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A COORDINATE FREE DESCRIPTION OF MAGNETOHYDROSTATIC EQUILIBRIA

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INTRODUCTION

This paper addresses the question what geometrical restrictions are imposed on static magnetic fields by the MagnetoHydro-Static (MHS) equations. This question is of obvious importance for the problem of coronal heating, since it has been argued by Parker (1972, 1979 and 1986; see references therein) that the MHS-equations in general overdetermine the magnetic field structure and that consequently the field needs to have some sort of symmetry to satisfy all the constraints imposed on it by the equations. The field in the solar corona is determined by the MHS equations and the boundary conditions at the corona/photosphere interface. The latter are the normal component of the magnetic field at the boundary (to ensure the continuity of the magnetic field) and the connectivity of the field lines, defined as the positions of all field line footpoints at the boundary (because the field lines are frozen in, Sturrock and Woodbury, 1967). These boundary conditions are completely arbitrary, because they are determined by the magnetic fields and the fluid motions in the photosphere and convection zone, that cannot be altered by the relatively weak forces from the coronal magnetic field. The general mathematical problem is therefore to determine the solutions of the MHS-equations in the corona subject to an arbitrary normal component of the magnetic field at the boundary and arbitrary connectivity.

It is very unlikely that these boundary conditions would conspire to satisfy any symmetry requirement that the MHS equations might impose. And even if they would at a given moment, only minor footpoint displacements - as a result of the photospheric velocity field - would destroy the symmetry. Hence the coronal magnetic field cannot be in static equilibrium at any time, and, according to Parker (1983), the force free condition will break down at some locations in the corona, where current sheets will form. In these sheets the dissipation of magnetic field is much larger than that calculated with classical resistivity and the resulting heating rate may be large enough to explain the observed non-thermal heating of the corona. This process is called **topological heating.**

Recently, however, Parker's hypothesis has been challenged by Van Ballegooijen (1985) and Antiochos (1986); see also their contributions in this chapter. Van Ballegooijen points out an error in Parker's (1972) original demonstration of the need for an ignorable coordinate and furthermore, by improving upon Parker's analysis, gives an algorithm for calculating solutions to the MHS equations, subject to arbitrary boundary conditions. Antiochos argues that the problem is generally well posed by showing that when the magnetic field is expressed in Euler potentials, the topology of the field in the corona is completely determined by the values of the potential at the boundary. Consequently there is no need for the formation of current sheets in

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these analyses.

Clearly the question whether there is the requirement of some sort of symmetry in the solutions of the MHS equations deserves further attention. This symmetry has to be of a subtler form than that of an ignorable coordinate as once proposed (Parker, 1972), since recently explicit analytical examples have been given of fully 3-D magnetostatic equilibria (Low, 1985). The latter solutions still show some form of symmetry, however.

The problem is the more intriguing because it has recently been shown quite convincingly by Tsinganos **et al.** (1984) and Moffat (1985) - in very different ways - that magnetostatic equilibria in Tokamak type structures do lead to topological dissipation, when they do not exhibit symmetry. However, in Tokamaks the boundary conditions are quite different: here the requirement is that the normal component of the field vanishes everywhere at the surface of the containment vessel and hence the field is self contained (see Martens, 1985, for a comparison). The field lines in a Tokamak are either infinite in length, or close in themselves, quite the contrary of the structure of closed coronal fields, where the field lines are anchored at both ends. If there is a difference between closed coronal magnetic fields and Tokamak-type fields with regard to their intrinsic symmetry, this difference must be caused by the nature of the boundary conditions. The difference then is probably related to the fact that the corona/photosphere interface takes up the stresses from the coronal field, while the containment vessel of a Tokamak obviously doesn't.

In this paper I will take up the issue of the geometrical constraints on magnetostatic equilibria from a somewhat different point of view. I will write the MHS equations in a general coordinate system - not necessarily orthogonal - and then choose the coordinates in such a way that the pressure gradients and current density vector are along coordinate lines, which makes their expressions very simple. I will then try to determine what constraints the MHS equations impose on the geometry of the solutions, that is expressed in the metric tensor. The first results do indicate some restrictions to the possible geometries of the solutions, but these do not seem to represent some sort of symmetry. The analysis of this paper cannot be regarded as completed, and more definite results will be published in the literature.

