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AN OPTIMAL OUTPUT FEEDBACK GAIN
VARIATION SCHEME FOR THE CONTROL OF
PLANTS EXHIBITING GROSS PARAMETER CHANGES

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PROJECT SUMMARY

This research was aimed at developing a novel concept for optimally designing output feedback controllers for plants whose dynamics exhibit gross changes over their operating regimes. This was, essentially, to formulate the design problem in such a way that the implemented feedback gains vary as the output of a dynamical system whose independent variable is a scalar parameterization of the plant operating point.

The results of this effort include derivation of necessary conditions for optimality for the general problem formulation, and for several simplified cases. The question of existence of a solution to the design problem was also examined, and it was shown that the class of gain variation schemes developed in this effort are capable of achieving gain variation histories which are arbitrarily close to the unconstrained gain solution for each point in the plant operating range. The theory was implemented in a feedback design algorithm, which was exercised in a numerical example.

The results of the research undertaken under this contract are applicable to the design of practical high-performance feedback controllers for plants whose dynamics vary significantly during operation. Many aerospace systems fall into this category.

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1. INTRODUCTION

Linear control theory has provided many techniques for the design of feedback controllers for plants with linearizable dynamics. In most of these methods, the feedback synthesis is performed for a plant model obtained by linearizing about some "representative" setpoint. Modelling errors and minor parameter variations are accommodated by taking measures to ensure that the control design is "robust." A robust controller, for our purposes, is one which performs adequately when implemented with plant dynamics which differ somewhat from the nominal design model, in the sense of their being a perturbation of the nominal plant. Robust design methods have been an active area of research in recent years [1 - 4], and have been brought to a fairly high level of maturity.

Many dynamical systems encountered in practice, however, exhibit gross, structured changes in their dynamics as they move about in their operating regimes. For these systems, it is difficult to design feedback controllers with globally satisfactory performance using design theory intended to accommodate only local plant perturbations. This problem motivated work which led to a number of approaches for using information about global plant parameter variations in designing the feedback gains. References [5 - 7] develop a design procedure in which a vector-valued cost function reflecting the plant dynamics at a number of operating points is Pareto-optimized. Implementation of the method leads to a nonlinear programming algorithm. In [8, 9] a design approach is described in which loci of permissible gain values are established for each of several operating points. Permissibility, here, means that the gains for each model place its closed-loop poles in some specified region of the complex plane. The feedback design is performed through examining tradeoffs over the intersection of these loci. References [10, 11] describe quadratic optimization procedures for stabilizing the members of a discrete set of linear plants by using output feedback gains which

minimize a scalar cost function reflecting the performance of all of the plants in the set. Finally, in [12 - 15], the problem of stabilizing a set of plants with a single compensator is examined in an algebraic context. A number of elegant results are derived, including global parameterization of stable compensators which stabilize a particular plant and, conversely, plants stabilizable by a given stable compensator [12, 13]. This approach to the global feedback design problem does not, however, appear to be developed to quite the level of practical applicability seen in the methods based on quadratic optimization [14, 15].

There is a serious practical difficulty with all of the work described above. The compromises required to make a fixed set of feedback gains perform adequately over a system's entire operating range can result in less satisfactory closed-loop performance at any single operating point than that attainable using a feedback scheme which can vary to accommodate changes in the plant. These latter feedback schemes fall into two broad categories. The first includes self-tuning regulators [16, 17] and controllers incorporating "online redesign" logic [18]. The second category consists of feedback structures whose gain elements are designed to have a particular functional dependence on parameters coordinatizing the instantaneous system operating point. The most typical of these latter approaches is gain scheduling.

In situations where the plant dynamics are relatively well-known, gain scheduling and similar gain variation strategies have an important conceptual advantage over those employing online tuning. The fact that the actual controller design is performed offline permits the designer as many testing and redesign iterations as are necessary to achieve the desired performance characteristics throughout the plant's operating range. Another consideration, which becomes important when there is a premium on the controller implementation's computational overhead, is that the structure of the gain's dependence on the operating point parameterization is chosen by the designer. This permits further flexibility in deciding the tradeoff between the controller's performance and the complexity of its implementation. These considerations make this class of gain variation scheme es-

pecially well-suited for aerospace applications and it is, in fact, the one most commonly encountered in aerospace industrial practice.

Neither of the two gain scheduling approaches common in industrial practice exploit these advantages efficiently. The first approach consists simply of defining a grid over the domain of the plant parameterization and designing a set of feedback gains for each grid point. In implementation, the gains are obtained by interpolating to the current operating point from these grid points. This brute force approach can be very expensive in terms of the control computer's memory requirements. The other approach is to implement the gain as some simple curve fit, often straight line, to gains calculated at selected parameterized operating points. This cures the problem of excessive computer storage requirements, but introduces grave difficulties of its own. The feedback gains thus implemented are only approximations of the designed gain values, and there is no a priori guarantee that the approximate gains will perform well. The practical result of this is that designing gain-scheduled feedback in this piecemeal fashion characteristically involves a great deal of uncertainty and "cut-and-try."

Recently, the quadratic optimization procedure developed in [11] has been extended to provide a much more orderly and rigorous procedure for designing scheduled-gain feedback [19, 20]. Essentially, the extension consists of redefining the plant models reflected in the problem cost function in such a way as to embed the gain schedule structure in their input and/or output matrices. This method can accommodate any gain schedule structure expressible as a polynomial in the parameters used to coordinatize the plant operating points. Reference [21] provides a full derivation of the procedure, and describes a nontrivial application to a self-repairing flight control problem.

The principal - and significant - advantage of this approach over defining gain schedules though the use of approximate curve fitting methods is that the scheduled gain is "exact" at each designed-for operating point, in the sense that the optimization is based on the scheduled gain's actual effect on the closed-loop dynamics of each plant model appearing in the design. Because of this, at

least at the operating points used in designing the controller, the effect of the scheduled feedback in implementation will be exactly as designed for, modulo effects due to modelling errors. The method's difficulties, on the other hand, are twofold. First, as in [10, 11], a discrete set of linear plant models is used to represent the plant's global parametric variations. The choice of these models is, essentially, a matter of heuristics and experienced judgment. Second, although the method can be used to design polynomial gain schedules of arbitrary finite order, the number of free parameters to optimize increases at least linearly with polynomial order unless further restrictions, again based on heuristics and judgment, are imposed on the form of the schedule.

This report develops an quadratic optimization-based approach for designing feedback gains which vary with system operating condition, applicable when the plant's location in its operating regime can be parameterized by a scalar function. The gain matrix is designed to be the output of a dynamical system having the plant operating point as its independent variable. This is a class of gain variation constraints which contains gain scheduling as a subset. In gain schedules, the gain is a function only of the independent variable; i.e., the system operating condition. The approach developed in this report can be used to design gain variation schemes in which that restriction is relaxed, to permit functional dependence on the instantaneous gain "state." Because of this, the gains can be designed to vary over the plant's operating regime in ways which would require infinite series representations, if implemented as polynomial gain schedules. On the other hand, the order and structure of the gain variation dynamics can be chosen by the designer in a tradeoff between computational overhead in the controller implementation, and the plant's requirement for complex gain variations. An additional positive feature of the theory developed in this report is that, as a direct consequence of the problem formulation, the design reflects the plant dynamics as they vary continuously across the operating regime, rather than only at an arbitrarily chosen collection of setpoints. In the sequel, we will refer to this class of gain variation schemes as PDGP schemes, short for Parameter-Dynamic Gain Propagation.

Section 2 first formulates the PDGP feedback optimization problem with only minimal assumptions made on the form of the gain propagation dynamics, then proposes and examines several candidate PDGP constraint expressions having features which make them attractive for practical implementation. The Section also briefly examines the question of existence of a solution to the general optimization problem. Section 3 derives necessary conditions for optimality for one of the PDGP constraint structures examined in Section 2, and uses these to develop a numerical design algorithm, which is exercised in a numerical example. Conclusions are presented in Section 4.

A significant byproduct of the numerical algorithm development has been the formulation of a novel, efficient iterative procedure for solving discrete Lyapunov equations. A key feature of the theory leading to this method is a least-squares-optimal approximation to certain symmetric sums of Kronecker products. This development is documented in Appendix A. The Lyapunov equation algorithm and its derivation are provided in Appendix B.

