NASA Contractor Report 178240

ICASE REPORT NO. 87-8

ICASE

RKH SPACE APPROXIMATIONS FOR THE FEEDBACK
OPERATOR IN A LINEAR HEREDITARY CONTROL SYSTEM

J. A. Reneke
R. E. Fennell

Contract No. NAS1-18107
February 1987

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

RKH Space Approximations for the Feedback Operator in a Linear Hereditary Control System*

J. A. Reneke and R. E. Fennell†
Department of Mathematical Sciences
Clemson University
Clemson S. C. 29634-1907

Abstract

Computational implementation of feedback control laws for linear hereditary systems requires the approximation of infinite dimensional feedback operators with finite dimensional operators. The dense subspaces of K-polygonal functions in reproducing kernel Hilbert spaces, RKH spaces, suggest finite dimensional approximations of the matrix representations of the control operators. A convergence theorem is developed for the approximations and the numerical implementation of the approximations is discussed.

* This research was supported in part by the following grants AFOSR-84-02336 and ONR-N00014-86-K-0693.

† Partially supported by NASA Contract No. NAS1-18107 while the second author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665-5225.

RKH Space Approximations for the Feedback
Operator in a Linear Hereditary Control System

1. Introduction. Linear hereditary systems are complicated by the fact that state space formulations require an infinite dimensional state space. For linear functional differential equations the usual choice of state space leads to a description by an abstract evolution equation and to approximations by semigroup techniques [2, 5, 8, 9]. We have advocated in several papers an alternate approach, i.e., something other than ordinary differential equations in infinite dimensional spaces [15, 16]. Our approach is to describe system dynamics by operator equations defined on a reproducing kernel Hilbert (RKH) space. This approach, motivated by an integral equation or an input-output operator description of the system dynamics is general enough to include linear functional differential equations and Volterra integral equations. In this approach one gives up the familiar guide provided by finite dimensional state space systems; but one gains the powerful methods associated with RKH spaces [1, 10, 13, 18]. In particular, matrix representations of continuous linear transformations and the approximations suggested by the convenient dense subspaces of K-polygonal functions are available in RKH spaces.

In this paper we apply these methods to the problem of constructing suitable finite dimensional approximations to the feedback operator of a linear hereditary system. We are concerned with systems as diagrammed in Figure 1, where A denotes the system input-output operator, D denotes a feedback control operator, and f and h denote, respectively, the system input and response. One should think of the "physical plant" as being modelled by the input-output operator A. We assume A is linear and causal but is not necessarily time invariant or memoryless.

In applications the feedback operator is determined by a control system designer to achieve some desired objective, i.e., stabililizing the system, tracking a desired trajectory, or obtaining a desired terminal condition for the system response. Such feedback operators may simply involve proportional/integral/derivative terms or more complex dynamic compensation. Synthesis methods for hereditary systems [3, 4, 8, 16] lead to feedback operators D which are linear and causal.

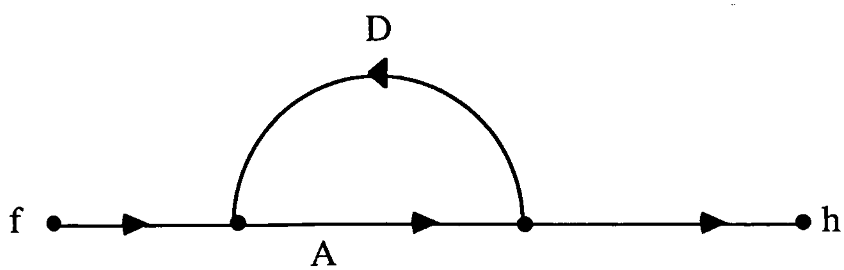

Figure 1. Feedback Control System.

In this paper we are not concerned with the synthesis of a feeback operator to achieve a desired objective. Motivated by problems of digital implementation of such control operators, we present methods to obtain finite dimensional (finite dimensional range) operator approximations to the system feedback operator. Digital implementation requires storage of a discrete representation of such approximations and rules for their evaluation. In Section 3 projection methods are used to obtain feedback operator approximations and show convergence of these and related system operator approximations as the sampling interval goes to zero.

In order that these approximations satisfy the physical requirements of causality a simple "hold" of one sampling period is introduced in their definition. Such delays in digital control systems result from computational and data processing considerations which can become a significant factor in the control of high order systems [12]. Although one normally expects to use equally spaced samples, this is not a requirement for our methods. Convergence of related operator approximations has been considered in our previous effort [15].

In Section 4 we discuss the computational implementation of our approximation scheme. A matrix representation of the approximate feedback operator is presented. If the feedback operator is time invariant, it is shown that evaluation of the approximate feedback operator can be achieved through a simple convolution formula and computer storage requirements can be greatly reduced. We illustrate the effect of implementing our feedback operator approximations by considering two numerical examples in Section 5. These examples indicate a slight deterioration in the control system performance as the holding time increases and point to several interesting problems for future research.

For finite dimensional state space systems, use of numerical integration rules and related approximation schemes to obtain digital filters is well understood [7]. The significance of our results lies in the fact that we have provided an analysis of a much broader class of feedback operator approximations and a compatible scheme for their digital representation and implementation. Examples show that our methods reduce to standard approximations in the state space case.

2. Preliminaries. We begin by presenting our abstract setting. One should refer to the text [19] by Willems for a basic discussion of the operator equation approach. For completeness, we present only necessary background information. Let r be a nonnegative number, S the number interval $[-r, \infty)$, d a positive integer, $X = R^d$, $<\bullet, \bullet>$ the usual inner product on X, and $| \bullet |$ the corresponding inner product norm. Let G be the class of functions from S into X to which f belongs provided 1) f is continuous on $[0, \infty)$ and 2) if $r > 0$ then f is continuous on $[-r, 0)$ and $f(0-)$ exists. Let $\{N_x$, x in S$\}$ be the family of pseudonorms defined on G by $N_x(f) = \mathrm{lub}\{ \, | f(t) |, -r \le t \le x\}$, f in G and x in S.

Let ***B*** denote the class of linear transformations of G to which B belongs only in case

1) $[Bf](x) = 0$ for each f in G and x in $[-r, 0]$ and
2) for each compact subinterval $[u, v]$ of $[0, \infty)$ there is a number b such that

$$| [Bf](t) - [Bf](s) | \le b \int_s^t N_x(f)dx, \text{ for each f in G and subinterval } [s, t] \text{ of } [u, v].$$

The nondecreasing function bI is called a variation function for B on [u, v].

We will reserve C to denote the element of ***B*** defined by

$$[Cf](t) = \begin{cases} 0 & -r \le t \le 0 \\ \int_0^t f(s)\, ds & 0 \le t \end{cases}$$

for f in G and t in S. The simple state space system given formally by $h(t) = f(t) + \alpha\int_0^t h(x)dx$, where α is a $d \times d$ matrix and f is in G, would be written as $h = f + \alpha Ch$. The general class of hereditary systems to be considered are of the form $h = f + Bh$ with B in $\boldsymbol{B}$. One should realize that our abstract setting is based upon an integral equation description of the system dynamics.

