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EQUATIONS OF MOTION OF A SPACE STATION WITH EMPHASIS ON THE EFFECTS OF THE GRAVITY GRADIENT

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16. ABSTRACT

The derivation of the equations of motion is based upon the principle of virtual work. As developed, these equations apply only to a space vehicle whose physical model consists of a rigid central carrier supporting several flexible appendages (not interconnected), smaller rigid bodies, and point masses. Clearly evident in the equations is the respect paid to the influence of the Earth's gravity field, considerably more than has been the custom in simulating vehicle motion. The effect of unpredictable crew motion is ignored.

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CORRIGENDA*

The upper case letter T, where it appears as a superscript to indicate the transpose of either a matrix or a vector, was supposed to have been of dimensions much smaller than those of the same letter used to designate a transformation (a rotation matrix). Prior to proofreading the first typewritten draft, the author was not aware that the typist was not equipped with the said diminutive character which appears so frequently in his handwritten work. In the interest of reducing time and effort spent on publication it was decided (regrettably) not to replace the larger size letter T, where it appears as a superscript, with a more suitable symbol at the typist's disposal.

* Technical Memorandum by L. P. Tuell entitled "Equations of Motion of a Space Station with Emphasis on the Effect of the Gravity Gradient."

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TECHNICAL MEMORANDUM

EQUATIONS OF MOTION OF A SPACE STATION WITH EMPHASIS ON THE EFFECT OF THE GRAVITY GRADIENT

SECTION 1 – MODEL DESCRIPTION

The class of space stations treated in this report does not include space stations with the most general configuration conceivable. The station model dealt with herein is comprised of a rigid central body supporting several flexible appendages, smaller rigid bodies, and point masses. The smaller rigid bodies include swivel engines and rotors while the point masses may be part of the mechanical analog of a consumable liquid aboard the carrier or may represent small bodies having a prescribed motion relative to the carrier (trim masses, for example, installed for the purpose of attitude control). Each flexible appendage admits 3-D bending and has at most two "rigid body" rotational degrees of freedom relative to the central body. The flexible appendages are not interconnected. The disturbing effects of crew motion on vehicle attitude will be ignored.

This particular choice of space station model was strongly influenced by the author's brief acquaintance with the SEPS vehicle of the early 1970's to which he applied certain of his previous developments [1] in describing its motion.

SECTION 2 – COORDINATE SYSTEMS

The formulations of this paper required the introduction of the following rectangular coordinate systems.

$X_N Y_N Z_N$ – The N-frame is a Newtonian frame whose existence is postulated (otherwise there would be no Newtonian mechanics). It is stipulated here that the N-frame is oriented as the S-frame defined below (this being merely for convenience, actually). The N-frame will enjoy brief recognition, its introduction being made solely for the purpose of preventing failure to include certain terms in the equations of motion (in particular, the vector translational equation). See Figure 1.

$X_S Y_S Z_S$ – The S-frame has origin at the geometric center of the reference ellipsoid with the X_S axis directed through the mean vernal equinox of a specified epoch, the Z_S axis directed as the Earth's mean spin vector, and the Y_S axis (lying in the mean equatorial plane) so directed as to make the system right-handed. The unit vectors $\vec{I}_S$, $\vec{J}_S$, and $\vec{K}_S$ which span the S-space are directed as the X_S, Y_S, and Z_S axes, respectively. The S-frame is coordinate system No. 4 of Reference 2.

$X_E Y_E Z_E$ – The origin of the "Earth fixed" or E-frame coincides with that of the S-frame and the Z_E axis is directed as the Z_S axis. The $X_E Y_E$ plane is the mean equatorial plane, the X_E axis being directed through the prime meridian (for all $t \geq t_o$) and the Y_E axis directed so that $X_E Y_E Z_E$ is right-handed.

$\tilde{x}\tilde{y}\tilde{z}$ — The $\tilde{B}$-frame (the structural axes system), also right-handed, is at rest relative to the rigid central carrier, and its origin and orientation may be arbitrarily chosen relative to the central body. Notice that this definition does not require that the origin of the $\tilde{B}$-frame be embedded in the central body, though such may be the case if deemed more convenient. (In the early Saturn vehicles, for example, the structural axes had origin 100 in. behind the engine gimbal plane.) The unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ spanning the $\tilde{B}$-space are directed as the $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ axes, respectively. See Figure 1.

xyz — The "body" axes system (herein designated the B-frame) has origin at the instantaneous center of mass (CM) of the entire vehicle system and is oriented always as $\tilde{x}\tilde{y}\tilde{z}$ (the $\tilde{B}$-frame). The origin of xyz has position vector $\vec{r}_{CM}$ relative to $\tilde{x}\tilde{y}\tilde{z}$. As a consequence of the redistribution of of mass due to bending, motion of internal parts, etc., the vector $\vec{r}_{CM}$ will vary in both direction and magnitude. The unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ also span the B-space. See Figure 1.

$x_i y_i z_i$ — The i-frame, a right-handed system, has origin at the "idealized" point of attachment of the i^{th} flexible appendage, the x_i axis serving as the axis of rotation about which flexible appendage i can rotate (through the angle θ_i) relative to the rigid central body. Bending displacements of points belonging to flexible appendage i are referred to axes $x_i y_i z_i$, that is, by the symbol $\vec{\Delta}^i \equiv \vec{\Delta}(\vec{r}_i, t)$ is meant the displacement due to bending of the point which had position vector $\vec{r}_i$ (referred to $x_i y_i z_i$) before deformation. The unit vectors $\vec{i}_i$, $\vec{j}_i$, and $\vec{k}_i$ span the i-space (i = 1, ..., NA), these being directed as the x_i, y_i, and z_i axes, respectively. It should be remarked here that the attachment point, being a point of the central body, does not itself undergo a "bending" displacement by virtue of the assumed rigidity of the central body. See Figure 1.

$x_i' y_i' z_i'$ — These axes have origin at the instantaneous CM of flexible appendage i whatever its configuration, deformed or undeformed. They are always oriented as $x_i y_i z_i$ (i = 1,... NA).

$x_{Ri}y_{Ri}z_{Ri}$ – The Ri-frame has origin at the CM of rotor i (i = 1, ... NR), the x_{Ri} axis being coincident with the axis of rotation and directed in accordance with the right-hand rule. The Ri-frame is fixed relative to the $\tilde{B}$-frame. It too is a right-handed system as are all the other systems of this report. Spanning the Ri-space are the unit vectors $\vec{i}_{RI}$, $\vec{j}_{Ri}$, and $\vec{k}_{Ri}$.

$x_{Ri}'y_{Ri}'z_{Ri}'$ – The (Ri)'-frame is fixed relative to rotor i and coincides with the Ri-frame at the instant the rotor begins to rotate and at such times at which the rotor completes a revolution. (Introduction of $x_{Ri}'y_{Ri}'z_{Ri}'$ was necessary because of the method of formulating the equations of motion. The angle, φ_{Ri}, through which $x_{Ri}'y_{Ri}'z_{Ri}'$ rotates relative to $x_{Ri}y_{Ri}z_{Ri}$ will not appear in the final results though its first and second time derivatives will.) The (Ri)' space is spanned by the unit vectors $\vec{i}_{Ri}'$, $\vec{j}_{Ri}'$ and $\vec{k}_{Ri}'$.

$x_{Gi}y_{Gi}z_{Gi}$ – The gimbal frame axes (Gi-frame) of the i^{th} single DOF (SDOF) control moment gyro (CMG) have origin at the CM of the gimbal frame, the x_{Gi} axis being coincident with the gimbal axis of rotation and the z_{Gi} axis directed as the spin vector of the gyro element. The x_{Gi} and z_{Gi} axes are directed in accordance with the right-hand rule. The direction of the x_{Gi} axis is invariant in the $\tilde{B}$- and B-frames. The unit vectors $\vec{i}_{Gi}$, $\vec{j}_{Gi}$, and $\vec{k}_{Gi}$ span the Gi-space.

$x_{gi}y_{gi}z_{gi}$ – The gyro element axes of the i^{th} SDOF CMG (the gi-frame) have the same origin as $x_{Gi}y_{Gi}z_{Gi}$, it being assumed here that the gimbal and gyro element have a common CM. The z_{gi} axis is directed as the gyro element spin vector. The gi-frame is fixed relative to the gyro element and coincides with the Gi-frame at each instant it completes a revolution. The unit vectors $\vec{i}_{gi}$, $\vec{j}_{gi}$, and $\vec{k}_{gi}$ span the gi-space. (As with the $x_{Ri}'y_{Ri}'z_{Ri}'$ frame it was the method of formulating the equations of motion that required introducing the $x_{gi}y_{gi}z_{gi}$ frame. The angle φ_{gi}, through which the gi-frame rotates at the constant rate $\dot{\varphi}_{gi} \equiv \omega_{gi}$ relative to the Gi-frame, will not appear in the final results, but ω_{gi} will be present in several terms.)

Associated with a two degree of freedom (2DOF) CMG are the following four coordinate systems and corresponding vector bases. All are right-handed and have the same origin, namely, the CM of the outer gimbal-inner gimbal-gyro element combination (it having been assumed that those three components of the CMG have a common CM.)

$x_{GB}y_{GB}z_{GB}$ – The gimbal base frame (GB-frame) is fixed relative to the $\tilde{B}$-frame and coincides with the OG-frame when the outer gimbal angle $\delta_{OG} = 0$. The GB-space is spanned by the unit vectors $\vec{i}_{GB}$, $\vec{j}_{GB}$, and $\vec{k}_{GB}$.

$x_{OG}y_{OG}z_{OG}$ – The outer gimbal space (OG-space) is spanned by the unit vectors $\vec{i}_{OG}$, $\vec{j}_{OG}$, and $\vec{k}_{OG}$. The z_{OG} axis (directed as $\vec{k}_{OG}$) is the outer gimbal rotation axis and its direction is invariant in the GB- and $\tilde{B}$-frames. The OG-frame is oriented as the IG-frame when the inner gimbal angle $\delta_{IG} = 0$.

$x_{IG}y_{IG}z_{IG}$ – The inner gimbal space (IG-space) is spanned by the unit vectors $\vec{i}_{IG}$, $\vec{j}_{IG}$, and $\vec{k}_{IG}$. The x_{IG} axis (directed as $\vec{i}_{IG}$) is the inner gimbal rotation axis. The gyro element spin vector is directed as $\vec{j}_{IG}$.

(The unit vector triads $\vec{i}_{\xi}$ $\vec{j}_{\xi}$ $\vec{k}_{\xi}$, ξ = GB, OG, IG, above are those of Reference 3 in different notation.)

$x_g y_g z_g$ – The gyro element space (g-space) is spanned by the unit vectors $\vec{i}_g$, $\vec{j}_g$, and $\vec{k}_g$. The y_g axis, directed as both $\vec{j}_g$ and $\vec{j}_{IG}$, is the gyro element spin axis. The g-frame (fixed relative to the gyro element) coincides with the IG-frame initially (that is, at the time designated t_o) and at each instant it completes a revolution.

(The seemingly absurd introduction of the axes $x_g y_g z_g$, never used in any of the literature browsed by the author, is necessary to the formulations of this paper, although the angle φ_g through which the g-frame rotates at the constant rate $\dot{\varphi}_g \equiv \omega_g$ relative to the IG-frame will not appear in the final results. The spin rate ω_g, however, will appear in certain terms of the moment equations, the outer gimbal equation, and the inner gimbal equation.)

The following is associated with the i^{th} swivel engine:

$x_{Ei}y_{Ei}z_{Ei}$ – The EI-frame has origin at the CM of swivel engine i, is fixed relative to swivel engine i, and if sensibly oriented has one of its axes directed as the engine thrust vector. The Ei-space is spanned by the unit vectors $\vec{i}_{Ei}$, $\vec{j}_{Ei}$, and $\vec{k}_{Ei}$ directed as x_{Ei}, y_{Ei}, and z_{Ei}, respectively.

SECTION 3 – TRANSFORMATIONS AND SUBSIDIARY RELATIONS

The vector bases defined in Section 2 give rise to the rotation matrices* defined below with important attendant relations.

T – The upper case letter T (when not used as a superscript to indicate the transpose of the matrix or column vector to which it is attached) will denote the rotation matrix defining the transformation $T(S\rightarrow B)$, the literal translation of the symbol $T(S\rightarrow B)$ being "transformation of the resolution of a vector on the S-vector basis to the resolution of that vector on the B-vector basis," or, more succinctly, "transformation from the S resolution to the B resolution." From the definitions of the $\tilde{B}$- and B-frames, it should be evident that $T(S\rightarrow B)$ and $T(S\rightarrow \tilde{B})$ are completely equivalent. (Insofar as consistency in notation is concerned, some will be eager to point out, in view of certain definitions to follow, that better notation might have been realized by attaching the subscripts SB to T. Though the author will be the first to agree that attaching the subscripts SB to T would, for the sake of consistency, be better, his frequent use of the letter T without subscripts in both years past and recently has left him almost impervious to change.)

The matrix T cannot be completely specified until an "Euler sequence" has been decided, that is, until one prescribes the sequence of rotations which the S-frame would have to undergo to assume the orientation of the B-frame (and hence of the $\tilde{B}$-frame). For a two, three, one sequence through the angles φ_p, φ_y, and φ_r, respectively, T is given by

$$T = [\varphi_r]_{(1)}\ [\varphi_y]_{(3)}\ [\varphi_p]_{(2)}$$

where

$$[\varphi_p]_{(2)} = \begin{bmatrix} \cos\varphi_p & 0 & -\sin\varphi_p \\ 0 & 1. & 0 \\ \sin\varphi_p & 0 & \cos\varphi_p \end{bmatrix} \tag{3-0.1}$$

*All rotation matrices herein are "proper" rotation matrices.

$$[\varphi_y]_{(3)} = \begin{bmatrix} \cos \varphi_y & \sin \varphi_y & 0 \\ -\sin \varphi_y & \cos \varphi_y & 0 \\ 0 & 0 & 1. \end{bmatrix} \tag{3-0.2}$$

$$[\varphi_r]_{(1)} = \begin{bmatrix} 1. & 0 & 0 \\ 0 & \cos \varphi_r & \sin \varphi_r \\ 0 & -\sin \varphi_r & \cos \varphi_r \end{bmatrix} \quad . \tag{3.03}$$

Following the method laid down in References 4 and 5, among others, it is shown in Reference 1 (and no doubt elsewhere) that corresponding to the Euler sequence above the angular velocity, $\vec{\omega}_B = [\omega_1, \omega_2, \omega_3]^T$, of the $\tilde{B}$-frame (or B-frame) relative to the S-frame is given in terms of the Euler angles and their first time derivatives by

$$\vec{\omega}_B \equiv \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \sin \varphi_y & 0 & 1. \\ \cos \varphi_y \cos \varphi_r & \sin \varphi_r & 0 \\ -\cos \varphi_y \sin \varphi_r & \cos \varphi_r & 0 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_p \\ \dot{\varphi}_y \\ \dot{\varphi}_r \end{bmatrix} \quad . \tag{3.04}$$

The vector $\vec{\omega}_B$ as given above has the B (and $\tilde{B}$) resolution. It is also shown in Reference 1 (and in literature not cited) that

$$\frac{d}{dt} T \equiv \dot{T} = \Omega T \quad , \tag{3-1}$$

the skew symmetric matrix Ω being given by

$$\Omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \quad . \tag{3-2}$$

A trivial consequence of the equation expressing $\dot{T}$ in terms of T and the components of $\vec{\omega}_B$, obtained by a simple transposition of its left and right members, is the relation

$$\dot{T}^T = T^T \Omega^T \tag{3-3}$$

which is used repeatedly in the development of the equations of motion. It is worthy of note in passing that the skew symmetric matrix Ω^T has the property that

$$\Omega^T \vec{V}_B \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} V_{B1} \\ V_{B2} \\ V_{B3} \end{bmatrix} = \vec{\omega}_B \times \vec{V}_B \tag{3-4}$$

where $\vec{V}_B = V_{B1}\vec{i} + V_{B2}\vec{j} + V_{B3}\vec{k}$ is an arbitrary vector expressed on the B (or $\tilde{B}$) vector basis.

With flexible appendage i ($1 \leq i \leq NA$) one associates the following rotation matrices.

$T_i = T(B \rightarrow i)_{\theta_i = 0}$ is a prescribed rotation matrix of constants, peculiar to a specific vehicle configuration, defining the transformation from the B (or $\tilde{B}$) resolution to the i resolution when $\theta_i = 0$, that is, when the i-frame is in its "null" orientation relative to the B-frame (or $\tilde{B}$-frame).

$$\mathcal{J}_i = [\theta_i]_{(1)} = \begin{bmatrix} 1. & 0 & 0 \\ 0 & \cos\theta_i & \sin\theta_i \\ 0 & -\sin\theta_i & \cos\theta_i \end{bmatrix} \tag{3-5}$$

$\tilde{T}_i = \mathcal{J}_i T_i = T(B \rightarrow i)$, the rotation matrix defining the transformation from the B (or $\tilde{B}$) resolution to the instantaneous i resolution. It is not difficult to establish the useful relations

$$\dot{\tilde{T}}_i^T = \tilde{T}_i^T \tilde{\Omega}_i^T \quad , \quad \ddot{\tilde{T}}_i^T = \tilde{T}_i^T \left(\tilde{\Omega}_i^{T^2} + \dot{\tilde{\Omega}}_i^T\right) \quad , \tag{3-6}$$

where

$$\tilde{\Omega}_i^T = \dot{\theta}_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1. \\ 0 & 1. & 0 \end{bmatrix} \quad . \tag{3-7}$$

It is easy to see further that with the notation $\vec{\tilde{\omega}}_i = \dot{\theta}_i \vec{i}_i$, the symbol $\vec{\tilde{\omega}}_i$ denoting the angular velocity of the i-frame relative to the B- or $\tilde{B}$- frame (and expressed on the i vector basis),

$$\tilde{\Omega}_i^T \vec{V}_i = \vec{\tilde{\omega}}_i \times \vec{V}_i \tag{3-8}$$

for any vector $\vec{V}_i$ expressed on the i vector basis.

Pertinent to the ith SDOF CMG are the following transformations, definitions, and consequential relations.

T_{Gio} = T(B→Gio), a prescribed rotation matrix of constants defining the transformation from the B (or $\tilde{B}$) resolution to the Gi resolution when the gimbal angle $\delta_{Gi} = 0$. The direction of $\vec{K}_{gi}$ (and hence of $\vec{K}_{Gi}$) when $\delta_{Gi} = 0$ and that of $\vec{i}_{Gi}$ determine the transformation T_{Gio}. (Here, by direction is meant direction in the B- (or $\tilde{B}$) frames.)

$$\tilde{T}_{Gi} = [\delta_{Gi}]_{(1)} = \begin{bmatrix} 1. & 0 & 0 \\ 0 & \cos\delta_{Gi} & \sin\delta_{Gi} \\ 0 & -\sin\delta_{Gi} & \cos\delta_{Gi} \end{bmatrix} \tag{3-9}$$

$T_{Gi} = \tilde{T}_{Gi} T_{Gio}$ = T (B→Gi), the transformation from the B (or $\tilde{B}$) resolution to the Gi resolution.

$\tilde{T}_{gi} = [\varphi_{gi}]_{(3)}$ = T(Gi→gi), the transformation from the Gi resolution to the gi resolution.

$T_{gi} = \tilde{T}_{gi} T_{Gi}$ = T(B→gi), the transformation from the B (or $\tilde{B}$) resolution to the gi resolution.

With the definitions:

$\vec{\omega}_{Gi} = \dot{\delta}_{Gi} \vec{i}_{Gi}$ = angular velocity of Gi frame (gimbal frame) relative to the B and $\tilde{B}$ frames expressed on the Gi vector basis (3-10)

$$\Omega_{Gi}^T = \dot{\delta}_{Gi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1. \\ 0 & 1. & 0 \end{bmatrix} \tag{3-11}$$

$$\vec{\omega}_{gi} = \dot{\varphi}_{gi}\,\vec{K}_{gi} = \omega_{gi}\,\vec{K}_{gi} = \text{angular velocity of the gyro element (gi frame) relative to the Gi frame} \tag{3-12}$$

$$\Omega^T_{gi} = \omega_{gi}\begin{bmatrix} 0 & -1. & 0 \\ 1. & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3-13}$$

there follows,

$$\dot{T}^T_{Gi} = T^T_{Gi}\ \Omega^T_{Gi} \tag{3-14}$$

$$\Omega^T_{Gi}\ \vec{V}_{Gi} = \vec{\omega}_{Gi} \times \vec{V}_{Gi} \quad \text{(for any vector } \vec{V}_{Gi} \text{ expressed on the Gi vector basis.)} \tag{3-15}$$

$$\Omega^T_{gi}\ \vec{V}_{gi} = \vec{\omega}_{gi} \times \vec{V}_{gi} \quad \text{(for any vector } \vec{V}_{gi} \text{ expressed on the gi vector basis.)} \tag{3-16}$$

$$\dot{\tilde{T}}^T_{gi} = \tilde{T}^T_{gi}\ \Omega^T_{gi} \tag{3-17}$$

$$\dot{T}^T_{gi} = T^T_{Gi}\ \Omega^T_{Gi}\ \tilde{T}^T_{gi} + T^T_{gi}\ \Omega^T_{gi}\quad . \tag{3-18}$$

Among other SDOF parameters presenting themselves in the derivation of the moment equation and gimbal equation are

$$I^{Gi} = \begin{bmatrix} I^{Gi}_{xx} & 0 & 0 \\ 0 & I^{Gi}_{yy} & 0 \\ 0 & 0 & I^{Gi}_{zz} \end{bmatrix} = \text{inertia matrix of gimbal referred to } x_{Gi}\ y_{Gi}\ z_{Gi}\ , \tag{3-19}$$

$$I^{gi} = \begin{bmatrix} I^{gi}_{xx} & 0 & 0 \\ 0 & I^{gi}_{yy} & 0 \\ 0 & 0 & I^{gi}_{zz} \end{bmatrix} = \text{inertia matrix of gyro element referred to } x_{gi}\ y_{gi}\ z_{gi}\ (I^{gi}_{xx} = I^{gi}_{yy} \text{ assumed})\ , \tag{3-20}$$

an important consequence of the assumption $I^{gi}_{xx} = I^{gi}_{yy}$ being

$$\tilde{T}_{gi}^{T} \; I^{gi} \; \tilde{T}_{gi} = I^{gi} \tag{3-21}$$

from which it follows that

$$T_{gi}^{T} \; I^{gi} \; T_{gi} = T_{Gi}^{T} \; I^{gi} \; T_{Gi} \quad . \tag{3-22}$$

Also appearing in the moment equation are the vectors $\vec{\omega}_{Gi}{}'$, $\dot{\vec{\omega}}_{Gi}{}'$, and $\vec{\omega}_{gi}{}'$, the prime indicating the B (or $\tilde{B}$) resolution. They are defined by

$$\vec{\omega}_{Gi}{}' = T_{Gi}^{T} \; \vec{\omega}_{Gi} = T_{Gio}^{T} \; \tilde{T}_{Gi}^{T} \; \vec{\omega}_{Gi} = T_{Gio}^{T} \; \vec{\omega}_{Gi} = T_{Gio}^{T} \begin{bmatrix} \dot{\delta}_{Gi} \\ 0 \\ 0 \end{bmatrix} \tag{3-23}$$

$$\dot{\vec{\omega}}_{Gi}{}' = T_{Gio}^{T} \begin{bmatrix} \ddot{\delta}_{Gi} \\ 0 \\ 0 \end{bmatrix} \tag{3-24}$$

$$\vec{\omega}_{gi}{}' = T_{gi}^{T} \; \vec{\omega}_{gi} = T_{Gi}^{T} \; \tilde{T}_{gi}^{T} \; \vec{\omega}_{gi} = \omega_{gi} \; T_{Gio}^{T} \begin{bmatrix} 0 \\ -\sin \delta_{Gi} \\ \cos \delta_{Gi} \end{bmatrix} \tag{3-25}$$

Transformations and subordinate relations pertinent to a 2DOF CMG follow.

$T_{BGB} = T(B \rightarrow GB)$, a prescribed rotation matrix of constants defining the transformation from the B (or $\tilde{B}$) resolution of a vector to its gimbal base (GB) resolution.

$T_{GBOG} = T(GB \rightarrow OG) = [\delta_{OG}]_{(3)}$, the transformation from the GB resolution to the OG resolution.

$T_{OGIG} = T(OG \rightarrow IG) = [\delta_{IG}]_{(1)}$, the transformation from the OG resolution to the IG resolution.

$T_{IGg} = T(IG \rightarrow g) = [\varphi_g]_{(2)}$, the transformation from the IG resolution to the g resolution.

$T_{BOG} = T(B \rightarrow OG) = T_{GBOG} T_{BGB} = [\delta_{OG}]_{(3)} \; T_{BGB} = T(B \rightarrow OG)$, the transformation from the B (or $\tilde{B}$) resolution to the OG resolution.

$T_{BIG} = T(B\rightarrow IG) = T(\tilde{B}\rightarrow IG) = T_{OGIG}T_{GBOG}T_{BGB} = [\delta_{IG}]_{(1)}\ [\delta_{OG}]_{(3)}\ T_{BGB}$, the transformation from the B (or $\tilde{B}$) resolution to the IG resolution.

$T_{Bg} = T(B\rightarrow g) = T_{IGg}T_{BIG}$, the transformation from the B (or $\tilde{B}$) resolution to the g resolution.