2. PDGP PROBLEM FORMULATION

This Section formulates the general PDGP optimization problem on which the numerical design algorithm of Section 3 is based. We consider a plant with input $u \in \mathcal{R}^m$ and output $y \in \mathcal{R}^\ell$ described by

$$\frac{d}{dt}x = f(x, u, \alpha) + \tilde{w}(t, \alpha) \tag{1}$$

$$y = g(x, \alpha) + \tilde{v}(t, \alpha) \tag{2}$$

where $x \in \mathcal{R}^n$ is the plant state, and $\tilde{w}$ and $\tilde{v}$ are white noise processes. The "plant parameter" α gives the location of the plant in its operating regime. It reflects quantities which vary slowly enough during plant operation to be modeled as constant in the feedback design, but whose values affect the plant's dynamic response. Examples of quantities which are often chosen as plant parameters include aging in process control, and Mach number in flight control problems. For this report, it is assumed that α varies on the domain $\alpha_o \leq \alpha \leq \alpha_f$. Assuming that the plant dynamics are linearizable throughout the domain of α, assign a locus of setpoints $\bar{x}$ and $\bar{u}$ as a function of α; that is, $\{\bar{x}(\alpha), \bar{u}(\alpha)\} : \mathcal{R} \rightarrow \mathcal{R}^n \times \mathcal{R}^m$. The plant perturbation dynamics are then described in discrete time by

$$x(k+1, \alpha) = A(\alpha)x(k, \alpha) + B(\alpha)u(k, \alpha) + w(k, \alpha) \tag{3}$$

$$y(k, \alpha) = C(\alpha)x(k, \alpha) + v(k, \alpha) \tag{4}$$

where k is the time sample index. In (3, 4), the variables w and v are discretizations of $\tilde{w}$ and $\tilde{v}$ with

$$E\{w(k,\alpha)w^T(s,\alpha)\} = W(\alpha)\delta_{ks} \tag{5}$$

$$E\{v(k,\alpha)v^T(s,\alpha)\} = V(\alpha)\delta_{ks} \tag{6}$$

$$E\{w(k,\alpha)v^T(s,\alpha)\} = 0 \tag{7}$$

The plant is to be stabilized for all $\alpha \in [\alpha_o, \alpha_f]$ by an output feedback control law

$$u(k,\alpha) = -G(\alpha)y(k,\alpha) \tag{8}$$

in which the feedback gain varies with α according to

$$\dot{\theta} = \Gamma(\theta,\alpha) \tag{9}$$

$$G = \Pi(\theta,\alpha) \tag{10}$$

where the "dot" notation on θ denotes differentiation with respect to α rather than time. Free parameters are $\theta(\bar{\alpha})$, the value of $\theta \in \mathcal{R}^q$ at some fixed $\bar{\alpha}$ in the domain of α. The PDGP feedback design problem, then, consists of choosing $\bar{\alpha}$ and the functional form of Γ and Π, then adjusting the value of $\theta(\bar{\alpha})$ to achieve the desired performance. The potential complexity of this problem motivates the use of optimization methods in the design process.

A cost function commonly used in discrete-time optimal output feedback design at a given operating point [11] is

$$c(\alpha) = \lim_{N\to\infty} \frac{1}{2(N+1)} \sum_{k=0}^{N} E\{x^T(k+1,\alpha)Q(\alpha)x(k+1,\alpha) + u^T(k,\alpha)R(\alpha)u(k,\alpha)\} \tag{11}$$

where $Q(\alpha) \geq 0$, $R(\alpha) \geq 0$. Reference [11] develops a number of sufficiency conditions for the existence of a G which minimizes (11) at a given operating point. Thinking of $c(\alpha)$ as the "α - instantaneous" system performance, it is reasonable to define a global cost over the operating regime by

$$J = \int_{\alpha_o}^{\alpha_f} c(\alpha)d\alpha \tag{12}$$

In the interest of clarity, notation indicating dependence on α will be suppressed in the sequel. The problem of minimizing (12) subject to (3, 4) and (9, 10) is more conveniently posed as that of minimizing a Lagrangian function. It is well known that, when the closed-loop dynamics defined by (3, 4) and (8) are asymptotically stable, c can be expressed as

$$c = tr\{KW\} + tr\{G^T(B^TKB + R)GV\} \tag{13}$$

where K satisfies the constraint

$$S(K,G) = \phi^T(G)K\phi(G) - K + Q + C^TG^TRGC = 0 \tag{14}$$

$$\phi(G) = A - BGC \tag{15}$$

The constraints (9, 10) and (14) are adjoined to J, using the expression (13), to form the Lagrangian

$$\mathcal{L} = \int_{\alpha_o}^{\alpha_f} c + tr\{S(K,G)\Lambda_K^T\} + tr\{[G - \Pi(\theta)]\Lambda_G^T\} + \lambda_\theta^T[\Gamma(\theta) - \dot{\theta}]d\alpha \tag{16}$$

where Λ_K, Λ_G and λ_θ are Lagrange multipliers associated with the constraints on K, G and θ.

Denoting the integrand of (16) as $\mathcal{H}$, the Euler-Lagrange equations for a stationary point of $\mathcal{L}$ are

$$\partial\mathcal{H}/\partial K = 0 \qquad \partial\mathcal{H}/\partial\Lambda_K = 0 \tag{17.a}$$

$$\partial\mathcal{H}/\partial G = 0 \qquad \partial\mathcal{H}/\partial\Lambda_G = 0 \tag{17.b}$$

$$\dot{\lambda}_\theta = -\partial\mathcal{H}/\partial\theta \qquad \dot{\theta} = \Gamma(\theta,\alpha) \tag{17.c}$$

Equations (17.a), expanded, are discrete Lyapunov equations:

$$\partial\mathcal{H}/\partial K = \phi\Lambda_K\phi^T - \Lambda_K + W + BGVG^TB^T = 0 \tag{18}$$

$$\partial\mathcal{H}/\partial\Lambda_K = S(K,G) = 0 \tag{19}$$

Equations (17.b), expanded, are

$$\partial\mathcal{H}/\partial G = \hat{K}G\hat{\Lambda}_K - B^TKA\Lambda_KC^T + \Lambda_G = 0 \tag{20.a}$$

$$\hat{K} = B^TKB + R \tag{20.b}$$

$$\hat{\Lambda}_K = C\Lambda_KC^T + V \tag{20.c}$$

$$\partial\mathcal{H}/\partial\Lambda_G = G - \Pi(\theta,\alpha) = 0 \tag{21}$$

and the equation for $\dot{\lambda}_\theta$ from (17.c) is

$$\dot{\lambda}_\theta = \frac{\partial}{\partial \theta}[tr\{\Pi(\theta,\alpha)\Lambda_G^T\} - \lambda_\theta^T \Gamma(\theta,\alpha)] \tag{22}$$

There remains the problem of optimally assigning boundary conditions on θ and λ_θ. There is no terminal cost in this problem, so $\lambda_\theta(\alpha_f)$ is chosen as

$$\lambda_\theta(\alpha_f) = 0 \tag{23}$$

The effect of the choice of $\theta(\alpha_o)$ on the cost is obtained by integrating $\lambda^T\dot{\theta}$ by parts in (16), which leads to the necessary condition

$$\partial \mathcal{L}/\partial\theta(\alpha_o) = \lambda_\theta(\alpha_o) = 0 \tag{24}$$

In summary, then, once a structure for Γ and Π is chosen, the PDGP optimization problem reduces to a matter of determining the optimal initial condition for θ. Since only the initial conditions are free in this problem, the key to effectively employing the above theory in designing a feedback gain variation scheme is to hold elements of θ constant, thus assigning them the role of static free parameters in the optimization. For example, a typical single-variable gain schedule takes the form

$$G(\alpha) = N + (\alpha - \alpha_o)M \tag{25}$$

This can be reexpressed as a PDGP scheme:

$$\dot{G} = M \qquad G(\alpha_o) = N \tag{26}$$

With the above notation, one could choose θ as

$$\theta = vec[G : M] \tag{27}$$

where $vec(.)$ is the column-stacking operator, so that

$$\Gamma = vec[M : 0] \tag{28}$$

$$\Pi = mat\{[I_{ml} : 0]\theta\} \tag{29}$$

where $mat(.)$ denotes column unstacking and I_{ml} in an identity matrix of dimension ml. Given the simplicity of (28, 29) however, it is easier to separate G and M, and assign Lagrange multipliers Λ_G and Λ_M. The Lagrangian for this problem, then, is

$$\mathcal{L} = \int_{\alpha_o}^{\alpha_f} c + tr\{S(K,G)\Lambda_k^T\} + tr\{[M - \dot{G}]\Lambda_G^T\} + tr\{-\dot{M}\Lambda_M^T\}d\alpha \tag{30}$$