We can obtain an equivalent class of models as follows. Let $\boldsymbol{A}$ denote the class of linear transformations of G to which A belongs only in case A - I is in $\boldsymbol{B}$ where I is the identity on G. If B is in $\boldsymbol{B}$ then I - B is a reversible map from G onto G and $(I - B)^{-1}$ is in $\boldsymbol{A}$. Similarly, an element A of $\boldsymbol{A}$ is a reversible map from G onto G and $I - A^{-1}$ is in $\boldsymbol{B}$. Thus we have equivalent system descriptions in terms of $B = I - A^{-1}$ or $A = (I - B)^{-1}$ [13]. For the system as diagrammed in Figure 1, if $B = I - A^{-1}$ and D is in $\boldsymbol{B}$ then $h = A(f + Dh) = (I - B)^{-1}(f + Dh)$. Thus $h = f + Bh + Dh$ or $h = (I - B - D)^{-1}f$. Here $(I - B - D)^{-1}$ exists since B + D belongs to $\boldsymbol{B}$.

Our analysis takes place in a reproducing kernel Hilbert space associated with the operators in $\boldsymbol{A}$ and $\boldsymbol{B}$. Let k denote the increasing function defined on S by

$$k(t) = \begin{cases} 1 + r + t & \text{if } -r \leq t < 0 \\ 1 + 2r + t & \text{if } 0 \leq t \end{cases}$$

for t in S. Let G_H denote the subspace of functions in G which are Hellinger integrable with respect to k, i.e., f is in G_H only in case for each compact subinterval [a, b] of S there is a number M such that

$$\sum_{p=1}^{n} |f(s_p) - f(s_{p-1})|^2 / (k(s_p) - k(s_{p-1})) \equiv \sum_{s} |df(s_{p-1}, s_p)|^2 / dk(s_{p-1}, s_p) \leq M$$

for each partition $\{s_p\}_0^n$ of [a, b]. Here $df(s_{p-1}, s_p) = f(s_p) - f(s_{p-1})$. The least such number M is denoted by $\int_a^b |df|^2 / dk$.

Note that elements of G_H are absolutely continuous on compact subintervals of $[0, \infty]$ and $[-r, 0)$ if $r > 0$ [10]. Furthermore, if each of f and g is in G_H and [a, b] is a compact subinterval of S then the integral $\int_a^b \langle df, dg\rangle / dk$ exists as a limit through refinement of partitions s of [a, b] of approximating sums of the form Σ_s $<df(s_{p-1}, s_p), dg(s_{p-1}, s_p)> / dk(s_{p-1}, s_p)$. Let G_∞ denote the subspace of G_H to which f belongs only in case $\int_{-r}^{\infty} | df |^2 / dk < \infty$ and Q_∞ the inner product for G_∞ given by $Q_\infty(f, g) = <f(-r), g(-r)> + \int_{-r}^{\infty} <df, dg> / dk$. The space $\{G_\infty, Q_\infty\}$ is a complete inner product space with norm N_∞ and has a reproducing kernel K given by $K(s, t)x = k(\min(s, t))x$ for (s, t) in $S \times S$ and x in X. Thus $K(, t)x$ belongs to G_∞ for each t in S and x in X and $<f(t), x> = Q_H(f, K(,t)x)$ for each f in G_∞, t in S and x in X.

It may be shown that the norm N_∞ restricted to $P_T G_H$ $T > 0$ is equivalent to the Sobolev norm $(\int_0^T \{ | f(t) |^2 + | f'(t) |^2\} dt)^{1/2}$. However use of the norm N_∞ or equivalently the kernel K results in simplified representations and approximations of continuous linear operators.

One can show [15] that an element B of ***B*** maps G into G_H and hence an element A of ***A*** maps G_H onto G_H. We will be concerned with the restrictions of elements of ***A*** and ***B*** to G_H but will not introduce any special notation for these restrictions.

Let $\{P_x$, x in S$\}$ denote the family of projections on G given by

$$[P_x f](t) = \begin{cases} f(t) & -r \le t \le x \\ f(x) & x \le t \end{cases}$$

for each f in G and (x, t) in $S \times S$. It is important to note that operators B in ***B*** are causal, i.e., $P_T B P_T$ for each $T \ge 0$.

To illustrate the properties of the kernel K let us compute the adjoint in $\{G_\infty, Q_\infty\}$ of $P_T C$, $T > 0$. If f is in G_∞, t is in S, and x is in X then

$$\begin{aligned} <[(P_T C)^* f](t), x> &= Q_\infty((P_T C)^* f, K(, t)x) \\ &= Q_\infty(f, P_T C K(, t)x) \end{aligned}$$

$$= \int_0^T <df, dCK(,t)x> / dk$$

$$= \int_0^T <df, K(,t)x>$$

$$= <f(T), K(T, t)x>$$

$$- <f(0), K(0, t)x> - \int_0^T <f, dK(,t)x> \ .$$

If $-r \le t \le 0$ then $<[P_TC)^*f](t),x> = <f(T) - f(0), k(t)x>$. If $0 \le t \le T$ then $<[(P_TC)^*f](t),x> = <f(T), k(t)x> - <f(0), k(0)x> - <[Cf](t), x>$. If $T \le t$ then $<[(P_TC)^*f](t), x> = <f(T), k(T)x> - <f(0), k(0)x> - <[Cf](T), x>$.

Thus

$$[(P_TC)^*f](t) = K(t, T)f(T) - K(t, 0)f(0) - [P_TCf](t) \ .$$

This expression will be used in the following section.

3. Matrix representations and the convergence theorem. One of the many advantages of an RKH space setting for our analysis is the tractability of the dense linear subspace defined in terms of the kernel K, the K-polygonal functions [10]. Consider the family of projections $\{\Pi_t\}$ from G into G_∞ with $t = \{t_p\}_0^n$ and increasing sequence in S defined by

$$[\Pi_t f](s) = \begin{cases} f(t_0) & -r \le s \le t_0 \\ (1/dk(t_{p-1}, t_p))\{dk(s, t_p)f(t_{p-1}) + dk(t_{p-1}, s)f(t_p)\} & t_{p-1} \le s \le t_p \\ t(t_n) & t_n < s \end{cases}$$

for each f in G and s in S. Elements of $\Pi_t G$ are called K-polygons. One can show that for each $T > -r$, the union of the subspaces $\Pi_t G_H$, $\{t_p\}_0^n$ a partition of [-r, T] is dense in $P_T G_H$ with respect to N_∞.