THE MHS EQUATIONS IN ARBITRARY COORDINATE SYSTEMS

The basic equations governing magnetostatic equilibria are well known

$$(\vec{\nabla} \times \vec{B}) \times \vec{B} - \vec{\nabla}P = \vec{0} \qquad (1),$$

$$\vec{\nabla}.\vec{B} = 0 \qquad (2).$$

There are no general solutions known to this deceivingly simple looking set of equations. The system is nonlinear because of the Lorentz-force term in Eq. (1): the sum of two solutions in general does not represent a third solution. All particular analytical solutions that are known to date have some sort of symmetry (Low, 1985).

An alternative notation of Eqs. (1) and (2) in an arbitrary orthogonal coordinate system is found after introduction of the metric tensor

$$g_{ij} = \begin{bmatrix} h_1{}^2 & 0 & 0 \\ 0 & h_2{}^2 & 0 \\ 0 & 0 & h_3{}^2 \end{bmatrix} \qquad (3),$$

where the h_i represent the length of the unit vectors. The components of the magnetic field are identified with the three independent components of an antisymmetric 3-D tensor B^{ij},

$$B^{12} = \frac{B_3}{h_1 h_2}, \qquad B^{23} = \frac{B_1}{h_2 h_3}, \qquad B^{31} = \frac{B_2}{h_1 h_3} \qquad (4).$$

The identification of the components of the magnetic field with those of an antisymmetric tensor, instead of with the components of a contravariant vector, will lead to a particulary simple formulation of the equations. Moreover, it accounts properly for the fact that the magnetic field is a pseudo-vector, instead of a real vector: by inspection of the expression for the Lorentz-force one finds that the magnetic field must remain the same under a mirror transformation of the the coordinates ($x \rightarrow -x$, etc.), since the Lorentz force and the current both will change sign.

The contravariant components of the current density vector are given by

$$j^i = \frac{(\vec{j}\cdot\hat{x}_i)}{h_i} \qquad (5),$$

while the covariant components of the pressure gradient are given in the same way.

The basic equations in this notation are

$$P_{\P k} = j^i B_{ki} \qquad (6),$$

$$j^i = B^{ij}{}_{\P\P j} = \frac{1}{\sqrt{g}}(\sqrt{g}B^{ij})_{\P j} \qquad (7),$$

$$\{B_{ij\P k}\} = 0 \qquad (8).$$

Here $_{\P\P i}$ denotes covariant differentiation and $_{\P i}$ ordinary differentiation with respect to the variable denoted by the index. $\{..\}$ means a summation over all permutations of the indices of the tensor within the brackets. For the antisymmetric magnetic tensor it reduces to

$$B_{12\P 3} + B_{23\P 1} + B_{31\P 2} = 0 \qquad (9).$$

FLUX SURFACE COORDINATES

Now I generalize the MHS equations (6) to (8) and suppose their validity in non-orthogonal coordinate systems. The expressions remain the same of course, only the metric tensor has off-diagonal elements. This allows one to choose a coordinate system that makes the equations particulary simple. First I choose the pressure gradient parallel to the first unit vector and hence

$$P = P(x_1) \qquad (10).$$

The isobaric surfaces are now by definition surfaces of constant x_1. Further I choose x_2 and x_3 perpendicular to x_1, but not necessarily perpendicular to each other. This means that the metric tensor has the form

$$g_{ij} = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & g_{23} \\ 0 & g_{23} & g_{33} \end{bmatrix} \qquad (11),$$

and has therefore 4 independent components. From the force balance equations (7) and Eq. (11) it can now easily be shown that

$$j^1 = 0$$

$$B^{23} = -B^{32} = 0$$

$$B_{23} = -B_{32} = 0 \qquad (12).$$

The equation expressing the divergencelessness of the magnetic field, Eq. (8), reduces to

$$B_{12,3} + B_{31,2} = 0 \qquad (13),$$

with the solution,

$$B_{12} = A_{,2}$$

$$B_{13} = A_{,3} \qquad (14),$$

and $A(x_1,x_2,x_3)$ an arbitrary function. Now that the three components of the magnetic field have been reduced to óne unknown function only óne component of the force-balance equations remains to be satisfied,

$$P_{,1} = j^2 A_{,2} + j^3 A_{,3} \qquad (15),$$

while the demand $j^1 = 0$ leads to a second constraint on the solutions (reminiscent of Low's (1980) compatibility relation).