and the Euler-Lagrange equations are (18, 19) and

$$\dot{G} = M \tag{31}$$

$$\dot{M} = 0 \tag{32}$$

$$-\partial\mathcal{H}/\partial G = \dot{\Lambda}_G = B^T K A \Lambda_K C^T - \hat{K} G \hat{\Lambda}_K \tag{33}$$

$$-\partial\mathcal{H}/\partial M = \dot{\Lambda}_M = -\Lambda_G \tag{34}$$

with boundary conditions

$$\Lambda_G(\alpha_o) = \Lambda_G(\alpha_f) = 0 \tag{35}$$

$$\Lambda_M(\alpha_o) = \Lambda_M(\alpha_f) = 0 \tag{36}$$

We now turn our attention to PDGP structures in which the gain matrix varies as a linear combination of states satisfying a system of linear homogeneous ordinary differential equations (ODEs). Two of the simplest Γ structures suitable for implementation in a flight control computer are

$$\dot{\theta} = N\theta M \tag{37}$$

$$\dot{\theta} = N\theta + \theta M \tag{38}$$

where $\theta \in \mathcal{R}^{p\times q}$, $N \in \mathcal{R}^{p\times p}$ and $M \in \mathcal{R}^{q\times q}$. These are both special cases of

$$vec\,\dot{\theta} = F\,vec\,\theta \tag{39}$$

Their implementational advantages over (39) lie in storage and computational considerations. In (39), $F \in \mathcal{R}^{pq\times pq}$ and p^2q^2 multiplications are required to calculate $\dot{\theta}$. In (37) and (38), N and M occupy $p^2 + q^2$ locations, and the derivative calculation requires $qp^2 + pq^2$ multiplications. This difference becomes important for large-order PDGS schemes. A suitable choice of Π structure for either (37) or (38) is

$$G = S\theta D + P \tag{40}$$

The elements of S, D and P can be either free parameters in the optimization, or fixed.

The chief characteristic of linear PDGP structures such as those defined in (37 - 40) is their analytical simplicity. This by no means implies that these structures are limited in the level of closed-loop performance they can attain, however. The following lemma quantifies this issue.

Lemma 1 : Assume that $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $Q(\alpha)$, $R(\alpha)$, $W(\alpha)$ and $V(\alpha)$ have continuous k^{th} derivatives on the interval $[\alpha_o, \alpha_f]$ for $k > 0$. Further, assume that the system (3, 4) is stabilizable by output feedback and that for each $\alpha \in [\alpha_o, \alpha_f]$ there exists a

$$G^*(\alpha) = arg \min_{G \in \mathcal{G}(\alpha)} \{c(\alpha)\} \tag{41}$$

for $c(\alpha)$ defined in (11), and where $\mathcal{G}(\alpha)$ is the set of asymptotically stabilizing gains for the plant at the operating point parameterized by the current value of α. Then, for $\theta \in \mathcal{R}^q$ given by (39) with $q \geq 2mlr$ for integer $r > 0$, and G given by

$$vec\, G = S\, vec\, \theta + P \tag{42}$$

there exist F, S, P and $\theta(\alpha_o)$ such that

$$\max_{\alpha \in [\alpha_o, \alpha_f]} \|G^*(\alpha) - G(\alpha)\| < b/(r+1)^{k+1} \tag{43}$$

where $b \geq 0$ is a constant.

Proof : See Appendix C.

The basic implication of this lemma is that by specifying gain dynamics of sufficiently large dimension in a linear PDGP problem, one can construct a gain variation which is arbitrarily close to the unconstrained variation of G^* over the operating regime. We note that the bound (43) was obtained by constructing an individual Fourier series expansion for each element of G and is, thus, extremely conservative.

In the next Section, the PDGP optimization problem will be solved for a slight simplification of the structure (38, 40). One of the attractive features of (38) is that it leads to a very simple transition expression:

$$\theta(\alpha + \Delta\alpha) = e^{N\Delta\alpha}\theta(\alpha)e^{M\Delta\alpha} \tag{44}$$

This can be easily exploited to permit use of the theory developed thus far in designing controllers for systems whose operating regimes are best coordinatized by a vector-valued parameter. For example, consider the case of an operating regime parameterized by two variables, say, p_1 and p_2. Now, define a locus of $\rho = (p_1(\alpha), p_2(\alpha))$ in the operating regime forming a path, with arc length α as the independent variable. This situation is illustrated in Figure 1, for ρ chosen to be a spiral. In order to move from point A to point B, one simply integrates (38) between $\alpha(A)$ and $\alpha(B)$. To move to point C, rather than laboriously propagate (38) around three revolutions of the spiral, it would be more economical to have $e^{N\Delta\alpha}$ and $e^{M\Delta\alpha}$ for $\Delta\alpha$ roughly corresponding to one full turn stored in the controller implementation. This would permit using (44) to "cut across" turns of the spiral to the turn containing C. The final correction in reaching C would then be performed, again using (38).

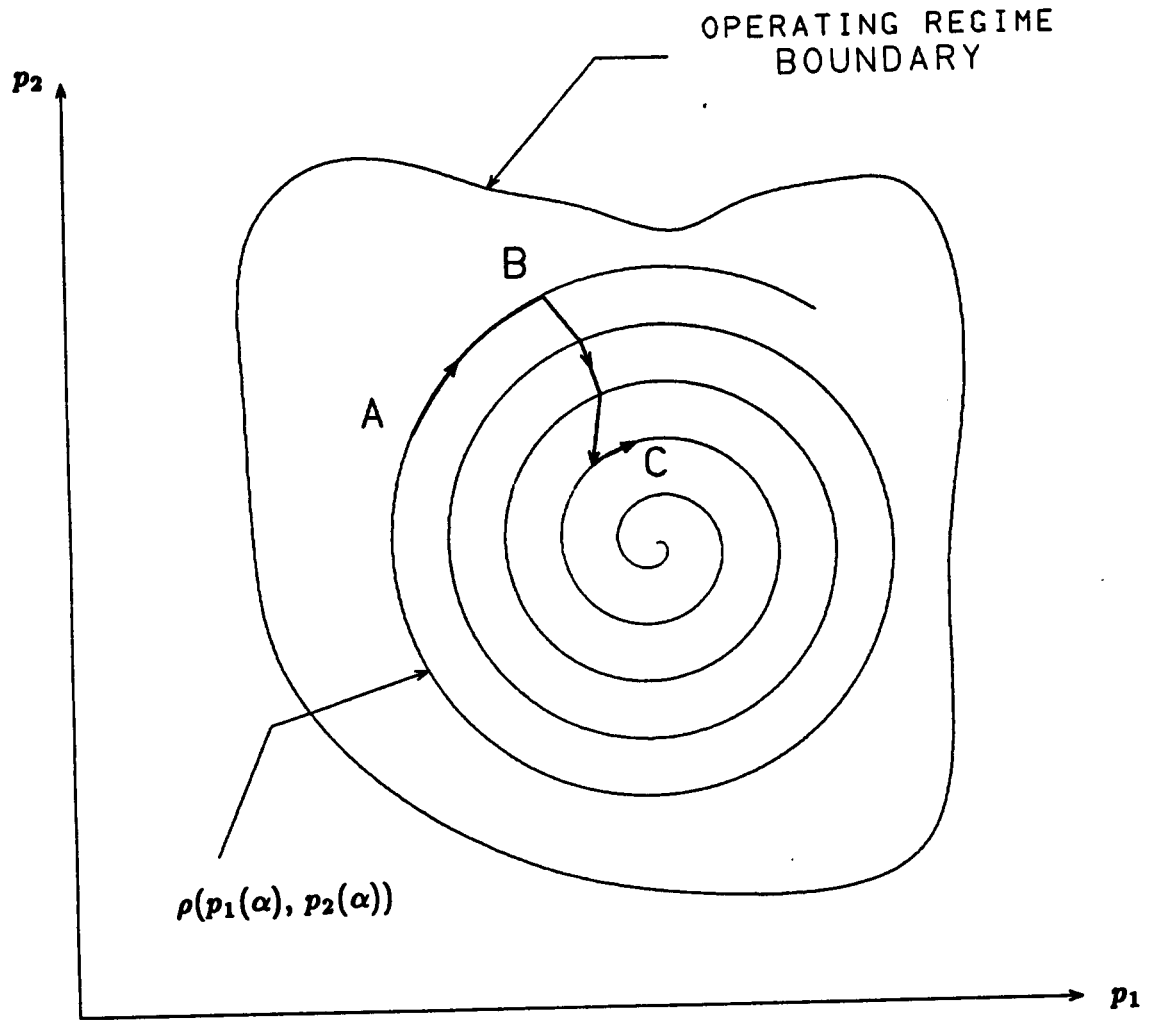

FIGURE 1. MULTIPARAMETER IMPLEMENTATION OF SINGLE-PARAMETER PDGP SCHEME

3. A LINEAR PDGP SCHEME

In this Section, necessary conditions for optimality in a simple linear PDGP structure are derived. These, in turn, are used in a computational algorithm for calculating locally optimal gain parameters. The algorithm is exercised in a numerical example. The class of gain propagation schemes considered take the form (38, 40) with S and D in (40) restricted to identity matrices, and $P = 0$; in other words,