Suppose that D is an element of $\boldsymbol{B}$, $\{t_p\}_0^n$ is a partition of some interval [-r, T], and $c = \text{mesh}(t) = \max (t_p - t_{p-1})$, p = 1, 2, ..., n. When T > 0 we assume that 0 is the partition. Let D_t

denote the linear transformation of G defined by

$$[D_t f](u) = \begin{cases} 0 & -r \le u \le c - r \\ [\Pi_t D \Pi_t f](u - c) & c - r \le u \end{cases}$$

for each f in G. At first it is natural to choose $\Pi_t D \Pi_t$ to approximate D [15] but $\Pi_t D \Pi_t$ does not satisfy the necessary causality requirement. Here D_t is causal and we show D_t approximates D. Thinking of D as a feedback operator and $\Pi_t f$ as sampled data, $D_t f$ represents an approximation to Df obtained from a translation of $\Pi_t D \Pi_t f$.

For linear functional differential equations the basic feedback operator arising from linear quadratic regulator design [3, 4, 6, 8] has the form $W(t) = -K_1(t)h(t) - \int_{t-r}^{t} K_2(t, \tau)h(\tau)d\tau$ where K_1 and K_2 denote matrix valued functions. Since our approach is based upon an integral equation description of the system dynamics the feedback operator D would be of the form $[Dh](t) = -\int_0^t \{K_1(\tau)h(\tau) + \int_{\tau-r}^{\tau} K_2(u, \tau)h(u)du\}\, d\tau$. With appropriate boundedness assumptions on K_1 and K_2, D belongs to $\boldsymbol{B}$.

To further illustrate the possibilities for the feedback operator D, we consider a finite dimensional state space example. Robust control system design methods for a plant $x_p' = \alpha_p x_p + ß_p u_p$, $z = \gamma_p x_p$ lead to a compensator $x_c' = \alpha_c x_c + ß_c z$, $y = \gamma_c x_c + \delta_c z$ and interconnection $u_p = r_{ref} - y$ where r_{ref} denotes a reference signal. Integration yields $x_p = x_p(0) + \alpha_p C x_p + ß_p C u_p$ or $x_p = (I - \alpha_p C)^{-1}(x_p(0) + ß_p C u_p)$, $x_c = (I - \alpha_c C)^{-1} ß_c C z$, and $y = \{\gamma_c (I - \alpha_c C)^{-1} ß_c C + \delta_c\} \gamma_p x_p$. Letting $u = ß_p C u_p$, we obtain $u = ß_p C r_{ref} - D x_p$ with $D = ß_p C\{\gamma_c (I - \alpha_c C)^{-1} ß_c C + \delta_c\} \gamma_p$. Here D belongs to $\boldsymbol{B}$ and our approximation metohds apply.

Theorem 1. The operator D_t is in $\boldsymbol{B}$. Furthermore, any variation function bk of D is also a variation function of D_t.

Proof. If bk is a variation function of D, f is in G, and $0 \le t_{p-1} \le u - c \le v - c \le t_p$ then

$$| [D_t f](v) - [D_t f](u) | = (1/dk(t_{p-1}, t_p)) \, | dk(t_{p-1}, v - c)[D\Pi_t f](t_p) + dk(v - c, t_p)[D\Pi_t f](t_{p-1}) - dk(t_{p-1}, u - c)[D\Pi_t f](t_p) - dk(u - c, t_p)[D\Pi_t f](t_{p-1}) |$$

$$\le b \, \frac{dk(u - c, v - c)}{dk(t_{p-1}\, t_p)} \int_{t(p-1)}^{t(p)} N_s(\Pi_t f) ds$$

$$\le b \;\; dk(u, v) \, N_{t(p)}(\Pi_t f)$$

$$\le b \;\; dk(u, v) \, N_{t(p)}(f)$$

$$\le b \int_u^v N_s(f) \, dk(s) \;\; .$$

If $0 \le t_{p-1} \le u - c \le t_p \le t_{q-1} \le v - c \le t_q$ then

$$| [D_t f](v) - [D_t f](u) |$$

$$\le | [D_t f](v) - [D_t f](t_{q-1} + c) |$$

$$+ \sum_{i=p+1}^{q-1} | [D_t f](t_i + c) - [D_t f](t_{i-1} + c) |$$

$$+ | [D_t f](t_p + c) - [D_t f](u) |$$

$$\le b \;\; dk(u, t_p + c) N_{t(p)}(f)$$

$$+ b \sum_{i=p+1}^{q-1} dk(t_{i-1} + c, t_i + c) N_{t(i)}(f)$$

$$+ b \;\; dk(t_{q-1} + c, v) N_{t(q)}(f)$$

$$\le b \int_u^v N_s(f) dk(s)$$

and we are through.

We are concerned in this paper with the problem of substituting the finite dimensional approximation D_t, i.e., D_t has finite dimensional range, for a given feedback operator D, see Figure

1. We begin by studing the convergence of nets D_t, t a partition of [-r, T]. In an application the feedback operator is chosen to achieve some desired objective, i.e., the plant h = f + Bh + Dh, or h = $(I - B - D)^{-1}f$ with $B = I - A^{-1}$ behaves satisfactorily. Since D_t represents a digital implementation of D, it is important to also consider convergence of the nets $(I - B - D_t)^{-1}$ to the system imput/output operator $(I - B - D)^{-1}$. In later sections we take up a computational implementation of these ideas and apply then to two examples.

A function L from $S \times S$ into the continuous linear transformations of X is called the matrix representation of D provided $<[Df](t), x> = Q_\infty(f, L(, t)x)$ for each f in G_∞, t in S, and x in X. A fundamental property of RKH spaces is that every continuous linear transformation between such spaces has a matrix representation. A problem is that D maps G_H into G_H rather than G_∞ into G_∞.

For T in S, P_TD is continuous from G_∞ into G_∞ with respect to N_∞ and so P_TD has a matrix representation. Note that $<[P_TDf](t), x> = Q_\infty(P_TDP_Tf, K(, t)x) = Q_\infty(P_Tf, (P_TD)^*K(, t)x)$, for $-r \le t \le T$, f in G_H, and x in X. Let $L(s, t)\, x = [(P_TD)^*K(, t)x](s)$ for $-r \le s, t \le T$ and x in X. To obtain L without having to compute $(P_TD)^*$ note that, for $1 \le i, j \le d$ and $-r \le s, t \le T$, $< e_i, L(s, t)e_j > = Q_\infty(P_TDK(, s)e_i, K(, t)e_j) = <[DK(, s)e_i](t), e_j>$, where e_i and e_j are standard basis elements of $X = R^d$. In light of this equation we speak of the matrix representation of D rather than of P_TD.

Note that L(s, t) = 0 for $-r \le t \le 0$. For L causality means that given $-r \le t \le s$, $<x, L(s, t)y> = <[DK(,s)x](t),y> = <[DK(,t)x](t),y> = <x, L(t, t)y>$ for each x and y in X, i.e., L(s, t) = L(t, t). Hence L(,t)x is in G_∞ for each t in S and x in X.

