NASA Contractor Report 178226

ICASE REPORT NO. 86-82

ICASE

A PRACTICAL ASSESSMENT OF SPECTRAL ACCURACY FOR HYPERBOLIC PROBLEMS WITH DISCONTINUITIES

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Contract No. NAS1-18107

December 1986

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NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665

A Practical Assessment of Spectral Accuracy for Hyperbolic Problems with Discontinuities

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ABSTRACT

Numerical experiments are performed to compare the accuracy obtained when physical and transform space filters are used to smooth the oscillations in Fourier collocation approximations to discontinuous solutions of a linear wave equation. High order accuracy can be obtained away from a discontinuity but the order is strongly filter dependent. Polynomial order accuracy is demonstrated when smooth high order Fourier filters are used. Spectral accuracy is obtained with the physical space filter of Gottlieb and Tadmor.

[1]This work was supported under the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665-5225.

1. Introduction

Spectral solutions to partial differential equations possess what is known as "spectral accuracy", that is, the error decays at a rate which depends only on the smoothness of the solution. If the solution is C^∞ on the domain then the error decays faster than any polynomial in the number of modes used (See Gottlieb, Hussaini and Orszag[2]). If the solution is analytic in a region about the real axis (which depends on the type of spectral expansion used), then the error decays exponentially [10]. In practical problems, error convergence which is faster than polynomial order is observed. Such behavior is seen, for example, when solving (smooth) two-dimensional nonlinear gas-dynamic problems[7].

If the exact solution of an equation has discontinuities, as is typical for hyperbolic PDEs, then the spectral solution is characterised by strong global oscillations. As a result, error convergence is poor even in smooth regions away from any discontinuities. The problem is compounded when solving nonlinear equations because these oscillations can lead to instabilities. The presence of global oscillations seems to indicate that spectral methods are inappropriate for the solution of hyperbolic equations when the solutions are discontinuous.

Global oscillations in the discontinuous solutions can be removed by filtering. The main issue is whether or not "spectral accuracy" can be restored away from the discontinuity. In other words, can filtering be performed so that the error depends only on the local smoothness of the solution?

The papers by Majda, McDonough and Osher[8] and Gottlieb and Tadmor[4] prove that for linear problems, proper smoothing allows spectral accuracy to be recovered in any region which does not include a discontinuity. The two approaches are different. Majda *et al* examined smoothing of Fourier approximations in frequency space. As a function of frequency, their filters were required to be unity in a neighborhood of zero frequency and smoothly decrease to zero at high frequencies. Gottlieb and Tadmor proposed smoothing in physical space with a highly oscillatory kernel whose support varies with position. Their procedure is applicable to spectral methods based on other polynomial expansions such as Chebyshev and Legendre methods. Both papers include numerical experiments which show that with filtering, the solutions away from a discontinuity are far better than without. Neither shows, however, that the error is in fact spectral.

There have been many attempts to apply filtering to the solution of nonlinear problems. Solutions of the Euler equations of gas-dynamics with shocks were presented in [1],[6],[3],[9],[11],[13]. These papers indicate that some sort of filtering or artificial viscosity does give solutions which "look good". However, no one has presented solutions for which the error decay is spectral. Hussaini *et al* [6] tested several filtering strategies on a periodic problem with a shock wave and complex flow structure. They found that the best global error decay away from the shock was only first order. Solutions computed with the upwind finite difference scheme of Van Leer [12] were better than the spectral solution.

In light of the fact that the papers by Majda et al and Gottlieb and Tadmor do not actually demonstrate spectral accuracy in practice and that the filtered solutions of nonlinear problems have systematically failed to exhibit spectral accuracy, we present numerical experiments on filtering. In particular we want to assess the usefulness of filtering for a realistic (rather than asymptotically large) number of grid points. We limit this paper to the study of Fourier collocation approximations to a linear model wave equation in order to test the effectiveness of filtering strategies for which theoretical evidence is available. We present results for both physical and transform space filters.

2. Filtering of Fourier Spectral Approximations

The Fourier collocation (also called pseudospectral) approximation to the linear model equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad 0 \le x \le 2\pi \tag{1}$$

with periodic boundary conditions $u(0,t) = u(2\pi,t)$ is constructed by approximating the derivatives u_x at a number of grid (collocation) points. A uniform grid $x_j = 2\pi j/N$ is specified and approximate solution values v_j are assigned to these points. A Fourier interpolant is passed through the N values of v_j

$$v(x) = \sum_{-N/2}^{N/2-1} a_k e^{ikx} \tag{2}$$

where the coefficients are computed by

$$a_k = \frac{1}{N} \sum_{j=0}^{N-1} v_j e^{ikx_j} \qquad k = \frac{N}{2}, \ldots, \frac{N}{2} - 1 \tag{3}$$

The derivative approximations are computed by evaluating $v'(x_j)$ and we require that the differential equation is satisfied by v_j at the grid points

$$\frac{\partial}{\partial t} v_j + \sum_{-N/2}^{N/2-1} ik a_k e^{ikx_j} = 0 \qquad j = 0,1,..,N-1 \tag{4}$$

Because these equations have constant coefficients, they can be solved exactly in time, so that any errors are due solely to the spatial approximation.

$$v_j(t) = -\sum_{-N/2}^{N/2-1} ika_k e^{ik(x_j-t)} \tag{5}$$

The initial condition for the differential initial value problem is u(x,o) = F(x). For the approximation we use $v_j = F(x_j)$. This is appropriate for the scalar eq. 1 but not for systems [8].