$$\dot{G} = NG + GM \qquad G(\alpha_o) = G_o \tag{45}$$

$$\dot{N} = 0 \tag{46}$$

$$\dot{M} = 0 \tag{47}$$

The Lagrangian for this problem is

$$\mathcal{L} = \int_{\alpha_o}^{\alpha_f} c + tr\{S(K,G)\Lambda_K^T\} + tr\{[NG + GM - \dot{G}]\Lambda_G^T\} + tr\{-\dot{M}\Lambda_M^T\} + tr\{-\dot{N}\Lambda_N^T\} d\alpha \tag{48}$$

and the Euler-Lagrange equations are (14) and (18), (45 - 47) and

$$\dot{\Lambda}_G = -\partial\mathcal{H}/\partial G = -N^T\Lambda_G - \Lambda_G M^T - \hat{K}G\hat{\Lambda}_K + B^T K A \Lambda_K C^T \tag{49}$$

$$\dot{\Lambda}_N = -\partial\mathcal{H}/\partial N = -\Lambda_G G^T \tag{50}$$

$$\dot{\Lambda}_M = -\partial \mathcal{H}/\partial M = -G^T \Lambda_G \tag{51}$$

The boundary conditions to be satisfied are

$$\Lambda_G(\alpha_o) = \Lambda_G(\alpha_f) = 0 \tag{52}$$

$$\Lambda_N(\alpha_o) = \Lambda_N(\alpha_f) = 0 \tag{53}$$

$$\Lambda_M(\alpha_o) = \Lambda_M(\alpha_f) = 0 \tag{54}$$

Solution of the necessary conditions for optimality in this problem consists, essentially, of determining G_o, N and M such that Λ_G, Λ_M and Λ_N have null initial conditions. The simplest general solution approach is to exploit the fact, from (24), that

$$\partial \mathcal{L}/\partial G_o = \Lambda_G(\alpha_o) \tag{55}$$

$$\partial \mathcal{L}/\partial N = \Lambda_N(\alpha_o) \tag{56}$$

$$\partial \mathcal{L}/\partial M = \Lambda_M(\alpha_o) \tag{57}$$

Equations (55 - 57) suggest converting the problem of satisfying the costate boundary conditions to one of locally minimizing the Lagrangian using a descent algorithm. For example, a simple steepest descent algorithm takes the form:

0. Determine an initial guess for $G^o(\alpha_f)$, N^o and M^o. Set $k = 0$.

1. Integrate (49-51) backwards from α_f to α_o, to obtain $\Lambda_G^k(\alpha_o)$, $\Lambda_N^k(\alpha_o)$ and $\Lambda_M^k(\alpha_o)$.

2. Increment G_o, N and M as

$$G_o^{k+1} = G_o^k - \xi_k \Lambda_G^k(\alpha_o) \tag{58}$$

$$N^{k+1} = N^k - \xi_k \Lambda_N^k(\alpha_o) \tag{59}$$

$$M^{(k+1)} = M_k - \xi \Lambda_M^k(\alpha_o) \tag{60}$$

. where $\xi_k \in (0, 1]$ is chosen to ensure a satisfactory cost decrease for the iteration. One standard criterion, given in [22], for determining ξ_k takes the following form for this problem:

$$\rho \xi_k \varphi_k \leq J^k - J^{k+1} \leq (1 - \rho) \xi_k \varphi_k \tag{61}$$

. where ρ is a fixed parameter in the interval $(0, 0.5]$, J^k is the value of J from (12) for the k^{th} iteration's parameter values, and

$$\varphi_k = \|\Lambda_G^k(\alpha_o)\|^2 + \|\Lambda_N^k(\alpha_o)\|^2 + \|\Lambda_M^k(\alpha_o)\|^2 \tag{62}$$

3. Set $k = k + 1$ and go to 1.

In order to put the above theory in perspective, we now consider a feedback design problem for a simple linear system with variable dynamics, given by

$$\dot{x} = A_c + u + w \tag{63}$$

$$y = x \tag{64}$$

where A_c is

$$A_c = \begin{bmatrix} 0 & 1 \\ -\omega^2(\alpha) & -.01 \end{bmatrix} \tag{65}$$

and ω^2 varies as

$$\omega^2(\alpha) = .0001 + (10 - .0001)\alpha \tag{66}$$

for $0 \leq \alpha \leq 1$. The system is sampled at 10 Hz, and it is assumed that the discretized process noise covariances are $W = I$ and $V = 0$. The penalty weights are $Q = R = I$.

Feedback gains were calculated using the PDGP formulation of this Section and, for comparison, the design approaches described in [11] and [19 - 21]. In [11, 19 - 21], the cost function is defined for a discrete collection of plant models, which are chosen by the designer to represent plant variation over the domain of α. The cost takes the form

$$J = \sum_{j=1}^{n_p} c(\alpha_j) \tag{67}$$

where n_p is the number of models in the collection, and $c(\alpha_j)$ is the quadratic regulation cost defined in (11). For the example problem, 11 plant models were chosen, spaced over $0 \leq \alpha \leq 1$ at intervals of $\Delta\alpha = 0.1$.

The necessary conditions for optimality consist of satisfying (18, 19) at each of the α_j and

$$\sum_{j=1}^{n_p} \hat{K}_j G(\hat{\Lambda}_K)_j - B_j^T K_j A_j (\Lambda_K)_j C_j^T = 0 \tag{68}$$

where the j subscripts denote evaluation at α_j. Equation (68) can be viewed as a discrete-parameter analog of (20.a). In [11], G does not vary with α. References [19 - 21] develop an extension of the theory in [11] in which the gain schedule structure

$$G(\alpha) = G_o + \alpha G_1 \tag{69}$$

is embedded into the optimization problem formulation. It should be noted that the theory in [11, 19 - 21] also accommodates vector-valued α, in addition to the scalar case.

Optimal feedback solutions for the fixed-gain, scheduled-gain and PDGP formulations are given in Table 1. In order to represent unconstrained variation of the optimal gains, "pointwise" optimal output feedback gains ($n_p = 1$) were calculated for each of the plants in the fixed and scheduled-gain design model. The numerical calculations were considered converged when the 2-norm of the gradients $\Lambda_G(0)$ or (68) dropped below one percent of the cost. In order to consistently compare the performance of the various feedback variation structures, the α-integral performances (12) of the fixed, scheduled and pointwise-optimal designs were calculated from their individual model costs, using Simpson's rule.

Unsuprisingly, the fixed-gain design gave the worst performance, 14.25, whereas the unconstrained pointwise-optimal design returned a cost of 13.24. For this example, the PDGP and scheduled-gain designs returned 13.38 and 13.36, respectively. These values can be considered virtually the same, given the rather casual accuracy of the Simpson's rule quadrature used to obtain the cost for the scheduled-gain design. The variation of the model cost $c(\alpha)$ for each of the gain solutions is displayed in Figure 2. The scheduled-gain design optimization apparently sacrificed the performance at $\alpha = 0$ in order to achieve a good match to pointwise optimal performance across $0.1 \leq \alpha \leq 1$. The PDGP scheme, on the other hand, resulted in less severely degraded performance at $\alpha = 0$, but allows the degradation to persist until approximately $\alpha = 0.12$; thereafter, it very slightly outperforms the scheduled gain.