For each positive number T, let N_{HT} denote the operator pseudonorm defined on the linear operators F of G_H which are continuous with respect to N_T by $N_{HT}(F) = \sup\{N_T(F_g)/N_\infty(P_{Tg}) \mid g \text{ in } G_H, P_{Tg} \ne 0\}$. Assume for the rest of the paper that T is a fixed positive number. We will discuss convergence of D_t and the control systems $\{(I - B - D_t)^{-1}\}$, t a partition of [-r, T], using N_{HT} and the matrix representations L and M of D and D_t, respectively.

Theorem 2. The operator $P_T D$ is the limit through refinement of the net of finite dimensional operators $\{D_t\}$, t a partition of [-r, T], with respect to the operator pseudonorm N_{HT}.

Lemma. If t is a partition of [-r, T], u is in the range of t, and x is in X then $N_\infty(L(\,,u)x - M(\,, u + c)x)^2 = N_\infty(L(\,,u)x)^2 - \Sigma_t \,|\, dL(\,, u)\, x \,|^2/dk$.

Proof. If y is in X, v is in S, and c = mesh(t) then

$$
\begin{aligned}
<y, M(v, u + c)x> &= <[D_t K(\,, v)y](u + c), x> \\
&= Q_\infty(\Pi_t D \Pi_t K(\,,v)y, K(\,, u)x) \\
&= Q_\infty(K(\,, v)y, \Pi_t L(\,, u)x) \\
&= <y, [\Pi_t L(\,, u)x](v)> \;.
\end{aligned}
$$

Thus

$$
\begin{aligned}
Q_\infty(L(\,, u)x, M(\,, u + c)x) &= Q_\infty(L(\,, u)x, \Pi_t L(\,, u)x) \\
&= \Sigma_t \,|\, dL(\,, u)x \,|^2 / dk \\
&= N_\infty(M(\,, u + c)x)^2 \;.
\end{aligned}
$$

Hence $N_\infty(L(\,, u)x - M(\,, u + c)x)^2 = N_\infty(L(\,, u)x)^2 - \Sigma_t \,|\, dL(\,, u)x \,|^2 / dk$.

Lemma. If [u, v] is a subinterval of [0, T] and x is in X then $N_\infty(L(\,, v)x - L(\,, u)x) \leq N_\infty(P_T D) \,|\, x \,|\, dk(u, v)^{1/2}$. Also, for each partition t of [-r, T], $N_\infty(M(\,, v)x - M(\,, u)x) \leq N_\infty(P_T D) \,|\, x \,|\, dk(u, v)^{1/2}$.

Proof. Note that $N_\infty(L(\,, v)x-L(\,, u)x) = N_\infty((P_T D)^*(K(\,, v) - K(\,, u))x) \leq N_\infty(P_T D) \,|\, x \,|\cdot dk(u,v)^{1/2}$ and $N_\infty(M(\,, v)x - M(\,, u)x) = N_\infty(\Pi_t (P_T D)^* \Pi_t (K(\,, v - c) - K(\,, u - c))x) \leq N_\infty(P_T D) \,|\, x \,|\, dk(u, v)^{1/2}$.

Proof of the theorem. If $\{s_p\}_0^m$ is a partition of [-r, T], $\{t_q\}_0^n$ is a refinement of s, c = mesh(t), f is in G_H, x is in X, and u is in $[s_{p-1}, s_p]$ for some positive integer p then

$$
\begin{aligned}
&| <[Df](u) - [D_tf](u), x> | \\
&\quad = | Q_\infty(f, L(, u)x - M(, u)x) | \\
&\quad \leq N_\infty(P_Tf) \{N_\infty(L(, u)x - L(, s_{p-1})x) \\
&\qquad + N_\infty(L(, s_{p-1})x - M(, s_{p-1} + c)x) + N_\infty(M(, s_{p-1} + c)x - M(, u)x)\} \\
&\quad \leq N_\infty(P_Tf)\{N_\infty(P_TD) | x | dk(s_{p-1}, u)^{1/2} + (N_\infty(L(, s_{p-1})x)^2 \\
&\qquad - \Sigma_t | dL(, s_{p-1}) x |^2/dk)^{1/2} + N_\infty(P_TD) | x | (dk(s_{p-1}, u) + dk(0, c))^{1/2}\} \\
&\quad \leq N_\infty(P_Tf)\{3N_\infty(P_TD) | x | (mesh(s))^{1/2} + (N_\infty(L(, s_{p-1})x)^2 \\
&\qquad - \Sigma_t | dL (, s_{p-1}) x |^2 /dk)^{1/2}\} \ .
\end{aligned}
$$

Suppose that $\varepsilon > 0$, $\{s_p\}_0^n$ is a partition of [-r, T] such that mesh(s) < $(\varepsilon/6(N_\infty(P_TD) + 1))^2/d$, t′ is a refinement of s such that if t refines t′, p = 1, 2, ..., n and i = 1, 2, ..., d then $N_\infty(L(, s_{p-1}))e_i)^2 - \Sigma_t | dL(, s_{p-1})e_i |^2 / dk < (\varepsilon/2)^2/d$. If t refines t′ and | x | = 1 then

$$
\begin{aligned}
&| < [Df](u) - [D_tf](u), x> | \\
&\quad \leq \sum_{i=1}^{d} | x_i | | <[Df](u) - [D_tf](u), e_i> | \\
&\quad \leq \sum_{i=1}^{d} | x_i | N_\infty(P_Tf) \varepsilon /d^{1/2} \\
&\quad \leq \varepsilon | x | N_\infty(P_Tf) \ ,
\end{aligned}
$$

i.e., $N_HT (D - D_t) \leq \varepsilon$.

<u>Theorem 3</u>. If B is in $\boldsymbol{B}$ then $(I - B - D)^{-1}$ is the limit through refinement of the net $\{(I - D - D_t)^{-1}\}$, t a partition of [-r, T], with respect to the operator pseudonorm N_{HT}.

<u>Lemma</u>. If bk is a variation function for D then $be^{bT}k$ is a variation function for $(I - D)^{-1} - I$ on [-r, T].