Fourier space filtering of the approximation for eq. 5 was proposed by Majda *et al* [8]. Here, the coefficients are modified by multiplying them by a filter function $\rho(2\pi k/N)$. For the linear, constant coefficient problem, the filtered solution can be written

$$v_j(t) = -\sum_{-N/2}^{N/2-1} ika_k\rho_k e^{ik(x_j-t)} \tag{6}$$

The filter function $\rho(\varphi)$ is a positive C^∞ cutoff function which vanishes near $\varphi = \pm\pi$. In a neighborhood of the origin, $\varphi = 0$, $\rho \equiv 1$. With such filtering, Majda *et al*[8] show that the error decay is spectral; that is, in any region R which does not include the discontinuity there is a constant C (which depends on λ and the distance from a discontinuity) such that

$$\sup_R | u(x) - v(x) | < CN^{-\lambda}$$

for all λ.

Gottlieb and Tadmor[4] chose to filter in physical space. The rationale was that the Fourier space filtering, which is localized in transform space, includes the contribution of all points in physical space, including points near a discontinuity. By filtering in physical space, the support of the filter can be defined so as to avoid these points. Smoothing is performed by convolving the collocation approximation with a regularization kernel

$$v*\Psi = \int_0^{2\pi} v(y)\Psi^{\theta,p}(x,y)dy \tag{7}$$

The kernel, ψ, is formed by the product of two factors. One is the Dirichlet kernel

$$D_p(y) = (1/2\pi)\sin((p + 1/2)y)/\sin(y/2) \tag{8}$$

which is the physical space representation of truncation of the infinite Fourier series to a finite one of order p. The other factor localizes this in physical space. $D_p(y)$ is multiplied by a C^{2s} cutoff function, ρ, which satisfies $\rho(0) = 1$ and vanishes outside of the interval $[-\pi,\pi]$. Finally, a free parameter θ is introduced so that the support of the kernel can be controlled. The resulting kernel is

$$\psi^{\theta,p}(y) = \theta^{-1}\rho(y/\theta)D_p(y/\theta) \tag{9}$$

For the collocation approximation, the convolution integral is replaced by the trapezoidal rule and the smoothed solution as a function of x is

$$v_{smooth}(x,t) = \frac{2\pi}{N}\sum_{n=0}^{N-1} v_n\Psi^{\theta,p}(x - y_n) \tag{10}$$

Because the kernel is localized in physical space, terms for which $(x_j - y_n)/\theta$ is large will not contribute to the sum. Near the boundaries, i.e., x_j near 0 and 2π, the sum must be modified to take into account the fact that the solution is actually periodic. Thus, $(x_j - y_{n-N})/\theta$ may be within the support of ψ. To include points within the support of ψ but not explicitly included in eq.10, the solution must be "wrapped around" to obtain accuracy near the boundaries.

The error obtained by this smoothing is spectral. If $p = (N/2)^{\beta}$ where $0 < \beta < 1$ then the error is bounded by

$$|v_{smooth} - v(x)| \leq C \, \|\rho\|_s \max_{\substack{|y-x|\leq\theta\pi \\ 0\leq k\leq s}} |D^k v(y)| \left[N^{-\beta s+1} + \frac{1}{\theta} N^{\beta} N^{-(1-\beta)s} \right] \tag{11}$$

In other words, the error depends on the smoothness of the cutoff function.

It is interesting to note that although the Dirichlet kernel has unit area on the interval, the regularization kernel does not. Thus, this smoothing does not preserve the mean of the function. If viewed in Fourier space, this means that the filter does not have amplitude identically one near the origin. As a consequence, even though the filter will asymptotically give spectral error decay, it can actually degrade the accuracy of a perfectly smooth function (such as a constant). A polynomial order Fourier space filtering will degrade the solution only in the higher modes, which will be quite small if the solution is smooth, .

Both filtering strategies provide a lot of freedom. First, the Fourier space cutoff filter is not immediately defined. Such C^{∞} functions exist, but are difficult to construct. Consequently, Majda *et al*[8] suggested the use of an exponential cutoff filter

$$\rho(\varphi) = \begin{cases} 1 & |\varphi| \le \varphi_c \\ \exp\left[-a(|\varphi| - \varphi_c)^{2m}\right] & \varphi_c < |\varphi| \le \pi \end{cases} \tag{12}$$

This filter satisfies the requirement that $\rho = 1$ in a neighborhood of 0 and decreases to zero at $\pm\pi$, but it is no longer smooth. Majda et al preferred the fourth order filter, m =2. There is still freedom in choosing the cutoff frequency φ_c and the coefficient, a.

The freedom in specifying the physical space filter comes through the specification of the cutoff function, ρ and the parameters β and θ. Gottlieb and Tadmor[4] suggest the use of the C^∞ cutoff function

$$\rho(\xi) = \begin{cases} \exp(\alpha\xi^2/(\xi^2-1)) & |\xi| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

The coefficient α is a free parameter. The parameter θ determines the support of the filter. Gottlieb and Tadmor suggest a variable sized support which avoids the discontinuity. For a discontinuity at the point s, $\theta = |x - s|$.

3. Results of Numerical Experiments

The effectiveness of physical and Fourier space filtering was tested on the model problem with the initial condition given by

$$F(x) = \frac{1}{1 + \cos^2(\frac{5\pi x}{4})} \qquad 0 < x \leq 2\pi \tag{14}$$

which has a jump discontinuity at x = 0. An example of the solution when the discontinuity has moved to x = π and the sharpened cosine filter (see below) has been applied is shown in Fig. 1. The circles represent the spectral solution with 128 points and the solid line is the exact solution.

It is important that the solution have high spectral content away from the discontinuity. If not, a filter may give results which are better than can be expected in general. For example, a first order filter can give excellent results away from a discontinuity if the solution is piecewise constant.