Figure 3 displays the gain variation histories for the elements of G. It would appear that the $1,1$ element of G is bears primary responsibility for adding damping to the system as ω^2 in (65) increases. Both the PDGP and scheduled-gain solutions closely approximate the trajectory of this element. The scheduled-gain and PDGP histories for the other elements, however, are significantly different, particularly for the off-diagonal terms. It must be remembered that the goal

TABLE 1. FEEDBACK GAIN VARIATION SOLUTIONS

FIXED: $$G = \begin{bmatrix} 1.9149 & -0.1468 \\ -0.0845 & 0.6476 \end{bmatrix}$$

SCHEDULED: $$G = G_o + \alpha G_1$$

$$G_o = \begin{bmatrix} 0.9263 & 0.0471 \\ 0.1670 & 0.9882 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 1.7343 & -0.2920 \\ -0.8645 & -0.7316 \end{bmatrix}$$

PDGP: $$\frac{dG}{d\alpha} = NG + GM$$

$$G(0) = \begin{bmatrix} 1.0693 & 0.0057 \\ -0.0076 & 0.8944 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.4188 & -0.2062 \\ 0.1956 & 3.5646 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.5246 & 0.3326 \\ -0.0015 & -4.4935 \end{bmatrix}$$

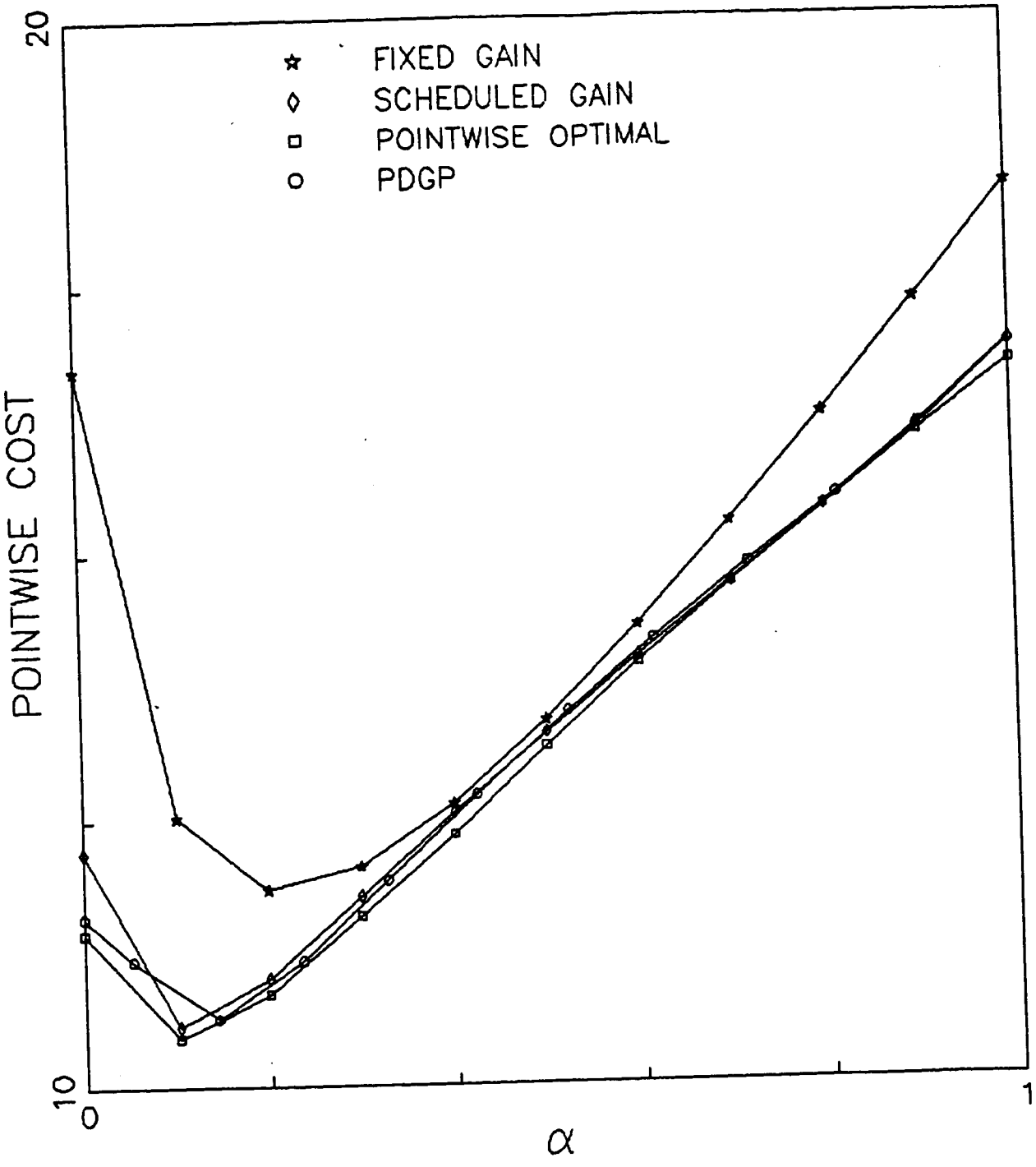

FIGURE 2. VARIATION OF $c(\alpha)$

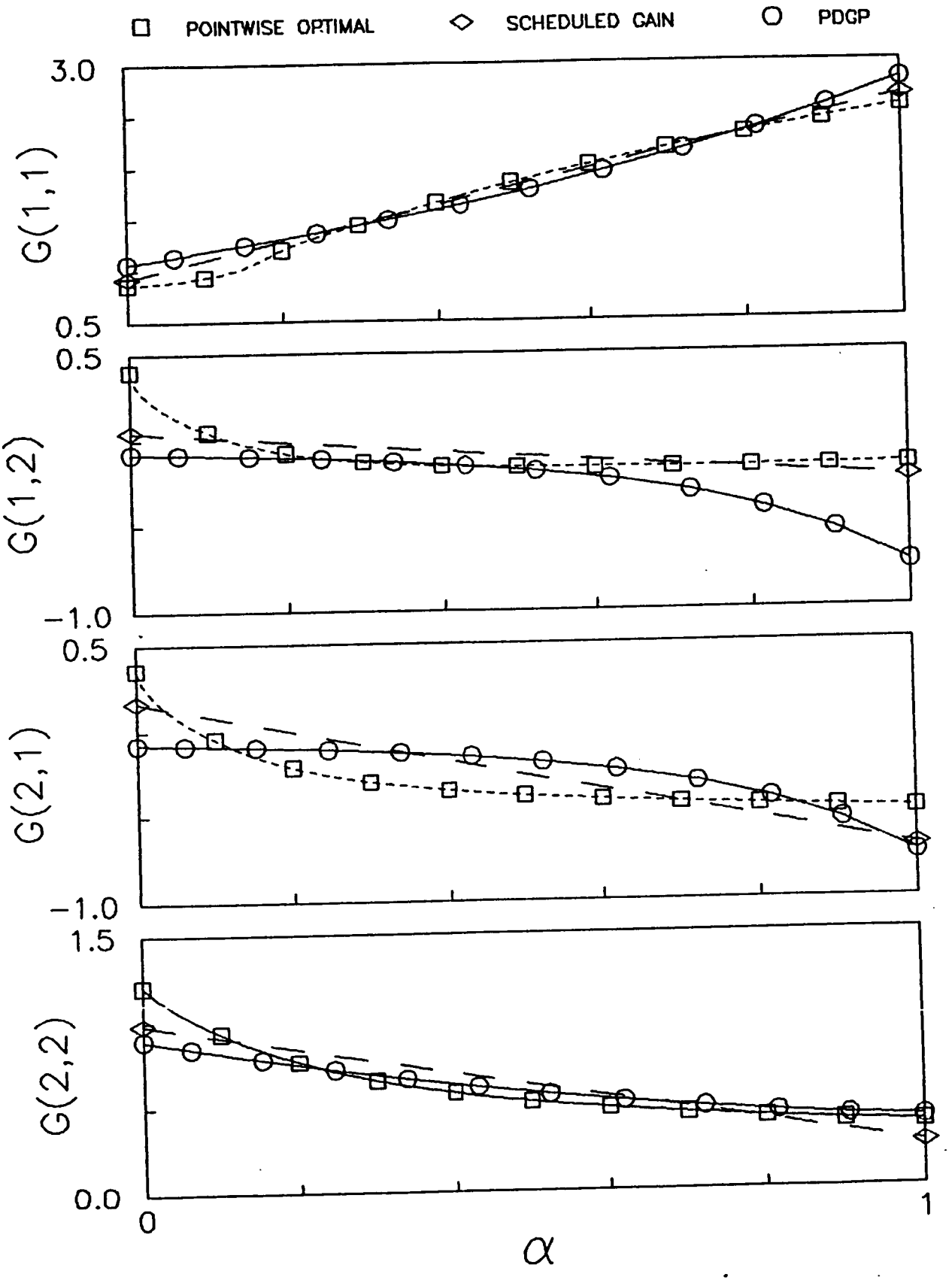

FIGURE 3. GAIN VARIATION HISTORIES

of the scheduled-goal of the scheduled-gain and PDGP methods considered here is not necessarily producing gains which provide a "best-fit" to the unconstrained variation of optimal G, but rather, to optimize the *system* performance over the domain of a α, subject to the gain schedule or PDGP dynamic constraints. On the other hand, as we know from Lemma 1, if a PDGP structure with more degrees of freedom had been chosen, the trajectory matches between the unconstrained and PDGP gains would have been closer.