With the symbols $\vec{\omega}_{OG}$, $\vec{\omega}_{IG}$, and $\vec{\omega}_g$ denoting the angular velocity of the OG-frame relative to the GB-frame (and hence relative to the $\tilde{B}$- and B-frames), the angular velocity of the IG-frame relative to the OG-frame, and the angular velocity of the g-frame relative to the IG-frame, respectively, one has

$$\begin{aligned} \vec{\omega}_{OG} &= \dot{\delta}_{OG}\ \vec{K}_{OG} \\ \vec{\omega}_{IG} &= \dot{\delta}_{IG}\ \vec{i}_{IG} = \dot{\delta}_{IG}\ \vec{i}_{OG} \\ \vec{\omega}_g &= \omega_g\ \vec{j}_g = \omega_g\ \vec{j}_{IG} \end{aligned} \tag{3-26}$$

and the associated skew symmetric matrices

$$\left.\begin{aligned} \Omega^T_{OG} &= \dot{\delta}_{OG} \begin{bmatrix} 0 & -1. & 0 \\ 1. & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \Omega^T_{IG} &= \dot{\delta}_{IG} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1. \\ 0 & 1. & 0 \end{bmatrix} \\ \Omega^T_g &= \omega_g \begin{bmatrix} 0 & 0 & 1. \\ 0 & 0 & 0 \\ -1. & 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{3-27}$$

having the properties

$\Omega^T_{OG}\ \vec{V}_{OG} = \vec{\omega}_{OG} \times \vec{V}_{OG}$ (for arbitrary $\vec{V}_{OG}$ expressed on the OG-vector basis),

$$\Omega_{IG}^T \vec{V}_{IG} = \vec{\omega}_{IG} \times \vec{V}_{IG} \text{ (for arbitrary } \vec{V}_{IG} \text{ expressed on the IG-vector basis)},$$

$$\Omega_g^T \vec{V}_g = \vec{\omega}_g \times \vec{V}_g \text{ (for arbitrary } \vec{V}_g \text{ expressed on the g-vector basis)}.$$

From the definitions above one can easily deduce that

$$\begin{aligned}
\dot{T}_{GBOG}^T &= T_{GBOG}^T \, \Omega_{OG}^T \\
\dot{T}_{BOG}^T &= T_{BOG}^T \, \Omega_{OG}^T \\
\dot{T}_{OGIG}^T &= T_{OGIG}^T \, \Omega_{IG}^T \\
\dot{T}_{BIG}^T &= T_{BIG}^T \left(T_{OGIG} \, \Omega_{OG}^T \, T_{OGIG}^T + \Omega_{IG}^T \right) \\
\dot{T}_{IGg}^T &= T_{IGg}^T \, \Omega_g^T \\
\dot{T}_{Bg}^T &= \left(T_{BOG}^T \, \Omega_{OG}^T \, T_{OGIG}^T + T_{BOG}^T \, T_{OGIG}^T \, \Omega_{IG}^T \right) T_{IGg}^T + T_{BIG}^T \, T_{IGg}^T \, \Omega_g^T \quad ,
\end{aligned} \tag{3-28}$$

these relations being of considerable utility in subsequent derivations. Regarding the derivations to follow, it should be remarked that the inertia matrices of outer gimbal, inner gimbal, and gyro element, denoted, respectively, by I^{OG}, I^{IG}, and I^g, are supposed diagonal with $I_{xx}^{OG} = I_{yy}^{OG}$, $I_{yy}^{IG} = I_{zz}^{IG}$, and $I_{xx}^g = I_{zz}^g$. The matrices I^{OG}, I^{IG}, and I^g are, incidentally, referred to $x_{OG}y_{OG}z_{OG}$, $x_{IG}y_{IG}z_{IG}$, and $x_g y_g z_g$, respectively. Abbreviations of certain combinations of these matrices appear in the moment equation. They are defined as follows.

$$\begin{aligned}
I_B^{OG+IG+g} &= T_{BOG}^T I^{OG} \, T_{BOG} + T_{BIG}^T \left(I^{IG} + I^g \right) T_{BIG} \\
I^{IG+g} &= I^{IG} + I^g \\
I_B^{IG+g} &= T_{BIG}^T \, I^{IG+g} \, T_{BIG} \\
I_B^g &= T_{Bg}^T \, I^g \, T_{Bg} = T_{BIG}^T \, I^g \, T_{BIG} \quad .
\end{aligned} \tag{3-28.1}$$

The trace of I^{ξ} is denoted by $Tr(I^{\xi})$, that is,

$$Tr(I^{\xi}) = I_{xx}^{\xi} + I_{yy}^{\xi} + I_{zz}^{\xi} \quad , \quad \xi = OG, IG, g \quad . \tag{3-28.2}$$

The first time derivatives of the vectors in equation (3-26), as measured in and expressed on the indicated vector bases, are obviously

$$\begin{aligned} \left(\frac{d}{dt}\right)_{OG} \vec{\omega}_{OG} &= \dot{\vec{\omega}}_{OG} = \ddot{\delta}_{OG} \vec{K}_{OG} \\ \left(\frac{d}{dt}\right)_{IG} \vec{\omega}_{IG} &= \dot{\vec{\omega}}_{IG} = \ddot{\delta}_{IG} \vec{i}_{IG} \\ \left(\frac{d}{dt}\right)_{g} \vec{\omega}_{g} &= \dot{\vec{\omega}}_{g} = \dot{\omega}_{g} \vec{j}_{g} = \vec{0} \quad (\omega_g = \text{constant}) \quad . \end{aligned} \tag{3-29}$$

The B (or $\tilde{B}$) resolutions of the vectors of equations (3-26) are given by

$$\begin{aligned} \vec{\omega}_{OG}' &= T_{BOG}^{T} \vec{\omega}_{OG} \\ \vec{\omega}_{IG}' &= T_{BIG}^{T} \vec{\omega}_{IG} \\ \vec{\omega}_{g}' &= T_{Bg}^{T} \vec{\omega}_{g} \end{aligned} \tag{3-30}$$

while the first time derivatives (as measured in the $\tilde{B}$- or B-frame) of the primed vectors of equations (3-30) are given by

$$\begin{aligned} \left(\frac{d}{dt}\right)_{B} \vec{\omega}_{OG}' &= \dot{\vec{\omega}}_{OG}' = T_{BOG}^{T} \dot{\vec{\omega}}_{OG} \\ \left(\frac{d}{dt}\right)_{B} \vec{\omega}_{IG}' &= \dot{\vec{\omega}}_{IG}' = T_{BIG}^{T} \dot{\vec{\omega}}_{IG} + \vec{\omega}_{OG}' \times \vec{\omega}_{IG}' \\ \left(\frac{d}{dt}\right)_{B} \vec{\omega}_{g}' &= \dot{\vec{\omega}}_{g}' = (\vec{\omega}_{OG}' + \vec{\omega}_{IG}') \times \vec{\omega}_{g}' \quad . \end{aligned} \tag{3-31}$$

Attention is called here to a remark concerning derivatives that appears as the first sentence of the paragraph following equation (4-19).

Pertinent to the i^{th} swivel engine is the transformation $T_{Ei} = T(B \to Ei)$, the transformation from the B (or $\tilde{B}$) resolution to the Ei resolution, whose time derivative is shown in Reference 1 to be given by

$$\dot{T}_{Ei} = \Omega_{Ei} T_{Ei} - T_{Ei} \Omega \tag{3-32}$$

where

$$\Omega_{Ei} = -\Omega_{Ei}^{T} = \begin{bmatrix} 0 & \omega_{Ei}^{(3)} & -\omega_{Ei}^{(2)} \\ -\omega_{Ei}^{(3)} & 0 & \omega_{Ei}^{(1)} \\ \omega_{Ei}^{(2)} & -\omega_{Ei}^{(1)} & 0 \end{bmatrix} , \tag{3-33}$$

the $\omega_{Ei}^{(j)}$, j = 1, 2, 3, being the components of the vector $\vec{\omega}_{Ei}$ which is here the angular velocity of the Ei-frame relative to the S-frame and is supposed expressed on the Ei vector basis, that is,

$$\vec{\omega}_{Ei} = T_{Ei} (\vec{\omega}_B + \vec{\omega}_{Ei}') \quad , \tag{3-34}$$

the symbol $\vec{\omega}_{Ei}'$ denoting the angular velocity of the Ei-frame relative to B (or $\tilde{B}$) frame and expressed on the B (or $\tilde{B}$) vector basis. Among other relations satisfied by $\dot{T}_{Ei}$ are the following,

$$\begin{aligned} \dot{T}_{Ei} &= T_{Ei} \Omega_{Ei}' \\ \dot{T}_{Ei} &= \tilde{\Omega}_{Ei} T_{Ei} \end{aligned} \tag{3-35}$$

where

$$\Omega_{Ei}' = -\Omega_{Ei}'^{T} = \begin{bmatrix} 0 & \omega_{Ei}'^{(3)} & -\omega_{Ei}'^{(2)} \\ -\omega_{Ei}'^{(3)} & 0 & \omega_{Ei}'^{(1)} \\ \omega_{Ei}'^{(2)} & -\omega_{Ei}'^{(1)} & 0 \end{bmatrix} , \tag{3-36}$$

and

$$\tilde{\Omega}_{Ei} = -\tilde{\Omega}^T_{Ei} = \begin{bmatrix} 0 & \tilde{\omega}^{(3)}_{Ei} & -\tilde{\omega}^{(2)}_{Ei} \\ -\tilde{\omega}^{(3)}_{Ei} & 0 & \tilde{\omega}^{(1)}_{Ei} \\ \tilde{\omega}^{(2)}_{Ei} & -\tilde{\omega}^{(1)}_{Ei} & 0 \end{bmatrix} , \tag{3-37}$$

the $\omega'^{(j)}_{Ei}$, j = 1, 2, 3, being the components of $\vec{\omega}'_{Ei}$ and the $\tilde{\omega}^{(j)}_{Ei}$, j = 1, 2, 3, those of the vector $\vec{\tilde{\omega}}_{Ei} = T_{Ei}\,\vec{\omega}'_{Ei}$ which is the angular velocity of the Ei-frame relative to the B (or $\tilde{B}$) frame expressed on the Ei-vector basis. An interesting consequence of the last two expressions for $\dot{T}_{Ei}$ is the similarity transformation,

$$\Omega'_{Ei} = T^T_{Ei}\,\tilde{\Omega}_{Ei}\,T_{Ei} \quad . \tag{3-38}$$

Complete specification of T_{Ei} and $\vec{\omega}'_{Ei}$ cannot be made until a "B to Ei Euler sequence" is prescribed, that sequence being dependent upon the point of application and direction of the thrust delivered by the engine, the engine actuator arrangement, the sign convention for engine deflections, etc.

The symbol I^{Ei} appearing in the moment equation denotes the inertia matrix of swivel engine i referred to the axes $x_{Ei}y_{Ei}z_{Ei}$ (defined in section 2).

Transformations pertaining to the i^{th} rotor are the following:

T_{Ri} = T(B→Ri), the rotation matrix of constants defining the transformation from the B (or $\tilde{B}$) resolution to the Ri resolution. Its elements are functions of the components of the unit vector $\vec{\Lambda}_{Ri} \equiv [\lambda_{Ri}, \mu_{Ri}, \nu_{Ri}]^T$ which specifies the positive direction (relative to the B-frame) of the spin axis (the x_{Ri} axis) of rotor i. A frame oriented as the B-frame could be carried into the Ri orientation by the 3, 2 sequence through the angles $\theta_{Ri} = \tan^{-1}(\mu_{Ri}/\lambda_{Ri})$, $0 \le \theta_{Ri} < 2\pi$, and $-\psi_{Ri}$ where $\psi_{Ri} = \tan^{-1}(\nu_{Ri}/\sqrt{1 - \nu_{Ri}^2})$, $-\pi/2 \le \psi_{Ri} \le \pi/2$, in which case T_{Ri} is determined by

$$T_{Ri} = [-\psi_{Ri}]_{(2)}\,[\theta_{Ri}]_{(3)} \quad . \tag{3-39}$$

$\mathcal{J}_{Ri}$ = T(Ri→(Ri)') = $[\varphi_{Ri}]_{(1)}$, the transformation from the Ri to the (Ri)' resolution.

$\tilde{T}_{Ri} = \mathcal{J}_{Ri} T_{Ri} = T(B \to (Ri)')$, the transformation from the B resolution to the (Ri)' resolution.

From the above it is a trivial matter to show that

$$\dot{\tilde{T}}^T_{Ri} = \tilde{T}^T_{Ri}\, \tilde{\Omega}^T_{Ri} \quad , \qquad \ddot{\tilde{T}}^T_{Ri} = \tilde{T}^T_{Ri}\, (\tilde{\Omega}^{T^2}_{Ri} + \dot{\tilde{\Omega}}^T_{Ri}) \quad , \tag{3-40}$$

The skew symmetric matrix $\tilde{\Omega}^T_{Ri}$ being given by

$$\tilde{\Omega}^T_{Ri} = \tilde{\omega}_{Ri} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1. \\ 0 & 1. & 0 \end{bmatrix} , \qquad \tilde{\omega}_{Ri} = \dot{\varphi}_{Ri} \quad , \tag{3-41}$$

with the obvious property that

$$\tilde{\Omega}^T_{Ri}\, \vec{V}_{(Ri)'} = \vec{\tilde{\omega}}_{Ri} \times \vec{V}_{(Ri)'} \quad , \tag{3-42}$$

where the vector $\vec{V}_{(Ri)'}$ is an arbitrary vector expressed on the (Ri)' vector basis and $\vec{\tilde{\omega}}_{Ri}$ denotes the angular velocity of the (Ri)'-frame relative to the B-frame expressed on the (Ri)' vector basis. By definition of $\vec{\tilde{\omega}}_{Ri}$ one can write

$$\vec{\tilde{\omega}}_{Ri} = \dot{\varphi}_{Ri}\, \vec{i}_{Ri}' = \tilde{\omega}_{Ri}\, \vec{i}_{Ri}' \quad , \tag{3-43}$$

the scaler $\dot{\varphi}_{Ri} \equiv \tilde{\omega}_{Ri}$ admitting of both positive and negative values to correspond to positive or negative rotations (in accordance with the right-hand rule) about the rotor spin axis. Obviously,

$$\dot{\vec{\tilde{\omega}}}_{Ri} \equiv \left(\frac{d}{dt}\right)_{(Ri)'} \vec{\tilde{\omega}}_{Ri} = \dot{\tilde{\omega}}_{Ri}\, \vec{i}_{Ri}' = \ddot{\varphi}_{Ri}\, \vec{i}_{Ri}' \tag{3-44}$$

and furthermore, if $\vec{\omega}_{Ri}'$ denotes the B (or $\tilde{B}$) resolution of the angular velocity of the (Ri)' frame relative to the B (or $\tilde{B}$) frame then

$$\vec{\omega}_{Ri}' = \tilde{T}^T_{Ri}\, \vec{\tilde{\omega}}_{Ri} \tag{3-45}$$

and it follows that

$$\dot{\vec{\omega}}_{Ri}' \equiv \left(\frac{d}{dt}\right)_B \vec{\omega}_{Ri}' = \tilde{T}_{Ri}^T \tilde{\Omega}_{Ri}^T \tilde{\vec{\omega}}_{Ri} + \tilde{T}_{Ri}^T \dot{\tilde{\vec{\omega}}}_{Ri} = \tilde{T}_{Ri}^T \dot{\tilde{\vec{\omega}}}_{Ri} = T_{Ri}^T \dot{\tilde{\vec{\omega}}}_{Ri} \quad . \tag{3-46}$$

The inertia matrix, I^{Ri}, of rotor i referred to $x_{Ri}'y_{Ri}'z_{Ri}'$ is herein assumed diagonal with $I_{yy}^{Ri} = I_{zz}^{Ri}$. Under these assumptions it follows that

$$\mathcal{J}_{Ri}^T \, I^{Ri} \, \mathcal{J}_{Ri} = I^{Ri} \tag{3-47}$$

and hence that

$$\tilde{T}_{Ri}^T \, I^{Ri} \, \tilde{T}_{Ri} = T_{Ri}^T \mathcal{J}_{Ri}^T \, I^{Ri} \, \mathcal{J}_{Ri} \, T_{Ri} = T_{Ri}^T \, I^{Ri} \, T_{Ri} \quad , \tag{3-48}$$

a fact used in arriving at equation (4-67) in the following section.

Passage from the S resolution to the E resolution is effected via the transformation

$$T_{SE} = T(S \rightarrow E) = [\alpha_P + \omega_e \, (t - t_o)]_{(3)} \quad , \tag{3-49}$$

the symbol α_P denoting the right ascension of the prime meridian at time t_o and ω_e the magnitude of the Earth's spin vector.

Not to be overlooked, and certainly not the least important of the several transformations discussed herein, is the relation between the unit vector triad $\vec{u}_R \vec{u}_\lambda \vec{u}_\delta$ associated with a point of the earth's exterior gravitational field and the triad $\vec{I}_S \vec{J}_S \vec{K}_S$, that being

$$\begin{bmatrix} \vec{u}_R \\ \vec{u}_\lambda \\ \vec{u}_\delta \end{bmatrix} = [-\delta]_{(2)} \; [\alpha]_{(3)} \begin{bmatrix} \vec{I}_S \\ \vec{J}_S \\ \vec{K}_S \end{bmatrix} \tag{3-50}$$

where α and δ are the right ascension and declination, respectively, of the field point. If $\vec{R}_s = [X_S, Y_S, Z_S]^T = X_S \vec{I}_S + Y_S \vec{J}_S + Z_S \vec{K}_S$ denotes the position vector of the field point referred to the S-frame then the angles α and δ are given by

$$\alpha = \tan^{-1}(Y_S/X_S) \quad , \quad 0 \leq \alpha < 2\pi \quad ,$$

$$\delta = \sin^{-1}(Z_S/R_S) \quad , \quad -\frac{\pi}{2} < \delta < \frac{\pi}{2} \quad , \quad R_S = |\vec{R}_S| = \sqrt{X_S{}^2 + Y_S{}^2 + Z_S{}^2} \qquad (3\text{-}51)$$

If λ is the longitude (positive east of the prime meridian) of the field point then α is also given by

$$\alpha = \alpha_P + \omega_e\,(t-t_o) + \lambda \quad . \qquad (3\text{-}52)$$

The unit vectors $\vec{u}_R$, $\vec{u}_\lambda$, and $\vec{u}_\delta$ can also be expressed in terms of $\vec{I}_S$, $\vec{J}_S$, and $\vec{K}_S$ as follows.

$$\vec{u}_R = \vec{R}_S/R_S$$

$$\vec{u}_\lambda = \frac{\vec{K}_S \times \vec{R}_S}{|\vec{K}_S \times \vec{R}_S|} \quad , \quad \vec{R}_S \neq \pm R_S\,\vec{K}_S \quad , \qquad (3\text{-}53)$$

$$\vec{u}_\delta = \vec{u}_R \times \vec{u}_\lambda \quad .$$

Relative to the rotating E-frame (the earth fixed frame $X_E Y_E Z_E$) the field point has position $\vec{R}_E = [X_E, Y_E, Z_E]^T = T(S\rightarrow E)\,\vec{R}_S$ so that λ and δ can also be computed in accordance with

$$\lambda = \tan^{-1}(Y_E/X_E) \quad , \quad 0 \leq \lambda < 2\pi$$

$$(3\text{-}54)$$

$$\delta = \sin^{-1}(Z_E/R) \quad , \quad -\frac{\pi}{2} \leq \delta < \frac{\pi}{2} \quad , \quad R = |\vec{R}_E| = |\vec{R}_S| \quad ,$$

and in terms of $\vec{I}_E$, $\vec{J}_E$, and $\vec{K}_E$ the unit vectors $\vec{u}_R$, $\vec{u}_\lambda$, and $\vec{u}_\delta$ are given by

$$\begin{bmatrix} \vec{u}_R \\ \vec{u}_\lambda \\ \vec{u}_\delta \end{bmatrix} = [-\delta]_{(2)}\,[\lambda]_{(3)} \begin{bmatrix} \vec{I}_E \\ \vec{J}_E \\ \vec{K}_E \end{bmatrix} \quad . \qquad (3\text{-}55)$$

In Reference 6, the acceleration due to the Earth's gravity at the point with spherical coordinates (R, λ, δ) referred to the Earth fixed frame $X_E Y_E Z_E$ is resolved into its radial, longitudinal, and latitudinal components, these being the $\vec{u}_R$ component, the $\vec{u}_\lambda$ component, and the $\vec{u}_\delta$ component, respectively.

SECTION IV. THE EQUATIONS OF MOTION

The coordinates defining system configuration include the following: The components of the vector $\vec{\tilde{R}}_s \equiv [\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s]^T$, the position, referred to the S-frame, of the origin of the $\tilde{B}$-frame (the structural axes $\tilde{x}\tilde{y}\tilde{z}$); the Euler angles φ_p, φ_y, and φ_r specifying the orientation of the $\tilde{B}$-frame (and also that of the B-frame) relative to the S-frame; the angles θ_i, i = 1, ... NA, the symbol θ_i denoting the angle through which the i^{th} flexible appendage* is permitted to rotate as a "whole" relative to the rigid central carrier; the generalized bending displacement coordinates η_j^i, j = 1, ... N_i, associated with the i^{th} flexible appendage, i = 1, ... NA; the displacement ξ_{pi}, i = 1, ... NP, of the "point" mass m_{pi} from its equilibrium position; the angular deflections β_{yi} and β_{pi} of the i^{th} swivel engine, i = 1, ... NSE; the gimbal angle δ_{Gi} of the i^{th} SDOF CMG, i = 1, ... NSDOF; the outer gimbal angle δ_{OGi} and inner gimbal angle δ_{IGi} of the i^{th} two DOF CMG, i, ... N2DOF.

The principal of virtual work finds itself of considerable utility in developing the equations of motion. That principle**, as applied to dynamic systems, states that in an arbitrary virtual displacement of the system (compatible with the constraints) the virtual work done by the inertial forces plus that done by the external forces equals the change in strain energy. On invoking the principle of virtual work and appealing to the independence of the generalized coordinates defined above, it follows that

$$-\int_{(m)} \frac{\partial \vec{R}_S^{\,T}}{\partial q} \left\{ \ddot{\vec{R}}_S + \frac{\mu_M \vec{R}_S^{(MOON)}}{|\vec{R}_S^{(MOON)}|^3} + \frac{\mu_S \vec{R}_S^{(SUN)}}{|\vec{R}_S^{(SUN)}|^3} \right\} dm + Q_q - \frac{\partial D}{\partial \dot{q}} = \frac{\partial U}{\partial q} \qquad (4\text{-}1)$$

where q may be any of the aforementioned coordinates.

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* Later in this section, attention will be directed to a system wherein each flexible appendage is allowed two rotational degrees of freedom.

** See Reference 16, pp. 114-115.

In the volume integral in the left member of equation (4-1), $\vec{R}_S$ denotes the position referred to the S-frame of a generic point of the vehicle, while $\ddot{\vec{R}}_S$ denotes the second time derivative, as measured by an observer in the S-frame, of the vector $\vec{R}_S$. (It should be remarked here that the number of dots above a vector symbol indicates the order of the time derivative of that vector as measured by an observer in the reference frame on which the vector is expressed.) The superscript T on $\vec{R}_S$ in equation (4-1) indicates that $\vec{R}_S^{\,T}$ is the transpose of the 3 x 1 column vector $\vec{R}_S$. It is to be understood that when used as a subscript on an integral sign as in equation (4-1) above, the letter m simply indicates that the integration extends over the volume occupied by the entire vehicle system, while if it appears as one factor of a product, it denotes the numerical value of the mass of the entire system.

The generalized force Q_q is such that the product $Q_q \delta q$ is the virtual work done by those forces not derivable from either the potential* function U (as herein defined) or the dissipation function D when only the coordinate q undergoes the virtual displacement δq, the forces alluded to being (in this paper) the gravitational forces of Earth, Moon, and Sun; direct solar radiation; aerodynamic forces; the torques applied to rotate the flexible appendages (relative to the rigid central carrier); and the torques applied to rotate the CMG gimbals. Accordingly, Q_q must be given by

$$Q_q = \int_{(m)} \frac{\partial \vec{R}_S^{\,T}}{\partial q} (d\, \vec{F}_{GRAVITY,S}) + \int_{(A)} \frac{\partial \vec{R}_S^{\,T}}{\partial q} (d\, \vec{F}_{AERO,S})$$

$$\int_{(A)} \frac{\partial \vec{R}_S^{\,T}}{\partial q} (d\, \vec{F}_{SOLAR,S}) + \mathcal{M}_q \quad , \tag{4-1A}$$

where

$$\vec{F}_{GRAVITY,\,S} = \vec{F}^{(EARTH)}_{GRAVITY,\,S} + \vec{F}^{(MOON)}_{GRAVITY,\,S} + \vec{F}^{(SUN)}_{GRAVITY,\,S}$$

= GRAVITATIONAL FORCE EXERTED BY EARTH, MOON, AND SUN ON THE ENTIRE SYSTEM

*The symbol U in equation (4-1) is better termed the "strain" energy function. Use of the term "potential function" should not mislead the reader to believe that "every" force for which there exists a potential function "should" be derivable from U.

$\vec{F}_{AERO,\ S}$ = AERODYNAMIC FORCE ON THE ENTIRE SYSTEM

$\vec{F}_{SOLAR,\ S}$ = RESULTANT OF DIRECT SOLAR RADIATION PRESSURE ON SURFACE AREA

the subscript S again being indicative of the S-resolution, and where, necessarily,

$$\mathcal{M}_q \equiv 0 \text{ if} \begin{cases} q \neq \theta_i, \ i = 1, \ \ldots \ NA \\ q \neq \delta_{Gi}, \ i = 1, \ \ldots \ NSDOF \\ q \neq \delta_{OGi}, \ i = 1, \ \ldots \ N2DOF \\ q \neq \delta_{IGi}, \ i = 1, \ \ldots \ N2DOF \end{cases} , \tag{4-1B}$$

the symbol $\mathcal{M}_q$ denoting the torque applied to impart a change in the coordinate q when q is any of the angular coordinates θ_i, δ_{Gi}, δ_{OGi}, or δ_{IGi} (for each i over the respective ranges). The significance of the subscript m on the first integral sign in the right member of (4-1A) has already been discussed. It should be self evident that the subscript A on the second and third integral signs in (4-1A) simply indicates that the integration extends over the surface area of the entire vehicle (actually that portion of the surface experiencing impingement of air molecules in the case of the second integral and that part of the surface exposed to direct solar radiation in the case of the third integral).

The most simple of all approximations to $\vec{F}^{(MOON)}_{GRAVITY,\ S}$ and $\vec{F}^{(SUN)}_{GRAVITY,\ S}$ will here be considered satisfactory. They are

$$\vec{F}^{(MOON)}_{GRAVITY,\ S} \cong m\mu_m \ (\vec{R}^{(MOON)}_S - \vec{R}^{CM}_S)/|\vec{R}^{(MOON)}_S - \vec{R}^{CM}_S|^3 \tag{4-1C}$$

$$\vec{F}^{(SUN)}_{GRAVITY,\ S} \cong m\mu_s \ (\vec{R}^{(SUN)}_S - \vec{R}^{CM}_S)/|\vec{R}^{(SUN)}_S - \vec{R}^{CM}_S|^3 \tag{4-1D}$$

where

μ_m = product of universal gravitational constant and mass of Moon

μ_s = product of universal gravitational constant and mass of Sun

$\vec{R}_s^{(MOON)}$ = position vector of moon referred to the S-frame

$\vec{R}_s^{(SUN)}$ = position vector of sun referred to the S-frame

$\vec{R}_s^{CM} = \vec{\tilde{R}}_s + T^T \vec{\tilde{r}}_{CM}$

$\vec{\tilde{r}}_{CM}$ = position of vehicle CM referred to the $\tilde{B}$-frame,

the components of $\vec{R}_s^{(MOON)}$ and $\vec{R}_s^{(SUN)}$ being presumed known tabular functions of time, that is, available from the ephemerides of moon and sun. The far more detailed approximation to the B-resolution of the vector $\vec{F}^{(EARTH)}_{GRAVITY,\ S}$, herein denoted by $\vec{F}_{gB}$, is provided by Appendix A.

The vector sum

$$\frac{\mu_M \vec{R}_S^{(MOON)}}{|\vec{R}_S^{(MOON)}|^3} + \frac{\mu_S \vec{R}_S^{(SUN)}}{|\vec{R}_S^{(SUN)}|^3} \tag{4-1E}$$

appearing within braces in the left member of equation (4-1) is merely an approximation to the acceleration of the S-frame relative to the Newtonian frame (the N-frame) and is realized by treating Earth, Moon, and Sun as point masses and ignoring all forces on the earth other than the gravitational forces of moon and sun. See Figure 1.