So far the non-orthogonality of the coordinates has not been used. I make a small digression from my main argument now to explore somewhat further the compatability constraint in orthogonal systems, $j^1 = 0$.

$$(\sqrt{g}B^{12})_{,2} + (\sqrt{g}B^{13})_{,3} = 0 \qquad (16).$$

In an orthogonal coordinate system I can now reduce the force balance equation and the compatability constraint to:

$$\left[\frac{A_{,2}h_3}{h_1h_2}\right]_{,2} + \left[\frac{A_{,3}h_2}{h_1h_3}\right]_{,3} = 0 \qquad (17),$$

and,

$$h_1h_2h_3P_{\partial 1} = A_{\partial 2}\left[\frac{A_{\partial 2}h_3}{h_1h_2}\right]_{\partial 1} + A_{\partial 3}\left[\frac{A_{\partial 3}h_2}{h_1h_3}\right]_{\partial 1} \qquad (18).$$

By trial and error one may find that it is extraordinarily difficult to obtain functions A that satisfy **both** constraints Eqs. (17) and (18). To give a short and very simple example I shall investigate the case where the isobaric surfaces are cylinders. We have $(x_1,x_2,x_3) = (r,\phi,z)$, $h_1=h_3=1$, $h_2=r$, and therefore Eqs. (17) and (18) reduce to

$$A_{\partial\phi\partial\phi} + r^2A_{\partial z\partial z} = 0 \qquad (19).$$

$$rP_{\partial r} = A_{\partial\phi}\left(\frac{A_{\partial\phi}}{r}\right)_{\partial r} + A_{\partial z}\left(A_{\partial z}r\right)_{\partial r} \qquad (20).$$

The only solution of Eq.(19) that is consistent with $P = P(r)$ is

$$A(r,\phi,z) = f(r)z + g(r)\phi \qquad (21),$$

and Eq. (20) takes the well known form

$$P_{\partial r} = \tfrac{1}{2}(B_\phi^{\,2} + B_z^{\,2})_{\partial r} + B_\phi^{\,2}/r \qquad (22),$$

after the identification

$$g(r) = B_z r\ , \qquad f(r) = -B_\phi \qquad (23).$$

The solutions of Eq. (22) are well known (Lüst and Schlüter, 1954). I conclude that the requirement of cylindrical isobares introduces the necessity of cylindrical symmetry of the magnetic field. However, the general question one would like to answer remains: what are the restrictions imposed on the function A by Eqs. (17) and (18) in any coordinate system?

NON-ORTHOGONAL COORDINATES

I will proceed with the main line of my argument now and specify further the choice of the coordinate system. I choose the direction of the second unit vector along the current density vector, i.e.

$$j^i = (0,j^2,0) \qquad (24).$$

It can be shown that with this choice of the second unit vector the coordinate system cannot be orthogonal anymore. In Fig. 1 a cylindrical surface is drawn that contains a set of field lines and a set of current density vectors. The first unit vector is by definition perpendicular to this surface, while the other two must lie within the surface. In the figure a current line is drawn which has the second unit vector everywhere parallel to it. In an orthogonal coordinate system the third unit vector must be everywhere perpendicular to the second one, but if one follows the third coordinate from point (x_1,x_2,x_3)

one finds that it intersects again the current line at the point $(x_1,x_2,x_3+\Delta x_3)$. However, one may also follow the current line from the point (x_1,x_2,x_3) and then the second intersection is reached at $(x_1,x_2+\Delta x_2,x_3)$. This point is physically the same as $(x_1,x_2,x_3+\Delta x_3)$ and hence it must have the same coordinates; if not, the coordinate system becomes double (and multi-) valued. This requirement can be met by having x_2 and x_3 as cyclic coordinates: $x_2 + \Delta x_2 \equiv x_2$, and $x_3 + \Delta x_3 \equiv x_3$ (just as in spherical coordinates). However, now one runs into the contradiction that two physically different points (the first and the second intersection) have identical coordinates; a situation which is also undesirable. I conclude that it is impossible to choose a consistent isobaric orthogonal coordinate system if $\vec{j}//x_2$.