Four Fourier space filters were applied to the solution of the linear model problem. Graphs of the amplitude vs. phase of these filters are shown on Fig. 2. The graph of the exponential cutoff filter, eq. 12, is for $\varphi_c = \pi/2$. The coefficient was chosen so that the highest mode was multiplied by 10^{-14}, a number on the order of the machine epsilon. For comparison purposes, the use of the Lanczos filter in eq. 6

$$\rho(\varphi) = \sin(\varphi)/\varphi \tag{15}$$

is equivalent to differentiating in physical space with the second order centered difference approximation

$$du/dx \approx (u_{j+1} - u_{j-1})/2\Delta x \tag{16}$$

Another second order filter is the raised cosine filter

$$\rho_c(\varphi) = (1 + \cos(\varphi))/2 \tag{17}$$

This filter is equivalent to the common physical space average

$$(u_{j+1} + 2u_j + u_{j-1})/4$$

It can be sharpened to higher order by standard techniques [5]. An eighth order sharpened raised cosine filter is

$$\rho = \rho_c^4(35 - 84\rho_c + 70\rho_c^2 - 20\rho_c^3) \tag{18}$$

Results for Fourier space filtering are presented in figures 3-5. These are graphs of the logarithm of the absolute value of the pointwise error when the discontinuity was near $x = \pi$. For each filter, errors are presented for N = 32(circles), N=64(squares) and N = 128(diamonds). To keep the errors in perspective, 32 points are required to obtain errors on the order of 10^{-5} for the closely related smooth problem with $F(x) = (1 + \cos^2(6\pi x/4))^{-1}$.

Figures 3 and 4 are graphs of errors smoothed with the Lanczos and sharpened raised cosine filter. As expected, the error near the discontinuity is O(1), regardless of the filter used. Since these are semi-log plots, a uniform spacing between the curves for different values of N indicates polynomial accuracy. Both filters give the expected results: The spacing between the curves for the Lanczos filtering is roughly 0.6 which corresponds to second order convergence. Eighth order accuracy is observed for the sharpened raised cosine filter.

Under the same circumstances, the exponential cutoff does not give spectral accuracy. Fig. 5 shows the results of the fourth order filter with a cutoff at 0.5π. Though the error appears to decrease faster than polynomial order in some areas, the error is not smooth. Furthermore, the region of large error about the jump is also quite large. If all the modes except k = 0 are smoothed ($\varphi_c = 0$) then the region of high error about the jump shrinks to that of the other polynomial order filters. The error decay, however, is strictly fourth order. These results are consistent with what was observed by Majda *et al*[8].

Spectral accuracy has been obtained with physical space filtering. Parameters close to those proposed by Gottlieb and Tadmor[4] were used. The errors do not appear to be sensitive to the parameter α and $\alpha = 10$ is illustrated. They are sensistive to β and the best results were obtained for $\beta = .9$.

Figure 6 is a graph of the errors for the physical space filter. Away from $x=\pi$ the error decay ranges between fourth and eighth order as the number of modes increases and hence is spectral. Still, this filter is outperformed by the eighth order sharpened cosine filter. This is partly due to the fourth order behavior for small N. But for fixed N the error varies as much as four orders of magnitude within the smooth region. Errors near $x=\pi$ are O(10) rather than O(1). This behavior near the jump is consistent with the error estimate, eq. 12, which gives a large error bound when θ is very small. In the smooth parts of the solution, the error is largest where the parameter θ is largest.

The quality of the physical space filter is very dependent on θ. For example, the solution can be substantially improved on this problem by <u>not</u> using variable support in x. Fig. 7 shows the errors for the constant value of $\theta = 1$, which gives the best results for this problem. The errors near the jump are now O(1). For each N the error is now more uniform over the smooth region and the error decay ranges between fourth and tenth order. Unfortunately, the improvement seen in the smooth part is problem dependent. For the initial condition $F(x) = (1 + (x/\pi)^2)^{-1}$, the errors away from the jump are an order of magnitude better using the variable θ than using a constant θ.

Keeping the physical space support of the filter fixed in space does have an advantage in the efficiency of the filtering method. The convolution represented by eq. 10 requires $O(N^2)$ operations to compute. Represented in transform space, this can be done with a fast Fourier transform in O(NlogN) operations, but only if θ is kept constant.

The results presented here show that the asymptotic decay of the error is very filter dependent. For coarse grids, e.g. N = 32, however, the differences are not great. Figs. 3-7 indicate that the error after any of the high order smoothings is only roughly a third of a second order finite difference method as modelled by the Lanczos filtering. The lack of clear superiority of the spectral method over a low order finite difference method for coarse meshes in the presence of discontinuities appears to be typical.

4. Conclusions

The results of this investigation show that it is possible to obtain high order solutions to the discontinuous linear model initial value problem. Even though the Fourier approximation is first order everywhere, away from a jump discontinuity it is possible to extract higher order information by filtering. The order of accuracy is very filter dependent. Smooth, high order polynomial filters, of which eq. 18 is an example, are easy to construct and their behavior is typical. Sacrificing smoothness for infinite order tangency at zero frequency as proposed by Majda *et al* does not provide satisfactory results. Spectral accuracy has been obtained with the filter of Gottlieb and Tadmor in the sense that the order of accuracy of the method increases as the number of points increases. For a coarse mesh, however, there is little difference between a second order finite difference method and a computationally more expensive spectral method.

Of the filters tested, the spectrally accurate physical space filter of Gottlieb and Tadmor gives the worst results near a discontinuity. This could be disastrous for a nonlinear problem. The performance of the filter can be improved by allowing the the support of the filter to include the discontinuity. For the problem described here, the best overall results away form the jump were obtained by keeping $\theta = 1$ everywhere, but this is not always the case.

Finally, we remark that the conclusions of this paper do not necessarily carry directly over to the solution of nonlinear problems. Though eighth order accuracy is observed for the filter eq. 18, Hussaini *et al*[6] observed only first order accuracy when it was applied to the solution of a nonlinear problem.

ACKNOWLEDGEMENTS

The author would like to thank M.Y. Hussaini and T.A. Zang for suggesting this investigation and D. Gottlieb and E. Tadmor for helpful discussions.