Subtracting m times the first term of the expression (4-1E) from both members of (4-1C) followed by a pre-multiplication by the rotation matrix T ≡ T (s→B) gives the vector

$$\vec{F}_{gB}^{(MOON)} = m\ \mu_M\ T \left\{ \frac{\vec{R}_S^{(MOON)} - \vec{R}_S^{CM}}{|\vec{R}_S^{(MOON)} - \vec{R}_S^{CM}|^3} - \frac{\vec{R}_S^{(MOON)}}{|\vec{R}_S^{(MOON)}|^3} \right\} \tag{4-1F}$$

appearing in the translational equation below. A similar combination of the second term in expression (4-1E) with the members of (4-1D) gives, after a pre-multiplication by T, the vector

$$\vec{F}_{gB}^{(SUN)} = m\ \mu_S\ T \left\{ \frac{\vec{R}_S^{(SUN)} - \vec{R}_S^{CM}}{|\vec{R}_S^{(SUN)} - \vec{R}_S^{CM}|^3} - \frac{\vec{R}_S^{(SUN)}}{|\vec{R}_S^{(SUN)}|^3} \right\}, \tag{4-1G}$$

also appearing in the translational equation.

One will notice that regarding the S-frame as Newtonian (as the author has seen some people do) would result in the omission of the expression (4-1E). Although the S-frame does not rotate relative to the N-frame, it still cannot qualify as Newtonian by virtue of its acceleration relative to the N-frame.

The author of this paper has no comment regarding the significance of the error introduced by the complete neglect of electromagnetic forces; reflection of solar radiation by the earth and its atmosphere; direct thermal radiation from the earth; micrometeorite impacts, the attraction of celestial bodies other than the Earth, Moon, and Sun; etc.

The thrust is embedded in the first volume integral and the generalized force Q_q in equation (4-1), the dominant part (often called the "momentum" component) belonging to the volume integral and the other part (referred to as the "pressure" component) belonging to the surface integrals which comprise a part of Q_q.

The major contribution of the i^{th} flexible appendage to the potential function U is the strain energy $U_{\Delta i}$ given by

$$U_{\Delta i} = \frac{1}{2} \sum_{K=1}^{N_i} M_K^i \, {\omega_K^i}^2 \, {\eta_K^i}^2 \quad , \quad i = 1, \ldots NA \quad , \tag{4-2}$$

small deformation theory having been assumed and thermal effects ignored. The symbol ω_K^i denotes the frequency of undamped free vibration in the K^{th} natural mode whose shape function, the 3 x 1 column $\vec{\varphi}_K^{(i)}$, is a function of position $\vec{r}_i \equiv [x_i, y_i, z_i]^T$, referred to the i-frame (defined in section 2), of points of flexible appendage i in its undeformed state. The generalized mass, M_K^i, associated with the K^{th} natural bending mode of flexible appendage i is defined by

$$M_K^i = \int_{(m_i)} \vec{\varphi}_K^{(i)T} \vec{\varphi}_K^{(i)} \, dm_i \quad , \quad K = 1, \ldots N_i, \quad i = 1, \ldots NA \quad . \tag{4-3}$$

A remark regarding the symbol m_i, similar to that regarding m, is in order here, that being that when m_i appears as a subscript on an integral sign as in equation

(4-3), it indicates that the integration* extends over the volume occupied by flexible appendage i, while elsewhere m_i denotes the numerical value of the mass of flexible appendage i.

An approximation to $D_{\Delta i}$, here defined as one-half the rate at which energy is dissipated through structural damping in flexible appendage i, is

$$D_{\Delta i} \approx \frac{1}{2} \sum_{K=1}^{N_i} 2\, M_K^i\, \zeta_K^i\, \omega_K^i\, \dot{\eta}_K^{i\,2} \quad , \quad i = 1, \ldots NA \quad , \tag{4-4}$$

the symbol ζ_K^i denoting the damping ratio associated with the K^{th} natural bending mode of flexible appendage i. The approximation (4-4) is based upon the assumption of the existence of a viscous damping coefficient distribution (per unit volume) which varies directly as density (Reference 1). Although it is generally agreed that the viscous approximation to structural damping is inadequate, the damping force derivable from equation (4-4) will herein be deemed satisfactory. On summing the expression for $D_{\Delta i}$ over i (i = 1, ... NA), one has an approximation to the contribution of structural damping in all of the flexible appendages to the dissipation function D.

An elastic restoring force $-K_{\theta i}\, \theta_i$ and viscous damping force $-C_{\theta i}\, \dot{\theta}_i$, both of which oppose the rotational motion (relative to the central carrier) of flexible appendage i, give rise to the contributions

$$U_{\theta i} = (1/2) K_{\theta i}\, \theta_i^2 \quad , \quad i = 1, \ldots NA \quad , \tag{4-5}$$

and

$$D_{\theta i} = (1/2) C_{\theta i}\, \dot{\theta}_i^2 \quad , \quad i = 1, \ldots NA \quad , \tag{4-6}$$

to U and D, respectively.

The restraining spring and viscous damper forces, $-K_{pi}\xi_{pi}$ and $-C_{pi}\dot{\xi}_{pi}$, which impede the displacement (ξ_{pi}) of the point mass m_{pi} account for the following terms in the expressions below for U and D.

*Context will make clear whether the integration extends over the undeformed or deformed appendage.

$$U_{pi} = (1/2) K_{pi} \xi_{pi}^2 \quad , \quad i = 1, \ldots NP \quad , \tag{4-7}$$

$$D_{pi} = (1/2) C_{pi} \dot{\xi}_{pi}^2 \quad , \quad i = 1, \ldots NP \quad . \tag{4-8}$$

To account for the energy loss through viscous friction in the gimbal bearings of the CMG's, one should include in the expression for D terms pertinent to the SDOF and 2DOF CMG's, these being, respectively,

$$D_{\delta Gi} = (1/2) C_{Gi} \dot{\delta}_{Gi}^2 \quad , \quad i = 1, \ldots NSDOF \quad , \tag{4-9}$$

and

$$D_{\delta OGi} = (1/2) C_{OGi} \dot{\delta}_{OGi}^2 \quad , \quad i = 1, \ldots N2DOF \quad , \tag{4-10}$$

$$D_{\delta IGi} = (1/2) C_{IGi} \dot{\delta}_{IGi}^2 \quad , \quad i = 1, \ldots N2DOF \quad . \tag{4-11}$$

The terms in U and D pertinent to swivel engine i are, respectively,

$$U_{SEi} = \frac{1}{2} [K_{PEi}(\beta_{pi} - \beta_{PCi})^2 + K_{YEi}(\beta_{Yi} - \beta_{YCi})^2] \tag{4-12}$$

and

$$D_{SEi} = \frac{1}{2} [C_{PEi} \dot{\beta}_{Pi}^2 + C_{YEi} \dot{\beta}_{Yi}^2] \quad , \quad i = 1, \ldots NSE \quad . \tag{4-13}$$

In equation (4-12), K_{PEi} and K_{YEi} are "effective" torsional spring constants having the dimension NEWTON*METERS/RADIAN. The C_{PEi} and C_{YEi} in equation (4-13) are "effective" viscous damping coefficients (in pitch and yaw, respectively) having the dimension NEWTON*METERS/(RAD/SEC), that of a viscous torsional damping coefficient. The angles β_{PCi} and β_{YCi} are engine deflection commands (in pitch and yaw, respectively). From equation (4-12) is derivable the moment about the swivel point imparted by the actuators, and from equation (4-13) the damping moment retarding engine deflection. From equation (4-2) and equations (4-4) through (4-13), there follows

$$U = \sum_{i=1}^{NA} U_{\Delta i} + \sum_{i=1}^{NA} U_{\theta_i} + \sum_{i=1}^{NP} U_{Pi} + \sum_{i=1}^{NSE} U_{SEi}$$

$$= \frac{1}{2}\sum_{i=1}^{NA}\sum_{k=1}^{N_i} M_K^i \,\omega_K^{i^2}\,\eta_K^{i^2} + \frac{1}{2}\sum_{i=1}^{NA} K_{\theta_i}\,\theta_i^2 + \frac{1}{2}\sum_{i=1}^{NP} K_{Pi}\,\xi_{Pi}^2$$

$$+ \frac{1}{2}\sum_{i=1}^{NSE} [K_{PEi}(\beta_{Pi}-\beta_{PCi})^2 + K_{YEi}(\beta_{Yi}-\beta_{YCi})^2] \quad , \qquad (4\text{-}14)$$

$$D = \sum_{i=1}^{NA} D_{\Delta i} + \sum_{i=1}^{NA} D_{\theta_i} + \sum_{i=1}^{NP} D_{Pi} + \sum_{i=1}^{NSDOF} D_{\delta_{Gi}} + \sum_{i=1}^{N2DOF} (D_{\delta_{OGi}} + D_{\delta_{IGi}})$$

$$+ \sum_{i=1}^{NSE} D_{SEi}$$

$$= \frac{1}{2}\sum_{i=1}^{NA}\sum_{K=1}^{N_i} 2\,M_K^i\,\zeta_K^i\,\omega_K^i\,\dot{\eta}_K^{i^2} + \frac{1}{2}\sum_{i=1}^{NA} C_{\theta_i}\,\dot{\theta}_i^2 + \frac{1}{2}\sum_{i=1}^{NP} C_{Pi}\,\dot{\xi}_{Pi}^2$$

$$+ \frac{1}{2}\sum_{i=1}^{NSDOF} C_{Gi}\,\dot{\delta}_{Gi}^2 + \frac{1}{2}\sum_{i=1}^{N2DOF} (C_{OGi}\,\dot{\delta}_{OGi}^2 + C_{IGi}\,\dot{\delta}_{IGi}^2)$$

$$+ \frac{1}{2}\sum_{i=1}^{NSE} (C_{PEi}\,\dot{\beta}_{Pi}^2 + C_{YEi}\,\dot{\beta}_{Yi}^2) \quad . \qquad (4\text{-}15)$$

In developing the equation of motion corresponding to a system coordinate q in accordance with equation (4-1), it is convenient, if not necessary, to first write the volume integral $\int_{(m)} (\partial \vec{R}_S/\partial q)^T \ddot{\vec{R}}_S \, dm$ as the sum of several integrals,

$$\int_{(m)} \frac{\partial \vec{R}_S^T}{\partial q} \ddot{\vec{R}}_S \, dm = \left(\int_{(m_o)} + \int_{(m_f)} + \sum_{i=1}^{NSE} \int_{(m_{Ei})} + \sum_{i=1}^{NP} \int_{(m_{Pi})} + \sum_{i=1}^{NA} \int_{(m_i)} \right.$$

$$+ \sum_{i=1}^{NR} \int_{(m_{Ri})} + \sum_{i=1}^{NSDOF} \int_{(m'_{Gi})} + \sum_{i=1}^{NSDOF} \int_{(m'_{gi})} + \sum_{i=1}^{N2DOF} \int_{(m_{OGi})}$$

$$\left. + \sum_{i=1}^{N2DOF} \int_{(m_{IGi})} + \sum_{i=1}^{N2DOF} \int_{(m_{gi})} \right) \frac{\partial \vec{R}_S^T}{\partial q} \ddot{\vec{R}}_S \, dm \quad , \quad (4\text{-}16)$$

each pertinent to a particular part of the vehicle and sharing with that part an intimate connection with one or more system coordinates. Equation (4-16), wherein the notation is self-explanatory, is complemented by the following definitions:

m_o = mass of the rigid central carrier

m_f = mass of liquid propellant (in the tanks and feedlines) plus the products of combustion in all the engines forward of their respective nozzle exit planes (the word "forward" as used here is that of the thrust delivered by an engine).

m_{Ei} = mass of the i^{th} swivel engine.

m_{Pi} = mass of the i^{th} "point" mass.

m_i = mass of the i^{th} flexible appendage.

m_{Ri} = mass of the i^{th} rotor (the i^{th} rotor may or may not be a reaction wheel).

m'_{Gi} = mass of gimbal of i^{th} SDOF CMG.

m'_{gi} = mass of gyro element of i^{th} SDOF CMG.

m_{OGi} = mass of outer gimbal of i^{th} 2 DOF CMG.

m_{IGi} = mass of inner gimbal of i^{th} 2 DOF CMG.

m_{gi} = mass of gyro element of i^{th} 2 DOF CMG.

The manipulations leading to the system equations of motion begin with the expressions for $\vec{R}_s$ and its requisite derivatives in terms of the generalized coordinates and their time derivatives. In general, the position of an arbitrary point of the vehicle relative to the S-frame (defined in section 2) is given by

$$\vec{R}_s = \tilde{\vec{R}}_s + T^T \tilde{\vec{r}} \quad , \tag{4-17}$$

the vector $\tilde{\vec{r}}$ being the position of the point referred to the $\tilde{B}$-frame (section 2) and $T \equiv T\ (S \to \tilde{B}) = T\ (S \to B)$ the rotation matrix defined in section 3. Direct differentiation then gives

$$\dot{\vec{R}}_s = \dot{\tilde{\vec{R}}}_s + T^T \left(\dot{\tilde{\vec{r}}} + \Omega^T \tilde{\vec{r}}\right) \tag{4-18}$$

$$\ddot{\vec{R}}_s = \ddot{\tilde{\vec{R}}}_s + T^T \left[\ddot{\tilde{\vec{r}}} + 2\,\Omega^T \dot{\tilde{\vec{r}}} + \left(\dot{\Omega}^T + \Omega^{T^2}\right) \tilde{\vec{r}}\right] \quad , \tag{4-19}$$

use having been made of equation (3-3) and the skew symmetric matrix Ω defined by (3-2) with the property (3-4).

To be recalled at this point is a remark made earlier to the effect that the number of dots above a vector symbol is indicative of the order of the time derivative of that vector as measured by an observer in the reference frame <u>on whose associated vector basis the vector is resolved</u>. Thus,

$$\dot{\vec{R}}_s \equiv \left(\frac{d}{dt}\right)_s \vec{R}_s \quad , \quad \ddot{\vec{R}}_s \equiv \left(\frac{d^2}{dt^2}\right)_s \vec{R}_s \quad , \quad \dot{\tilde{\vec{R}}}_s \equiv \left(\frac{d}{dt}\right)_s \tilde{\vec{R}}_s \quad , \quad \ddot{\tilde{\vec{R}}}_s \equiv \left(\frac{d^2}{dt^2}\right)_s \tilde{\vec{R}}_s \ ,$$

while

$$\dot{\tilde{\vec{r}}} \equiv \left(\frac{d}{dt}\right)_{\tilde{B}} \tilde{\vec{r}} \quad , \quad \ddot{\tilde{\vec{r}}} \equiv \left(\frac{d^2}{dt^2}\right)_{\tilde{B}} \tilde{\vec{r}} \quad .$$

In view of this convention and the property (3-4) of Ω, it should be evident that the expression within braces in the right member of equation (4-19) is completely equivalent to the more familiar "textbook like" expression

$$\left(\frac{d^2}{dt^2}\right)_s \tilde{\vec{r}} = \left(\frac{d^2}{dt^2}\right)_{\tilde{B}} \tilde{\vec{r}} + 2\,\vec{\omega}_B \times \left(\frac{d}{dt}\right)_{\tilde{B}} \tilde{\vec{r}} + \left[\left(\frac{d}{dt}\right)_{\tilde{B}} \vec{\omega}_B\right] \times \tilde{\vec{r}} + \vec{\omega}_B \times (\vec{\omega}_B \times \tilde{\vec{r}}) \quad , \tag{4-20}$$

all vectors in equation (4-20) having the $\tilde{B}$-resolution. Thus, with the "derivative convention" of this report, one can dispense with the cumbersome notation of equation (4-20) and pass from equation (4-17) to equation (4-18) and from equation (4-18) to equation (4-19) as if in contempt of (yet in complete accord with) the differential operator relation

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_{\tilde{B}} + \vec{\omega}_B \times \quad .$$

Clearly, the only system coordinates (and derivatives thereof) in evidence in equations (4-17), (4-18), and (4-19) are the components of $\tilde{\vec{R}}_s$ and the Euler angles (the Euler angles entering through T^T and Ω^T which are completely determined once an Euler sequence has been prescribed). Consequently, it is necessary to complement those equations by expressions for $\tilde{\vec{r}}$, $\dot{\tilde{\vec{r}}}$, and $\ddot{\tilde{\vec{r}}}$ in terms of the other system coordinates and their time derivatives. Each expression alluded to pertains to a particular subdivision of the vehicle system, the manner of "subdividing" the system being dictated by the subscripts in equation (4-16). In what follows, context should make clear the intended meaning of the "mass symbols" m_o, m_f, m_{Ei}, ... m_{gi}. For example, the symbol m_o in the paragraph immediately following is an abbreviation for the "rigid central carrier."

At points of m_o, one has simply

$$\begin{aligned} \tilde{\vec{r}} &\equiv \tilde{\vec{r}} \\ \dot{\tilde{\vec{r}}} &= \ddot{\tilde{\vec{r}}} = \vec{0} \quad , \end{aligned} \tag{4-21}$$

the vanishing of $\dot{\tilde{\vec{r}}}$ and $\ddot{\tilde{\vec{r}}}$ being a consequence of the assumed rigidity of m_o and the definition of the $\tilde{B}$-frame as one at rest relative to m_o.

At points of the fluid mass m_f, the vector $\dot{\tilde{\vec{r}}}$ is a function of both position and time, that is,

$$\dot{\vec{\tilde{r}}} = \dot{\vec{\tilde{r}}}\,(\vec{\tilde{r}},\ t) = \begin{bmatrix} \dot{\tilde{x}}\ (\tilde{x},\ \tilde{y},\ \tilde{z},\ t) \\ \dot{\tilde{y}}\ (\tilde{x},\ \tilde{y},\ \tilde{z},\ t) \\ \dot{\tilde{x}}\ (\tilde{x},\ \tilde{y},\ \tilde{z},\ t) \end{bmatrix} \tag{4-22}$$

so that

$$\ddot{\vec{\tilde{r}}} \equiv \begin{bmatrix} \ddot{\tilde{x}} \\ \ddot{\tilde{y}} \\ \ddot{\tilde{z}} \end{bmatrix} = \begin{bmatrix} (\tilde{\nabla}\dot{\tilde{x}}) \cdot \dot{\vec{\tilde{r}}} + \partial\dot{\tilde{x}}/\partial t \\ (\tilde{\nabla}\dot{\tilde{y}}) \cdot \dot{\vec{\tilde{r}}} + \partial\dot{\tilde{y}}/\partial t \\ (\tilde{\nabla}\dot{\tilde{z}}) \cdot \dot{\vec{\tilde{r}}} + \partial\dot{\tilde{z}}/\partial t \end{bmatrix} \tag{4-23}$$

where

$$\tilde{\nabla} \equiv \vec{i}\ \partial/\partial\tilde{x} + \vec{j}\ \partial/\partial\tilde{y} + \vec{k}\ \partial/\partial\tilde{z} \quad .$$

Via the relations

$$\rho\,\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\,(\rho f) - f\,\frac{\partial\rho}{\partial t}$$

$$\frac{\partial\rho}{\partial t} = -\ \tilde{\nabla} \cdot (\rho\dot{\vec{\tilde{r}}}) \text{ (the continuity equation)}$$

$$\tilde{\nabla} \cdot (f\rho\ \dot{\vec{\tilde{r}}}) = f\ \tilde{\nabla} \cdot (\rho\dot{\vec{\tilde{r}}}) + (\tilde{\nabla} f) \cdot (\rho\dot{\vec{\tilde{r}}}) \quad ,$$

where f is a scalar function of position ($\vec{\tilde{r}}$) and time t, and ρ is density (also a function of position and time), one can go further to show that

$$\rho\ \ddot{\vec{\tilde{r}}} = \frac{\partial}{\partial t}\,(\rho\dot{\vec{\tilde{r}}}) + \begin{bmatrix} \tilde{\nabla} \cdot (\dot{\tilde{x}}\ \rho\ \dot{\vec{\tilde{r}}}) \\ \tilde{\nabla} \cdot (\dot{\tilde{y}}\ \rho\ \dot{\vec{\tilde{r}}}) \\ \tilde{\nabla} \cdot (\dot{\tilde{z}}\ \rho\ \dot{\vec{\tilde{r}}}) \end{bmatrix} \quad , \tag{4-24}$$

an important result (intimately related to thrust) finding its application in the derivation of the translational and rotational equations.

At points of m_{Ei}

$$\vec{\tilde{r}} = \vec{\tilde{r}}_{Ei} - \ell_{Ei} \vec{\Lambda}_{Ei} + T_{Ei}^{T} \vec{r}_{Ei} \tag{4-25}$$

$$\dot{\vec{\tilde{r}}} = -\ell_{Ei} \dot{\vec{\Lambda}}_{Ei} + \dot{T}_{Ei}^{T} \vec{r}_{Ei} \quad , \quad \left(\frac{d}{dt}\right)_{Ei} \vec{r}_{Ei} \equiv \vec{0} \quad , \tag{4-26}$$

$$\ddot{\vec{\tilde{r}}} = -\ell_{Ei} \ddot{\vec{\Lambda}}_{Ei} + \ddot{T}_{Ei}^{T} \vec{r}_{Ei} \tag{4-27}$$

$\vec{\tilde{r}}_{Ei}$ being the position (referred to the $\tilde{B}$-frame) of the swivel point, the positive scalar ℓ_{Ei} the distance between the swivel point and the engine CM; the unit vector $\vec{\Lambda}_{Ei}$ (having the $\tilde{B}$-resolution) directed as the thrust $\vec{F}_{Ei}$; $\vec{r}_{Ei}$ the position of a generic point of the engine referred to $x_{Ei}\ y_{Ei}\ z_{Ei}$ (defined in section 2); and T_{Ei} the rotation matrix whose definition appears among those in section 3.

Pertinent to the point mass m_{pi}, one has

$$\vec{\tilde{r}} = \vec{\tilde{r}}_{pi} + \xi_{pi} \vec{\Lambda}_{pi} \tag{4-28}$$

$$\dot{\vec{\tilde{r}}} = \dot{\xi}_{pi} \vec{\Lambda}_{pi} \tag{4-29}$$

$$\ddot{\vec{\tilde{r}}} = \ddot{\xi}_{pi} \vec{\Lambda}_{pi} \tag{4-30}$$

$\vec{\tilde{r}}_{pi}$ being the equilibrium position of m_{pi} and ξ_{pi} its displacement in the direction of the unit vector $\vec{\Lambda}_{pi}$ if $\xi_{pi} > 0$ and in the direction of $-\vec{\Lambda}_{pi}$ if $\xi_{pi} < 0$. The unit vector $\vec{\Lambda}_{pi}$ has a constant direction in the $\tilde{B}$-frame. (If the mass m_{pi} is a part of the mechanical analog of a consumable liquid, liquid propellant for example, then as the liquid is depleted, the vector $\vec{\tilde{r}}_{pi}$ will vary which is to say that its time derivatives will not vanish. However, even in such a case the approximations $\dot{\vec{\tilde{r}}}_{pi} \approx \vec{0}$, $\ddot{\vec{\tilde{r}}}_{pi} \approx \vec{0}$ will be made herein.)

For $dm \in m_i$, that is, at points of flexible appendage i,

$$\vec{\tilde{r}} = \vec{\tilde{r}}_i + \tilde{T}_i^T [\vec{r}_i + \vec{\Delta} (\vec{r}_i, t)] \tag{4-31}$$

$$\dot{\vec{\tilde{r}}} = \dot{\tilde{T}}_i^T [\vec{r}_i + \vec{\Delta} (\vec{r}_i, t)] + \tilde{T}_i^T \dot{\vec{\Delta}}(\vec{r}_i, t) \tag{4-32}$$

$$\ddot{\vec{r}} = \ddot{\tilde{T}}_i^T [\vec{r}_i + \vec{\Delta}(\vec{r}_i, t)] + 2\,\dot{\tilde{T}}_i^T\,\dot{\vec{\Delta}}(\vec{r}_i, t) + \tilde{T}_i^T\,\ddot{\vec{\Delta}}(\vec{r}_i, t) \tag{4-33}$$

In equation (4-31) $\vec{\tilde{r}}_i$ denotes the position of the point of attachment of flexible appendage i; the matrix $\tilde{T}_i$ is as defined in section 3; $\vec{r}_i$ denotes position relative to the i-frame (section 2) when the appendage is in its undeformed state; and $\vec{\Delta}(\vec{r}_i, t)$ is the displacement, due to deformation of the appendage, experienced by the point whose position (referred to the i-frame) before deformation was $\vec{r}_i$. The generalized bending displacement coordinates η_j^i, $j = 1, \ldots N_i$, are to be so determined that $\vec{\Delta}(\vec{r}_i, t)$ is given by

$$\vec{\Delta}(\vec{r}_i, t) = \sum_{j=1}^{N_i} \eta_j^i\,\vec{\varphi}_j^{(i)} = \Phi^{(i)}\,\vec{\eta}^{(i)} \quad , \tag{4-34}$$

the $3 \times N_i$ modal matrix $\Phi^{(i)}$ having for its j^{th} column the 3×1 column $\vec{\varphi}_j^{(i)}$ whose elements are functions of $\vec{r}_i$, and the $N_i \times 1$ column $\vec{\eta}^{(i)}$ having η_j^i as its j^{th} element.

Denoting the position (referred to the $\tilde{B}$-frame) of the CM of rotor i by $\vec{\tilde{r}}_{Ri}$ and position referred to the (Ri)'frame (section 2) by $\vec{r}\,'_{Ri}$, the position (relative to the $\tilde{B}$-frame) of a differential element of mass belonging to m_{Ri} must be given by

$$\vec{\tilde{r}} = \vec{\tilde{r}}_{Ri} + \tilde{T}_{Ri}^T\,\vec{r}_{Ri}\,' \tag{4-35}$$

from which

$$\dot{\vec{\tilde{r}}} = \dot{\tilde{T}}_{Ri}^T\,\vec{r}\,'_{Ri} \quad , \quad \ddot{\vec{\tilde{r}}} = \ddot{\tilde{T}}_{Ri}^T\,\vec{r}\,'_{Ri} \quad , \tag{4-36}$$

it being presumed that $\vec{\tilde{r}}_{Ri}$ is a constant vector on the $\tilde{B}$ basis and that the rotor is rigid so that $(d/dt)_{(Ri)'}\,\vec{r}_{Ri}\,' \equiv \vec{0}$. The "rotors" of this paper may or may not include reaction wheels but do not include the rotating elements of a CMG.

The position of a differential mass $dm \in m'_{Gi}$, referred to the $\tilde{B}$-frame, is

$$\vec{\tilde{r}} = \vec{\tilde{r}}_{Gi} + T_{Gi}^T\,\vec{r}_{Gi} \tag{4-37}$$

where $\vec{\tilde{r}}_{Gi}$ is the position of the CM of m'_{Gi}, $\vec{r}_{Gi}$ denotes position relative to the Gi-frame (section 2), and T_{Gi} the rotation matrix whose definition (section 3) is found among those pertinent to the i^{th} SDOF CMG. With $\vec{\tilde{r}}_{Gi}$ regarded as a constant vector (in the $\tilde{B}$-frame) and m'_{Gi} regarded rigid, there follows

$$\dot{\vec{\tilde{r}}} = \dot{T}_{Gi}^{T} \; \vec{r}_{Gi} \quad , \quad \ddot{\vec{\tilde{r}}} = \ddot{T}_{Gi}^{T} \; \vec{r}_{Gi} \quad . \tag{4-38}$$

Similarly, for $dm \in m'_{gi}$

$$\vec{\tilde{r}} = \vec{\tilde{r}}_{gi} + T_{gi}^{T} \; \vec{r}_{gi} \tag{4-39}$$

$$\dot{\vec{\tilde{r}}} = \dot{T}_{gi}^{T} \; \vec{r}_{gi} \quad , \quad \ddot{\vec{\tilde{r}}} = \ddot{T}_{gi}^{T} \; \vec{r}_{gi} \quad , \tag{4-40}$$

$\vec{\tilde{r}}_{gi}$ being the position of the CM of m'_{gi} (also assumed rigid); $\vec{r}_{gi}$ the position referred to the gi-frame (section 2); and T_{gi} as defined in section 3. (The assumption that m'_{Gi} and m'_{gi} have a common CM implies, of course, that $\vec{\tilde{r}}_{gi} = \vec{\tilde{r}}_{Gi}$. It is assumed further that the gimbal axis passes through the CM.)