4. CONCLUSIONS

This report has documented a research effort which developed and examined a novel optimization-based approach to designing multivariable output feedback gains which vary to accommodate changes in the plant dynamics being regulated: Parameter-Dynamic Gain Propagation, or PDGP. The key feature of the approach is that the designed gains vary as the output of a dynamical system which, as its independent variable, uses a parameter which coordinatizes the plant dynamics in a manner analogous to the parameter used in traditional gain scheduling schemes.

The general optimization problem was formulated, and necessary conditions for optimality were derived. Conditions for the existence of a solution to the problem were also obtained. Several simplified PDGP structures were examined, with special attention being paid to a restricted version of the general linear structure, for which a numerical optimization algorithm was derived and implemented. This latter was tested and demonstrated in a numerical example.

APPENDIX A

The work described in this appendix was performed under NASA Contract NAS1-17493 and is included here for completeness.

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LEAST SQUARES APPROXIMATION OF SYMMETRIC KRONECKER PRODUCT SUMS

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APPENDIX A.

LEAST SQUARES APPROXIMATION

OF SYMMETRIC KRONECKER PRODUCT SUMS

A situation which occurs with some frequency in optimization analyses involving matrix-valued quantities is that of having to invert positive definite matrices having the general form

$$H = \sum_{i=1}^{\eta} A_i \otimes B_i + \sum_{j=1}^{\nu} [C_j \otimes C_j + C_j^T \otimes C_j^T] \tag{A.1}$$

where $A_i \geq 0$, $B_i \geq 0$ for $i = 1, ..., \eta$, and $C_j, j = 1, ..., \nu$ are general square matrices. Numerical inversion of H is usually very cumbrous unless the terms in the Kronecker products are of relatively small dimension. Furthermore, one of the most common situations for which a requirement for H^{-1} exists is in Newton or quasi-Newton iterative schemes, where H is the Hessian of a scalar cost function which is to be minimized. In these cases, the exact value of H^{-1} is not actually needed, insofar as the inverse Hessian is only used to exploit "curvature" information and to provide scaling in modifying the gradient-based descent step in the problem's free parameters. This motivates the notion of approximating H by a quantity $\tilde{H}$, having the structure

$$\tilde{H} = S \otimes P \tag{A.2}$$

so that its inverse is

$$\tilde{H}^{-1} = S^{-1} \otimes P^{-1} \tag{A.3}$$

When the C_j are zero, a straightforward approach to solving this approximation problem is to calculate S and P to minimize the Euclidean norm of $H - S \otimes P$, given by

$$J = tr\{H^T H - (S^T \otimes P^T)\, H - H^T (S \otimes P) + S^T S \otimes P^T P\} \tag{A.4}$$

The necessary conditions for a minimum in J are

$$\partial J/\partial S = 0 \qquad \partial J/\partial P = 0 \tag{A.5}$$

or

$$\partial J/\partial S = \|P\|^2 S - \sum_{i=1}^{\eta} A_i tr\{B_i P\} = 0 \tag{A.6}$$

$$\partial J/\partial P = \|S\|^2 P - \sum_{i=1}^{\eta} B_i tr\{A_i S\} = 0 \tag{A.7}$$

Equations (A.6) and (A.7) can be solved iteratively, using the following simple algorithm:

0.) Choose any $P^{(0)} > 0$. Set $k = 0$.

1.) Set $k = k + 1$.

2.) From (A.7), evaluate

$$S^{(k)} = \frac{1}{\|P^{(k-1)}\|^2} \sum_{i=1}^{\eta} A_i tr\{B_i P^{(k-1)}\} \tag{A.8}$$

3.) From (A.8), evaluate

$$P^{(k)} = \frac{1}{\|S^{(k)}\|^2} \sum_{i=1}^{\eta} B_i tr\{A_i S^{(k)}\} \tag{A.9}$$

4.) Go to 1.

Numerical experience [21] indicates that this algorithm is quite robustly convergent, and a formal proof of algorithmic convergence is under development. It can easily be seen from (A.6) and (A.7) that

$$\partial^2 J/\partial vec S \partial vec^T S = \|P\|^2 I_{n^2 \times n^2} \tag{A.10}$$

$$\partial^2 J/\partial vec P \partial vec^T P = \|S\|^2 I_{n^2 \times n^2} \tag{A.11}$$

so that (A.8) and (A.9) are a quasi-Newton iteration.

In order for the approximation (A.2) to be of practical value, it is important that $S > 0$ and $P > 0$, so that the inverse in (A.3) is positive definite. Furthermore, since the algorithm is only exercised through a finite number of iterations, it is desirable to know for what k one can guarantee that $S^{(k)} > 0$ and $P^{(k)} > 0$. These issues are settled in the following lemma:

Lemma2 : If $H > 0$ then $S^{(k)} > 0$ and $P^{(k)} > 0$ for all $k > 0$.

Proof : See Appendix C.

Unfortunately, a direct extension of the above theory to the case where the C_j are not zero does not generally result in $\tilde{H} > 0$. In order to treat this case, it is necessary to constrain $S \geq 0$ and $P \geq 0$. This is done for A_i, B_i and $C_j \in \mathcal{R}^{n \times n}$ by enforcing

$$S = \bar{S}\bar{S}^T \qquad P = \bar{P}\bar{P}^T \tag{A.12}$$

where $\bar{S} \in \mathcal{R}^{n \times n_s}$ and $\bar{P} \in \mathcal{R}^{n \times n_p}$, where n_s and n_p may be freely chosen. For this case, we seek a minimum of

$$J = tr\{H^T H - \bar{S}\bar{S}^T \otimes \bar{P}\bar{P}^T H - H^T \bar{S}\bar{S}^T \otimes \bar{P}\bar{P}^T + \bar{S}\bar{S}^T \bar{S}\bar{S}^T \otimes \bar{P}\bar{P}^T \bar{P}\bar{P}^T\} \tag{A.13}$$

The necessary conditions for a local minimum of J are

$$\partial \bar{J}/\partial \bar{S} = 4tr\{(\bar{P}^T\bar{P})^2\}\bar{S}\bar{S}^T\bar{S} - 4\sum_{i=1}^{\eta} tr\{\bar{P}^T B_i \bar{P}\}A_i\bar{S} - 4\sum_{j=1}^{\nu} tr\{\bar{P}^T C_j \bar{P}\}(C_j^T + C_j)\bar{S} = 0 \qquad \text{(A.14)}$$

$$\partial \bar{J}/\partial \bar{P} = 4tr\{(\bar{S}^T\bar{S})^2\}\bar{P}\bar{P}^T\bar{P} - 4\sum_{i=1}^{\eta} tr\{\bar{S}^T A_i \bar{S}\}B_i\bar{P} - 4\sum_{j=1}^{\nu} tr\{\bar{S}^T C_j \bar{S}\}(C_j^T + C_j)\bar{P} = 0 \qquad \text{(A.15)}$$

Equations (A.14) and (A.15) could be incorporated into a steepest descent algorithm for performing the minimization of J. This is not particularly attractive, however, because of the known slow convergence of steepest descent methods. It is preferable, instead, to seek a numerically inexpensive means of incorporating some, if not all, of the second derivative information into the iterations. The simplest is to use the increments

$$\delta\bar{S}_{ij} = -\frac{\partial J/\partial \bar{S}_{ij}}{\partial^2 J/\partial \bar{S}_{ij}^2} \qquad \delta\bar{P}_{ij} = -\frac{\partial J/\partial \bar{P}_{ij}}{\partial^2 J/\partial \bar{P}ij^2} \qquad \text{(A.16)}$$

where $(.)_{ij}$ is the ij^{th} element of the matrix. This is equivalent to using the inverse of the main diagonals of $\partial^2 J/\partial vecS\partial vec^T S$ and $\partial^2 J/\partial vecP\partial vec^T P$ in the step calculations. This approach has been used in other general optimization problems [23], and has provided a significant enhancement in algorithmic performance over steepest descent in those applications. The main diagonal elements of the Hessian matrix are given by

$$\frac{\partial^2 J}{\partial \bar{S}_{ij}^2} = 4tr\{(\bar{P}^T\bar{P})^2\}[(\bar{S}^T\bar{S})_{jj} + (\bar{S}\bar{S}^T)_{ij} + \bar{S}_{ij}^2]$$
$$- 4\sum_{k=1}^{\eta} tr\{\bar{P}^T B_k \bar{P}\}(A_k)_{ii}$$
$$- 4\sum_{m=1}^{\nu} tr\{\bar{P}^T C_m \bar{P}\}(C_m^T + C_m)_{ii} \qquad \text{(A.17)}$$

$$\frac{\partial^2 J}{\partial \bar{P}_{ij}^2} = 4tr\{(\bar{S}^T\bar{S})^2\}[(\bar{P}^T\bar{P})_{jj} + (\bar{P}\bar{P}^T)_{ii} + \bar{P}_{ij}^2]$$
$$- 4\sum_{k=1}^{\eta} tr\{\bar{S}^T A_k \bar{S}\}(B_k)_{ii}$$
$$- 4\sum_{m=1}^{\nu} tr\{\bar{S}^T C_m \bar{S}\}(C_m^T + C_m)_{ii} \qquad \text{(A.18)}$$

The algorithm for solving (A.14) and (A.15) then takes the form

0.) Choose $\bar{P}^{(0)} > 0$, $\bar{S}^{(0)} > 0$. Set $k = 0$.