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FIGURE CAPTIONS

1. Graph of the solution of eq. 1 with initial condition, eq. 14. The solid line is the exact solution, circles are computed solutions smoothed with the sharpened raised cosine filter.

2. Fourier space filters. a: raised cosine; b: Lanczos; c: sharpened raised cosine; d: exponential cutoff.

3. Pointwise errors for the solution filtered with the second order Lanczos filter.

4. Pointwise errors for the solution filtered with the eighth order sharpened raised cosine filter.

5. Pointwise errors for the solution filtered with the exponential cutoff filter.

6. Pointwise errors for the solution filtered with the variable support physical space filter.

7. Pointwise errors for the solution filtered with a constant support physical space filter.

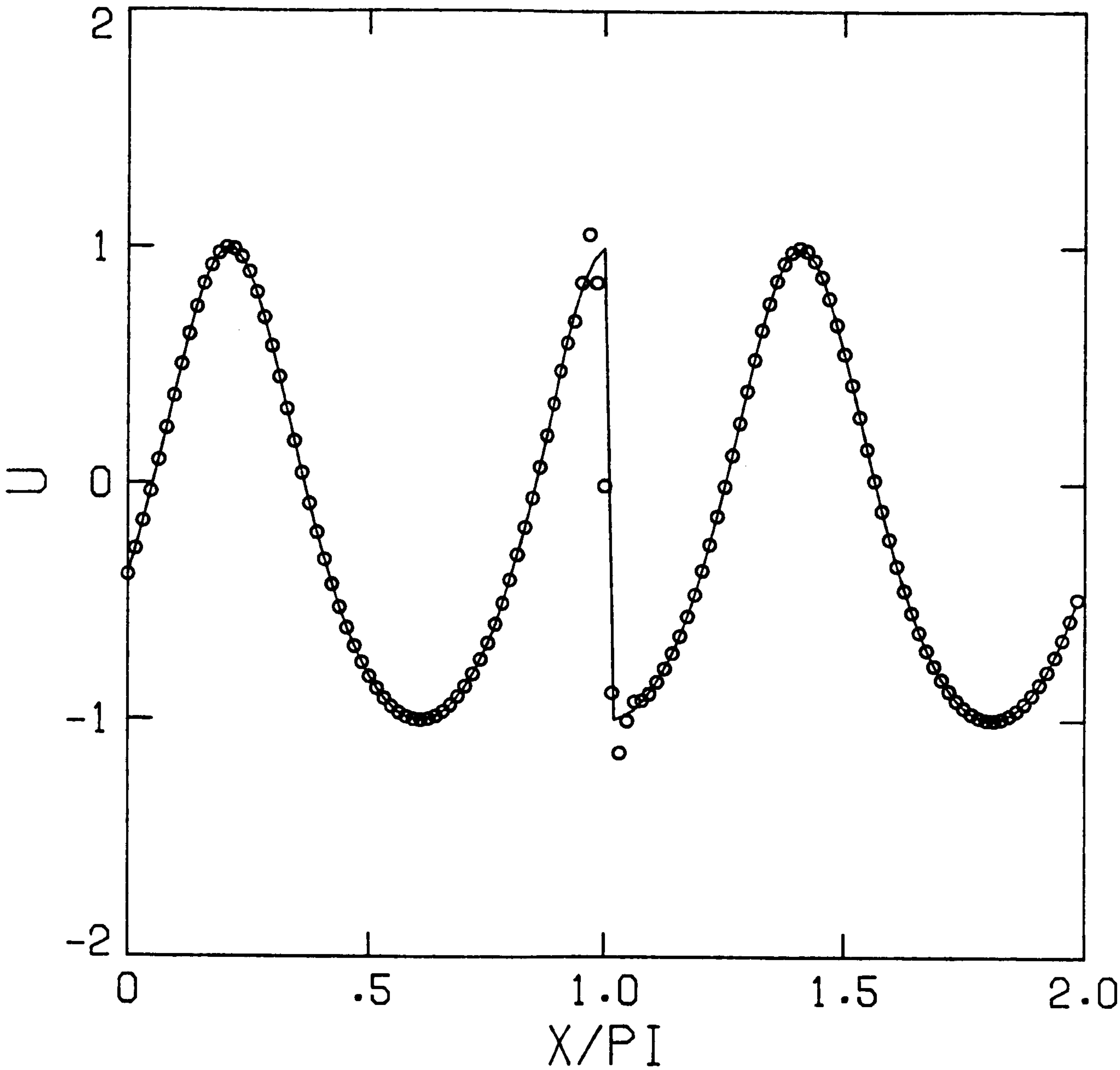
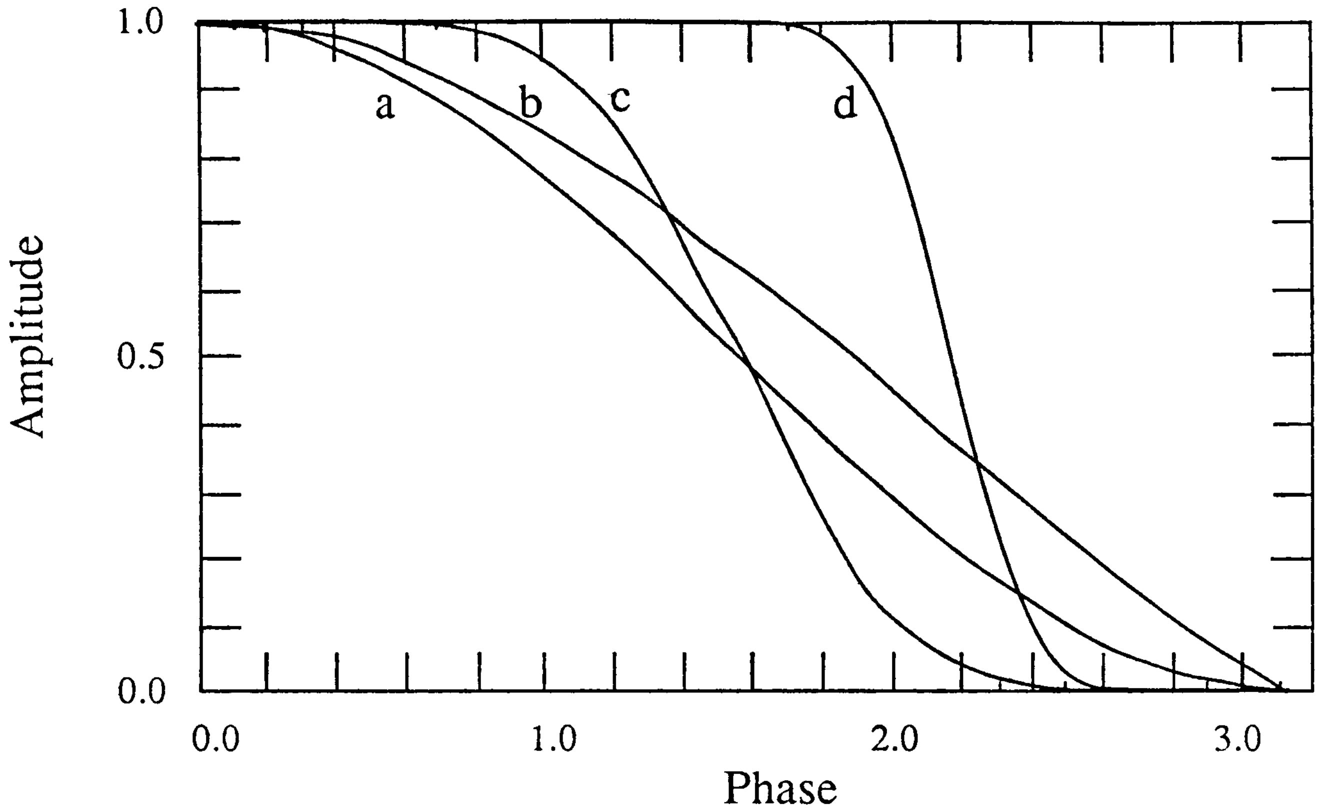


