, \left[\frac{r}{2}\right]$ as a factor.

For H_{n-2m}, the representation

$$H_{n-2m} = \sum_{k=1}^{m+1} (z_{2k-1} + z_{2k})\,\Phi_{2k-1}\,, \tag{36a}$$

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also applies, where Φ^{2i-1} ($i = 1, 2, \ldots, m + 1$) are functions of the roots that for all $i \neq k$ do not contain the sum $z_{2k-1} + z_{2k}$ as a factor.

These representations of the Hurwitz determinants, which provide a certain clarification of the structure of the Hurwitz determinants with regard to their construction from the roots of equations, comparable to the root representation of Orlando to [30] for H_{n-1}, together with the last theorem lead directly to the following result:

"The following conditions are necessary and sufficient in order for [exactly] ν roots of the equation $F(z) = 0$ to be symmetric to the origin:"

$$H_n = H_{n-1} = \cdots = H_{n-\nu+1} = 0;\ |H_{n-\nu} \cdot H_{n-(\nu+1)}| \neq 0.$$

This result allows us to replace conditions (11) with Hurwitz determinants.

Elsewhere [6], the special case was reported in which the 2k uppermost Hurwitz determinants vanish if $F(z) = 0k$ has purely imaginary root pairs.

We also arrive at the following coupling rule for the vanishing of Hurwitz determinants of the main and secondary sequences:

"If the r uppermost Hurwitz determinants of the main sequence

$$H_{n-1},\ H_{n-3},\ H_{n-5},\ \ldots,\ H_{n-(2r-1)},$$

vanish, then the r uppermost Hurwitz determinants of the secondary sequence

$$H_n,\ H_{n-2},\ H_{n-4},\ \ldots,\ H_{n-(2r-2)}$$

also vanish. The converse applies without restrictions if one also assumes $a_n \neq 0$. However, if the absolute term vanishes, only the vanishing of $H_{n-(2r-1)}$ cannot be concluded."

REFERENCES

1. Bader, W.: "Contribution on the realization of a.c. resistances with prescribed frequency dependencies," Arch. Elektrotech. 34, 293 (1940).

2. Bashkow, T.R. and C.A. Desoer: "A network proof of a theorem on Hurwitz polynomials and its generalization," Quart. Appl. Math. 44, 423 (1957).

3. Bilharz, H.: "Remark on a theorem of Hurwitz," Z. angew. Math. Mech. 24, 77 (1944).

4. Bückner, H.: "A formula for an integral occurring in the theory of linear servomechanisms and control systems," Quart. appl. Math. 40, 205 (1952); Note on my paper "A formula for an integral occurring in the theory of linear servomechanisms and control systems," Quart. appl. Math. 42, 206 (1954).

5. Cohn, A.: "The number of roots of an algebraic equation in a circle," Math. Z. 14, 140 (1922).

6. Cremer, L.: "A new method to evaluate the stability of linear control systems," Z. angew. Math. Mech. 25/27, 160 (1947).

8. Cremer, L: "Reducing the number of stability criteria given positive coefficients of the characteristic equation," Z. angew. Math. Mech. 33, 221 (1953).

9. Effertz, F.H.: "Restricted functions, frequency characteristics of electrical networks and algebraic stability criteria," Z. angew. Math. Mech. 33, 281 (1953).

10. Effertz, F.H.: "On the synthesis of networks containing two kinds of elements," Proc. Symp. Mod. Netw. Synthesis 5, 145 (1955).

11. Effertz, F.H.: "A parametric representation for stability interfaces, and stability study of harmonically linearized systems using Cremer-Leonhard locus families," Regelungstechnik [Control Engineering], Verlag Oldenbourg, Munich, 1957, p. 152. /350

12. Effertz, F.H.: "A criterion for a class of monotonic dynamic systems," Regelungstechnik [Control Engineering], Verlag Oldenbourg, Munich, 1957, p. 248.

13. Effertz, F.H.: "Construction of logarithmic root and phase angle loci of control systems," Regelungstechnik [Control Engineering], Verlag Oldenbourg, Munich, 1957, p. 284.