1.) Set $k = k + 1$.

2.) Evaluate $\delta\bar{S}^{(k)}$, $\delta\bar{P}^{(k)}$ using (A.16).

3.) Increment $\bar{S}^{(k)}$ and $\bar{P}^{(k)}$ as

$$\bar{S}^{(k+1)} = \bar{S}^{(k)} + \bar{\xi}_k \delta\bar{S}^{(k)} \qquad \text{(A.19)}$$

$$\bar{P}^{(k+1)} = \bar{P}^{(k)} + \bar{\xi}_k \delta\bar{P}^{(k)} \qquad \text{(A.20)}$$

where ξ_k is chosen to ensure that each iteration brings $S^{(k+1)}$ and $P^{(k+1)}$ closer to convergence.

4.) Go to 1.

We are deliberately vague in Step 3 of the above algorithm. It would be very straightforward to base the choice of ξ_k on achieving some improvement in the cost J; for example, choosing ξ_k to satisfy

$$J(\bar{S}^{(k+1)}, \bar{P}^{(k+1)}) < J(\bar{S}^{(k)}, \bar{P}^{(k)}) \tag{A.21}$$

Unfortunately, there is no information about the value of J directly available from the calculations in (A.16). Also, it can be seen by comparing (A.13) with (A.14 - A.18) that evaluation of (A.21) would add a very significant increment of numerical overhead to the algorithm, particularly when n is large, and n_s and n_p are relatively small.

It would be much more efficient to base the test for improvement on quantities such as gradient norms or, better yet, the gradient traces. Work is in progress for determining an efficient, but reliable, improvement test. In testing the above algorithm numerically on a 20^{th}-order problem (H was 400^{th}-order), Step 3 was omitted entirely and ξ_k was fixed at $\xi = 0.25$, with good success.

Note that the numerical overhead for evaluating (A.16) varies significantly with n_s and n_p. From (A.14 - A.16) and $n_s = n_p = n$, we see that one pass through Step 2 of the algorithm requires $O(4(\eta + \nu + 5/4)n^3)$ multiplications. Unfortunately, when $n_s < n$ or $n_p < n$, the resulting approximation (A.2, A.12) is rank deficient. For treating cases where a sequence $\{H_k : k = 1, ...\}$ must be approximated, in which H_k is "close" to H_{k-1}, an efficient algorithmic approach would consist of approximating H_1 using $n_s = n_p = n$, then approximating subsequent members of the sequence for $n_s = n_p << n$ by

$$\tilde{H}_{k+1} = (\beta \tilde{S}_k + \bar{S}_{k+1}\bar{S}_{k+1}^T) \otimes (\beta \tilde{P}_k + \bar{P}_{k+1}\bar{P}_{k+1}^T) \tag{A.22}$$

$$\tilde{S}_k = \bar{S}_1\bar{S}_1^T + \sum_{j=1}^{k} \bar{S}_j\bar{S}_j^T \tag{A.23}$$

$$\tilde{P}_k = \bar{P}_1\bar{P}_1^T + \sum_{j=1}^{k} \bar{P}_j\bar{P}_j^T \tag{A.24}$$

where

$$\bar{S}_k, \bar{P}_k = arg \min_{S,P} \|\tilde{H}_k - H_k\| \cdot \tag{A.25}$$

and $0 < \beta < 1$ is chosen to prevent the norm of $S_k \otimes P_k$ from growing larger than that of H_k. Note that, if $\tilde{S}_1 > 0$ and $\tilde{P}_1 > 0$, then each subsequent $\tilde{S}_k > 0$ and $\tilde{P}_k > 0$. This procedure can be approximated in a roundabout manner by sequentially combining the first and second algorithms in this Appendix, but a direct method of calculating $\bar{S}_k$, $\bar{P}_k$ would be much more satisfying. Although it is unlikely that the derivation of such a method poses significant difficulty, time and the scope of the project did not permit pursuing this issue.

APPENDIX B.

AN EFFICIENT ITERATIVE PROCEDURE FOR SOLVING DISCRETE LYAPUNOV EQUATIONS

In the numerical algorithm described in Section 3, discrete Lyapunov equations (14) and (18) are solved for K and Λ_K at each step in α over the integration interval $[\alpha_f, \alpha_o]$. It can plainly be seen that, regardless of the solution procedure used, this is one of the dominant numerical overhead items in the algorithm.

The most efficient procedure to date [24] for solving the equation

$$A^T X A - X + C = 0 \tag{B.1}$$

where $C = C^T$ is to:

1.) Transform A into upper real Schur form $\tilde{A}$ via

$$\tilde{A} = U^T A U \tag{B.2}$$

2.) Transform C to $\tilde{C}$ via

$$\tilde{C} = U^T C U \tag{B.3}$$

3.) Partition $\tilde{A}$ according to the dimensions of the nonzero main diagonal blocks. This naturally breaks the problem of solving

$$\tilde{A}^T \tilde{X} \tilde{A} - \tilde{X} + C = 0 \tag{B.4}$$

into 1×1 and 2×2 subproblems which can be solved sequentially.

4.) Back-transform $\tilde{X}$ to X using

$$X = U\tilde{X}U^T \tag{B.5}$$

The virtue of this procedure is the operation in Step 3, since partition-wise solution of (B.4) requires only minimal numerical overhead, compared with solution of (B.1) as a linear system of order $n(n+1)/2$. Unfortunately, Step 1 can be quite expensive, since generally, the QR algorithm is used to calculate U. Because of this, the overall algorithm requires $2(3+2\sigma)n^3$ floating point multiplications, where σ is the number of QR iterations required for satisfactory convergence. This number tends to increase when the eigenvalues of A have similar magnitudes. Because of this, when a solution to (B.1) is required for A which is asymptotically stable in discrete time, as in feedback optimization problems, σ can become large. Another drawback of the above algorithm becomes apparent when (B.1) must be solved repetitively for a series of similar, though not identical A matrices. In this case, new U matrices must be calculated for each A.

These considerations motivate investigating procedures which make use of "local" information. A successful algorithm of this type would exploit the fact that solution $X(A_1)$, for $A = A_1$ in (B.1), is "close" to $X(A_2)$ when A_1 is close to A_2. A straightforward point-of-departure is to consider a Newton algorithm for minimizing (driving to zero) the function

$$J = tr\{[A^TX^TA - X^T + Q^T][A^TXA - X + Q]\} \tag{B.6}$$

J in (B.6) is the squared Euclidean norm of the error in (B.1) when X is not the solution. The gradient of J is

$$\partial J/\partial X = 2[AA^TXAA^T - A^TXA - AXA^T + AQA^T + X - Q] \tag{B.7}$$

or

$$\partial J/\partial\, vec\, X = 2[AA^T \otimes AA^T - A^T \otimes A^T - A \otimes A + I]\, vec\, X + vec\,[AQA^T - Q] \quad \text{(B.8)}$$

and its Hessian is

$$\frac{\partial^2 J}{\partial\, vec\, X\, \partial\, vec^T X} = 2[AA^T \otimes AA^T - A^T \otimes A^T - A \otimes A + I] \quad \text{(B.9)}$$

The Newton algorithm for this problem consists of calculating a sequence $\{X_k\}$ of approximations to the solution X, where

$$vec\, X_{k+1} = vec\, X_k - \xi_k \left[\left(\frac{\partial^2 J}{\partial\, vec\, X\, \partial\, vec^T X}\right)^1 \frac{\partial J}{\partial\, vec\, X}\right]\Bigg|_{X=X_k} \quad \text{(B.10)}$$

and $0 < \xi_k \leq 1$ is a parameter chosen to ensure satisfactory improvement in J from iteration to iteration. The algorithm, based on a Taylor series for $\partial J/\partial X$ about zero, does satisfy the requirement for exploiting local information, but is, nonetheless, quite impractical for all but very small problems. This is because, for $A \in \mathcal{R}^{n\times n}$, mere construction of the Hessian matrix requires $2n^4$ multiplications, and its inversion requires $O(n^6/3)$ multiplications. After inversion, an additional $n^2(n^2+1)/2$ multiplications are required to evaluate (B.10). The $6n^3$ multiplications per iteration required to calculate $\partial J/\partial X$ ($15n^3/2$ on the first iteration) also impose an unattractively high computational burden if any significant number of iterations are anticipated.