Proof. If $h = f + Dh$ then by the Gronwall inequality $N_u(h) \le e^{bu}N_u(f)$ for each u in [0, T]. Thus for each subinterval [u, v] of [0, T]

$$
\begin{aligned}
&| [(I - D)^{-1}f](v) - f(v) - [(I - D)^{-1}f](u) + f(u) | \\
&\quad \le b \int_u^v N_s((I - D)^{-1}f)dk(s) \\
&\quad \le b \int_u^v e^{bs}N_s(f)dk(s) \\
&\quad \le be^{bT} \int_u^v N_s(f)\, dk(s) \ .
\end{aligned}
$$

Proof of the theorem. Assume b_1k and b_2k are variation functions for B and D, respectively. Then $N_T((I - B - D_t)^{-1}) \le 1 + be^{bT}(k(T) - k(0))$, where $b = b_1 + b_2$. Hence for each f in G_H.

$$
\begin{aligned}
&N_T((I - B - D_t)^{-1}f - (I - B - D)^{-1}f) \\
&\quad \le N_T((I - B - D_t)^{-1})N_{HT}(D - D_t)N_\infty(P_T(I - B - D)^{-1})N_\infty(P_Tf) \\
&\quad \le \{1 + be^{bT}(k(T) - k(0))\}N_{HT}(D - D_t)N_\infty(P_T(I - B - D)^{-1})\, N_\infty(P_Tf),
\end{aligned}
$$

.i.e., $N_{HT}((I - B - D_t)^{-1} - (I - B - D)^{-1}) \le \{1 + be^{bT}(k(T) - k(0))\}N_\infty(P_T(I - B - D)^{-1})N_{HT}(D - D_t)$. Thus $(I - B - D)^{-1}$ is the limit with respect to the operator pseudonorm N_{HT} of the net $\{(I - B - D_t)^{-1}\}$, t a partition of [0, T].

Thus $N_T(I - B - D_t)^{-1}f - (I - B - D)^{-1}f)$ converges to zero with respect to refinements for each f in G_{HT} i.e., the solution of $h = f + Bh + D_th$ converges uniformly on [-r, T] to the solution of $h = f + Bh + Dh$.

4. Computational implementation. In this section we elaborate on the discrete structure of the approximation D_t which facilitiates a digital implementation. Recall that L and M denote the matrix representations of D and D_t respectively. In this paper we are not concerned with computing L(s, t) or M(s, t) for various values of s and t in S but with the effect of using the approximations D_t in

place of D. We would expect that most D's met in practice would have representations in terms of familiar integrals and we can obtain the various values of L and M using some existing numerical quadrature code.

In addition, the extensive literature [2, 3, 8, 9] on numerical approximations for hereditary systems is available for evaluations of those L and M arising from linear quadratic regulator designs.

If $\{t_p\}_0^n$ is an increasing sequence in S, $t_{p-1} \leq s \leq t_p$, and f is in G then $[\Pi_t\dot{D}\Pi_t f](s) = (dk(s, t_p)[D\Pi_t f](t_{p-1}) + dk(t_{p-1}, s)[D\Pi_t f](t_p))/dk(t_{p-1}, t_p)$. Thus to evaluate $\Pi_t D\Pi_t$ we need to know $[D\Pi_t f[(t_p)$ for p = 0, 1, 2, ..., n. Let f_t denote the (n + 1)d vector given by $f_t = \mathrm{col}(f(t_0), f(t_1), ..., f(t_n))$ and L_t the $(n + 1)d \times (n + 1)d$ block matrix whose (p, q) block is given by $L_t(p, q)_{ij} = \langle L(t_p, t_q)e_j, e_i\rangle$ for $0 \leq p, q \leq n$ and $1 \leq i, j \leq d$. Define K_t in an analogous way.

We have $\Pi_t f = \Sigma_{q=0}^n K(, t_q)c_q$ where $c_q = [(K_t)^{-1} f_t](q)$ for q = 0, 1, 2, ..., n. Hence $D\Pi_t f = \Sigma_{q=0}^n DK(, t_q)c_q = \Sigma_{q=0}^n L(t_q,)^*c_q$ and, for p = 0, 1, ..., n $[D\Pi_t f](t_p) = [(L_t)^*(K_t)^{-1}f_t](p)$.

Let M_t denote the $(n + 1)d \times (n +1)d$ block matrix defined by

$$M_t(p, q) = \begin{cases} 0 & p = 0 \\ [(L_t)^*(K_t)^{-1}](p - 1, q) & 1 \leq p \leq n, \quad 0 \leq q \leq n \end{cases} .$$

Then $[D_t f](t_p) = [M_t f_t](p)$.

The matrices $(L_t)^*(K_t)^{-1}$ and M_t denote discrete representations of the operators $\Pi_t D\Pi_t$ and D_t respectively. Let us look closer at the structure of these matrices.

One can show that $(K_t)^{-1}$ is symmetric block tridiagonal. For p = 1, 2, ..., n. the p-th block in the first subdiagonal is $-(1/dk(t_{p-1}, t_p))I$. The first main diagonal block is $(1/k(t_0) + 1/dk(t_0, t_1))I$, the last main diagonal block is $(1/dk(t_{n-1}, t_n))I$, and, for p = 2, ..., n the p-th main diagonal block is $(1/dk(t_{p-2}, t_{p-1}) + 1/dk(t_{p-1}, t_p))I$. If $0 \leq p < q \leq n$ then $[(L_t)^*(K_t)^{-1}](p, q) = (L_t(p, p))^*\{[(K_t)^{-1}](q -1, q) + [(K_t)^{-1}](q, q) + [(K_t)^{-1}](q + 1), q)\} = 0$, i.e., $(L_t)^*(K_t)^{-1}$ is block lower triangular. Clearly M_t is also block lower triangular.

If D is a time invariant operator, i.e., $[Dg](t + b) = [Df](t)$ with $g(t) = f(t - b)$ for each f in $(I - P_0)G_H$, $b \geq 0$, $t \geq 0$ then the structure of the matrices simplifies further. In this case $L(v, w) - L(u, w) = L(v + b, w + b) - L(a + b), w + b)$ for $0 \leq u \leq v \leq w$ and $b \geq 0$ [15]. Let $t_p = - r + pc$, $p = 0, \ldots, n$ with $r = p'c$. It follows that

$$
\begin{aligned}
(L_t)^*(K_t)^{-1}(p, q) &= 0 && 0 \leq q \leq p \leq p' \\
&= \{L_t(p'+1, p'+1)^* - L_t(p', p'+1)\}/(r+c) && p = q = p' + 1 \\
&= \{L_t(p'+1, p'+1)^* - L_t(p', p'+1)^*\}/c && p = q > p' + 1 \\
&= \{L_t(p'+1, p)^* - L_t(p', p)^*\}/(r+c) && \\
&\quad - \{L_t(p'+1, p-1)^* - L_t(p', p-1)^*\}/c && p > q = p' + 1 \\
&= \{L_t(p'+1, p'+1+j)^* - L_t(p', p'+1+j)^*\}/c && p = q + j > q > p' \\
&\quad - \{L_t(p'+1, p'+j)^* - L_t(p', p'+j)^*\}/c \ .
\end{aligned}
$$