Figure 1



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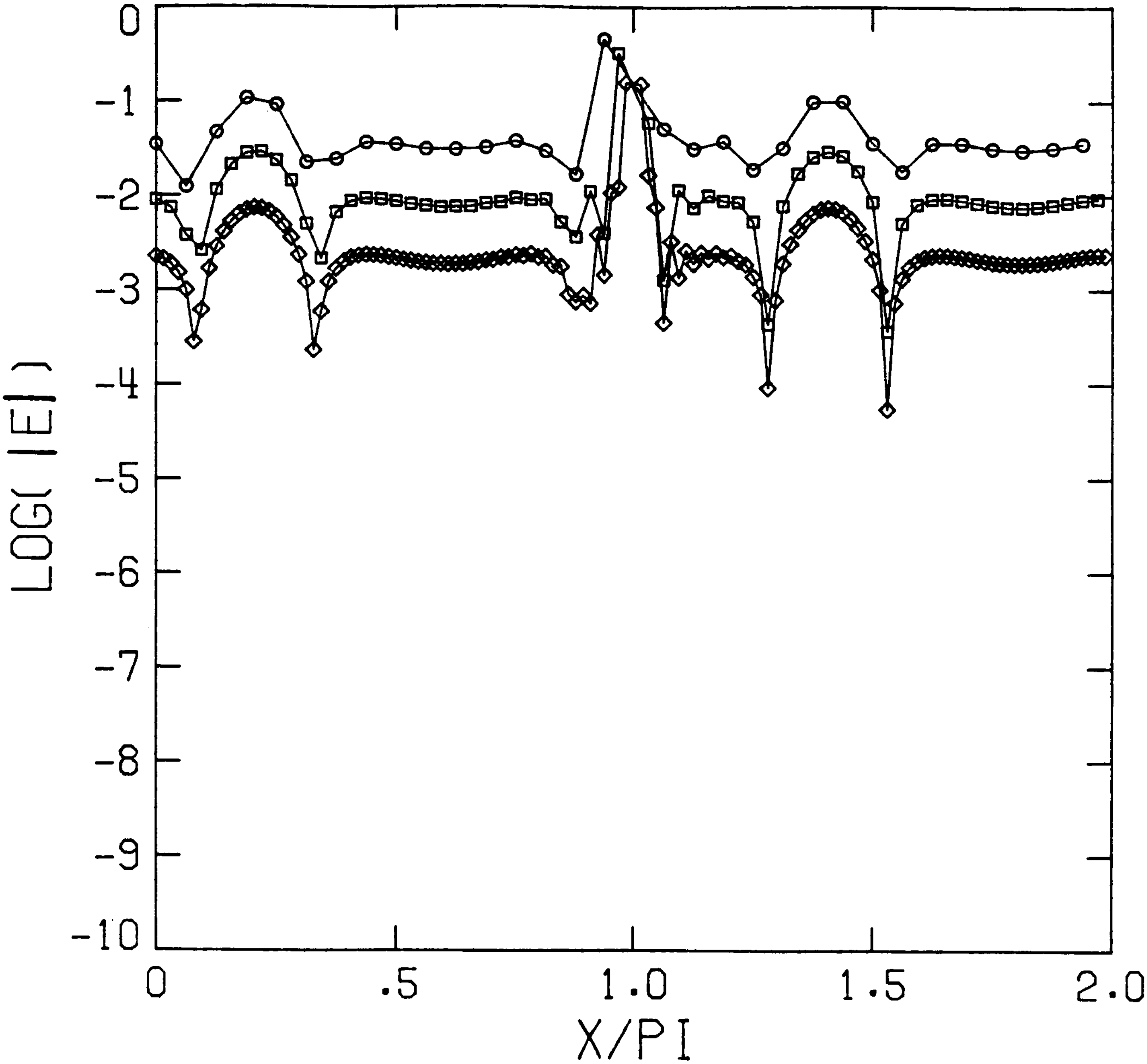