In the definitions and consequent relations pertaining to a two DOF CMG (Sections 2 and 3) the additional subscript i (suggestive of the i^{th} two DOF CMG) was omitted on all symbols incident thereto not only for "convenience" but deemed unnecessary by the author in view of the self-evident fact that the orientation of the associated coordinate systems and the structure of the relevant transformation matrices change from CMG to CMG. In keeping with the aforementioned definitions and relations, the subscript i will be suppressed on all symbols in this paragraph, it being understood that they apply to a two DOF CMG. Thus, under the assumption that the outer gimbal, inner gimbal, and gyro element of the two DOF CMG in question are rigid with a common CM located at $\vec{\tilde{r}}_{OG} = \vec{\tilde{r}}_{IG} = \vec{\tilde{r}}_{g}$ relative to the $\tilde{B}$-frame, there follows:

$$dm \in m_{OG} \rightarrow \begin{cases} \vec{\tilde{r}} = \vec{\tilde{r}}_{OG} + T_{BOG}^{T} \, \vec{r}_{OG} \\ \dot{\vec{\tilde{r}}} = \dot{T}_{BOG}^{T} \, \vec{r}_{OG} \\ \ddot{\vec{\tilde{r}}} = \ddot{T}_{BOG}^{T} \, \vec{r}_{OG} \end{cases} \tag{4-41}$$

$$dm \in m_{IG} \rightarrow \begin{cases} \vec{\tilde{r}} = \vec{\tilde{r}}_{IG} + T^T_{BIG}\, \vec{r}_{IG} \\ \dot{\vec{\tilde{r}}} = \dot{T}^T_{BIG}\, \vec{r}_{IG} \\ \ddot{\vec{\tilde{r}}} = \ddot{T}^T_{BIG}\, \vec{r}_{IG} \end{cases} \tag{4-42}$$

$$dm \in m_g \rightarrow \begin{cases} \vec{\tilde{r}} = \vec{\tilde{r}}_g + T^T_{Bg}\, \vec{r}_g \\ \dot{\vec{\tilde{r}}} = \dot{T}^T_{Bg}\, \vec{r}_g \\ \ddot{\vec{\tilde{r}}} = \ddot{T}^T_{Bg}\, \vec{r}_g \end{cases} \tag{4-43}$$

where $\vec{r}_\xi$ denotes position relative to the ξ-frame, ξ = OG, IG, g, and $T_{B\xi} \equiv (TB \rightarrow \xi)$ denotes the rotation matrix defining the transformation from the B (or $\tilde{B}$) resolution to the ξ-resolution, ξ = OG, IG, g.

Turning to the development of the equations of motion, attention will first be focused on the vector equivalent of the three scalar equations corresponding to the components $\tilde{X}_s$, $\tilde{Y}_s$, and $\tilde{Z}_s$ of $\vec{\tilde{R}}_s$. By mere inspection of equations (4-1), (4-17), (4-19), and the expressions for U, D, and Q_q ($q = \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s$), it should be obvious that the vector translational equation, before further simplification, reads as follows:

$$\int_{(m)} \{\ddot{\vec{\tilde{R}}}_s + T^T\, [\ddot{\vec{\tilde{r}}} + 2\,\Omega^T\, \dot{\vec{\tilde{r}}} + (\dot{\Omega}^T + \Omega^{T^2})\, \vec{\tilde{r}}]\}\, dm = \vec{F}_{GRAVITY,S}$$

$$- \frac{m\, \mu_m\, \vec{R}_s^{(MOON)}}{|\vec{R}_s^{(MOON)}|^3} - \frac{m\, \mu_s\, \vec{R}_s^{(SUN)}}{|\vec{R}_s^{(SUN)}|^3} + \vec{F}_{AERO,\,S} + \vec{F}_{SOLAR,\,S}\;, \tag{4-44}$$

a result that one might have expected without resorting to equation (4-1). An immediate simplification of equation (4-44) is

$$m\,T\,\ddot{\vec{\tilde{R}}}_s = -m\,(\dot{\Omega}^T + \Omega^{T^2})\,\vec{\tilde{r}}_{CM} - \int_{(m)} (\ddot{\vec{\tilde{r}}} + 2\,\Omega^T\,\dot{\vec{\tilde{r}}})\,dm + \vec{F}_{gB} + \vec{F}_{gB}^{(SUN)}$$

$$+ \vec{F}_{gB}^{(MOON)} + \vec{F}_{AERO} + \vec{F}_{SOLAR}\;. \tag{4-45}$$

All vectors in the right member of equation (4-45) have the $\tilde{B}$ (and also the B) resolution. The vector $\vec{F}_{gB}$ is defined in appendix A, while $\vec{F}_{gB}^{(MOON)}$ and $\vec{F}_{gB}^{(SUN)}$ are given by equations (4-1F) and (4-1G), respectively. Integral expressions for $\vec{F}_{AERO}$ and $\vec{F}_{SOLAR}$ are given in appendices B and C, respectively. The vector $\vec{r}_{cm}$, the position of the CM of the entire system relative to the $\tilde{B}$-frame, is defined in appendix D. Via the relations (4-21) through (4-43), one can argue that the integrand of the integral in (4-45) vanishes except at points of m_f, m_{Ei} (i = 1, ... NSE), m_i (i = 1, ... NA), and at the point masses m_{pi} (i = 1, ... NP).

Integrating both members of equation (4-24) over V_f, the volume occupied by m_f, and applying the divergence theorem gives

$$\int_{(V_f)} \rho \ddot{\vec{\tilde{r}}} \, dV \equiv \int_{(m_f)} \ddot{\vec{\tilde{r}}} \, dm = \int_{(V_f)} \frac{\partial}{\partial t} (\rho \dot{\vec{\tilde{r}}}) \, dV - \int_{(A_f)} \dot{\vec{\tilde{r}}} (\rho \dot{\vec{\tilde{r}}} \cdot \vec{n}_f) \, dA \quad , \qquad (4\text{-}46)$$

where A_f is the area of the surface bounding V_f and $\vec{n}_f$ is a unit vector normal to A_f (directed inward) at a generic point of A_f. The integrand of the surface integral in equation (4-46) vanishes at all points of A_f except on A_E (the sum of the exit areas of all engines, both swiveled and fixed, associated with m_f). Thus,

$$- \int_{(m_f)} (\ddot{\vec{\tilde{r}}} + 2 \Omega^T \dot{\vec{\tilde{r}}}) \, dm = - 2 \Omega^T \int_{(m_f)} \dot{\vec{\tilde{r}}} \, dm - \int_{(V_f)} \frac{\partial}{\partial t} (\rho \dot{\vec{\tilde{r}}}) \, dV$$

$$+ \int_{(A_E)} \dot{\vec{\tilde{r}}} (\rho \dot{\vec{\tilde{r}}} \cdot \vec{n}) \, dA \quad , \qquad (4\text{-}47)$$

the symbol $\vec{n}$ denoting the unit normal vector (directed inward) at a generic point of A_E. The surface integral in equation (4-47) will be recognized as the "momentum" component of the thrust associated with m_f. The thrust, $\vec{F}_T$, is defined by

$$\vec{F}_T = \int_{(A_E)} \dot{\vec{\tilde{r}}} (\rho \dot{\vec{\tilde{r}}} \cdot \vec{n}) \, dA + \int_{(A_E)} (P - P_o) \vec{n} \, dA \quad , \qquad (4\text{-}48)$$

the symbols P and P_o in equation (4-48) denoting, respectively, local static pressure and free stream static pressure. On dropping the rightmost surface integral (the "pressure" component of $\vec{F}_T$) in equation (4-48) as negligible and discarding also the first two terms in the right member of equation (4-47), the first being the Coriolis force and the second representing the unsteadiness of the fluid flow, one has the approximation

$$- \int_{(m_f)} (\ddot{\vec{\tilde{r}}} + 2\,\Omega^T\,\dot{\vec{\tilde{r}}})\,dm \approx \vec{F}_T \quad . \tag{4-49}$$

Manipulations (omitted here) will show that

$$- \int_{(m_{Ei})} (\ddot{\vec{\tilde{r}}} + 2\,\Omega^T\,\dot{\vec{\tilde{r}}})\,dm = m_{Ei}\,\ell_{Ei}\,(\ddot{\vec{\Lambda}}_{Ei} + 2\,\Omega^T\,\dot{\vec{\Lambda}}_{Ei}) \;, \; i = 1, \ldots NSE, \tag{4-50}$$

$$- \int_{(m_{Pi})} (\ddot{\vec{\tilde{r}}} + 2\,\Omega^T\,\dot{\vec{\tilde{r}}})\,dm = -\,m_{Pi}\,(\ddot{\xi}_{Pi}\,\vec{\Lambda}_{Pi} + 2\,\dot{\xi}_{Pi}\,\Omega^T\,\vec{\Lambda}_{Pi}) \;, \; i = 1, \ldots NP \;, \tag{4-51}$$

$$\begin{aligned} - \int_{(m_i)} (\ddot{\vec{\tilde{r}}} + 2\,\Omega^T\,\dot{\vec{\tilde{r}}})\,dm = &-m_i\,\tilde{T}_i^T\,\{\vec{i}_i \times [\vec{\ell}_i^{\,(0)} + \Psi^{(i)}\,\vec{\eta}^{(i)}]\,\ddot{\theta}_i \\ &+ \vec{i}_i \times [\vec{i}_i \times (\vec{\ell}_i^{\,(0)} + \Psi^{(i)}\,\vec{\eta}^{(i)})]\,\dot{\theta}_i^2 \\ &+ \vec{i}_i \times [\Psi^{(i)}\,\dot{\vec{\eta}}^{(i)}]\,\dot{\theta}_i\} \\ &- 2\,m_i\,\Omega^T\,\tilde{T}_i^T\,\{\vec{i}_i \times [\vec{\ell}_i^{\,(0)} + \Psi^{(i)}\,\vec{\eta}^{(i)}]\,\dot{\theta}_i \\ &+ \Psi^{(i)}\,\dot{\vec{\eta}}^{(i)}\} - m_i\,\tilde{T}_i^T\,\Psi^{(i)}\,\ddot{\vec{\eta}}^{(i)} \;, \; i = 1, \ldots NA. \end{aligned} \tag{4-52}$$

In arriving at equation (4-52), use was made of equations (3-6), (3-7), and (3-8). The symbol $\vec{\ell}_i^{\,(0)}$ denotes the position, relative to $x_i y_i z_i$ (the i-frame defined in section 2), of the CM of appendage i when $\vec{\eta}^{(i)} \equiv \vec{0}$, that is, when the appendage i is in its undeformed state. The $3 \times N_i$ matrix $\Psi^{(i)}$ has for its j^{th} column the 3×1 column matrix

$$\vec{\psi}_j^{(i)} = (1/m_i) \int_{(m_i)} \vec{\varphi}_j^{(i)} \, dm \quad , \quad j = 1, \dots N_i, \tag{4-52.1}$$

which is to say, obviously, that

$$\Psi^{(i)} = (1/m_i) \int_{(m_i)} \Phi^{(i)} \, dm \quad . \tag{4-52.2}$$

In view of equations (4-49) through (4-52), the translational equation (4-45) can be made to assume the form

$$\begin{aligned} m\, T\, \ddot{\vec{\tilde{R}}}_s = & -m\, (\dot{\Omega}^T + \Omega^{T^2})\, \vec{\tilde{r}}_{cm} + \vec{F}_{gB} + \vec{F}_{gB}^{(SUN)} + \vec{F}_{gB}^{(MOON)} + \vec{F}_T + \vec{F}_{AERO} \\ & + \vec{F}_{SOLAR} + \sum_{i=1}^{NSE} m_{Ei}\, \ell_{Ei}\, (\ddot{\vec{\Lambda}}_{Ei} + 2\, \Omega^T\, \dot{\vec{\Lambda}}_{Ei}) \\ & - \sum_{i=1}^{NP} m_{Pi}\, (\ddot{\xi}_{Pi}\, \vec{\Lambda}_{Pi} + 2\, \dot{\xi}_{Pi}\, \Omega^T\, \vec{\Lambda}_{Pi}) \\ & - \sum_{i=1}^{NA} m_i\, \tilde{T}_i^T\, \{\vec{i}_i \times [\vec{\ell}_i^{(0)} + \Psi^{(i)}\, \vec{\eta}^{(i)}]\, \ddot{\theta}_i + \vec{i}_i \times [\Psi^{(i)}\, \dot{\vec{\eta}}^{(i)}]\, \dot{\theta}_i \\ & \qquad + \vec{i}_i \times [\vec{i}_i \times (\vec{\ell}_i^{(o)} + \Psi^{(i)}\, \vec{\eta}^{(i)})]\, \dot{\theta}_i^2\} \\ & - 2\, \Omega^T \sum_{i=1}^{NA} m_i\, \tilde{T}_i^T\, \{\vec{i}_i \times [\vec{\ell}_i^{(0)} + \Psi^{(i)}\, \vec{\eta}^{(i)}]\, \dot{\theta}_i + \Psi^{(i)}\, \dot{\vec{\eta}}^{(i)}\} \\ & - \sum_{i=1}^{NA} m_i\, \tilde{T}_i^T\, \Psi^{(i)}\, \ddot{\vec{\eta}}^{(i)} \quad . \end{aligned} \tag{4-53}$$

Note the absence of time derivatives of $\vec{\tilde{r}}_{cm}$ in equation (4-53). Therein lies the reason for choosing the components of $\vec{\tilde{R}}_s$ as three of the system coordinates instead of the components of the position vector, $\vec{R}_s^{CM}$, of the CM of the entire system referred to the S-frame. Should one want the differential equation satisfied by $\vec{R}_s^{CM}$, he has only to make the substitution $\vec{\vec{R}}_s = \vec{R}_s^{CM} - T^T \vec{\vec{r}}_{cm}$ in equation (4-53).

Definite expressions for $\vec{\Lambda}_{Ei}$ and its time derivatives in terms of β_{Pi}, β_{Yi} and their time derivatives cannot be written until one specifies the actuator arrangement, the sign convention for β_{Pi} and β_{Yi} and the direction of the thrust $\vec{F}_{Ei}$ when β_{Pi} and β_{Yi} are zero. If the use of swiveled engines proves to be impractical, as is likely to be the case, then, clearly, the terms reflecting their effect can be deleted without destroying the validity of the equations in which they appear.

The effect of the rotors will not become apparent until the vector rotational equation has been written, this problem being addressed in the next and several succeeding paragraphs.

The "customary" starting point in the approach to the moment equation is the generalized principle* of angular momentum (sometimes also called the generalized angular momentum equation) or special case thereof. Writing the moment equation via equation (4-1), as will be done in this report, requires no prior knowledge of the aforementioned principle; however, the use of equation (4-1) will require a few more manipulations, the extra preliminary manipulations serving in fact to establish a relation from which it is possible to deduce the principle of angular momentum (see Reference 8, pp. 20-21 of Chapter 2, paying particular attention to equations 14, 17, and 18).

Supposing the moment reference point** (MRP) to be the CM of the entire system, one should write (with $\vec{r}$ denoting position relative to the B-frame defined in section 2)

$$\vec{R}_s = \vec{R}_s^{CM} + T^T \vec{r}$$

and begin the manipulations with

$$\frac{\partial \vec{R}_s^T}{\partial \varphi_k} = \vec{r}^T \frac{\partial T}{\partial \varphi_k} \quad , \qquad k = p, y, r \quad .$$

*See References 8 and 14 (among others).

**The well known special case of the generalized angular momentum equation would apply here, that being the case wherein the linear moment relative to the MRP vanishes identically.

Clearly, the manipulations can proceed no further until the matrix T is definitely specified by a prescribed Euler sequence. Fortunately, as extensive (yet simple) manipulations of the author have shown, the form (4-57) of the moment equation is the same no matter which of the 3! = 6 distinct Euler sequences below is selected to define the matrix T (see equations (3-0.1), (3-0.2), (3-0.3)).

$$
T = \begin{cases}
[\varphi_y]_{(3)}\ [\varphi_p]_{(2)}\ [\varphi_r]_{(1)} & \text{(a 1,2,3 sequence through } \varphi_r,\ \varphi_p, \text{ and } \varphi_y) \\
[\varphi_p]_{(2)}\ [\varphi_y]_{(3)}\ [\varphi_r]_{(1)} & \text{(a 1,3,2 sequence through } \varphi_r,\ \varphi_y, \text{ and } \varphi_p) \\
[\varphi_y]_{(3)}\ [\varphi_r]_{(1)}\ [\varphi_p]_{(2)} & \text{(a 2,1,3 sequence through } \varphi_p,\ \varphi_r, \text{ and } \varphi_y) \\
[\varphi_r]_{(1)}\ [\varphi_y]_{(3)}\ [\varphi_p]_{(2)} & \text{(a 2,3,1 sequence through } \varphi_p,\ \varphi_y, \text{ and } \varphi_r) \\
[\varphi_r]_{(1)}\ [\varphi_p]_{(2)}\ [\varphi_y]_{(3)} & \text{(a 3,2,1 sequence through } \varphi_y,\ \varphi_p, \text{ and } \varphi_r) \\
[\varphi_p]_{(2)}\ [\varphi_r]_{(1)}\ [\varphi_y]_{(3)} & \text{(a 3,1,2 sequence through } \varphi_y,\ \varphi_r, \text{ and } \varphi_p)
\end{cases}
$$

Selecting the 2, 3, 1 sequence (merely for the sake of being definite) so that

$$
\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = T \begin{bmatrix} \vec{I}_s \\ \vec{J}_s \\ \vec{K}_s \end{bmatrix} = [\varphi_r]_{(1)}\ [\varphi_y]_{(3)}\ [\varphi_p]_{(2)} \begin{bmatrix} \vec{I}_s \\ \vec{J}_s \\ \vec{K}_s \end{bmatrix} ,
$$

it can be shown with little effort that

$$
\frac{\partial \vec{R}_s^{\,T}}{\partial \varphi_p} = - \vec{r}^{\,T}\ T \begin{bmatrix} 0 & 0 & 1. \\ 0 & 0 & 0 \\ -1. & 0 & 0 \end{bmatrix}
$$

$$
\frac{\partial \vec{R}_s^{\,T}}{\partial \varphi_y} = - \vec{r}^{\,T}\ T\ [\varphi_p]^T_{(2)} \begin{bmatrix} 0 & -1. & 0 \\ 1. & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} [\varphi_p]_{(2)}
$$

$$\frac{\partial \vec{R}_s^T}{\partial \varphi_r} = - \vec{r}^T [\varphi_r]_{(1)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1. \\ 0 & 1. & 0 \end{bmatrix} [\varphi_y]_{(3)} [\varphi_p]_{(2)} \quad .$$

Further scratchwork (not repeated here) has shown that summing corresponding members of the three equations whose forms are those assumed by equation (4-1) when $q = \varphi_p,\ \varphi_y$, and φ_r, followed by some rearrangement, gives

$$\left\{ \vec{i} + (\vec{K}_s')_B + (\vec{J}_s)_B \right\}^T \left\{ \int_{(m)} \vec{r} \times T\ \ddot{\vec{R}}_s\ dm - \int_{(m)} \vec{r} \times T \left(d\ \vec{F}^{(EARTH)}_{GRAVITY,s} \right) \right.$$

$$- \int_{(m)} \vec{r} \times T \left(d\ \vec{F}^{(SUN)}_{GRAVITY,s} \right) - \int_{(m)} \vec{r} \times T \left(d\ \vec{F}^{(MOON)}_{GRAVITY,s} \right)$$

$$\left. - \int_{(A)} \vec{r} \times T\ (d\ \vec{F}_{AERO,s}) - \int_{(A)} \vec{r} \times T\ (d\ \vec{F}_{SOLAR,s}) \right\} = 0 \ . \qquad (4\text{-}54)$$

In equation (4-54), the subscript B indicates the B-resolution, that is,

$$(\vec{J}_s)_B = \text{B-RESOLUTION OF } \vec{J}_s = T\ \vec{J}_s = T \begin{bmatrix} 0 \\ 1. \\ 0 \end{bmatrix}$$

$$(\vec{K}_s')_B = [\varphi_r]_{(1)}\ [\varphi_y]_{(3)}\ \vec{K}_s' = [\varphi_r]_{(1)}\ [\varphi_y]_{(3)} \begin{bmatrix} 0 \\ 0 \\ 1. \end{bmatrix} = \text{B-RESOLUTION OF } \vec{K}_s'$$

where

$$\begin{bmatrix} \vec{I}_s' \\ \vec{J}_s' \\ \vec{K}_s' \end{bmatrix} = [\varphi_p]_{(2)} \begin{bmatrix} \vec{I}_s \\ \vec{J}_s \\ \vec{K}_s \end{bmatrix} \quad .$$

Recalling the definition of $T \equiv T\,(S \rightarrow B)$, it should be obvious that

$$\int_{(m)} \vec{r} \times T \; d\left(\vec{F}^{(EARTH)}_{GRAVITY,S}\right) = \int_{(m)} \vec{r} \times d\,\vec{F}_{gB} = \vec{M}_{gB} \quad \text{[see (A-28) and (A-29)]}$$

$$= \text{TORQUE DUE TO EARTH'S GRAVITY FIELD}$$

$$\int_{(m)} \vec{r} \times T \; d\left(\vec{F}^{(SUN)}_{GRAVITY,S}\right) = \vec{M}^{(SUN)}_{gB} = \text{TORQUE DUE TO SUN'S GRAVITY FIELD}$$

$$\int_{(m)} \vec{r} \times T \; d\left(\vec{F}^{(MOON)}_{GRAVITY,S}\right) = \vec{M}^{(MOON)}_{gB} = \text{TORQUE DUE TO MOON'S GRAVITY FIELD}$$

$$\int_{(A)} \vec{r} \times T\,(d\,\vec{F}_{AERO,S}) = \vec{M}_{AERO} = \text{AERODYNAMIC TORQUE (APPENDIX B)}$$

$$\int_{(A)} \vec{r} \times T\,(d\,\vec{F}_{SOLAR,S}) = \vec{M}_{SOLAR} = \text{SOLAR RADIATION TORQUE (APPENDIX C)}$$

Treating both Sun and Moon as point masses results in the following approximations to $\vec{M}^{(SUN)}_{gB}$ and $\vec{M}^{(MOON)}_{gB}$, these being found by developments similar to that of equation (A-7).

$$\vec{M}^{(SUN)}_{gB} \approx \frac{3\,\mu_s}{R^3_{SUN}}\,\vec{u}_{SUN} \times (\square\,\vec{u}_{SUN}) \tag{4-55}$$

$$\vec{M}^{(MOON)}_{gB} \approx \frac{3\,\mu_M}{R^3_{MOON}}\,\vec{u}_{MOON} \times (\square\,\vec{u}_{MOON}) \tag{4-56}$$

In equation (4-55)

$$R_{SUN} = |\vec{R}_{SUN}| \;, \quad \vec{R}_{SUN} = T\,(\vec{R}^{CM}_s - \vec{R}^{(SUN)}_s) \;, \quad \vec{u}_{SUN} = \vec{R}_{SUN}/R_{SUN} \;,$$

while in equation (4-56)

$$R_{MOON} = |\vec{R}_{MOON}| \quad , \quad \vec{R}_{MOON} = T\,(\vec{R}_s^{CM} - \vec{R}_s^{(MOON)}) \quad , \quad \vec{u}_{MOON} = \vec{R}_{MOON}/R_{MOON} \ .$$

The symbol $\mathcal{I}$ denotes the inertia matrix of the entire system (in its instantaneous deformed configuration) referred to the B-frame which has origin at the instantaneous CM of the entire system. It follows from equations (4-54), (4-55), and (4-56) that

$$\int_{(m)} \vec{r} \times T\ddot{\vec{R}}_s \, dm = \vec{M}_{gB} + \vec{M}_{gB}^{(SUN)} + \vec{M}_{gB}^{(MOON)} + \vec{M}_{AERO} + \vec{M}_{SOLAR} \ . \tag{4-57}$$

A strong advocate of the direct use of either the principle of angular momentum or equation (4-57) might insist that equation (4-57) be invoked at the outset in complete disregard of the manipulations leading to it. The author of this report would then hasten to point out that some of the readers of this report are likely to be the uninitiated who will find it instructive to base the development of the equations of motion upon the principle of virtual work.

Alternatives to equation (4-1) in the approach to the moment equation are the relations (4-58) and (4-59), both of which are established in Reference 1 (see also Reference 15 on quasi-coordinates).

$$\int_{(m)} \vec{L}_\omega (\widetilde{KE}) \, dm + (T\dot{\vec{R}}_s^{CM}) \times \int_{(m)} \dot{\vec{r}} \, dm = \vec{M}_{EXT} \quad , \tag{4-58}$$

$$\int_{(m)} \vec{L}_\omega (\widetilde{KE}) \, dm + \vec{V} \times \int_{(m)} \nabla_V \, (\widetilde{KE}) \, dm = \vec{M}_{EXT} \ . \tag{4-59}$$

The operators $\vec{L}_\omega$ and ∇_V are defined by

$$\vec{L}_\omega(\ ) = \left(\frac{d}{dt}\right)_B \nabla_\omega(\ ) + \vec{\omega}_B \times \nabla_\omega(\ ) = \left(\frac{d}{dt}\right)_B \nabla_\omega(\ ) + \Omega^T \nabla_\omega(\ )$$

$$\nabla_V = \begin{bmatrix} \frac{\partial}{\partial V_1} \\ \frac{\partial}{\partial V_2} \\ \frac{\partial}{\partial V_3} \end{bmatrix}_{(B)} \equiv \vec{i}\,\frac{\partial}{\partial V_1} + \vec{j}\,\frac{\partial}{\partial V_2} + \vec{k}\,\frac{\partial}{\partial V_3}$$

where

$$\nabla_\omega \equiv \begin{bmatrix} \frac{\partial}{\partial \omega_1} \\ \frac{\partial}{\partial \omega_2} \\ \frac{\partial}{\partial \omega_3} \end{bmatrix}_{(B)} = \vec{i}\,\frac{\partial}{\partial \omega_1} + \vec{j}\,\frac{\partial}{\partial \omega_2} + \vec{k}\,\frac{\partial}{\partial \omega_3} \quad ,$$

and

$$\vec{V} \equiv \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}_{(B)} = V_1\vec{i} + V_2\vec{j} + V_3\vec{k} = T\dot{\vec{R}}_s^{CM} \quad .$$

In both equation (4-58) and (4-59), the symbol $\widetilde{KE}$ denotes the kinetic energy per unit mass. However, in equation (4-58), it is necessary to write

$$\widetilde{KE} = (1/2)\ \dot{\vec{R}}_s^T\ \dot{\vec{R}}_s = 1/2 \left\{\dot{\vec{R}}_s^{CM} + T^T\ (\dot{\vec{r}} + \Omega^T\ \vec{r})\right\}^T \left\{\dot{\vec{R}}_s^{CM} + T^T\ (\dot{\vec{r}} + \Omega^T\ \vec{r})\right\}$$

while in equation (4-59), one must write

$$\widetilde{KE} = (1/2)\left\{\vec{V} + \dot{\vec{r}} + \Omega^T\ \vec{r}\right\}^T \left\{\vec{V} + \dot{\vec{r}} + \Omega^T\ \vec{r}\right\} .$$

The vector $\vec{M}_{EXT}$ is the B-resolution of the resultant external moment about the system CM, and if comprised of the moments of only those external forces accounted for in this paper, it is identically the right member of equation (4-57). Complete equivalence of the left members of equations (4-57), (4-58), and (4-59) is also established in Reference 1.