14. Effertz, F.H.: "Representation of the quadratic control surface as a function of Routhian sample functions and of the initial values of the control system," Regelungstechnik [Control Engineering], Verlag Oldenbourg, Munich, 1957, p. 298.

15. Effertz, F.H.: "A method to calculate stability criteria given positive coefficients of the characteristic equation," Z. angew. Math. Mech. 37, 487 (1957).

16. Effertz, F.H.: "A calculation method for stability conditions," Regelungstechnik [Control Engineering], Verlag Oldenbourg, Munich, 1957, p. 368.

17. Effertz, F.H.: "On the relation between the stability boundary surfaces of linear and nonlinear servomechanisms and the realizability boundary surfaces of some classes of frequency characteristics of electrical networks," in: W. Cauer, Synthesis of Linear Communication Networks, Vol. II, McGraw-Hill, New York, 1958, pp. 840-856.

18. Frank, E.: "On the zeros of polynomials with complex coefficients," Bull. Amer. Math. Soc. 52, 144 (1946).

19. Frank, E.: "On the real parts of the zeros of complex polynomials and applications to continued fraction expansions of analytic functions. Transact. Amer. Math. Soc. 62, 272 (1947).

20. Fujiwara, M.: "The roots of algebraic equations," Tôkohu math. J. 8, 78 (1915).

21. Fujiwara, M.: "Algebraic equations whose roots lie in a circle or a half-plane," Math. Z. 24, 161 (1926).

22. Fuller, A.T.: "Stability criteria for linear systems and realizability criteria for R.C. networks," Proc. Cambridge Phil. Soc. 53, 878 (1957).

23. Hermite, Ch.: "On the number of roots of an algebraic equation between given limits," Crelles J. 52, 39 (1856); Oeuvres, I, 397.

24. Hurwitz, A.: "The conditions under which an equation has only roots with negative real parts," Math. Ann. 46, 273 (1895).

25. Krylov, N. and N. Bogoljubov: Einführung in die nichtlineare Mechanik [Introduction to Nonlinear Mechanics], Kiev, 1937; Princeton, 1949.

26. Liénard-Chipart: "The sign of the real part of roots of an algebraic equation," J. Math. pur. appl. 10, 291 (1914).

27. Magnus, K.: "A method to examine nonlinear vibration and control systems," VDI-Forschungsheft 451 (1955).

28. Netto, E.: Vorlesungen über Algebra [Lectures on Algebra]. Vol. 1. Leipzig, 1896.

29. Noether, M.: "The common factor of two binary forms," Erl. Ber. 1895.

30. Orlando, L.: "The Hurwitz problem relative to the real parts of the roots of an algebraic equation," Math. Ann. 71, 233 (1911).

31. Perron, O.: Die Lehre von den Kettenbrüchen [Continued Fraction Theory], Teubner, Leipzig, 1929.

32. Routh, E.J.: A Treatise on the Stability of a Given State of Motion, London, 1877.

33. Routh, E.J.: Die Dynamik der Systeme starrer Körper [The Dynamics of Systems of Rigid Bodies]. German Edition, Vol. II, Leipzig, 1898, p. 222.

34. Scheibner, W.: "Mathematical remarks," Leipziger Ber. 40, 1 (n.d.).

35. Schmeidler, W.: Vorträge über Determinanten und Matrizen [Lectures on Determinants and Matrices], Berlin, 1949.

36. Schur, J.: "The algebraic equations having only roots with negative real parts," Z. angew. Math. Mech. 1, 307 (1921).

37. Sherman, S., J. Di Paola and H.F. Frissel: "The simplification of flutter calculations by use of an extended form of the Routh-Hurwitz discriminant," J. Aeronaut. Sci. 12, 385 (1945).

38. Vahlen, K. Th.: "Root enumeration in stability questions," Z. angew. Math. Mech. 14, 65 (1934).

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16. Abstract

This paper critically examines the standard algebraic criteria for the stability of linear control systems and their proofs, reveals important previously unnoticed connections, and presents new representations. Algebraic stability criteria have also acquired significance for stability studies of non-linear differential equation systems by the Krylov-Bogoljubov-Magnus Method, and allow realization conditions to be determined for classes of broken rational functions as frequency characteristics of electrical networks.

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