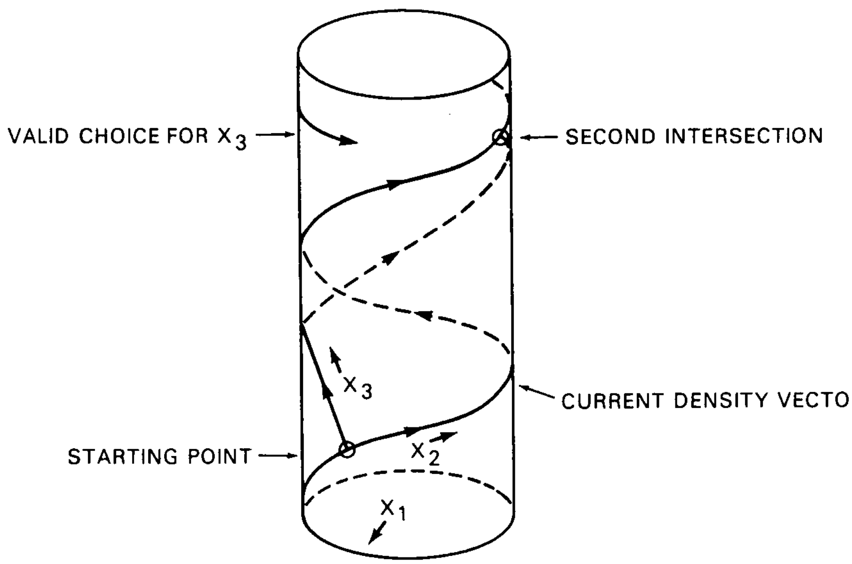

Figure 1. A demonstration of the inconsistency of an orthogonal coordinate system with one unit vector parallel to the pressure gradient and another with the current density vector.

A valid choice for the third coordinate is indicated in Fig. 1. It is clear that as one follows the third unit vector, the line will close in itself and there is no inconsistency. With this choice the third vector is not perpendicular anymore to the second and hence the term $g_{23}=g_{32}$ in the metric tensor must be nonzero.

Eqs. (14), (15) and (16) were derived for the metric Eq. (11). By the special choice for the second coordinate one finds in addition, because $j^3=0$,

$$(\sqrt{g}B^{31})_{,1} = 0 \qquad (25).$$

Eqs. (16) and (25) are satisfied when

$$(\sqrt{g}B^{21}) = f(x_1,x_3) + m_{,3}(x_2,x_3) \qquad (26),$$

and

$$(\sqrt{g}B^{31}) = -m_{\P 2}(x_2,x_3) \tag{27},$$

with f and m arbitrary functions. The force balance equation (15) now reduces to

$$\sqrt{g}P_{\P 1} = A_{\P 2}f_{\P 1} \tag{28}.$$

The contravariant components of the field tensor may be eliminated with

$$A_{\P 2} = g_{11}g_{22}B^{12} + g_{11}g_{23}B^{13} \tag{29},$$

$$A_{\P 3} = g_{11}g_{32}B^{12} + g_{11}g_{33}B^{13} \tag{30},$$

and one is left with three equations, (26), (27) and (28), for the function A. For a given geometry g_{ij}, and arbitrary functions f, m and P, the function A is clearly overdetermined and consequently some restrictions must apply to the possible choices of the geometry. In this stage of the research it is not clear yet what these restrictions are, although it seems that the restrictions do not necessarily impose an ignorable coordinate.

More work along the lines of this paper is needed to shed light on the geometries that are consistent with MHS-equilibria. In particular the restrictions that the boundary conditions impose on the possible solutions will be investigated.

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