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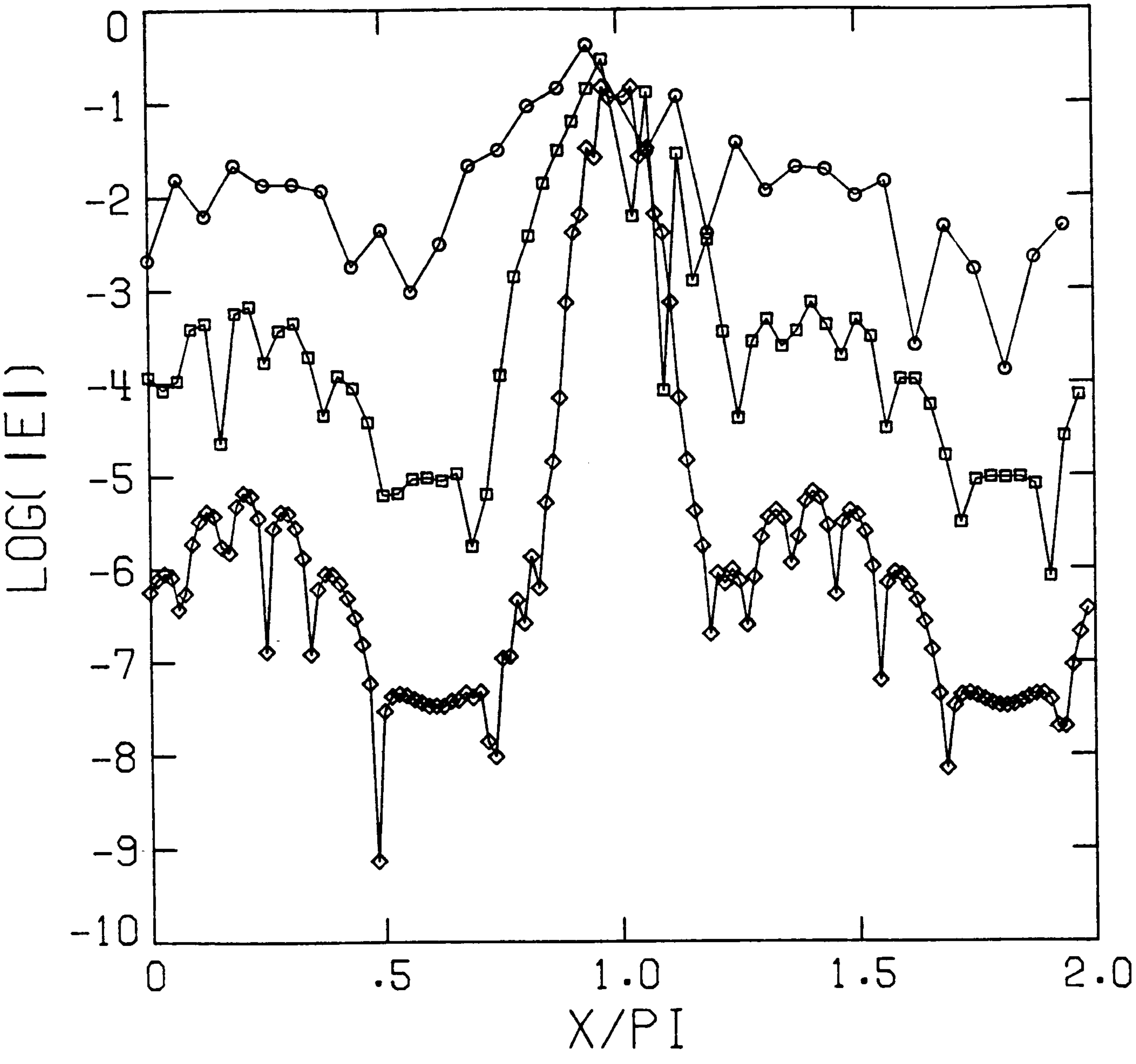
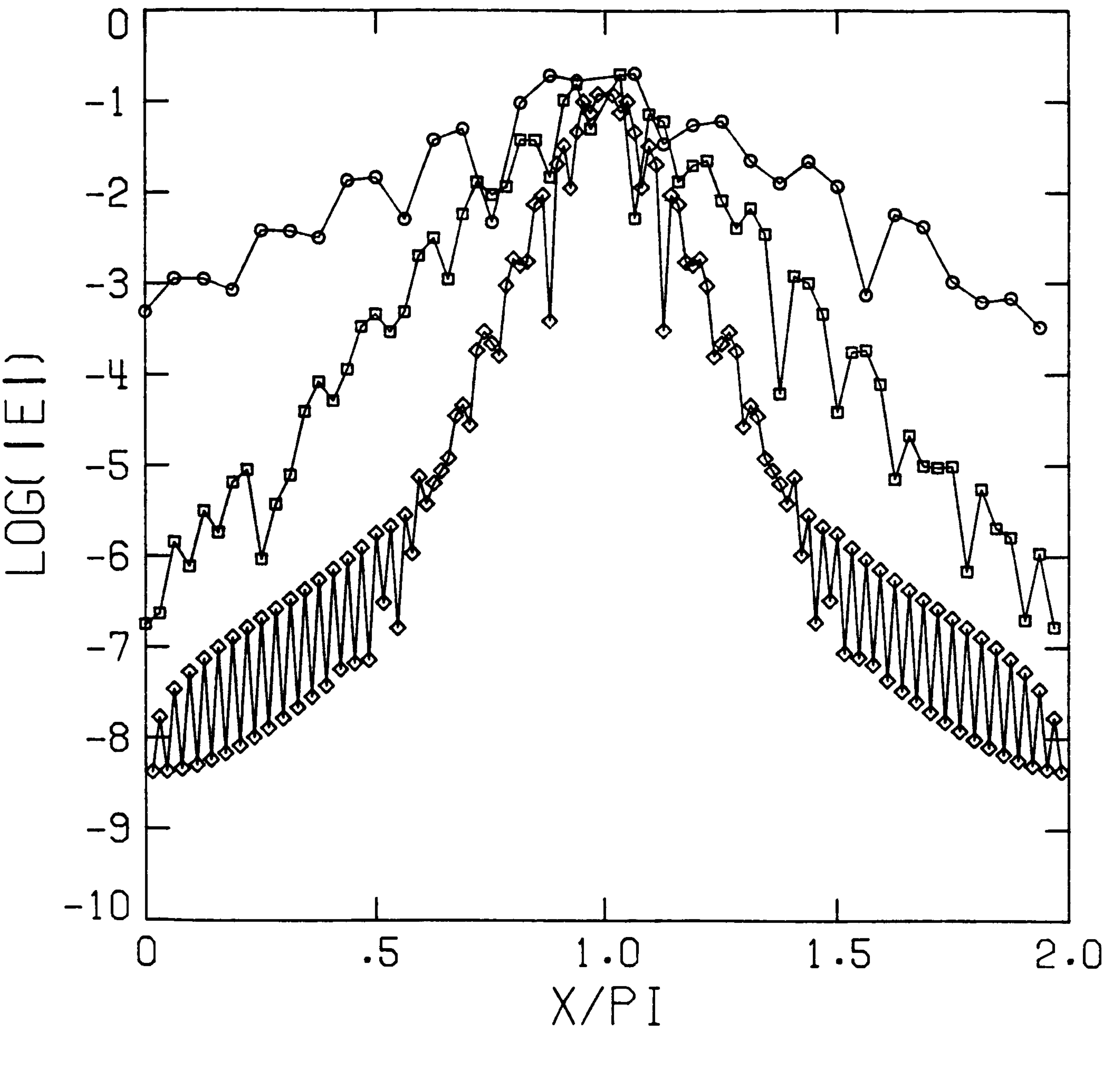