If we assume that solutions to (B.1) are required for a sequence of closely spaced A's and C's, a numerically inexpensive approximation to the Newton step (B.10) can be constructed, based on the discussion in Section A. Denote $\partial^2 J/\partial\, vec\, X \partial\, vec^T X$ for the k^{th} A and C by H_k in (A.22 - A.25). Given the form (A.22) of S_k and P_k, and the fact that S_{k-1}^{-1} and P_{k-1}^{-1} are available, S_k^{-1} and P_k^{-1} can be calculated using the standard formula

$$(S_k^{-1})^j = (S_k^{-1})^{j-1} + \frac{(S_k^{-1})^{j-1}\bar{S}_k^j \bar{S}_k^{j^T} (S_k^{-1})^{j-1}}{1 + \bar{S}_k^{j^T} (S_k^{-1})^{j-1} \bar{S}_k^j} \qquad j = 1, \ldots, n_s \quad \text{(B.11)}$$

$$(S_k^{-1})^o = S_{k-1}^{-1} \tag{B.12}$$

where $\bar{s}_k^j$ is the j^{th} column of $\bar{S}_k$, and similarly for P_k^{-1}. Updating S_k^{-1} and P_k^{-1} in this manner requires only $O(3n_s n^2)$ multiplications. Since it is assumed that the A_k and C_k (and, hence, the X_k) are closely spaced, evaluation of the gradient $\partial J/\partial X$ can be accomplished via numerical differencing, eliminating $6n^3$ multiplications per iteration. Overall, then, one concludes that for small n_s, n_p, only some finite multiple of n^2 multiplications per iteration are required. The decomposition giving H_k is not included in this estimate, since the derivation was not finalized, but, for small n_s, n_p, should not affect the estimate materially.

APPENDIX C.

PROOFS

This Appendix contains proofs of assertions made in Section 2 and Appendix A.

Proof of Lemma 1 :

Consider the problem of minimizing J from (12) without any constraints on the α-dynamics of G. The Lagrangian for this problem is

$$\mathcal{L} = \int_{\alpha_o}^{\alpha_f} c + tr\{S(G,K)\Lambda_K^T\}d\alpha \tag{C.1}$$

Note that, in minimizing $\mathcal{L}$, G assumes the role of a control variable. The Minimum principle gives the necessary condition

$$\partial\mathcal{H}/\partial G = \hat{K}G\hat{\Lambda}_K - B^T K A \Lambda_K C^T = 0 \tag{C.2}$$

The necessary conditions on K and Λ_K are, again, (18) and (19). From (18, 19) and the Lemma's assumptions, it is trivial to show that K and Λ_K are k-times continuously differentiable functions of α. Therefore, $G^*(\alpha)$, given by

$$G^* = \hat{K}^+ B^T K A \Lambda_K C^T \hat{\Lambda}_K^+ \tag{C.3}$$

where $(.)^+$ denotes the Moore-Penrose inverse, has a continuous k^{th} derivative in α.

Denote the ij^{th} element of $G^*(\alpha)$ by $g_{ij}^*(\alpha)$, which has the Fourier series representation

$$g_{ij}^*(t) = a_o + \sum_{n=1}^{\infty} a_n cos(n\pi t - \omega_n) \tag{C.4}$$

with

$$t = (\alpha - \alpha_o)/(\alpha_f - \alpha_o) \tag{C.5}$$

over the interval $[\alpha_o, \alpha_f]$. One can form an approximation to $g^*_{ij}(t)$ by truncating the series (C.4) at the r^{th} term:

$$\tilde{g}_{ij}(t) = a_o + \sum_{n=1}^{r} a_n cos(n\pi t - \omega_n) \tag{C.6}$$

so that

$$g^*_{ij}(t) - \tilde{g}_{ij}(t) = \sum_{n=r+1}^{\infty} a_n cos(n\pi t - \omega_n) \tag{C.7}$$

It is well known that the a_n in (C.4) approach zero at least as rapidly as γ/n^{k+1} for some constant $\gamma \geq 0$; therefore,

$$g^*_{ij}(t) - \tilde{g}_{ij}(t) < \sum_{n=r+1}^{\infty} \gamma_{ij}/n^{k+1} \tag{C.8}$$

Since $k \geq 0$, (C.8) can be rewritten

$$g^*_{ij}(t) - \tilde{g}_{ij}(t) < \frac{\gamma_{ij}}{(r+1)^{k+1}} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2 \gamma_{ij}}{6(r+1)^{k+1}} \tag{C.9}$$

The truncated series (C.6) has a minimal realization as a linear system of order $2r$, so that a linear system of order $2mlr$ suffices to realize the first r trigonometric terms in the Fourier expansion of each of the elements in $G^*(t)$. Therefore, (43) holds for

$$b = \frac{\pi^2}{6} \max_{i,j} \gamma_{ij} \tag{C.10}$$

Proof of Lemma 2 :

Denote a set of eigenvectors of $A_i \in \mathcal{R}^{p\times p}$ as v_{ni}, $n = 1,...,p$ and those of $B_i \in \mathcal{R}^{q\times q}$ as w_{mi}, $m = 1,...,q$. Suppose that

$$\sum_{i=1}^{\nu} A_i \otimes B_i > 0 \tag{C.11}$$

but that

$$det\{\sum_{i=1}^{\nu} A_i\} = 0 \tag{C.12}$$

Since each $A_i \geq 0$, (C.12) implies that there is some v_n such that

$$A_i v_n = 0 \tag{C.13}$$

for $i = 1,...,\eta$; that is, all of the A_i share a zero eigenvalue with eigenvector v_n. Therefore, each $A_i \otimes B_i$ satisfies

$$(A_i \otimes B_i)v_n \otimes w_{mi} = A_i v_n \otimes B_i w_{mi} = 0 \tag{C.14}$$

for $m = 1,...,q$, so that

$$\sum_{i=1}^{\nu}(A_i \otimes B_i)v_n \otimes w_{mi} = 0 \tag{C.15}$$

which contradicts (C.11), thus implying

$$\sum_{i=1}^{\nu} A_i > 0 \tag{C.16}$$

Similar reasoning leads to

$$\sum_{i=1}^{\nu} B_i > 0 \tag{C.17}$$

Note that if (C.16) holds, then for $\alpha_i > 0$, $i = 1, ..., \eta$,

$$\sum_{i=1}^{\nu} \alpha_i A_i > 0 \tag{C.18}$$

Since $P^{(0)} > 0$ by assumption, then

$$\alpha_i = tr\{B_i P^{(0)}\} / \|P^{(0)}\|^2 > 0 \tag{C.19}$$

which, from (A.9), implies that $S^{(1)} > 0$. This, in turn, implies that $P^{(1)} > 0$, from (A.10).

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16. Abstract

This research was aimed at developing a novel concept for optimally designing output feedback controllers for plants whose dynamics exhibit gross changes over their operating regimes. This was, essentially, to formulate the design problem in such a way that the implemented feedback gains vary as the output of a dynamical system whose independent variable is a scalar parameterization of the plant operating point.

The results of this effort include derivation of necessary conditions for optimality for the general problem formulation, and for several simplified cases. The question of existence of a solution to the design problem was also examined, and it was shown that the class of gain variation schemes developed in this effort are capable of achieving gain variation histories which are arbitrarily close to the unconstrained gain solution for each point in the plant operating range. The theory was implemented in a feedback design algorithm, which was exercised in a numerical example.

The results of the research undertaken under this contract are applicable to the design of practical high-performance feedback controllers for plants whose dynamics vary significantly during operation. Many aerospace systems fall into this category.

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