Equation (4-57) with its right member replaced by $\vec{M}_{EXT}$ is the same as equation (17) of Chapter 3 in Reference 8 wherein the symbol $\vec{a}$ denotes the acceleration distribution instead of $T\ddot{\vec{R}}_s$.

On writing

$$T\ddot{\vec{R}}_s = T\ddot{\vec{R}}_s^{CM} + \ddot{\vec{r}} + 2\,\Omega^T\,\dot{\vec{r}} + (\dot{\Omega}^T + {\Omega^T}^2)\,\vec{r}$$

and recognizing the relations

$$\int_{(m)} \vec{r}\,dm = \vec{0} \quad \text{(by definition of the CM)}$$

$$\int_{(m)} \vec{r} \times \dot{\Omega}^T\,\vec{r}\,dm = \Box\,\dot{\vec{\omega}}_B$$

$$\int_{(m)} \vec{r} \times {\Omega^T}^2\,\vec{r}\,dm = \vec{\omega}_B \times \Box\,\vec{\omega}_B = \Omega^T\,\Box\,\vec{\omega}_B$$

$$\int_{(m)} \{\vec{r} \times \Omega^T\,\dot{\vec{r}} + \dot{\vec{r}} \times \Omega^T\,\vec{r}\}\,dm = \dot{\Box}\,\vec{\omega}_B$$

$$2\int_{(m)} \vec{r} \times \Omega^T\,\dot{\vec{r}}\,dm = \dot{\Box}\,\vec{\omega}_B + \vec{\omega}_B \times \int_{(m)} \vec{r} \times \dot{\vec{r}}\,dm = \dot{\Box}\,\vec{\omega}_B + \Omega^T \int_{(m)} (\vec{r} \times \dot{\vec{r}})\,dm \quad ,$$

use being made of the vector identities $\vec{a} \times [\vec{b} \times (\vec{b} \times \vec{a})] = -\vec{b} \times [\vec{a} \times (\vec{a} \times \vec{b})]$ and $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ in establishing the third and fifth, one can show that equation (4-57) may be put in the form

$$\Box\,\dot{\vec{\omega}}_B + \dot{\Box}\,\vec{\omega}_B + \Omega^T\,\Box\,\vec{\omega}_B + \int_{(m)} \vec{r} \times \ddot{\vec{r}}\,dm + \Omega^T \int_{(m)} \vec{r} \times \dot{\vec{r}}\,dm = \vec{M}_{EXT} \quad . \qquad (4\text{-}60)$$

Before dealing with the two integrals in equation (4-60) in detail, their integrands will be transformed via the substitution $\vec{r} = \vec{\tilde{r}} - \vec{\tilde{r}}_{CM}$. Thus, the moment equation will be rewritten, temporarily, as

$$\square \dot{\vec{\omega}}_B + \dot{\square} \vec{\omega}_B + \Omega^T \square \vec{\omega}_B + \int_{(m)} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \ddot{\vec{\tilde{r}}}\, dm + \Omega^T \int (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \dot{\vec{\tilde{r}}}\, dm = \vec{M}_{EXT} \quad , \tag{4-61}$$

where, obviously, the condition $\int_{(m)} \vec{r}\, dm = \vec{0}$ has been imposed.

To express the two integrals in equation (4-61) in terms of the system coordinates and known system parameters, the system will be subdivided in the manner indicated by the subscripts on the integral signs in equation (4-16) and substitutions made from equations (4-21) through (4-43). Manipulations leading to the results that follow will not be repeated here.

It obviously follows from equation (4-21) that

$$\int_{(m_o)} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \ddot{\vec{\tilde{r}}}\, dm + \Omega^T \int_{(m_o)} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \dot{\vec{\tilde{r}}}\, dm = \vec{0} \quad . \tag{4-62}$$

On substituting from equations (4-22) through (4-24) and again drawing from References 1 and 8, the author has arrived at the <u>approximation</u>

$$\int_{(m_f)} \vec{r} \times \ddot{\vec{\tilde{r}}}\, dm \approx - \int_{(A_E)} \vec{r} \times \dot{\vec{\tilde{r}}} (\rho\, \dot{\vec{\tilde{r}}} \cdot \vec{n})\, dA \approx -\vec{M}_T \quad , \tag{4-63}$$

the symbol $\vec{M}_T$ denoting the moment of the thrust about the system CM and defined by

$$\vec{M}_T = \int_{(A_E)} \vec{r} \times \dot{\vec{\tilde{r}}} (\rho\, \dot{\vec{\tilde{r}}} \cdot \vec{n})\, dA + \int_{(A_E)} \vec{r} \times (p - p_o)\, \vec{n}\, dA \quad . \tag{4-64}$$

The approximations (4-63) are consequences of the complete neglect of the volume integral $\int_{(V_f)} \vec{r} \times \frac{\partial}{\partial t} (\rho\, \dot{\vec{\tilde{r}}})\, dV$ and the rightmost surface integral in equation (4-64), the volume integral being encountered in the manipulations pertinent to $\int_{(m_f)} \vec{r} \times \ddot{\vec{\tilde{r}}}\, dm$. Also made in this report is the approximation

$$\Omega^T \int_{(m_f)} \vec{r} \times \dot{\vec{\tilde{r}}}\, dm \approx \vec{0} \quad . \tag{4-65}$$

In each of the vector products appearing in equations (4-63) through (4-65), the factor on the left has been written $\vec{r}$ instead of its equivalent $\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}$ to better emphasize the fact that the moment is about the system CM.

By referring to equations (4-25) through (4-27) and to the definitions and relations of sections 2 and 3 pertaining to swivel engine i, one can show (as in Reference 1) that

$$\begin{aligned} \int_{(m_{Ei})} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \ddot{\vec{\tilde{r}}}\, dm + \Omega^T \int_{(m_{Ei})} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \dot{\vec{\tilde{r}}}\, dm &= - m_{Ei}\, \ell_{Ei}\, \vec{q}_{Ei} \times \ddot{\vec{\Lambda}}_{Ei} \\ &\quad - m_{Ei}\, \ell_{Ei}\, \vec{\omega}_B \times (\vec{q}_{Ei} \times \dot{\vec{\Lambda}}_{Ei}) + T_{Ei}^T\, I^{Ei}\, T_{Ei}\, \dot{\vec{\omega}}'_{Ei} + (\vec{\omega}_B + \vec{\omega}'_{Ei}) \\ &\quad \times T_{Ei}^T\, I^{Ei}\, T_{Ei}\, \vec{\omega}'_{Ei} \quad , \quad i = 1, \ldots \text{NSE} \quad , \end{aligned} \tag{4-66}$$

where

$$\vec{q}_{Ei} = \vec{\tilde{r}}_{Ei} - \vec{\tilde{r}}_{CM} - \ell_{Ei}\, \vec{\Lambda}_{Ei} \quad .$$

A relation similar to equation (4-66) holds for the i^{th} rotor, as consideration of equation (4-35), equation (4-36) and the definitions and relations of sections 2 and 3 concerning rotor i will show that (Reference 1 again)

$$\begin{aligned} \int_{(m_{Ri})} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \ddot{\vec{\tilde{r}}}\, dm + \Omega^T \int_{(m_{Ri})} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \dot{\vec{\tilde{r}}}\, dm &= T_{Ri}^T\, I^{Ri}\, T_{Ri}\, \dot{\vec{\omega}}'_{Ri} \\ &\quad + (\vec{\omega}_B + \vec{\omega}'_{Ri}) \times T_{Ri}^T\, I^{Ri}\, T_{Ri}\, \vec{\omega}'_{Ri} \ , \\ &\qquad i = 1, \ldots \text{NR} \quad . \end{aligned} \tag{4-67}$$

From the content of the paragraph containing equations (4-28) through (4-30), there follows

$$\int_{(m_{Pi})} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \ddot{\vec{\tilde{r}}}\, dm + \Omega^T \int_{(m_{Pi})} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \dot{\vec{\tilde{r}}}\, dm = m_{Pi} \{(\vec{\tilde{r}}_{Pi} - \vec{\tilde{r}}_{CM})$$

$$\times (\ddot{\xi}_{Pi} \vec{\Lambda}_{Pi}) + \vec{\omega}_B \times [(\vec{\tilde{r}}_{Pi} - \vec{\tilde{r}}_{CM}) \times (\dot{\xi}_{Pi} \vec{\Lambda}_{Pi})]\} , \quad i = 1, \ldots NP \quad . \tag{4-68}$$

Substituting from equations (4-31) through (4-34) and manipulating in accordance with the relevant definitions and relations of sections 2 and 3 leads to

$$\int_{(m_i)} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \ddot{\vec{\tilde{r}}}\, dm + \Omega^T \int_{(m_i)} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \dot{\vec{\tilde{r}}}\, dm = \tilde{T}_i^T I^i \tilde{T}_i \dot{\vec{\tilde{\omega}}}_i' + (\vec{\omega}_B + \vec{\tilde{\omega}}_i')$$

$$\times \tilde{T}_i^T I^i \tilde{T}_i \vec{\tilde{\omega}}_i' + m_i (\vec{\tilde{r}}_i - \vec{\tilde{r}}_{CM}) \times \tilde{T}_i^T [(\dot{\tilde{\Omega}}_i^T + \tilde{\Omega}_i^{T^2}) (\vec{\ell}_i^{(0)} + \Psi^{(i)} \vec{\eta}^{(i)})$$

$$+ 2 \tilde{\Omega}_i^T \Psi^{(i)} \dot{\vec{\eta}}^{(i)} + \Psi^{(i)} \ddot{\vec{\eta}}^{(i)}] + \vec{\omega}_B \times \{m_i (\vec{\tilde{r}}_i - \vec{\tilde{r}}_{CM})$$

$$\times \tilde{T}_i^T [\tilde{\Omega}_i^T (\vec{\ell}_i^{(0)} + \Psi^{(i)} \vec{\eta}^{(i)}) + \Psi^{(i)} \dot{\vec{\eta}}^{(i)}]\}$$

$$+ \tilde{T}_i^T \sum_{j=1}^{N_i} \left\{ \ddot{\eta}_j^i \int_{(m_i)} \vec{r}_i \times \vec{\varphi}_j^{(i)}\, dm + \eta_j^i \sum_{K=1}^{N_i} \ddot{\eta}_K^i \int_{(m_i)} \vec{\varphi}_j^{(i)} \times \vec{\varphi}_K^{(i)}\, dm \right.$$

$$\left. + 2 \dot{\eta}_j^i \int_{(m_i)} \vec{r}_i \times (\vec{\tilde{\omega}}_i \times \vec{\varphi}_j^{(i)})\, dm + 2 \eta_j^i \sum_{K=1}^{N_i} \dot{\eta}_K^i \int_{(m_i)} \vec{\varphi}_j^{(i)} \times (\vec{\tilde{\omega}}_i \times \vec{\varphi}_K^{(i)})\, dm \right\}$$

$$+ \vec{\omega}_B \times \left\{ \tilde{T}_i^T \sum_{j=1}^{N_i} \left[\dot{\eta}_j^i \int_{(m_i)} \vec{r}_i \times \vec{\varphi}_j^{(i)}\, dm + \eta_j^i \sum_{K=1}^{N_i} \dot{\eta}_K^i \int_{(m_i)} \vec{\varphi}_j^{(i)} \times \vec{\varphi}_K^{(i)}\, dm \right] \right\},$$

$$i = 1, \ldots NA \quad , \tag{4-69}$$

where

$$\vec{\tilde{\omega}}_i{}' = \tilde{T}_i^T \vec{\tilde{\omega}}_i \quad , \quad \dot{\vec{\tilde{\omega}}}_i{}' = \tilde{T}_i^T \dot{\vec{\tilde{\omega}}}_i \quad ,$$

and the symbol I^i denotes the inertia matrix of flexible appendage i, in its instantaneous deformed state, referred to the axes $x_i y_i z_i$ defined in section 2.

For convenience in writing the moment equation below, the integrals in equation (4-69) will be expressed more compactly by introducing the symbols $\vec{C}^i_{oj}$, $\vec{C}^i_{jK}$, $\mathcal{J}^i_{rj}$, and $\mathcal{J}^i_{jK}$ defined by

$$\vec{C}^i_{oj} = \int_{(m_i)} \vec{r}_i \times \vec{\varphi}_j^{(i)} \, dm \tag{4-69.1}$$

$$\vec{C}^i_{jK} = \int_{(m_i)} \vec{\varphi}_j^{(i)} \times \vec{\varphi}_K^{(i)} \, dm \tag{4-69.2}$$

$$(i = 1, \ldots NA, \; j = 1, \ldots N_i, \; K = 1, \ldots N_i)$$

$$\mathcal{J}^i_{rj} = \int_{(m_i)} \mathcal{S}(\vec{r}_i)\, \mathcal{S}(\vec{\varphi}_j^{(i)}) \, dm \tag{4-69.3}$$

$$\mathcal{J}^i_{jK} = \int_{(m_i)} \mathcal{S}(\vec{\varphi}_j^{(i)})\, \mathcal{S}(\vec{\varphi}_K^{(i)}) \, dm \tag{4-69.4}$$

where the operator $\mathcal{S}$ is such that when applied to the vector $\vec{A} \equiv [A_1, A_2, A_3]^T$, the result is the skew symmetric matrix

$$\mathcal{S}(\vec{A}) = \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix} \quad . \tag{4-69.5}$$

The integrals in equations (4-69.1) through (4-69.4) also appear among the time independent integrals of Appendix F which arise in deriving the bending equations. Definitions (4-69.3) and (4-69.4) permit one to write

$$\int_{(m_i)} \vec{r}_i \times (\vec{\tilde{\omega}}_i \times \vec{\varphi}_j^{(i)})\, dm = -\mathcal{J}^{\,i}_{r_j}\, \vec{\tilde{\omega}}_i \quad , \qquad \int_{(m_i)} \vec{\varphi}_j^{(i)} \times (\vec{\tilde{\omega}}_i \times \vec{\varphi}_k^{(i)})\, dm = -\mathcal{J}^{\,i}_{jK}\, \vec{\tilde{\omega}}_i \quad .$$

The contribution of the i^{th} SDOF CMG to the two integrals in equation (4-61) is, in view of equations (4-37) through (4-40) and (3-9) through (3-25),

$$\int_{(m_{Gi}'+m_{gi}')} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \ddot{\vec{\tilde{r}}}\, dm + \Omega^T \int_{(m_{Gi}'+m_{gi}')} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \dot{\vec{\tilde{r}}}\, dm$$

$$= (I^{Gi}_{xx} + I^{gi}_{xx})\, [\dot{\vec{\omega}}_{Gi}' + \vec{\omega}_B \times \vec{\omega}_{Gi}'] + I^{gi}_{zz}\, [(\vec{\omega}_B + \vec{\omega}_{Gi}') \times \vec{\omega}_{gi}'] \quad . \tag{4-70}$$

The right member of equation (4-70) is a simplification of the expression (4-71),

$$(T^T_{Gi}\, I^{Gi}\, T_{Gi} + T^T_{gi}\, I^{gi}\, T_{gi})\, \dot{\vec{\omega}}_{Gi}' + (\vec{\omega}_B + \vec{\omega}_{Gi}') \times (T^T_{Gi}\, I^{Gi}\, T_{Gi} + T^T_{gi}\, I^{gi}\, T_{gi})\, \vec{\omega}_{Gi}'$$

$$+ (\vec{\omega}_B + \vec{\omega}_{gi}') \times T^T_{gi}\, I^{gi}\, T_{gi}\, \vec{\omega}_{gi}' + 2\, \vec{\omega}_{gi}' \times T^T_{gi}\, I^{gi}\, T_{gi}\, \vec{\omega}_{Gi}'$$

$$+ [\mathrm{Tr}\, (I^{gi})]\, (\vec{\omega}_{Gi}' \times \vec{\omega}_{gi}') \quad , \tag{4-71}$$

the first three terms of which one might have anticipated after inspection of equation (4-67) and certain terms of (4-66) and (4-69). By $\mathrm{Tr}\,(I^{gi})$ is meant the trace of I^{gi}, that is, $\mathrm{Tr}(I^{gi}) = I^{gi}_{xx} + I^{gi}_{yy} + I^{gi}_{zz}$.

In this paragraph, as in that segment of section 3 pertaining to a 2 DOF CMG and in that paragraph of section 4 containing equations (4-41) through (4-43), an additional subscript i (or superscript i as the case may be) will not be attached to any symbol relating to a 2 DOF CMG. Manipulating in accordance with the definitions and relations of the aforementioned segments of this paper leads to the contribution of a 2 DOF CMG to the two integrals in equation (4-61), that being

$$\int_{(m_{OG}+m_{IG}+m_g)} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \ddot{\vec{\tilde{r}}}\, dm + \Omega^T \int_{(m_{OG}+m_{IG}+m_g)} (\vec{\tilde{r}} - \vec{\tilde{r}}_{CM}) \times \dot{\vec{\tilde{r}}}\, dm$$

$$= I_B^{OG+IG+g}\, \dot{\vec{\omega}}_{OG}' + (\vec{\omega}_B + \vec{\omega}_{OG}') \times I_B^{OG+IG+g}\, \vec{\omega}_{OG}'$$

$$+ I_B^{I\,G+g}\, (\dot{\vec{\omega}}_{IG}' + \vec{\omega}_{IG}' \times \vec{\omega}_{OG}') + (\vec{\omega}_B + \vec{\omega}_{IG}')$$

$$\times I_B^{I\,G+g}\, \vec{\omega}_{IG}' + 2\, \vec{\omega}_{IG}' \times I_B^{I\,G+g}\, \vec{\omega}_{OG}'$$

$$- [\mathrm{Tr}\, (I^{IG+g})]\, (\vec{\omega}_{IG}' \times \vec{\omega}_{OG}') + (\vec{\omega}_B + \vec{\omega}_g') \times I_B{}^g\, \vec{\omega}_g'$$

$$+ 2\, \dot{\vec{\omega}}_g' \times I_B^g\, (\vec{\omega}_{OG}' + \vec{\omega}_{IG}')$$

$$- [\mathrm{Tr}\, (I^g)]\, [\vec{\omega}_g' \times (\vec{\omega}_{OG}' + \vec{\omega}_{IG}')] \quad . \tag{4-72}$$

The factors $\mathrm{Tr}(I^{\xi})$, $\xi = IG + g,\ g$, in equation (4-72) could be replaced by $\mathrm{Tr}(I_B^{\xi})$, $\xi = IG + g,\ g$, since the trace is invariant under a similarity transformation. See equations (3-28.1).

On assembling the several contributions to the integrals in equation (4-61), as given by equations (4-62), (4-63), (4-66), (4-67), (4-68), (4-69), (4-70), and (4-72), one can recast the moment equation in the somewhat more familiar form

$$\square\, \dot{\vec{\omega}}_B + \dot{\square}\, \vec{\omega}_B + \vec{\omega}_B \times \square\, \vec{\omega}_B = \vec{M}_T + \vec{M}_{gB} + \vec{M}_{gB}^{(MOON)} + \vec{M}_{gB}^{(SUN)} + \vec{M}_{AERO} + \vec{M}_{SOLAR}$$

$$+ \sum_{i=1}^{NSE} m_{Ei}\, \ell_{Ei}\, \{\vec{q}_{Ei} \times \ddot{\vec{\Lambda}}_{Ei} + \vec{\omega}_B \times (\vec{q}_{Ei} \times \dot{\vec{\Lambda}}_{Ei})\}$$

$$- \sum_{i=1}^{NSE} \{T_{Ei}^T\, I^{Ei}\, T_{Ei}\, \dot{\vec{\omega}}_{Ei}' + (\vec{\omega}_B + \vec{\omega}_{Ei}') \times T_{Ei}^T\, I^{Ei}\, T_{Ei}\, \vec{\omega}_{Ei}'\}$$

(4-73)

(continued on next page)

$$- \sum_{i=1}^{NR} \{T_{Ri}^T \, I^{Ri} \, T_{Ri} \, \dot{\vec{\omega}}_{Ri}' + (\vec{\omega}_B + \vec{\omega}_{Ri}') \times T_{Ri}^T \, I^{Ri} \, T_{Ri} \, \vec{\omega}_{Ri}'\}$$

$$- \sum_{i=1}^{NP} m_{Pi} \{(\tilde{\vec{r}}_{Pi} - \tilde{\vec{r}}_{CM}) \times (\ddot{\xi}_{Pi} \, \vec{\Lambda}_{Pi}) + \vec{\omega}_B \times [(\tilde{\vec{r}}_{Pi} - \tilde{\vec{r}}_{CM})$$

$$\times (\dot{\xi}_{Pi} \, \vec{\Lambda}_{Pi})]\}$$

$$- \sum_{i=1}^{NA} \{\tilde{T}_i^T \, I^i \, \tilde{T}_i \, \dot{\tilde{\vec{\omega}}}_i' + (\vec{\omega}_B + \tilde{\vec{\omega}}_i') \times \tilde{T}_i^T \, I^i \, \tilde{T}_i \, \tilde{\vec{\omega}}_i'\}$$

$$- \sum_{i=1}^{NA} \{m_i \, (\tilde{\vec{r}}_i - \tilde{\vec{r}}_{CM}) \times \tilde{T}_i^T \, [(\dot{\tilde{\Omega}}_i^T + \tilde{\Omega}_i^{T^2}) \, (\vec{\ell}_i^{(0)} + \psi^{(i)} \, \vec{\eta}^{(i)})$$

$$+ 2 \, \tilde{\Omega}_i^T \, \psi^{(i)} \, \dot{\vec{\eta}}^{(i)} + \psi^{(i)} \, \ddot{\vec{\eta}}^{(i)}]\}$$

$$- \vec{\omega}_B \times \sum_{i=1}^{NA} \{m_i \, (\tilde{\vec{r}}_i - \tilde{\vec{r}}_{CM}) \times \tilde{T}_i^T \, [\tilde{\Omega}_i^T \, (\vec{\ell}_i^{(0)} + \psi^{(i)} \, \vec{\eta}^{(i)})$$

$$+ \psi^{(i)} \, \dot{\vec{\eta}}^{(i)}]\}$$

$$- \sum_{i=1}^{NA} \tilde{T}_i^T \sum_{j=1}^{N_i} \{\ddot{\eta}_j^i \, \vec{C}_{oj}^i + \eta_j^i \sum_{K=1}^{N_i} \ddot{\eta}_K^i \, \vec{C}_{jK}^i - 2 \, \dot{\eta}_j^i \, \mathcal{J}_{rj}^i \, \tilde{\vec{\omega}}_i$$

$$- 2 \, \eta_j^i \sum_{K=1}^{N_i} \dot{\eta}_K^i \, \mathcal{J}_{jK}^i \, \tilde{\vec{\omega}}_i\}$$

$$- \vec{\omega}_B \times \sum_{i=1}^{NA} \left\{\tilde{T}_i^T \sum_{j=1}^{N_i} \left[\dot{\eta}_j^i \, \vec{C}_{oj}^i + \eta_j^i \sum_{K=1}^{N_i} \dot{\eta}_K^i \, \vec{C}_{jK}^i\right]\right\}$$

(4-73)

(continued on next page)

$$- \sum_{i=1}^{NSDOF} \{ (I_{xx}^{Gi} + I_{xx}^{gi}) (\dot{\vec{\omega}}'_{Gi} + \vec{\omega}_B \times \vec{\omega}'_{Gi}) + I_{zz}^{gi} (\vec{\omega}_B + \vec{\omega}'_{Gi}) \times \vec{\omega}'_{gi} \}$$

$$- \sum_{i=1}^{N2DOF} \{ I_B^{OG+IG+g} \dot{\vec{\omega}}'_{OG} + (\vec{\omega}_B + \vec{\omega}'_{OG}) \times I_B^{OG+IG+g} \vec{\omega}'_{OG}$$

$$+ I_B^{IG+g} (\dot{\vec{\omega}}'_{IG} + \vec{\omega}'_{IG} \times \vec{\omega}'_{OG}) + (\vec{\omega}_B + \vec{\omega}'_{IG})$$

$$\times I_B^{IG+g} \vec{\omega}'_{IG} + 2 \vec{\omega}'_{IG} \times I_B^{IG+g} \vec{\omega}'_{OG}$$

$$- [\mathcal{T}r (I^{IG+g})] (\vec{\omega}'_{IG} \times \vec{\omega}'_{OG}) + (\vec{\omega}_B + \vec{\omega}'_g)$$

$$\times I_B^g \vec{\omega}'_g + 2 \vec{\omega}'_g \times I_B^g (\vec{\omega}'_{OG} + \vec{\omega}'_{IG})$$

$$- [\mathcal{T}r (I^g)] [\vec{\omega}'_g \times (\vec{\omega}'_{OG} + \vec{\omega}'_{IG})] \}_{(i)} \,. \qquad \text{(4-73) (Conc.)}$$

One expecting to see terms such as $-T_{Ei}^T I^{Ei} T_{Ei} \dot{\vec{\omega}}_B$, $-T_{Ri}^T I^{Ri} T_{Ri} \dot{\vec{\omega}}_B$ etc., and $-\vec{\omega}_B \times T_{Ei}^T I^{Ei} T_{Ei} \vec{\omega}_B$, $-\vec{\omega}_B \times T_{Ri}^T I^{Ri} T_{Ri} \vec{\omega}_B$, etc., in the right member of equation (4-73) should not be alarmed since their negatives are embedded in the left member of equation (4-73). Recall that $\square$ is the inertia matrix of the entire system referred to the B-frame defined in Section 2. An expression for $\square$ is given in Appendix E.