Thus the storage requirements for $(L_t)^*K_t)^{-1}$ and M_t are significantly reduced. One need only store $(L_t)^*(K_t)^{-1}(p,q)$ for $p = p' < q \leq n$, $p = p' + 1 \leq q \leq n$, $0 \leq q \leq p' < p$. Here $[D_tf](t_p) = 0$ for $p \leq p' + 1$ and for $p' + 1 < p \leq n$

$$
\begin{aligned}
[D_tf](t_p) &= \sum_{q=0}^{p-1} (L_t)^*(K_t)^{-1}(p-1, q)f(t_q) \\
&= \sum_{q=0}^{p'+1} (L_t)^*(K_t)^{-1}(p-1, q)f(t_q) + \sum_{j=0}^{p-1-(p'+2)} a_j f(t_{p-1-j})
\end{aligned}
$$

where $a_j = ((L_t)^*(K_t)^{-1}(n, n-j)$ for $j = 0, \ldots, n - (p'+1)$. This expression simplifies further if $r = 0$, which would be the case if D is of the form $[Df](t) = \int_0^t g(t - s)f(s)\, ds$ for example. If $r = 0$ then $p' = 0$ and

$$
[D_tf](t_p) = \sum_{j=0}^{p-1} a_j f((p-1-j)c)
$$

for $p' + 1 < p \leq n$. Thus in the time invariant case evaluation of $[D_tf](t_p)$ involves a convolution sum and, consequently, computation time can be reduced through use of an appropriate fast algorithm.

In the examples to follow, we solve the system $h = \Pi_{t'}f + (B + D_t)_{t'} h$ with t' a refinement of t to obtain approximate solutions of the system $h = f + Bh + D_t h$.

5. Numerical examples. In order to simulate $h = f + Bh + D_t h$ with $t = \{t_p\}_0^n$ a partition of $[-r, T]$ one should choose, depending on the nature of B, a differential equation solver, see for example [12]. On the interval $[-r, 0]$ we have $h = f$. Let $t_{p'} = 0 < t_{p'+1}$. On the interval $[0, t_{p'+1}]$ we know f and $D_t h$ and so can solve for h. Proceeding in this manner one steps across the interval $[0, T]$.

The following examples, were chosen to illustrate the effect of implementing the approximations D_t in a feedback control system. Because the examples are simple we chose to base our calculations on the corollary to Theorem 3.

Example 1. Consider the time invariant system $h = f + Bh + Dh$ with $[Bh](t) = -4 \int_0^t (t-\tau)h(t)\, d\tau + \int_0^t h(\tau-.25)\, dt$ and $[Dh(t) = -4 \int_0^t h(\tau)\, d\tau$ for $t \geq 0$. Here the feedback operator D stabilizes the system $h = f + Bh$. The effect of this stabilizing feedback is illustrated in Figure 2. In Figure 2 h denotes the approximate solution of the system with $D = 0$ and h_2, h_8, h_{32} denote approximate solutions of the feedback system with $c = t_p - t_{p-1} = 1/128, 1/32, 1/8$ respectively. It is interesting to note that stability is maintained for relatively large values of the "holding time" c. Here we obtain $N_2(\bar{h}-h_p) \leq .031036, .061612, .12045, .22959, .41532$ for $c = p/256$, $p = 2, 4, 8, 16, 32$ respectively, where $\bar{h}$ denotes the solution of $h = f + Bh + Dh$.

Example 2. Let $[Bh](t) = -4\int_0^t(t-\tau)h(\tau)\, d\tau - \int_0^t h(\tau-.25)\, d\tau$. For a given disturbance f, we wish to choose a control u to drive the response of the system $h = f + Bh + u$ to zero over a terminal interval $[1.25, 1.5]$. Optimization techniques [14] can be used to obtain such a control in the form $u = Dh + g$, i.e., u is the sum of a feedback term and an open loop term. Here $D = A^{-1}C^2$ with $A^{-1} = I - B$ and g depends on the known disturbance f and known parameters. The effects of

More specific knowledge of the feedback operator leads to further simplifications. For example, if $[Dh](t) = \int_0^t \kappa h(\tau)d\tau$, which is the case for a state space system $h'(t) = \alpha h(t) + \beta v(t)$ with feedback $v(t) = \kappa h(t)$, then $[D_t h](c) = 0$ and for $p > 1$ $[D_t h](pc) = \kappa (c/2)\{h(0) + 2h(c) + \ldots + 2h((p-2)c) + h((p-1)c)$, i.e., the approximation scheme reduces to the trapezoid rule. Implementation of this approximation yields $h'(t) = \alpha h(t) + \beta\kappa \{h((p-1)c) + h(pc)\}/2$ for $pc < t \leq (p+1)c$, i.e., a simple "first order hold" approximation for the feedback operator.

Before illustrating the effect of implementing the approximation D_t in a feedback control system, we present a corollary to Theorem 3 which provides a method to obtain approximate solutions to the system $h = f + Dh$ with D in $\boldsymbol{B}$.

For $t = \{t_p\}_0^n$ a partition of [-r, T], if $h = \Pi_t f + D_t h$ then h is in $\Pi_t G_H$ and for $0 \leq p \leq n$ $h(t_p) = f(t_p) + [D_t h](t_p)$ or $h_t(p) = f_t(p) + [M_t h_t](p)$. Thus $h_t = f_t + M_t h_t$ or $h_t = ((I - D_t)^{-1}\Pi_t f)_t = (I_{(n+1)d} - M_t)^{-1} f_t$.

Corollary to Theorem 3. If f is in G_H, $h = (I - D)^{-1} f$, and ε is a positive number then there is a partition s of [0, T] such that if $\{t_p\}_0^n$ refines s and $0 \leq p \leq n$ then $| h(t_p) - [(I_{(n+1)d} - M_t)^{-1} f_t](p) | < \varepsilon$.

Proof. Using Theorem 3 with B = 0, there is a partition s of [0, T] such that if $\{t_p\}_0^n$ refines s then $N_{HT}((I - D)^{-1} - (I - D_t)^{-1})$ is smaller than either 1 or $(1/2)\ \varepsilon/(N_\infty(f) + 1)$ and $N_\infty(f - \Pi_t f) < 1/2\ \varepsilon/(N_{HT}(I - D)^{-1}) + 1)$. Thus

$$
\begin{aligned}
&N_T((I - D)^{-1}f - (I - D_t)^{-1}\Pi_t f) \\
&\quad \leq N_{HT}((I - D)^{-1} - (I - D_t)^{-1})N_\infty(f) \\
&\qquad + N_{HT}((I - D_t)^{-1})N_\infty(f - \Pi_t f) \\
&\quad \leq 1/2\ \varepsilon + N_{HT}((I - D)^{-1} + 1)N_\infty(f - \Pi_t f) \\
&\quad < \varepsilon\ .
\end{aligned}
$$

Since $((I - D_t)^{-1}\Pi_t f)_t = (I_{(n+1)d} - M_t)f_t$ we are through.

implementing this control and the approximations $u = D_t h + g$ with $f(t) = 1 + t$ for $-.25 \leq t \leq 1.5$ are illustrated in Figure 3. Here h denotes the approximate solution of the system with D = 0 and h_2, h_{16}, and h_{32} denote approximate solutions with $c = t_p - t_{p-1} = 1/128$, 1/16. 1/8 respectively. Note that the desired terminal condition holds when c = 1/128 and there is a slight deterioration in the control system performance for larger values of the "holding time" c. For the example we obtain $N_{1.5}(h - h_p) \leq .0032262, .0072059, .015469, .032368, .067435, .14337$ for c = p/256 and p = 2, 4, 8, 16, 32 respectively, where $\bar{h}$ denotes the solution of $h = f + Bh + Dh + g$.