Figure 4



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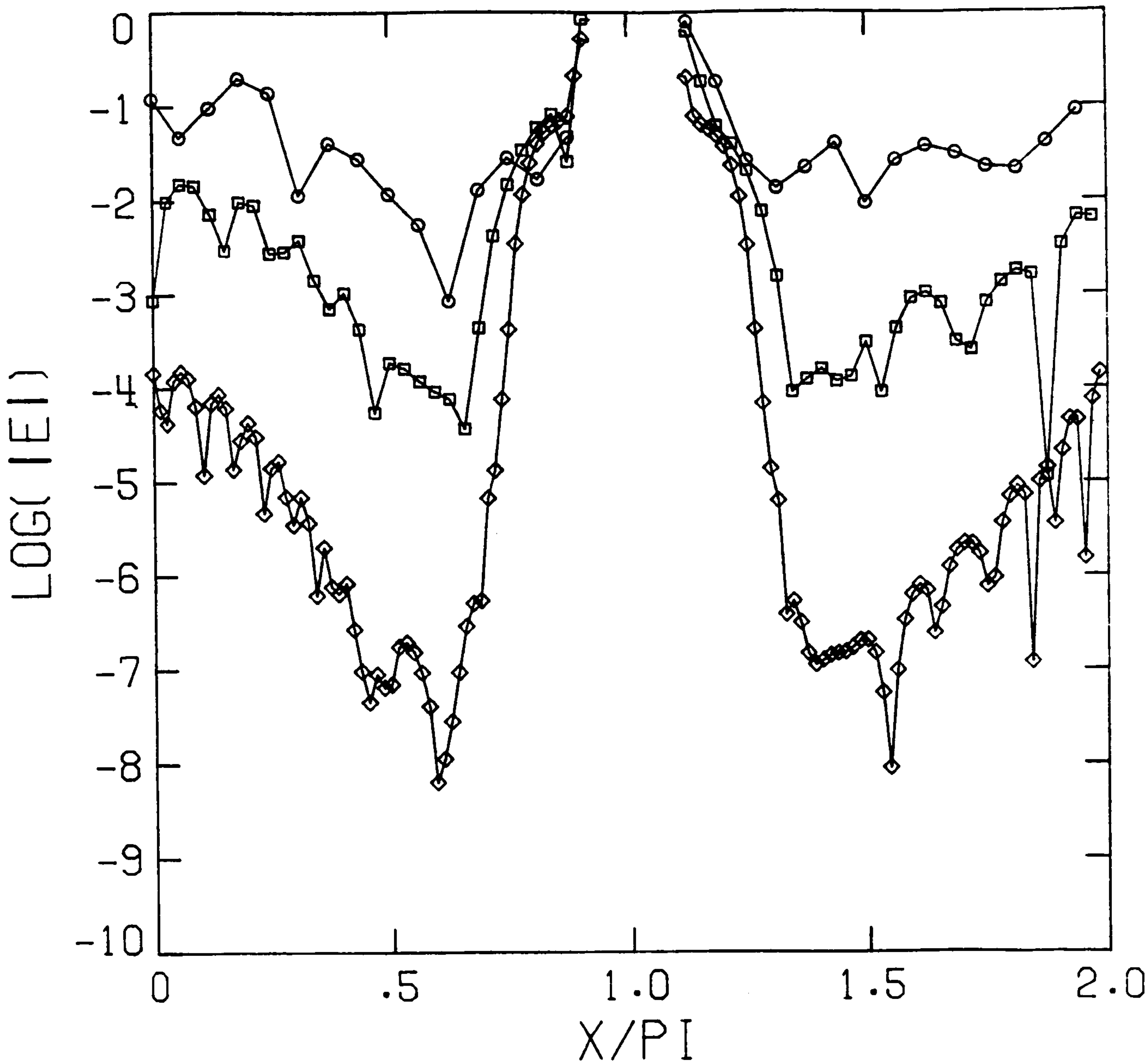