The primed vectors associated with the CMG's in equation (4-73) have the B-resolution as do the vectors $\vec{\omega}'_{Ei}$, $\vec{\omega}'_{Ri}$, and $\vec{\omega}'_i$ (see Section 3). Their introduction made it possible to write the moment equation more compactly, not to mention imparting a certain "consistency of appearance" in the presence of the other primed vectors. The explicit dependence of the CMG terms upon the gimbal angles and their time derivatives and the gyro element spin rates is shown by the following:

$$(I_{xx}^{Gi} + I_{xx}^{gi}) (\dot{\vec{\omega}}'_{Gi} + \vec{\omega}_B \times \vec{\omega}'_{Gi}) = (I_{xx}^{Gi} + I_{xx}^{gi}) \left\{ \ddot{\delta}_{Gi} T_{Gio}^T \begin{bmatrix} 1. \\ 0 \\ 0 \end{bmatrix} + \dot{\delta}_{Gi} \vec{\omega}_B \times \left(T_{Gio}^T \begin{bmatrix} 1. \\ 0 \\ 0 \end{bmatrix} \right) \right\}, \qquad \text{(4-73.1)}$$

$$I_{zz}^{gi}\,(\vec{\omega}_B + \vec{\omega}'_{Gi}) \times \vec{\omega}'_{gi} = -I_{zz}^{gi}\,\omega_{gi}\left\{\dot{\delta}_{Gi}\,T_{Gio}^T\begin{bmatrix}0\\ \cos\delta_{Gi}\\ \sin\delta_{Gi}\end{bmatrix} - \vec{\omega}_B \times \left(T_{Gio}^T\begin{bmatrix}0\\ -\sin\delta_{Gi}\\ \cos\delta_{Gi}\end{bmatrix}\right)\right\}, \tag{4-73.2}$$

$$I_B^{OG+IG+g}\,\dot{\vec{\omega}}'_{OG} = \ddot{\delta}_{OG}\,T_{BGB}^T\begin{bmatrix}(I_{zz}^g - I_{yy}^g)\sin\delta_{OG}\sin\delta_{IG}\cos\delta_{IG}\\ (I_{yy}^g - I_{zz}^g)\cos\delta_{OG}\sin\delta_{IG}\cos\delta_{IG}\\ I_{zz}^{OG} + I_{zz}^{IG} + I_{yy}^g\sin^2\delta_{IG} + I_{zz}^g\cos^2\delta_{IG}\end{bmatrix}, \tag{4-73.3}$$

$$(\vec{\omega}_B + \vec{\omega}'_{OG}) \times I_B^{OG+IG+g}\,\vec{\omega}'_{OG} = \vec{\omega}_B \times \left\{\dot{\delta}_{OG}\,T_{BGB}^T\begin{bmatrix}(I_{zz}^g - I_{yy}^g)\sin\delta_{OG}\sin\delta_{IG}\cos\delta_{IG}\\ (I_{yy}^g - I_{zz}^g)\cos\delta_{OG}\sin\delta_{IG}\cos\delta_{IG}\\ I_{zz}^{OG} + I_{zz}^{IG} + I_{yy}^g\sin^2\delta_{IG} + I_{zz}^g\cos^2\delta_{IG}\end{bmatrix}\right.$$

$$+ \dot{\delta}_{OG}^2\,T_{BGB}^T\begin{bmatrix}(I_{zz}^g - I_{yy}^g)\cos\delta_{OG}\sin\delta_{IG}\cos\delta_{IG}\\ (I_{zz}^g - I_{yy}^g)\sin\delta_{OG}\sin\delta_{IG}\cos\delta_{IG}\\ 0\end{bmatrix}, \tag{4-73.4}$$

$$I_B^{I\,G+g}\,(\dot{\vec{\omega}}'_{IG} + \vec{\omega}'_{IG} \times \vec{\omega}'_{OG}) = (I_{xx}^{IG} + I_{xx}^g)\,\ddot{\delta}_{IG}\,T_{BGB}^T\begin{bmatrix}\cos\delta_{OG}\\ \sin\delta_{OG}\\ 0\end{bmatrix} \tag{4-73.5}$$

$$(\vec{\omega}_B + \vec{\omega}'_{IG}) \times I_B^{I\,G+g}\,\vec{\omega}'_{IG} = (I_{xx}^{IG} + I_{xx}^g)\,\dot{\delta}_{IG}\left\{\vec{\omega}_B \times T_{BGB}^T\begin{bmatrix}\cos\delta_{OG}\\ \sin\delta_{OG}\\ 0\end{bmatrix}\right\} \tag{4-73.6}$$

$$\{2\vec{\omega}'_{IG} \times I_B^{IG+g}\,\vec{\omega}'_{OG} - [Tr(I^{IG+g})]\,(\vec{\omega}'_{IG} \times \vec{\omega}'_{OG})\} = \dot{\delta}_{OG}\,\dot{\delta}_{IG}\,T^T_{BGB} \begin{bmatrix} -\sin\delta_{OG}\,[I^{IG}_{xx} + I^g_{yy}\cos(2\delta_{IG}) \\ + 2\,I^g_{zz}\sin^2\delta_{IG}] \\ \hline \cos\delta_{OG}\,[I^{IG}_{xx} + I^g_{yy}\cos(2\delta_{IG}) \\ + 2\,I^g_{zz}\sin^2\delta_{IG}] \\ \hline (I^g_{yy} - I^g_{zz})\sin(2\delta_{IG}) \end{bmatrix} \tag{4-73.7}$$

$$(\vec{\omega}_B + \vec{\omega}'_g) \times I_B^g\,\vec{\omega}'_g = I^g_{yy}\,\omega_g \left\{\vec{\omega}_B \times T^T_{BGB} \begin{bmatrix} -\sin\delta_{OG}\cos\delta_{IG} \\ \cos\delta_{OG}\cos\delta_{IG} \\ \sin\delta_{IG} \end{bmatrix}\right\} \tag{4-73.8}$$

$$\{2\vec{\omega}'_g \times I_B^g\,(\vec{\omega}'_{OG} + \vec{\omega}'_{IG}) - [Tr(I^g)]\,[\vec{\omega}'_g \times (\vec{\omega}'_{OG} + \vec{\omega}'_{IG})]\}$$

$$= -\omega_g\,I^g_{yy}\,T^T_{BGB}\left\{\dot{\delta}_{OG}\cos\delta_{IG}\begin{bmatrix} \cos\delta_{OG} \\ \sin\delta_{OG} \\ 0 \end{bmatrix} + \dot{\delta}_{IG}\begin{bmatrix} -\sin\delta_{IG}\sin\delta_{OG} \\ \sin\delta_{IG}\cos\delta_{OG} \\ -\cos\delta_{IG} \end{bmatrix}\right\} \tag{4-73.9}$$

The relations (4-73.1) and (4-73.2) hold under the assumption that I^{Gi} and I^{gi} are diagonal and $I^{gi}_{xx} = I^{gi}_{yy}$, while equations (4-73.3) through (4-73.9) are valid if I^{OG}, I^{IG}, and I^g are diagonal with $I^{OG}_{xx} = I^{OG}_{yy}$, $I^{IG}_{yy} = I^{IG}_{zz}$, and $I^g_{zz} = I^g_{xx}$.

In view of equation (4-53), it seems that the selection of the system CM as the MRP, rather than the origin of the $\tilde{x}\tilde{y}\tilde{z}$ axes (the $\tilde{B}$-frame), in deriving the moment equation, (4-73), is an inconsistency. A more practical and wiser choice of MRP could be the origin of the $\tilde{B}$-frame, a point to be discussed subsequent to writing equation

(4-73.14) below. Equation (4-73.14) is the form of the moment equation when moments are referenced to the origin of the $\tilde{B}$-frame. The author's derivation of (4-73.14) begins by invoking (4-57) which is here rewritten more concisely as

$$\int \vec{r} \times T\ddot{\vec{R}}_s \, dm = \vec{\mathcal{M}}^{CM}_{EXT} \tag{4-73.10}$$

where the symbol $\vec{\mathcal{M}}^{CM}_{EXT}$ denotes the resultant moment of the external forces about the system CM (and has the B resolution). Via the relations

$$\vec{r} = \vec{\tilde{r}} - \vec{\tilde{r}}_{CM}$$

$$\vec{\tilde{r}}_{CM} = (1/m) \int_{(m)} \vec{\tilde{r}} \, dm \qquad \text{(see Appendix D)}$$

$$T\ddot{\vec{R}}_s = T\ddot{\vec{\tilde{R}}}_s + \ddot{\vec{\tilde{r}}} + 2\Omega^T \dot{\vec{\tilde{r}}} + (\dot{\Omega}^T + {\Omega^T}^2)\, \vec{\tilde{r}}$$

one may pass from equation (4-73.10) to (4-73.11)

$$\int_{(m)} \vec{\tilde{r}} \times \{\ddot{\vec{\tilde{r}}} + 2\Omega^T \dot{\vec{\tilde{r}}} + (\dot{\Omega}^T + {\Omega^T}^2)\, \vec{\tilde{r}}\}\, dm = \vec{\mathcal{M}}^{CM}_{EXT} + \vec{\tilde{r}}_{CM} \times \int_{(m)} \{\ddot{\vec{\tilde{r}}} + 2\Omega^T \dot{\vec{\tilde{r}}} + (\dot{\Omega}^T + {\Omega^T}^2)\, \vec{\tilde{r}}\}\, dm \tag{4-73.11}$$

With $\vec{\mathcal{F}}_{EXT}$ denoting the B resolution of the sum of the external forces, it follows from equation (4-73.11) and either equation (4-44) or (4-45) that

$$\int_{(m)} \vec{\tilde{r}} \times \{\ddot{\vec{\tilde{r}}} + 2\Omega^T \dot{\vec{\tilde{r}}} + (\dot{\Omega}^T + {\Omega^T}^2)\, \vec{\tilde{r}}\}\, dm = \vec{\mathcal{M}}^{CM}_{EXT} + \vec{\tilde{r}}_{CM} \times \vec{\mathcal{F}}_{EXT} - m\, \vec{\tilde{r}}_{CM} \times T\, \ddot{\vec{\tilde{R}}}_s \tag{4-73.12}$$

But

$$\vec{\mathcal{M}}^{CM}_{EXT} + \vec{\tilde{r}}_{CM} \times \vec{\mathcal{F}}_{EXT} = \vec{\tilde{\mathcal{M}}}_{EXT} = \text{sum of external moments about origin of } \tilde{x}\tilde{y}\tilde{z}$$

$$\int_{(m)} \vec{\tilde{r}} \times \dot{\Omega}^T \vec{\tilde{r}}\, dm = \tilde{\square}\dot{\vec{\omega}}_B \quad , \quad (\tilde{\square} = \text{inertia matrix of entire system referred to } \tilde{x}\tilde{y}\tilde{z})$$

$$\int_{(m)} \vec{\tilde{r}} \times \Omega^{T^2} \vec{\tilde{r}}\, dm = \vec{\omega}_B \times \tilde{\square}\vec{\omega}_B = \Omega^T \tilde{\square}\vec{\omega}_B$$

$$2 \int_{(m)} \vec{\tilde{r}} \times \Omega^T \dot{\vec{\tilde{r}}}\, dm = \dot{\tilde{\square}}\vec{\omega}_B + \Omega^T \int_{(m)} \dot{\vec{\tilde{r}}} \times \dot{\vec{\tilde{r}}}\, dm$$

so that equation (4-73.12) may be written

$$\tilde{\square}\dot{\vec{\omega}}_B + \dot{\tilde{\square}}\vec{\omega}_B + \Omega^T \tilde{\square}\vec{\omega}_B + m\, \mathcal{S}(\vec{\tilde{r}}_{CM})\, T\, \ddot{\vec{\tilde{R}}}_s = \mathcal{M}_{EXT} - \int_{(m)} \vec{\tilde{r}} \times \ddot{\vec{\tilde{r}}}\, dm - \Omega^T \int_{(m)} \vec{\tilde{r}} \times \dot{\vec{\tilde{r}}}\, dm \,. \tag{4-73.13}$$

Dealing with the integrals in the right member of equation (4-73.13) in the manner of arriving at equations (4-62) through (4-69) and (4-70) through (4-72), there follows

$$\tilde{\square}\dot{\vec{\omega}}_B + \dot{\tilde{\square}}\vec{\omega}_B + \Omega^T \tilde{\square}\vec{\omega}_B + m\, \mathcal{S}(\vec{\tilde{r}}_{CM})\, T\, \ddot{\vec{\tilde{R}}}_s = \vec{\tilde{\mathcal{M}}}_{EXT} + \vec{\tilde{\mathcal{M}}}_T$$

$$+ \{\text{other terms attributed to the integrals in (4-73.13)}\} \,. \tag{4-73.14}$$

The vector $\vec{\tilde{\mathcal{M}}}_T$ is the resultant engine thrust moment about the origin of the $\tilde{B}$-frame, while the terms alluded to within braces include all the sums appearing in the right member of equation (4-73) modified, where necessary, by suppressing the symbol $\vec{\tilde{r}}_{CM}$. The vector $\vec{q}_{Ei}$ in the sum pertinent to the swivel engines should be replaced by $\vec{\tilde{q}}_{Ei} = \vec{\tilde{r}}_{Ei} - \ell_{Ei} \vec{\Lambda}_{Ei}$ (recall that $\vec{q}_{Ei} = \vec{\tilde{r}}_{Ei} - \vec{\tilde{r}}_{CM} - \ell_{Ei} \vec{\Lambda}_{Ei}$).

If $\square$ is known, then one can compute $\tilde{\square}$ in accordance with the relation

$$\tilde{\square} = \square + m\, \mathcal{S}(\vec{\tilde{r}}_{CM})\, \mathcal{S}(-\vec{\tilde{r}}_{CM}) \,. \tag{4-73.15}$$

However, the author does not recommend the use of equation (4-73.15) even if equation (4-73.14) is chosen to be the moment equation. With little effort, equation (E-4) of Appendix E can be transformed into one for determining $\tilde{\square}$ directly. Such a transformation* is effected by merely replacing the symbol I^f with $\tilde{I}^f$, the symbol $\tilde{I}^f$ denoting the inertia matrix of the fluid referred to the $\tilde{B}$-frame, and replacing each symbol $\vec{q}_\xi$ by the symbol $\vec{\tilde{q}}_\xi$ where $\vec{\tilde{q}}_\xi$ is defined by

$$\vec{\tilde{q}}_\xi = \vec{q}_\xi + \vec{\tilde{r}}_{CM} \quad , \quad \xi = 0,\ i,\ Ei,\ Ri,\ Pi,\ Gi,\ gi,\ OG,\ IG,\ g \quad , \tag{4-73.16}$$

the $\vec{q}_\xi$ (ξ = 0, i, Ei, Ri, Pi, Gi, gi, OG, IG, g) being exactly as defined in appendix E. From the definitions of the $\vec{q}_\xi$ and the $\vec{\tilde{q}}_\xi$, it should be obvious that the expressions for the $\vec{\tilde{q}}_\xi$ are devoid of dependence upon $\vec{\tilde{r}}_{CM}$, and hence, that the resulting expression for $\tilde{\square}$ will be neither explicitly nor implicitly dependent upon $\vec{\tilde{r}}_{CM}$. A differentiation with respect to time would then yield an expression for $\dot{\tilde{\square}}$ which is free of both $\vec{\tilde{r}}_{CM}$ and $\dot{\vec{\tilde{r}}}_{CM}$. Computation wise, the use of such expressions for $\tilde{\square}$ and $\dot{\tilde{\square}}$ would be far more expedient than the direct application of equations (E-4) and (E-5) followed by an application of equation (4-73.15) to find $\tilde{\square}$ and $\dot{\tilde{\square}}$. In fact, computation of $\square$ and $\dot{\square}$ would proceed more efficiently by first computing $\tilde{\square}$ and $\dot{\tilde{\square}}$ and then $\square$ and $\dot{\square}$ from

$$\square = \tilde{\square} - m\,\mathcal{S}(\vec{\tilde{r}}_{CM})\,\mathcal{S}(-\vec{\tilde{r}}_{CM})$$

$$\dot{\square} = \dot{\tilde{\square}} - m\,\mathcal{S}(\dot{\vec{\tilde{r}}}_{CM})\,\mathcal{S}(-\vec{\tilde{r}}_{CM}) - m\,\mathcal{S}(\vec{\tilde{r}}_{CM})\,\mathcal{S}(-\dot{\vec{\tilde{r}}}_{CM}) - \dot{m}\,\mathcal{S}(\vec{\tilde{r}}_{CM})\,\mathcal{S}(-\vec{\tilde{r}}_{CM}) \quad .$$

From a programming standpoint, it should be evident that the choice of the origin of the $\tilde{B}$-frame as MRP offers certain computational advantages. In that regard, still another point favoring that choice should be made. Consider recasting the system equations of motion in the form

$$A\ddot{\vec{X}} = \vec{F}\,(t,\ \vec{X},\ \dot{\vec{X}})$$

as a first step in rendering them amenable to some numerical integration scheme. (As one submatrix of the partitioned column matrix $\ddot{\vec{X}}$, the author has in mind the 3x1 column $\ddot{\vec{\psi}}_B \equiv [\ddot{\psi}_1,\ \ddot{\psi}_2,\ \ddot{\psi}_3]^T$ whose elements are the second time derivatives of the

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*The reader should realize that here the author is <u>not</u> changing the appearance of (E-4) via the substitution of $\vec{\tilde{q}}_\xi - \vec{\tilde{r}}_{CM}$ for $\vec{q}_\xi$, ξ = 0, i, Ei, ..., g. If such a substitution were made, after replacing I^f by $\tilde{I}^f$, the resulting right member of (E-4) would define neither $\square$ nor $\tilde{\square}$; however, if I^f is left intact, the substitution would result in an expression which still defines $\square$, not $\tilde{\square}$. It should be clear that the result of making the "replacements" alluded to above is the same as that obtainable by invoking the "generalized transfer theorem" expressed by equation (E-3).

quasi-coordinates $\psi_i = \int_o^t \omega_i \, dt$, i = 1, 2, 3, which is to say, obviously, that $\ddot{\vec{\psi}}_B = \dot{\vec{\omega}}_B$. Also, the author has here supposed that equation (4-74), not (4-76), governs the motion of the point mass m_{pi}.) With the combination of equations (4-53) and (4-73) designated as system translational and rotational equations, it will be found that the mass matrix A is non-symmetric! But, that the matrix A corresponding to the combination of equations (4-53) and (4-73.14) is a symmetric matrix. The time consumed and storage space required for constructing and inverting the symmetric matrix should be less than that for the non-symmetric matrix. The author would definitely prefer equation (4-73.14) to equation (4-73).

Clearly, the moment equation must be accompanied by the equations giving the Euler angle rates, $\dot{\varphi}_K$, K = P, Y, r, in terms of the components of $\vec{\omega}_B$ and the φ_K, K = P, Y, r. The structure of these equations will, of course, depend upon the "Euler sequence" of rotations selected to pass from the S-orientation to the B-orientation. For a 2, 3, 1 sequence through the angles φ_P, φ_Y, φ_r, respectively, the $\dot{\varphi}_K$, K = P, Y, r, are determined by equation (3-0.4) except in those cases where φ_Y is an odd integral multiple of $\pi/2$, thereby making the matrix in equation (3-0.4) singular. Attention will not now be diverted to methods for circumventing the numerical difficulty encountered at such singularities.

With the coordinate ξ_{pi} assuming the role of q in equation (4-1), the result of some manipulation is the equation governing the motion of the point mass m_{pi},

$$m_{Pi}\,\ddot{\xi}_{Pi} + C_{Pi}\,\dot{\xi}_{Pi} + K_{Pi}\,\xi_{Pi} - m_{Pi}\,|\vec{\omega}_B \times \vec{\Lambda}_{Pi}|^2\,\xi_{Pi} + m_{Pi}\,\vec{\Lambda}_{Pi}^T\,\{T\,(\ddot{\vec{R}}_s - \vec{A}_{Pi})$$
$$+ (\dot{\Omega}^T + \Omega^{T^2})\,\vec{r}_{Pi}\} = 0 \quad , \quad i = 1, \ldots NP \quad . \tag{4-74}$$

The only symbol in equation (4-74) not heretofore defined is $\vec{A}_{pi}$ which is given by

$$\vec{A}_{pi} = [\alpha_{pi}]^T_{(3)}\,[-\delta_{pi}]^T_{(2)}\,\vec{A}_g\,(R_{pi}, \lambda_{pi}, \delta_{pi}) + \mu_s \left\{ \frac{\vec{R}_s^{(SUN)} - \vec{R}_s^{pi}}{|\vec{R}_s^{(SUN)} - \vec{R}_s^{pi}|^3} \right.$$
$$\left. - \frac{\vec{R}_s^{(SUN)}}{|\vec{R}_s^{(SUN)}|^3} \right\} + \mu_M \left\{ \frac{\vec{R}_s^{(MOON)} - \vec{R}_s^{pi}}{|\vec{R}_s^{(MOON)} - \vec{R}_s^{pi}|^3} - \frac{\vec{R}_s^{(MOON)}}{|\vec{R}_s^{(MOON)}|^3} \right\} \quad . \tag{4-75}$$

The definition (4-75) must be complemented by those following:

$$\vec{R}_s^{pi} \equiv [X_s^{pi},\ Y_s^{pi},\ Z_s^{pi}]^T = \vec{\tilde{R}}_s + T^T\ (\vec{\tilde{r}}_{pi} + \xi_{pi}\ \vec{\Lambda}_{pi})$$

= position vector, referred to the S-frame, of m_{pi}

$$R_{pi} = |\vec{R}_s^{pi}|$$

$$\alpha_{pi} = \tan^{-1}\ (Y_s^{pi}/X_s^{pi}) = \text{right ascension of } m_{pi}\ (0 \le \alpha_{pi} < 2\pi)$$

$$\delta_{pi} = \sin^{-1}\ (Z_s^{pi}/R_{pi}) = \text{declination of } m_{pi}\ (-\pi/2 \le \delta_{pi} \le \pi/2)$$

$$\lambda_{pi} = \alpha_{pi} - \alpha_P - \omega_\ell(t-t_o) = \text{east longitude of } m_{pi}\ (0 \le \lambda_{pi} < 2\pi)$$

(If the expression for λ_{pi} is negative then λ_{pi} is to be replaced by its positive equivalent modulo 2π.)

$\vec{A}_g\ (R_{pi},\ \lambda_{pi},\ \delta_{pi}) = \vec{A}_g$ evaluated at point with spherical coordinates

$(R_{pi},\ \lambda_{pi},\ \delta_{pi})$ referred to the E-frame (Section 2)
(See equations (A-2) through (A-5) of Appendix A and that part of Section 3 on the unit vectors $\vec{u}_R$, $\vec{u}_\lambda$ and $\vec{u}_\delta$)

It will be observed that in writing equation (4-74), the vector $\vec{\tilde{r}}_{pi}$ was regarded as a constant vector as implied by equation (4-29).

To allow for the possibility that m_{pi} is a "trim" mass installed for attitude control, one could add to the right member of equation (4-74) the force $F_{pi}\ \vec{\Lambda}_{pi}$, the scalar function of time F_{pi} being determined by some control law.

Introducing the symbols ω_{pi} and ζ_{pi}, satisfying $m_{pi}\ \omega_{pi}^2 = K_{pi}$ and $2\ m_{pi}\ \zeta_{pi}\ \omega_{pi} = C_{pi}$, and denoting, respectively, the undamped natural frequency and critical damping ratio associated with m_{pi}, permits one to rewrite equation (4-74) as

$$\ddot{\xi}_{pi} + 2\,\zeta_{pi}\,\omega_{pi}\,\dot{\xi}_{pi} + (\omega_{pi}^2 - |\vec{\omega}_B \times \vec{\Lambda}_{pi}|^2)\,\xi_{pi} + \vec{\Lambda}_{pi}^T\,\{T\,(\ddot{\vec{R}}_s - \vec{A}_{pi})$$

$$+ (\dot{\Omega}^T + \Omega^{T^2})\,\vec{\tilde{r}}_{pi}\} = 0 \quad , \quad i = 1, \ldots NP \quad . \tag{4-76}$$

Despite the title of this report, the influence of gravity on the motion of swivel engine i will be accounted for only in part, while that of the aerodynamic force and solar radiation pressure will be neglected entirely. In approximating the effect of the gravity field, the strength of the field is assumed constant throughout the region occupied by m_{Ei} and equal to that at its CM. The resulting equation corresponding to the engine deflection β_{pi} is, before further simplification

$$\frac{\partial \vec{\omega}'^T_{Ei}}{\partial \dot{\beta}_{pi}} \left\{ T^T_{Ei}\, I^{Ei}\, T_{Ei}\,(\dot{\vec{\omega}}_B + \dot{\vec{\omega}}'_{Ei} + \vec{\omega}_B \times \vec{\omega}'_{Ei}) + (\vec{\omega}_B + \vec{\omega}'_{Ei}) \times T^T_{Ei}\, I^{Ei}\, T_{Ei}\,(\vec{\omega}_B + \vec{\omega}'_{Ei}) \right\}$$

$$- m_{Ei}\,\ell_{Ei}\,\frac{\partial \vec{\Lambda}^T_{Ei}}{\partial \beta_{pi}} \left\{ T\,(\ddot{\vec{R}}_s - \vec{A}_{Ei}) + \left(\frac{d^2}{dt^2}\right)_{(s)} (\vec{\tilde{r}}_{Ei} - \ell_{Ei}\,\vec{\Lambda}_{Ei}) \right\}$$

$$+ C_{pEi}\,\dot{\beta}_{pi} + K_{pEi}\,(\beta_{pi} - \beta_{PCi}) \approx 0 \quad , \quad i = 1, \ldots NSE \quad . \tag{4-77}$$

The companion equation, corresponding to β_{Yi}, can be written immediately by merely replacing the subscript P in equation (4-77) with the subscript Y. (This is not to say that the final form of the equation satisfied by β_{Yi} can be obtained from that for β_{pi} by replacing the subscript P with Y. It is only at the stage of development indicated by equation (4-77) that such a change in subscript may be made.) In writing equation (4-77), the author has made use of the fact that $\partial \dot{\vec{R}}_s / \partial \dot{q} = \partial \vec{R}_s / \partial q$ and $\partial \dot{\vec{\Lambda}}_{Ei} / \partial \dot{\beta}_{pi} = \partial \vec{\Lambda}_{Ei} / \partial \beta_{pi}$. The equation defining $\vec{A}_{Ei}$ is quite similar to equation (4-75), that being

$$\vec{A}_{Ei} = [\alpha_{Ei}]^T_{(3)}\,[-\delta_{Ei}]^T_{(2)}\,\vec{A}_g\,(R_{Ei}, \lambda_{Ei}, \delta_{Ei}) + \mu_s \left\{ \frac{\vec{R}_s^{(SUN)} - \vec{R}_s^{Ei}}{|\vec{R}_s^{(SUN)} - \vec{R}_s^{Ei}|^3} \right.$$

$$\left. - \frac{\vec{R}_s^{(SUN)}}{|\vec{R}_s^{(SUN)}|^3} \right\} + \mu_m \left\{ \frac{\vec{R}_s^{(MOON)} - \vec{R}_s^{Ei}}{|\vec{R}_s^{(MOON)} - \vec{R}_s^{Ei}|^3} - \frac{\vec{R}_s^{(MOON)}}{|\vec{R}_s^{(MOON)}|^3} \right\} \tag{4-78}$$

where

$$\vec{R}_s^{\,Ei} \equiv [X_s^{Ei},\ Y_s^{Ei},\ Z_s^{Ei}]^T = \vec{\tilde{R}}_s + T^T\,(\vec{\tilde{r}}_{Ei} - \ell_{Ei}\,\vec{\Lambda}_{Ei})$$

= position vector, referred to the S-frame, of the CM of m_{Ei}

$$R_{Ei} = |\vec{R}_s^{\,Ei}|$$

$$\alpha_{Ei} = \tan^{-1}\,(Y_s^{Ei}/X_s^{Ei}) = \text{right ascension of CM of } m_{Ei}\ (0 \le \alpha_{Ei} < 2\pi)$$

$$\delta_{Ei} = \sin^{-1}\,(Z_s^{Ei}/R_{Ei}) = \text{declination of CM of } m_{Ei}\ (-\pi/2 \le \delta_{Ei} \le \pi/2)$$

$$\lambda_{Ei} = \alpha_{Ei} - \alpha_P - \omega_\ell(t-t_o) = \text{east longitude* of CM of } m_{Ei}\ (0 \le \lambda_{Ei} < 2\pi)$$

$\vec{A}_g\,(R_{Ei},\ \lambda_{Ei},\ \delta_{Ei}) = \vec{A}_g$ evaluated at the point with spherical coordinates $(R_{Ei},\ \lambda_{Ei},\ \delta_{Ei})$ referred to the E-frame.