6. Concluding Remarks. We have presented a theory for the approximation of feedback control operators which arise in the analysis of linear hereditary systems. The reproducing kernel Hilbert space setting introduced facilitates the analysis and digital representation of such approximations. Advances in on board computational capabilities indicate that storage of our discrete representations of a feedback operator is feasible. Furthermore, in the design of feedback controllers for infinite dimensional systems one need not be restricted to the search for simple feedback gains as in the finite dimensional case.

Although we have not addressed control design issues in this paper, such problems have been discussed using our abstract setting within a number of contexts [11, 16]. We are presently investigating the robustness of control designs based upon an input/output description of the systems. The results and examples presented in this paper indicate that such designs should have good robustness properties.

A unified discussion of our approach to system analysis, approximation, and control will appear in the lecture notes, "Structured Hereditary Systems" [17].

References

1. N. Aronzajn, Theory of reproducing kernels, Trans Amer. Math. Soc., 68(1950), 337-404.

2. H. T. Banks and J. A. Burns, Hereditary control problems: numerical methods based on averaging approximations, SIAM J. Control Opt. 16(1978), 169-208.

3. H. T. Banks and A. Manitius, Application of abstract variational theory to hereditary systems - A survey, IEEE Trans. Auto. Cont., AC-19(1974), 524-533.

4. H. T. Banks, I. G. Rosen and K. Ito, A spline based technique for computing Raccati operators and feedback controls in regulator problems for delay equations, SIAM J. Sci. Stat. Comput. 5(1984), 830-855.

5. R. F. Curtain and A. J. Pritchard, Infinite Dimensional Linear Systems Theory, Springer-Verlag, Berlin and New York 1978.

6. M. C. Delfour, Linear optimal control of systems with state and control variable delays, Automatica, 20(1984), 69-77.

7. G. F. Franklin and J. D. Powell, Digital Control of Dynamic Systems, Addison-Wesley Publishing Co., Reading Mass., 1980.

8. J. S. Gibson, Linear quadratic optimal control of hereditary systems: infinite dimensional Riccati equations and numerical approximations, SIAM J. Control Opt. 21 (1983), 95-139.

9. K. Kunisch, Approximation schemes for linear quadratic optimal control problems associated with delay equations, SIAM J. Control Opt. 20(1982), 506-540.

10. J. S. Mac Nerney, Hellinger integrals in inner product spaces, J. Elisha Mitchell Sci. Soc., 76(1960), 252-273.

11. R. B. Minton and J. A. Reneke, An RKH space approach to state feedback control for a class of linear stochastic systems, SIAM J. Control Opt., 24(1986), 700-714.

12. T. Mita, Optimal digital feedback control systems counting computation time of control laws, IEEE Trans. Auto. Contr. AC-30(1985), 542-548.

13. R. E. Moore, Computational Functional Analysis, Ellis Honwood Limited, Chester, England, 1985.

14. Neves, K. W. Automatic integration of functional differential equations: an approach, ACM Trans. Math. Software, 1(1975), 357-368.

15. Reneke, J. A. and R. E. Fennell, An indentification problem for hereditary systems, J. Applicable Anal., 11(1981), 167-183.

16. Reneke, J. A. and R. E. Fennell, Decentralized control strategies for hereditary systems, Proceedings IEEE Conference and Control, Dec. 1985. 1479-1483.

17. Reneke, J. A. and R. E. Fennell and R. B. Minton, Structured Hereditary Systems, to appear, Marcel Dekker Inc., Series on Pure and Applied Mathematics.

18. Weinert, H. L., Editor, Reproducing Kernel Hilbert Spaces - Applications in Statistical Signal Processing, Hutchenson Ross Publishing Co., Strougsburg, PA., 1982.