Figure 6

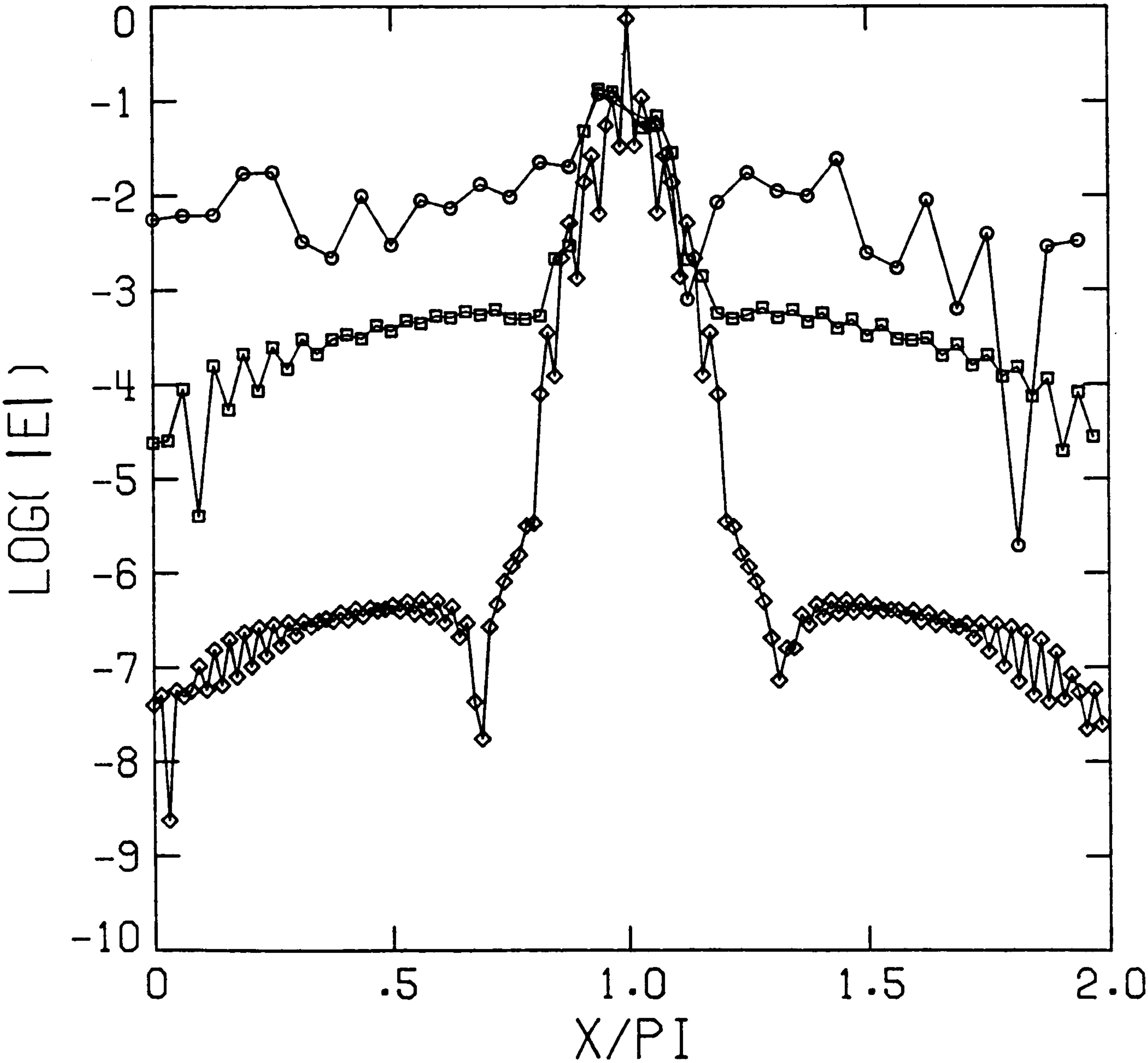

Figure 7

Standard Bibliographic Page

1. Report No. NASA CR-178226 ICASE Report No. 86-82	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A PRACTICAL ASSESSMENT OF SPECTRAL ACCURACY FOR HYPERBOLIC PROBLEMS WITH DISCONTINUITIES		5. Report Date December 1986
		6. Performing Organization Code
7. Author(s) David A. Kopriva		8. Performing Organization Report No. 86-82
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No.
		11. Contract or Grant No. NAS1-18107
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code 505-90-21-01

15. Supplementary Notes

Langley Technical Monitor: J. C. South

Submitted to J. Comp. Phys.

Final Report

16. Abstract

Physical and transform space filtering has been applied to the Fourier spectral collocation solution of the constant coefficient scalar wave equation with a discontinuous initial condition. High order accuracy can be extracted from the unfiltered solution. Smooth, high order Fourier space filtering gives expected polynomial order solutions away from the discontinuity. Spectral accuracy is observed with the physical space filter of Gottlieb and Tadmor.

17. Key Words (Suggested by Authors(s)) spectral methods, hyperbolic equations	18. Distribution Statement 64 - Numerical Analysis 61 - Computer Programming Unclassified - unlimited

19. Security Classif.(of this report) Unclassified	20. Security Classif.(of this page) Unclassified	21. No. of Pages 23	22. Price A02

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1987