Obviously, one can proceed no further until the components of $\vec{\Lambda}_{Ei} \equiv [\tilde{\lambda}_{Ei}, \tilde{\mu}_{Ei}, \tilde{\nu}_{Ei}]^T$ are given explcitly in terms of β_{pi} and β_{Yi}, and such expressions cannot be determined until the engine deflection sign convention, actuator arrangement, etc., have been decided. However, it is not difficult to deduce that

$$\vec{\omega}'_{Ei} = \frac{1}{1 - \tilde{\nu}_{Ei}^{\,2}} \begin{bmatrix} \tilde{\mu}_{Ei}\,\dot{\tilde{\nu}}_{Ei} \\ -\tilde{\lambda}_{Ei}\,\dot{\tilde{\nu}}_{Ei} \\ \tilde{\lambda}_{Ei}\,\dot{\tilde{\mu}}_{Ei} - \tilde{\mu}_{Ei}\,\dot{\tilde{\lambda}}_{Ei} \end{bmatrix} \tag{4-79}$$

whatever the structure of the scalar functions defining the direction cosines $\tilde{\lambda}_{Ei}$, $\tilde{\mu}_{Ei}$, and $\tilde{\nu}_{Ei}$. The symbol $\vec{\omega}'_{Ei}$, denoting the angular velocity of swivel engine i relative to the $\tilde{B}$-frame, was first introduced in Section 3 where the author failed to point out that the x_{Ei} axis of the Ei-frame (which is fixed relative to swivel engine i) is directed as the thrust vector and hence as $\vec{\Lambda}_{Ei}$.

*It should be evident that if the expression for λ_{Ei} is negative, it is to be replaced by its positive equivalent (modulo 2π). A similar remark holds for any symbol denoting the longitude of a point.

The only special case of equation (4-77) to be considered in this paper is that which applies to an engine whose thrust vector, $\vec{F}_{Ei}$, makes an acute angle with the $\tilde{x}$-axis, whose pitch actuator causes the $\tilde{x}\tilde{z}$-projection of the thrust vector to rotate about an axis parallel to the $\tilde{y}$-axis, and whose yaw actuator causes the $\tilde{x}\tilde{y}$-projection of the thrust vector to rotate about an axis parallel to the $\tilde{z}$-axis. For such an arrangement, it is easily shown (as in Reference 1) that $\vec{\Lambda}_{Ei}$ is given by

$$\vec{\Lambda}_{Ei} \equiv \begin{bmatrix} \tilde{\lambda}_{Ei} \\ \tilde{\mu}_{Ei} \\ \tilde{\nu}_{Ei} \end{bmatrix} = \frac{1}{\tilde{K}_{Ei}} \begin{bmatrix} 1. \\ \tan \tilde{\beta}_{yi} \\ -\tan \tilde{\beta}_{pi} \end{bmatrix}$$

where

$$\tilde{K}_{Ei} = (1 + \tan^2 \tilde{\beta}_{yi} + \tan^2 \tilde{\beta}_{pi})^{\frac{1}{2}}$$

$$\tilde{\beta}_{yi} = \beta_{yi} + \gamma_{yEi} + \varepsilon_{yEi}$$

$$\tilde{\beta}_{pi} = \beta_{pi} + \gamma_{pEi} + \varepsilon_{pEi}$$

γ_{Ei} = engine cant angle $(0 \leq \gamma_{Ei} < \pi/2)$

γ_{yEi} = engine cant in yaw (a known function of γ_{Ei} and the polar coordinates of the point of application of $\vec{F}_{Ei}$, the pole to which the polar coordinates are referred being the point P'_{Ei} defined below), $-\pi/2 < \gamma_{yEi} < \pi/2$

Λ_{Eio} - the line of action of $\vec{F}_{Ei}$ when $\beta_{yi} = \beta_{pi} = \varepsilon_{yEi} = \varepsilon_{pEi} = 0$

$\tilde{P}_{Ei}$ - the point at which line Λ_{Eio} intersects the plane $\tilde{y} = 0$ (This point need not be a point of the x-axis though such was the case for the early Saturn vehicles.)

$\tilde{P}'_{Ei}$ - the projection of $\tilde{P}_{Ei}$ upon the plane containing the point of application of $\vec{F}_{Ei}$ and parallel to the $\tilde{y}\tilde{z}$-plane.

ε_{yEi} - engine misalignment in yaw ($-\pi/2 << \varepsilon_{yEi} << \pi/2$)

γ_{pEi} - engine cant in pitch (also a known function of γ_{Ei} and the polar coordinates of the point of application of $\vec{F}_{Ei}$), $-\pi/2 < \gamma_{pEi} < \pi/2$

ε_{pEi} - engine misalignment in pitch ($-\pi/2 << \varepsilon_{pEi} << \pi/2$) .

The expressions for $\tilde{\beta}_{Yi}$ and $\tilde{\beta}_{pi}$ in Reference 1 included terms representing the contribution of bending, it being assumed there that the carrier vehicle experienced 3D bending at the swivel point. In this paper, however, the swivel point, supposed a point of the central carrier, suffers no displacement due to bending by virtue of the assumed rigidity of the carrier.

Again, drawing from the author's previous work (Reference 1), one has (bending or no bending) for this special case

$$\frac{\partial \vec{\omega}'_{Ei}}{\partial \dot{\beta}_{pi}} = \begin{bmatrix} -\tilde{\mu}_{Ei}(1-\tilde{\mu}_{Ei}{}^{2})/\tilde{\lambda}_{Ei} \\ (1-\tilde{\mu}_{Ei}{}^{2})(1-\tilde{\nu}_{Ei}{}^{2}) \\ 0 \end{bmatrix}$$

$$\frac{\partial \vec{\Lambda}_{Ei}}{\partial \beta_{pi}} = \begin{bmatrix} \tilde{\nu}_{Ei}(1-\tilde{\mu}_{Ei}{}^{2}) \\ \tilde{\mu}_{Ei}\tilde{\nu}_{Ei}(1-\tilde{\mu}_{Ei}{}^{2})/\tilde{\lambda}_{Ei} \\ -(1-\tilde{\mu}_{Ei}{}^{2})(1-\tilde{\nu}_{Ei}{}^{2})/\tilde{\lambda}_{Ei} \end{bmatrix}$$

$$\frac{\partial \vec{\omega}'_{Ei}}{\partial \dot{\beta}_{yi}} = \begin{bmatrix} -\tilde{\mu}_{Ei}{}^{2}\tilde{\nu}_{Ei}/\tilde{\lambda}_{Ei} \\ \tilde{\mu}_{Ei}\tilde{\nu}_{Ei} \\ 1. \end{bmatrix}$$

$$\frac{\partial \vec{\Lambda}_{Ei}}{\partial \beta_{yi}} = \begin{bmatrix} -\tilde{\mu}_{Ei}(1-\tilde{\nu}_{Ei}{}^{2}) \\ (1-\tilde{\mu}_{Ei}{}^{2})(1-\tilde{\nu}_{Ei}{}^{2})/\tilde{\lambda}_{Ei} \\ -\tilde{\mu}_{Ei}\tilde{\nu}_{Ei}(1-\tilde{\nu}_{Ei}{}^{2})/\tilde{\lambda}_{Ei} \end{bmatrix}$$

$$\dot{\vec{\omega}}'_{Ei} = \frac{1}{1 - \tilde{\nu}_{Ei}^{2}} \begin{bmatrix} \tilde{\mu}_{Ei}\, \ddot{\tilde{\nu}}_{Ei} + \dot{\tilde{\mu}}_{Ei}\, \dot{\tilde{\nu}}_{Ei} \\ - \dot{\tilde{\lambda}}_{Ei}\, \dot{\tilde{\nu}}_{Ei} - \lambda_{Ei}\, \ddot{\tilde{\nu}}_{Ei} \\ \tilde{\lambda}_{Ei}\, \ddot{\tilde{\mu}}_{Ei} - \tilde{\mu}_{Ei}\, \ddot{\tilde{\lambda}}_{Ei} \end{bmatrix} + \frac{2\, \tilde{\nu}_{Ei}\, \dot{\tilde{\nu}}_{Ei}}{(1 - \tilde{\nu}_{Ei}^{2})^{2}} \begin{bmatrix} \tilde{\mu}_{Ei}\, \dot{\tilde{\nu}}_{Ei} \\ - \tilde{\lambda}_{Ei}\, \dot{\tilde{\nu}}_{Ei} \\ \tilde{\lambda}_{Ei}\, \dot{\tilde{\mu}}_{Ei} - \tilde{\mu}_{Ei}\, \dot{\tilde{\lambda}}_{Ei} \end{bmatrix}$$

and, with no bending at the swivel point,

$$\dot{\tilde{\lambda}}_{Ei} = - \tilde{\mu}_{Ei}\, (1 - \tilde{\nu}_{Ei}^{2})\, \dot{\beta}_{yi} + \tilde{\nu}_{Ei}\, (1 - \tilde{\mu}_{Ei}^{2})\, \dot{\beta}_{pi}$$

$$\dot{\tilde{\mu}}_{Ei} = \frac{\tilde{\mu}_{Ei}}{\tilde{\lambda}_{Ei}}\, \dot{\tilde{\lambda}}_{Ei} + \frac{1}{\tilde{\lambda}_{Ei}}\, (1 - \tilde{\nu}_{Ei}^{2})\, \dot{\beta}_{yi}$$

$$\dot{\tilde{\nu}}_{Ei} = \frac{\tilde{\nu}_{Ei}}{\tilde{\lambda}_{Ei}}\, \dot{\tilde{\lambda}}_{Ei} - \frac{1}{\tilde{\lambda}_{Ei}}\, (1 - \tilde{\mu}_{Ei}^{2})\, \dot{\beta}_{pi}$$

$$\begin{aligned} \ddot{\tilde{\lambda}}_{Ei} = {} & - \tilde{\mu}_{Ei}\, (1 - \tilde{\nu}_{Ei}^{2})\, \ddot{\beta}_{yi} + \tilde{\nu}_{Ei}\, (1 - \tilde{\mu}_{Ei}^{2})\, \ddot{\beta}_{pi} + \dot{\beta}_{yi}\, [2\, \tilde{\mu}_{Ei}\, \tilde{\nu}_{Ei}\, \dot{\tilde{\nu}}_{Ei} \\ & - \dot{\tilde{\mu}}_{Ei}\, (1 - \tilde{\nu}_{Ei}^{2})] + \dot{\beta}_{pi}\, [\dot{\tilde{\nu}}_{Ei}\, (1 - \tilde{\mu}_{Ei}^{2}) - 2\, \tilde{\mu}_{Ei}\, \tilde{\nu}_{Ei}\, \dot{\tilde{\mu}}_{Ei}] \end{aligned}$$

$$\ddot{\tilde{\mu}}_{Ei} = \frac{1}{\tilde{\lambda}_{Ei}}\, [\tilde{\mu}_{Ei}\, \ddot{\tilde{\lambda}}_{Ei} + (1 - \tilde{\nu}_{Ei}^{2})\, \ddot{\beta}_{yi} - 2\, \tilde{\nu}_{Ei}\, \dot{\tilde{\nu}}_{Ei}\, \dot{\beta}_{yi}]$$

$$\ddot{\tilde{\nu}}_{Ei} = \frac{1}{\tilde{\lambda}_{Ei}}\, [\tilde{\nu}_{Ei}\, \ddot{\tilde{\lambda}}_{Ei} - (1 - \tilde{\mu}_{Ei}^{2})\, \ddot{\beta}_{pi} + 2\, \tilde{\mu}_{Ei}\, \dot{\tilde{\mu}}_{Ei}\, \dot{\beta}_{pi}]$$

$$\left(\frac{d^2}{dt^2}\right)_{(s)} (\vec{\tilde{r}}_{Ei} - \ell_{Ei}\, \vec{\Lambda}_{Ei}) = - \ell_{Ei}\, \ddot{\vec{\Lambda}}_{Ei} - 2\, \ell_{Ei}\, \Omega^{T}\, \dot{\vec{\Lambda}}_{Ei} + (\dot{\Omega}^{T} + {\Omega^{T}}^{2})\, (\vec{\tilde{r}}_{Ei} - \ell_{Ei}\, \vec{\Lambda}_{Ei})$$

Now, "if" the angles $\tilde{\beta}_{pi}$ and $\tilde{\beta}_{Yi}$ are sufficiently small as to permit the approximations $\tan \tilde{\beta}_{pi} \approx \tilde{\beta}_{pi}$, $\tan \tilde{\beta}_{yi} \approx \tilde{\beta}_{yi}$, and further, if such products as $\xi^{\ell_1} \eta^{\ell_2}$ are negligible when $\ell_1 + \ell_2 \geq 2$, ξ and η being any members of the following list,

$\tilde{\mu}_{Ei}$, $\dot{\tilde{\mu}}_{Ei}$, $\ddot{\tilde{\mu}}_{Ei}$, $\tilde{\nu}_{Ei}$, $\dot{\tilde{\nu}}_{Ei}$, $\ddot{\tilde{\nu}}_{Ei}$, β_{pi}, $\dot{\beta}_{pi}$, $\ddot{\beta}_{pi}$, β_{yi}, $\dot{\beta}_{yi}$, $\ddot{\beta}_{yi}$, ω_1, ω_2, ω_3,

$\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$,

then considerable manipulation will show that equation (4-77) becomes

$$\begin{aligned}(I_{yy}^{Ei} + m_{Ei}\,\ell_{Ei}^{2})\,(\ddot{\beta}_{pi} + \dot{\omega}_2) + I_{yz}^{Ei}\,(\ddot{\beta}_{yi} + \dot{\omega}_3) + I_{xy}^{Ei}\,\dot{\omega}_1 \\ + m_{Ei}\,\ell_{Ei}\,[\ddot{\tilde{X}}_B^{Ei}\,(\beta_{pi} + \gamma_{pEi} + \varepsilon_{pEi}) + \ddot{\tilde{Z}}_B^{Ei} + \tilde{y}_{Ei}\,\dot{\omega}_1 - \tilde{x}_{Ei}\,\dot{\omega}_2] \\ + C_{PEi}\,\dot{\beta}_{pi} + K_{pEi}\,(\beta_{pi} - \beta_{pCi}) \approx 0 \quad , \quad i = 1, \ldots \text{NSE} \quad ,\end{aligned} \tag{4-80}$$

where

$$[\ddot{\tilde{X}}_B^{Ei}, \ddot{\tilde{Y}}_B^{Ei}, \ddot{\tilde{Z}}_B^{Ei}]^T = T\,(\ddot{\vec{R}}_s - \vec{A}_{Ei}) \quad .$$

Similar approximations made in the equation corresponding to β_{Yi} lead to

$$\begin{aligned}(I_{zz}^{Ei} + m_{Ei}\,\ell_{Ei}^{2})\,(\ddot{\beta}_{yi} + \dot{\omega}_3) + I_{xz}^{Ei}\,\dot{\omega}_1 + I_{yz}^{Ei}\,(\ddot{\beta}_{pi} + \dot{\omega}_2) \\ + m_{Ei}\,\ell_{Ei}\,[\ddot{\tilde{X}}_B^{Ei}\,(\beta_{yi} + \gamma_{yEi} + \varepsilon_{yEi}) - \ddot{\tilde{Y}}_B^{Ei} - \tilde{x}_{Ei}\,\dot{\omega}_3 + \tilde{z}_{Ei}\,\dot{\omega}_1] \\ + C_{yEi}\,\dot{\beta}_{yi} + K_{yEi}\,(\beta_{yi} - \beta_{yCi}) \approx 0 \quad , \quad i = 1, \ldots \text{NSE} \quad .\end{aligned} \tag{4-81}$$

Since a CMG is a "small" component, it is here supposed that the effect of gravity on the motion of a CMG gimbal is altogether insignificant. With that in mind and with $q = \delta_{Gi}$ in equation (4-1), there follows

$$\begin{aligned}(I_{xx}^{Gi} + I_{xx}^{gi})\,\ddot{\delta}_{Gi} + C_{Gi}\,\dot{\delta}_{Gi} + (I_{xx}^{Gi} + I_{xx}^{gi})\,[1., 0, 0]\,T_{Gio}\,\dot{\vec{\omega}}_B \\ + [1., 0, 0]\,\{(T_{Gio}\,\vec{\omega}_B) \times \tilde{T}_{Gi}^{T}\,(I^{Gi} + I^{gi})\,\tilde{T}_{Gi}\,T_{Gio}\,\vec{\omega}_B\} \\ + I_{zz}^{gi}\,\omega_{gi}\,[0, \cos\delta_{Gi}, \sin\delta_{Gi}]\,T_{Gio}\,\vec{\omega}_B = \mathcal{M}_{\delta_{Gi}} \quad , \quad i = 1, \ldots \text{NSDOF} \quad ,\end{aligned} \tag{4-82}$$

as the equation governing the motion of the gimbal of the i^{th} SDOF CMG.

Writing the equations satisfied by the outer gimbal angle δ_{OG} and inner gimbal angle δ_{IG} of a 2DOF CMG requires considerably more manipulation than that required to arrive at equation (4-82). The equation of motion of the outer gimbal is

$$
\begin{aligned}
&(I^{OG}_{zz} + I^{IG}_{zz} + I^{g}_{yy} \sin^2 \delta_{IG} + I^{g}_{zz} \cos^2 \delta_{IG}) \ddot{\delta}_{OG} + I^{g}_{yy} \omega_g \dot{\delta}_{IG} \cos \delta_{IG} \\
&+ 2 \dot{\delta}_{OG} \dot{\delta}_{IG} (I^{g}_{yy} - I^{g}_{zz}) \sin \delta_{IG} \cos \delta_{IG} \\
&+ \begin{bmatrix} (I^{g}_{zz} - I^{g}_{yy}) \sin \delta_{OG} \sin \delta_{IG} \cos \delta_{IG} \\ (I^{g}_{yy} - I^{g}_{zz}) \cos \delta_{OG} \sin \delta_{IG} \cos \delta_{IG} \\ I^{OG}_{zz} + I^{IG}_{zz} + I^{g}_{yy} \sin^2 \delta_{IG} + I^{g}_{zz} \cos^2 \delta_{IG} \end{bmatrix}^T T_{BGB} \dot{\vec{\omega}}_B \\
&+ \begin{bmatrix} I^{g}_{yy} \omega_g \cos \delta_{IG} \cos \delta_{OG} + \dot{\delta}_{IG} \sin \delta_{OG} [I^{IG}_{xx} - I^{g}_{yy} \cos (2\delta_{IG}) + 2 I^{g}_{zz} \cos^2 \delta_{IG}] \\ I^{g}_{yy} \omega_g \cos \delta_{IG} \sin \delta_{OG} - \dot{\delta}_{IG} \cos \delta_{OG} [I^{IG}_{xx} - I^{g}_{yy} \cos (2\delta_{IG}) + 2 I^{g}_{zz} \cos^2 \delta_{IG}] \\ (I^{g}_{yy} - I^{g}_{zz}) \dot{\delta}_{IG} \sin (2\delta_{IG}) \end{bmatrix}^T T_{BGB} \vec{\omega}_B \\
&+ [0, 0, 1.]\ T_{BGB} \{\vec{\omega}_B \times [T^T_{BOG} (I^{OG} + I^{IG}) T_{BOG} \vec{\omega}_B]\} \\
&+ [0, 0, 1.]\ \{(T_{BOG} \vec{\omega}_B) \times (T^T_{OGIG} I^{g} T_{OGIG} T_{BOG} \vec{\omega}_B)\} = \mathcal{M}_{\delta_{OG}} - C_{OG} \dot{\delta}_{OG} ,
\end{aligned}
\tag{4-83}
$$

while that of the inner gimbal is

$$(I_{xx}^{IG} + I_{xx}^{g})\,\ddot{\delta}_{IG} - I_{yy}^{g}\,\omega_g\,\dot{\delta}_{OG}\cos\delta_{IG} + (I_{zz}^{g} - I_{yy}^{g})\,\dot{\delta}_{OG}^{2}\sin\delta_{IG}\cos\delta_{IG}$$

$$+ [\cos\delta_{OG}, \sin\delta_{OG}, 0]\; T_{BGB}\,\{(I_{xx}^{IG} + I_{xx}^{g})\,\dot{\vec{\omega}}_B$$

$$+ \vec{\omega}_B \times [T_{BIG}^{T}\,(I^{IG} + I^{g})\,T_{BIG}\,\vec{\omega}_B]\}$$

$$+ \begin{bmatrix} -I_{yy}^{g}\,\omega_g \sin\delta_{IG}\sin\delta_{OG} - \dot{\delta}_{OG}\sin\delta_{OG}\,[I_{xx}^{IG} - I_{yy}^{g}\cos(2\delta_{IG}) \\ + 2\,I_{zz}^{g}\cos^2\delta_{IG}] \\ \hline I_{yy}^{g}\,\omega_g \sin\delta_{IG}\cos\delta_{OG} + \dot{\delta}_{OG}\cos\delta_{OG}\,[I_{xx}^{IG} - I_{yy}^{g}\cos(2\delta_{IG}) \\ + 2\,I_{zz}^{g}\cos^2\delta_{IG}] \\ \hline -I_{yy}^{g}\,\omega_g\cos\delta_{IG} - (I_{yy}^{g} - I_{zz}^{g})\,\dot{\delta}_{OG}\sin(2\delta_{IG}) \end{bmatrix}^{T} T_{BGB}\,\vec{\omega}_B$$

$$+ C_{IG}\,\dot{\delta}_{IG} - \mathcal{M}_{\delta_{IG}} = 0 \quad . \tag{4-84}$$

In equations (4-83) and (4-84), as in certain other paragraphs, neither subscript i nor superscript i has been attached to any symbol relating to a 2DOF CMG.

Appealing again to equation (4-1) with $q = \theta_i$ and manipulating in the light of the definitions and relations pertaining to flexible appendage i, it is found that the coordinate θ_i must satisfy

$$
\begin{aligned}
& I^i_{xx}\,\ddot{\theta}_i + C_{\theta_i}\,\dot{\theta}_i + K_{\theta_i}\,\theta_i + \vec{i}_i^{\,T}\,\{I^i\,\tilde{T}_i\,\dot{\vec{\omega}}_B + (\tilde{T}_i\,\vec{\omega}_B)\times(I^i\,\tilde{T}_i\,\vec{\omega}_B)\} \\
& + \vec{i}_i^{\,T}\left\{\sum_{j=1}^{N_i}\ddot{\eta}^i_j\,\vec{C}^i_{oj} + \sum_{j=1}^{N_i}\sum_{K=1}^{N_i}\eta^i_j\,\ddot{\eta}^i_K\,\vec{C}^i_{jK}\right\} \\
& - 2\,\vec{i}_i^{\,T}\left\{\sum_{j=1}^{N_i}\dot{\eta}^i_j\,\mathcal{J}^i_{rj} + \sum_{j=1}^{N_i}\sum_{K=1}^{N_i}\eta^i_j\,\dot{\eta}^i_K\,\mathcal{J}^i_{jK}\right\}\tilde{T}_i\,\vec{\omega}_B \\
& - 2\,\dot{\theta}_i\,\vec{i}_i^{\,T}\left\{\sum_{j=1}^{N_i}\dot{\eta}^i_j\,\mathcal{J}^i_{rj} + \sum_{K=1}^{N_i}\sum_{j=1}^{N_i}\eta^i_K\,\dot{\eta}^i_j\,\mathcal{J}^i_{Kj}\right\}\vec{i}_i \\
& + \vec{i}_i^{\,T}\left\{\vec{\ell}_i^{\,(0)} + \Psi^{(i)}\,\vec{\eta}^{(i)}\right\}\times\left\{m_i\,\tilde{T}_i\left[T\left(\ddot{\vec{R}}_s + \frac{\mu_M\,\vec{R}_s^{\,(MOON)}}{|\vec{R}_s^{\,(MOON)}|^3} + \frac{\mu_s\,\vec{R}_s^{\,(SUN)}}{|\vec{R}_s^{\,(SUN)}|^3}\right.\right.\right. \\
& \left.\left.\left. - \frac{\mu_M(\vec{R}_s^{\,(MOON)} - \vec{R}_s^{\,CM_i})}{|\vec{R}_s^{\,(MOON)} - \vec{R}_s^{\,CM_i}|^3} - \frac{\mu_s(\vec{R}_s^{\,(SUN)} - \vec{R}_s^{\,CM_i})}{|\vec{R}_s^{\,(SUN)} - \vec{R}_s^{\,CM_i}|^3}\right) + (\dot{\Omega}^T + \Omega^{T^2})\,\tilde{\vec{r}}_i\right]\right\} \\
& - \vec{i}_i^{\,T}\,\{\vec{M}_{g_i} + [\vec{\ell}_i^{\,(0)} + \Psi^{(i)}\,\vec{\eta}^{(i)}]\times\vec{F}_{gi}\} - \vec{i}_i^{\,T}\int_{(A_i)}\left[\vec{r}_i + \sum_{j=1}^{N_i}\eta^i_j\,\vec{\varphi}_j^{\,(i)}\right]\times\tilde{T}_i\,d\vec{F}_{SOLAR} \\
& - \vec{i}_i^{\,T}\int_{(A_i)}\left[\vec{r}_i + \sum_{j=1}^{N_i}\eta^i_j\,\vec{\varphi}_j^{\,(i)}\right]\times\tilde{T}_i\,d\vec{F}_{AERO} - \mathcal{M}_{\theta_i} = 0\ , \quad i = 1, \ldots NA\ .
\end{aligned}
\tag{4-85}
$$

In equation (4-85), the symbols $\vec{F}_{gi}$ and $\vec{M}_{gi}$ denote, respectively, the force and moment exerted by the Earth's gravity field upon flexible appendage i, the reference point for $\vec{M}_{gi}$ being the instantaneous CM of the appendage (in its deformed state). The vectors $\vec{F}_{gi}$ and $\vec{M}_{gi}$ are given by equations (A-30) and (A-31) in Appendix A. Expressions for the differentials $d\vec{F}_{AERO}$ and $d\vec{F}_{SOLAR}$ are available from Appendices B and C. The subscript A_i on the integral signs indicates (as one should expect) that the integration extends over the surface of appendage i. The surface integrals are, in general, amenable only to numerical methods. On observing the absence of time derivatives of $\tilde{\vec{r}}_i$, it should be recalled that $\tilde{\vec{r}}_i$ is a constant vector on the $\tilde{B}$-frame to imply that $\dot{\tilde{\vec{r}}}_i = \vec{0} = \ddot{\tilde{\vec{r}}}_i$ and, hence, that $(d^2/dt^2)_S\, \tilde{\vec{r}}_i = (\dot{\Omega}^T + \Omega^{T^2})\, \tilde{\vec{r}}_i$. Equation (E-1) of Appendix E provides an expression for the inertia matrix I^i in terms of that of the undeformed appendage and the mode shape functions and generalized bending displacement coordinates pertinent to the appendage. By equation (E-2), one can compute the matrix $\square^i$ necessary to the computation of both $\vec{F}_{gi}$ and $\vec{M}_{gi}$. (The terms reflecting the dependence of $\vec{F}_{gi}$ on $\square^i$ can probably be safely neglected, however, not all terms showing the dependence of $\vec{M}_{gi}$ on $\square^i$ can be discarded.)