19. J. C. Willems, The Analysis of Feedback Systems, The MIT Press, Cambridge, Mass., 1971.

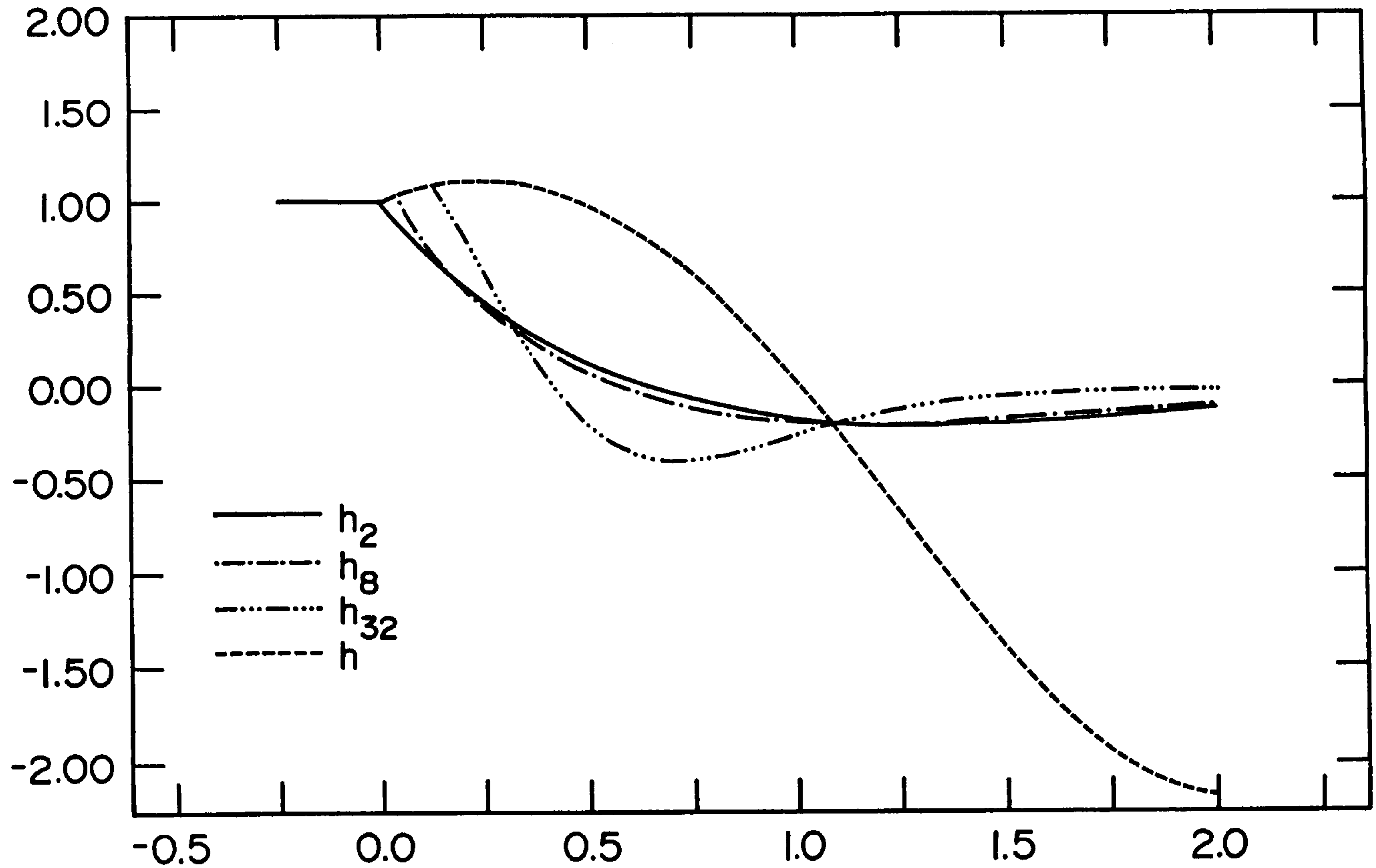

Figure 2. Approximate solutions for Example I.

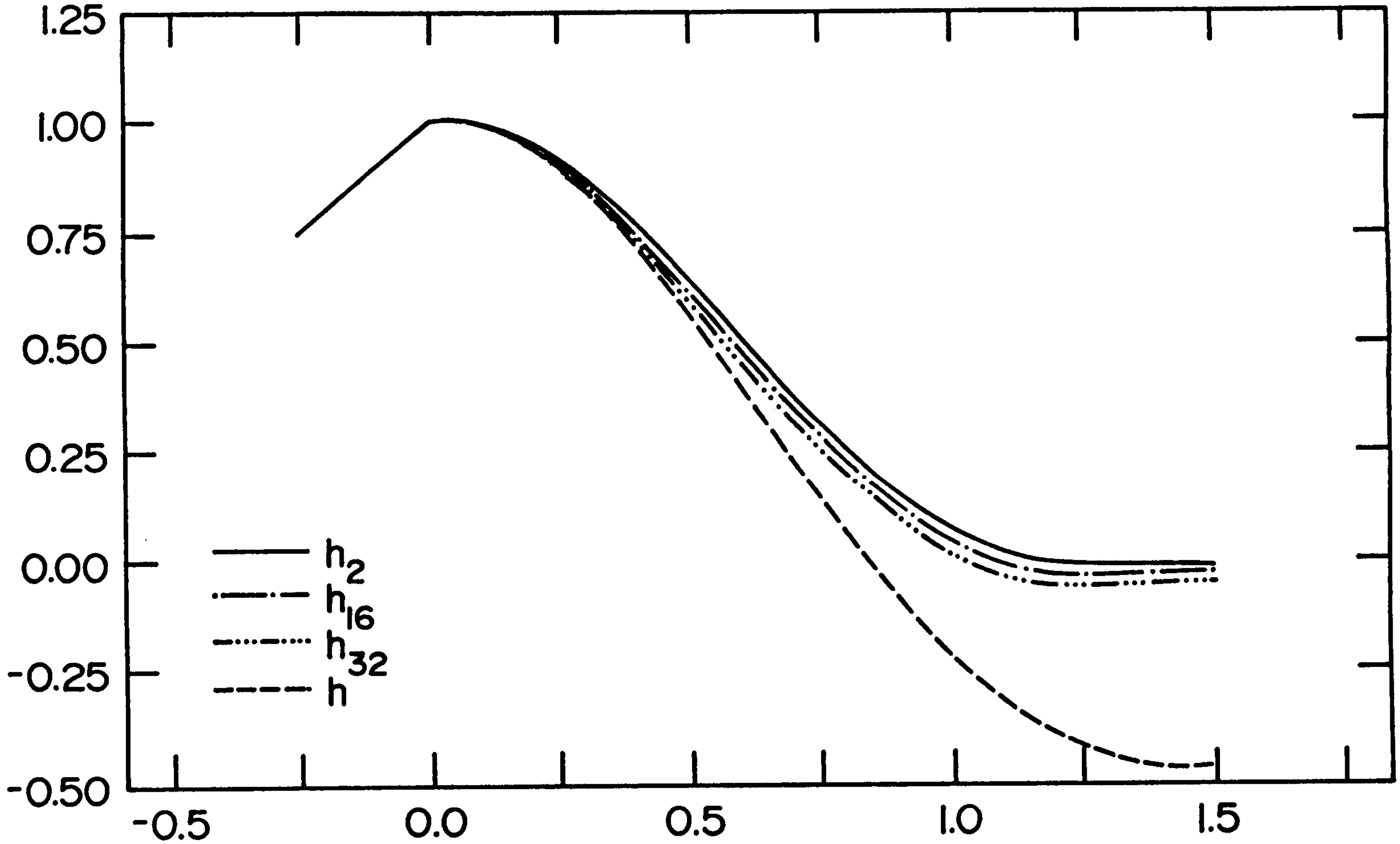

Figure 3. Approximate solutions for Example 2.

Standard Bibliographic Page

1. Report No. NASA CR-178240 ICASE Report No. 87-8	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle RKH SPACE APPROXIMATIONS FOR THE FEEDBACK OPERATOR IN A LINEAR HEREDITARY CONTROL SYSTEM		5. Report Date February 1987
		6. Performing Organization Code
7. Author(s) J. A. Reneke and R. E. Fennell		8. Performing Organization Report No. 87-8
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No.
		11. Contract or Grant No. NAS1-18107
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code 505-90-21-01

15. Supplementary Notes

Langley Technical Monitor: J. C. South

Submitted to Internat. J. Control

Final Report

16. Abstract

Computational implementation of feedback control laws for linear hereditary systems requires the approximation of infinite dimensional feedback operators with finite dimensional operators. The dense subspaces of K-polygonal functions in reproducing kernel Hilbert spaces, RKH spaces, suggest finite dimensional approximations of the matrix representations of the control operators. A convergence theorem is developed for the approximations and the numerical implementation of the approximations is discussed.

17. Key Words (Suggested by Authors(s)) hereditary systems, operator approximations, matrix representations	18. Distribution Statement 64 - Numerical Analysis 66 - Systems Analysis Unclassified - unlimited

19. Security Classif.(of this report) Unclassified	20. Security Classif.(of this page) Unclassified	21. No. of Pages 23	22. Price A02

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1987