To find the j^{th} bending equation associated with flexible appendage i, one invokes equation (4-1) with q equal to the modal coordinate η^i_j. The result of much manipulation is

$$
\begin{aligned}
& M^i_j\,(\ddot{\eta}^i_j + 2\,\zeta^i_j\,\omega^i_j\,\dot{\eta}^i_j + \omega^{i^2}_j\,\eta^i_j) + 2\,(\vec{\omega}^T_B\;\tilde{T}^T_i + \dot{\theta}_i\,\vec{i}^T_i) \sum_{K=1}^{N_i} \vec{C}^i_{Kj}\,\dot{\eta}^i_K \\
& + \dot{\vec{\omega}}^T_B\;\tilde{T}^T_i \left(\vec{C}^i_{oj} + \sum_{K=1}^{N_i} \eta^i_K\,\vec{C}^i_{Kj}\right) + \vec{\omega}^T_B\,\tilde{T}^T_i \left(\mathcal{J}^i_{jr} + \sum_{K=1}^{N_i} \mathcal{J}^i_{jK}\,\eta^i_K\right) \tilde{T}_i\,\vec{\omega}_B \\
& + \ddot{\theta}_i\,\vec{i}^T_i \left(\vec{C}^i_{oj} + \sum_{K=1}^{N_i} \eta^i_K\,\vec{C}^i_{Kj}\right) + \dot{\theta}^2_i\,\vec{i}^T_i \left(\mathcal{J}^i_{jr} + \sum_{K=1}^{N_i} \mathcal{J}^i_{jK}\,\eta^i_K\right) \vec{i}_i \\
& + 2\,\dot{\theta}_i\,\vec{\omega}^T_B\;\tilde{T}^T_i \left(\mathcal{J}^i_{jr} + \sum_{K=1}^{N_i} \mathcal{J}^i_{jK}\,\eta^i_K\right) \vec{i}_i + m_i \left[T\,\ddot{\tilde{\vec{R}}}_s + (\dot{\Omega}^T + \Omega^{T^2})\,\tilde{\vec{r}}_i\right]^T \tilde{T}^T_i\;\vec{\psi}^{(i)}_j \\
& = Q^{(G)}_{\eta^i_j} + Q^{(G,MOON)}_{\eta^i_j} + Q^{(G,SUN)}_{\eta^i_j} + Q^{(SOLAR)}_{\eta^i_j} + Q^{(AERO)}_{\eta^i_j}\;, \qquad (4\text{-}86)
\end{aligned}
$$

$$
j = 1, \ldots N_i \quad , \quad i = 1, \ldots NA \quad .
$$

Definitions of the symbols $\vec{C}^i_{oj}$, $\vec{C}^i_{Kj}$, $\mathcal{J}^i_{jr}$, and $\mathcal{J}^i_{jK}$ will be found in Appendix F.

Of the several terms in the right member of equation (4-86), the generalized force $Q^{(G)}_{\eta^i_j}$, attributed to the Earth's gravity field, is of particular interest. In fact, the primary objective of the assignment out of which this paper grew was to determine the effect of gravity on bending. The author has made the following approximation to $Q^{(G)}_{\eta^i_j}$.

$$
\begin{aligned}
Q^{(G)}_{\eta^i_j} \approx\ & m_i\, \vec{\psi}_j^{(i)T}\, \tilde{T}_i\, T\, [\alpha_{oi}]^T_{(3)}\, [-\delta_{oi}]^T_{(2)}\, \vec{A}_g\, (R_{oi}, \lambda_{oi}, \delta_{oi}) \\
& + \left\{ \frac{-\mu_E}{R_{oi}^3} + \frac{3\,\mu_E\, a^2\, J_{2o}\, (5 \sin^2 \delta_{oi} - 1)}{2\, R_{oi}^5} \right\} \left\{ M^i_j\, \eta^i_j + D^i_{oj} - m_i\, \vec{\psi}_j^{(i)T} \left(\vec{\ell}_i^{(o)} + \psi^{(i)} \vec{\eta}^{(i)} \right) \right\} \\
& + \left\{ \frac{3\mu_E}{R_{oi}^3} + \frac{15\,\mu_E\, a^2\, J_{2o}\, (1 - 7 \sin^2 \delta_{oi})}{2\, R_{oi}^5} \right\} \vec{u}^T_{Roi} \left\{ D^i_{rj} + \sum_{K=1}^{N_i} \eta^i_K\, D^i_{Kj} \right. \\
& \qquad \left. - m_i \left(\vec{\ell}_i^{(o)} + \psi^{(i)}\, \vec{\eta}^{(i)} \right) \vec{\psi}_j^{(i)T} \right\} \vec{u}_{Roi} \\
& - \frac{3\,\mu_E\, a^2\, J_{2o}}{R_{oi}^5}\, \vec{n}_i^T \left\{ D^i_{rj} + \sum_{K=1}^{N_i} \eta^i_K\, D^i_{Kj} - m_i \left(\vec{\ell}_i^{(o)} + \psi^{(i)}\, \vec{\eta}^{(i)} \right) \vec{\psi}_j^{(i)T} \right\} \vec{n}_i \\
& + \frac{15\,\mu_E\, a^2\, J_{2o} \sin \delta_{oi}}{R_{oi}^5}\, \vec{n}_i^T \left\{ D^i_{rj} + \sum_{K=1}^{N_i} \eta^i_K\, D^i_{Kj} - m_i \left(\vec{\ell}_i^{(o)} + \psi^{(i)}\, \vec{\eta}^{(i)} \right) \vec{\psi}_j^{(i)T} \right. \\
& \qquad \left. + \left[D^i_{rj} + \sum_{K=1}^{N_i} \eta^i_K\, D^i_{Kj} - m_i \left(\vec{\ell}_i^{(o)} + \psi^{(i)}\, \eta^{(i)} \right) \vec{\psi}_j^{(i)T} \right]^T \right\} \vec{u}_{Roi} \\
& + \frac{3\mu_E}{R_{oi}^4} \left\{ D^i_{jrr} + \sum_{K=1}^{N_i} \eta^i_K \left(D^i_{jrK} + D^i_{jKr} \right) + \sum_{K=1}^{N_i} \sum_{\ell=1}^{N_i} \eta^i_K\, \eta^i_\ell\, D^i_{jK\ell} \right. \\
& \qquad \left. + \frac{1}{2} \left[D^i_{rrj} + \sum_{K=1}^{N_i} \eta^i_K \left(D^i_{Krj} + D^i_{rKj} \right) + \sum_{K=1}^{N_i} \sum_{\ell=1}^{N_i} \eta^i_K\, \eta^i_\ell\, D^i_{\ell Kj} \right] \right\} \vec{u}_{Roi}
\end{aligned}
\tag{4-87}
$$

(4-87)
(Continued)

$$- \frac{3\mu_E}{R_{oi}^4} (\vec{\ell}_i^{(o)} + \psi^{(i)} \vec{\eta}^{(i)})^T \left\{ (M_j^i \eta_j^i + D_{oj}^i) \begin{bmatrix} 1. & 0 & 0 \\ 0 & 1. & 0 \\ 0 & 0 & 1. \end{bmatrix} + D_{jr}^i + \sum_{K=1}^{N_i} \eta_K^i (D_{Kj}^i + D_{jK}^i) \right.$$

$$- m_i \vec{\psi}_j^{(i)} (\vec{\ell}_i^{(o)} + \psi^{(i)} \vec{\eta}^{(i)})^T + \frac{1}{2} D_{rj}^i$$

$$\left. - \frac{m_i}{2} (\vec{\ell}_i^{(o)} + \psi^{(i)} \vec{\eta}^{(i)}) \vec{\psi}_j^{(i)T} \right\} \vec{u}_{Roi} \quad . \qquad (4\text{-}87)$$

(Concluded)

By definitions given elsewhere (Sections 2 and 3 and Appendix A), the product

$$\tilde{T}_i \; T \; [\alpha_{oi}]_{(3)}^T \; [-\delta_{oi}]_{(2)}^T \; \vec{A}_g \; (R_{oi}, \lambda_{oi}, \delta_{oi})$$

which appears as a factor of the first term in the right member of equation (4-87) should be recognized as the i-resolution of the gravitational acceleration due to the Earth at the point with spherical coordinates $(R_{oi}, \lambda_{oi}, \delta_{oi})$ referred to the E-frame, that point being the CM of flexible appendage i. The reader's attention is here called to that paragraph of appendix A which contains equation (A-30). The dominant term of the several terms comprising the expression

$$m_i \vec{\psi}_j^{(i)T} \; \tilde{T}_i \; T \; [\alpha_{oi}]_{(3)}^T \; [-\delta_{oi}]_{(2)}^T \; \vec{A}_g \; (R_{oi}, \lambda_{oi}, \delta_{oi}) \qquad (4\text{-}88)$$

is

$$\frac{-\mu_E m_i}{R_{oi}^2} \; \vec{\psi}_j^{(i)T} \; \vec{u}_{Roi}$$

as equations (A-3), (A-4), and (A-5) clearly show. All terms through those in R_{oi}^{-5} were not retained as the author first intended. Many were dropped because they could not be expressed as the sum of products of functions of time and integrals whose integrands are independent of time. Still others were discarded because of being either too unwieldy or of questionable importance. It is the author's contention that the approximation (4-87) is more than adequate for any practical application, and in fact, that most people would neglect all terms with the exception of expression (4-88) or its dominant part. Retention of only the terms belonging to (4-88) is equivalent to ignoring the variation of the gravity field over the appendage and assuming its magnitude and direction at all points of the appendage to be the same as that at the appendage CM. Definitions of the D's with subscripts and superscripts in equation (4-87) are among those of the time independent integrals in Appendix F.

The gravitational fields of the moon and Sun, ignoring their variation over appendage i, make the following contributions to the j^{th} bending equation.

$$Q_{\eta_j^i}^{(G,MOON)} \approx \vec{\psi}_j^{(i)^T} \left\{ m_i \, \mu_M \, \tilde{T}_i \, T \left(\frac{\vec{R}_s^{(MOON)} - \vec{R}_s^{CM_i}}{|\vec{R}_s^{(MOON)} - \vec{R}_s^{CM_i}|^3} - \frac{\vec{R}_s^{(MOON)}}{|\vec{R}_s^{(MOON)}|^3} \right) \right\}, \quad (4\text{-}89)$$

$$Q_{\eta_j^i}^{(G,SUN)} \approx \vec{\psi}_j^{(i)^T} \left\{ m_i \, \mu_s \, \tilde{T}_i \, T \left(\frac{\vec{R}_s^{(SUN)} - \vec{R}_s^{CM_i}}{|\vec{R}_s^{(SUN)} - \vec{R}_s^{CM_i}|^3} - \frac{\vec{R}_s^{(SUN)}}{|\vec{R}_s^{(SUN)}|^3} \right) \right\} . \quad (4\text{-}90)$$

In writing the mathematical definitions of the generalized forces $Q_{\eta_j^i}^{(SOLAR)}$ and $Q_{\eta_j^i}^{(AERO)}$, one can go only as far as the integral expressions.

$$Q_{\eta_j^i}^{(SOLAR)} = \int_{(A_i)} \vec{\varphi}_j^{(i)^T} \, \tilde{T}_i \, d\vec{F}_{SOLAR} \quad (4\text{-}91)$$

and

$$Q_{\eta_j^i}^{(AERO)} = \int_{(A_i)} \vec{\varphi}_j^{(i)^T} \, \tilde{T}_i \, d\vec{F}_{AERO} \quad , \quad (4\text{-}92)$$

the differentials $d\vec{F}_{AERO}$ and $d\vec{F}_{SOLAR}$ (here supposed having the B-resolution) being those defined in Appendices B and C. Analytical evaluation of the integrals in (4-91) and (4-92), which are both geometry dependent and time dependent, is, in general, impossible.

With slight modifications, the equations of motion in the preceding paragraphs can be made applicable to the configuration of Figure 2 provided the boom is regarded as rigid. The necessity of the addition of a few terms to certain of those equations is a consequence of allowing each solar panel an additional rotational degree of freedom, that being the freedom to rotate about an axis parallel to the axis of the boom to which it is attached. The symbol φ_i will here denote the angle through which the solar panel designated "flexible appendage i" in Figure 2 rotates about an axis parallel to the boom axis, which axis is parallel to the y_i-axis when $\theta_i = \varphi_i = 0$. Introduction of the coordinate φ_i naturally subjects the expressions for $\mathcal{J}_i$, $\vec{\tilde{\omega}}_i$ and $\tilde{\Omega}_i^T$ of Section 3 to change, the revised expressions being*

$$\mathcal{J}_i = [\theta_i]_{(1)} [\varphi_i]_{(2)}$$

$$\vec{\tilde{\omega}}_i = \dot{\theta}_i \vec{i}_i + \dot{\varphi}_i (\cos \theta_i \vec{j}_i - \sin \theta_i \vec{k}_i)$$

$$\tilde{\Omega}_i^T = \dot{\varphi}_i \begin{bmatrix} 0 & \sin \theta_i & \cos \theta_i \\ -\sin \theta_i & 0 & 0 \\ -\cos \theta_i & 0 & 0 \end{bmatrix} + \dot{\theta}_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1. \\ 0 & 1. & 0 \end{bmatrix} .$$

The rotation matrix $\tilde{T}_i$ is still defined by $\tilde{T}_i = \mathcal{J}_i T_i$, it being understood now that $T_i = [T(B\rightarrow i)]_{\theta_i = \varphi_i = 0}$, and the relations (3-6) and (3-8) continue to hold. Equation (4-73), the moment equation, remains valid as it stands. As for the other equations of motion, the appearance of only (4-53), (4-85), and (4-86) need to be altered, while (4-80) and (4-81) should be deleted since there are no swiveled engines (the RCS thrusters are fixed relative to the boom). To account for the elastic restoring force $-K_{\varphi_i} \varphi_i$ and viscous damping force $-C_{\varphi_i} \dot{\varphi}_i$, both of which resist a change in φ_i, one should add to equation (4-14) the expression $(1/2) \sum_i K_{\varphi_i} \varphi_i^2$ and to (4-15) the expression $(1/2) \sum_i C_{\varphi_i} \dot{\varphi}_i^2$, the index i having the appropriate range.

*As evident in the expression for $\mathcal{J}_i$, it is here supposed that passage <u>from</u> the "null" orientation of the i-frame <u>to</u> its instantaneous orientation is effected by a 2, 1 sequence through φ_i and θ_i.

After modification, equations (4-53), (4-85), and (4-86) read as (4-53)', (4-85)', and (4-86)', respectively.

$$m\,T\,\ddot{\vec{\tilde{R}}}_s = -m\,(\dot{\tilde{\Omega}}^T + \tilde{\Omega}^{T^2})\,\vec{\tilde{r}}_{CM} + \vec{F}_{gB} + \vec{F}_{gB}^{(SUN)} + \vec{F}_{gB}^{(MOON)} + \vec{F}_T$$

$$+ \vec{F}_{AERO} + \vec{F}_{SOLAR} + \sum_{i=1}^{NSE} m_{Ei}\,\ell_{Ei}\,(\ddot{\vec{\Lambda}}_{Ei} + 2\,\Omega^T\,\dot{\vec{\Lambda}}_{Ei})$$

$$- \sum_{i=1}^{NP} m_{pi}\,(\ddot{\xi}_{pi}\,\vec{\Lambda}_{pi} + 2\,\dot{\xi}_{pi}\,\Omega^T\,\vec{\Lambda}_{pi})$$

$$- \sum_{i=1}^{NA} m_i \left\{\tilde{T}_i^T\left[\Psi^{(i)}\,\ddot{\vec{\eta}}^{(i)} + 2\,\tilde{\Omega}_i^T\,\Psi^{(i)}\,\dot{\vec{\eta}}^{(i)} + (\dot{\tilde{\Omega}}_i^T + \tilde{\Omega}_i^{T^2})\left(\vec{\ell}_i^{(o)} + \Psi^{(i)}\,\vec{\eta}^{(i)}\right)\right]\right.$$

$$\left. + 2\,\Omega^T\,\tilde{T}_i^T\left[\Psi^{(i)}\,\dot{\vec{\eta}}^{(i)} + \tilde{\Omega}_i^T\,(\vec{\ell}_i^{(o)} + \Psi^{(i)}\,\vec{\eta}^{(i)})\right]\right\}, \qquad (4\text{-}53)'$$

$$I_{xx}^i\,\ddot{\theta}_i + C_{\theta_i}\,\dot{\theta}_i + K_{\theta_i}\,\theta_i + \vec{i}_i^T\left\{I^i\,\tilde{T}_i\,\dot{\vec{\omega}}_B + (\tilde{T}_i\,\vec{\omega}_B)\times(I^i\,\tilde{T}_i\,\vec{\omega}_B)\right\}$$

$$+ \vec{i}_i^T\left\{\sum_{j=1}^{N_i}\ddot{\eta}_j^i\,\vec{C}_{oj}^i + \sum_{j=1}^{N_i}\sum_{K=1}^{N_i}\eta_j^i\,\ddot{\eta}_K^i\,\vec{C}_{jK}^i\right\}$$

$$- 2\,\vec{i}_i^T\left\{\sum_{j=1}^{N_i}\dot{\eta}_j^i\,\mathcal{J}_{rj}^i + \sum_{j=1}^{N_i}\sum_{K=1}^{N_i}\eta_j^i\,\dot{\eta}_K^i\,\mathcal{J}_{jK}^i\right\}\tilde{T}_i\,\vec{\omega}_B$$

$$-2\dot{\theta}_i\,\vec{i}_i^T\left\{\sum_{j=1}^{N_i}\dot{\eta}_j^i\,\mathcal{J}_{rj}^i + \sum_{K=1}^{N_i}\sum_{j=1}^{N_i}\eta_K^i\,\dot{\eta}_j^i\,\mathcal{J}_{Kj}^i\right\}\vec{i}_i$$

$$+ \vec{i}_i^T\,\{\vec{\ell}_i^{(o)} + \Psi^{(i)}\,\vec{\eta}^{(i)}\}\times\left\{m_i\,\tilde{T}_i\left[T\left(\ddot{\vec{\tilde{R}}}_s + \frac{\mu_M\,\vec{R}_s^{(MOON)}}{|\vec{R}_s^{(MOON)}|^3} + \frac{\mu_S\,\vec{R}_s^{(SUN)}}{|\vec{R}_s^{(SUN)}|^3}\right.\right.\right.$$

$$- \frac{\mu_M (\vec{R}_s^{(MOON)} - \vec{R}_s^{CM_i})}{|\vec{R}_s^{(MOON)} - \vec{R}_s^{CM_i}|^3} - \frac{\mu_s (\vec{R}_s^{(SUN)} - \vec{R}_s^{CM_i})}{|\vec{R}_s^{(SUN)} - \vec{R}_s^{CM_i}|^3} \Bigg) + (\dot{\tilde{\Omega}}^T + \tilde{\Omega}^{T^2}) \tilde{\vec{r}}_i \Bigg] \Bigg\}$$

$$- \vec{i}_i^T \Bigg\{ \vec{M}_{gi} + [\vec{\ell}_i^{(o)} + \psi^{(i)} \vec{\eta}^{(i)}] \times \vec{F}_{gi} + \int_{(A_i)} (\vec{r}_i + \Phi^{(i)} \vec{\eta}^{(i)}) \times \tilde{T}_i \, d\vec{F}_{SOLAR}$$

$$+ \int_{(A_i)} (\vec{r}_i + \Phi^{(i)} \vec{\eta}^{(i)}) \times \tilde{T}_i \, d\vec{F}_{AERO} \Bigg\}$$

$$+ (I_{xy}^i \cos \theta_i - I_{xz}^i \sin \theta_i) \ddot{\varphi}_i + \dot{\theta}_i \dot{\varphi}_i (I_{xy}^i \sin \theta_i + I_{xz}^i \cos \theta_i)$$

$$+ \dot{\varphi}_i^2 [I_{yz}^i (\cos^2 \theta_i - \sin^2 \theta_i) + (I_{yy}^i - I_{zz}^i) \sin \theta_i \cos \theta_i]$$

$$+ 2 \dot{\varphi}_i [0, \sin \theta_i, \cos \theta_i] \{I^i - \tfrac{1}{2} \mathcal{T}_r (I^i) I_{(3x3)}\} \tilde{T}_i \vec{\omega}_B$$

$$- 2 \dot{\varphi}_i \vec{i}_i^T \left\{ \sum_{j=1}^{N_i} \dot{\eta}_j^i \mathcal{J}_{rj}^i + \sum_{j=1}^{N_i} \sum_{K=1}^{N_i} \eta_j^i \dot{\eta}_K^i \mathcal{J}_{jK}^i \right\} \begin{bmatrix} 0 \\ \cos \theta_i \\ -\sin \theta_i \end{bmatrix} - \mathcal{M}_{\theta_i} = 0 \quad , \qquad \text{(4-85)' (Conc.)}$$

$$M_j^i (\ddot{\eta}_j^i + 2 \zeta_j^i \omega_j^i \dot{\eta}_j^i + \omega_j^{i^2} \eta_j^i) + 2 (\vec{\omega}_B^T \tilde{T}_i^T + \tilde{\vec{\omega}}_i^T) \sum_{K=1}^{N_i} \dot{\eta}_K^i \vec{C}_{Kj}^i + \dot{\vec{\omega}}_B^T \tilde{T}_i^T (\vec{C}_{oj}^i$$

$$+ \sum_{K=1}^{N_i} \eta_K^i \vec{C}_{Kj}^i) + \vec{\omega}_B^T \tilde{T}_i^T (\mathcal{J}_{jr}^i + \sum_{K=1}^{N_i} \eta_K^i \mathcal{J}_{jK}^i) \tilde{T}_i \vec{\omega}_B + \dot{\tilde{\vec{\omega}}}_i^T (\vec{C}_{oj}^i + \sum_{K=1}^{N_i} \eta_K^i \vec{C}_{Kj}^i)$$

$$+ \tilde{\vec{\omega}}_i^T (\mathcal{J}_{jr}^i + \sum_{K=1}^{N_i} \eta_K^i \mathcal{J}_{jK}^i) \tilde{\vec{\omega}}_i + 2 \vec{\omega}_B^T \tilde{T}_i^T (\mathcal{J}_{jr}^i + \sum_{K=1}^{N_i} \eta_K^i \mathcal{J}_{jK}^i) \tilde{\vec{\omega}}_i + m_i [T \ddot{\tilde{\vec{R}}}_s$$

$$+ (\dot{\tilde{\Omega}}^T + \tilde{\Omega}^{T^2}) \tilde{\vec{r}}_i]^T \tilde{T}_i^T \vec{\psi}_j^i = Q_{\eta_j^i}^{(G)} + Q_{\eta_j^i}^{(G,MOON)} + Q_{\eta_j^i}^{(G,SUN)} + Q_{\eta_j^i}^{(SOLAR)} + Q_{\eta_j^i}^{(AERO)}$$

$$j = 1, \ldots N_i \quad , \quad i = 1, \ldots NA \quad . \qquad \text{(4-86)'}$$

At least one more remark regarding equation (4-86)' is in order, that being that it can be made to apply to a flexible appendage which is denied rotational degrees of freedom (such as θ_i and φ_i) by merely setting $\vec{\omega}_i$ and $\dot{\vec{\omega}}_i$ to $\vec{0}$ and replacing $\tilde{T}_i$ by T_i. When θ_i and φ_i are identically zero, the rotation matrix $\mathcal{J}_i$ becomes the 3x3 identity matrix and $\tilde{T}_i$ is the same as T_i. The allied equations (4-87) through (4-92) apply without change provided that $\tilde{T}_i$ is replaced by T_i when $\theta_i \equiv 0$ and $\varphi_i \equiv 0$.

The system coordinate φ_i must satisfy

$$
\begin{aligned}
&(I^i_{yy} \cos^2 \theta_i + I^i_{zz} \sin^2 \theta_i - 2 I^i_{yz} \sin \theta_i \cos \theta_i) \ddot{\varphi}_i + C_{\varphi_i} \dot{\varphi}_i + K_{\varphi_i} \varphi_i \\
&\quad + (I^i_{xy} \cos \theta_i - I^i_{xz} \sin \theta_i) \ddot{\theta}_i + 2 \dot{\theta}_i \dot{\varphi}_i [I^i_{yz} (\sin^2 \theta_i - \cos^2 \theta_i) \\
&\quad + (I^i_{zz} - I^i_{yy}) \sin \theta_i \cos \theta_i] - 2 \dot{\theta}_i [0, \sin \theta_i, \cos \theta_i] \{I^i - \tfrac{1}{2} \mathcal{T}_r (I^i) I_{(3x3)}\} \tilde{T}_i \vec{\omega}_B \\
&\quad - \dot{\theta}_i^2 (I^i_{xy} \sin \theta_i + I^i_{xz} \cos \theta_i) + [0, \cos \theta_i, -\sin \theta_i] \Bigg\{ I^i \tilde{T}_i \dot{\vec{\omega}}_B + (\tilde{T}_i \vec{\omega}_B) \\
&\quad \times I^i \tilde{T}_i \vec{\omega}_B + \sum_{j=1}^{N_i} \ddot{\eta}^i_j \vec{C}^i_{oj} + \sum_{j=1}^{N_i} \sum_{K=1}^{N_i} \eta^i_K \ddot{\eta}^i_j \vec{C}^i_{Kj} - 2 \Bigg(\sum_{j=1}^{N_i} \dot{\eta}^i_j \mathcal{J}^i_{rj} \\
&\quad + \sum_{j=1}^{N_i} \sum_{K=1}^{N_i} \eta^i_j \dot{\eta}^i_K \mathcal{J}^i_{jK} \Bigg) (\tilde{T}_i \vec{\omega}_B + \vec{\tilde{\omega}}_i) - \vec{M}_{gi} - (\vec{\ell}_i^{(o)} + \Psi^{(i)} \vec{\eta}^{(i)}) \times \vec{F}_{gi} \\
&\quad - \int_{(A_i)} (\vec{r}_i + \Phi^{(i)} \vec{\eta}^{(i)}) \times \tilde{T}_i \, d\vec{F}_{SOLAR} - \int_{(A_i)} (\vec{r}_i + \Phi^{(i)} \vec{\eta}^{(i)}) \times \tilde{T}_i \, d\vec{F}_{AERO} \\
&\quad + (\vec{\ell}_i^{(o)} + \Psi^{(i)} \vec{\eta}^{(i)}) \times \Bigg[m_i \tilde{T}_i \Bigg(T \Bigg(\ddot{\vec{R}}_s + \frac{\mu_M \vec{R}_s^{(MOON)}}{|\vec{R}_s^{(MOON)}|^3} + \frac{\mu_S \vec{R}_s^{(SUN)}}{|\vec{R}_s^{(SUN)}|^3} \Bigg) \\
&\quad - T \Bigg(\frac{\mu_M (\vec{R}_s^{(MOON)} - \vec{R}_s^{CM_i})}{|\vec{R}_s^{(MOON)} - \vec{R}_s^{CM_i}|^3} + \frac{\mu_S (\vec{R}_s^{(SUN)} - \vec{R}_s^{CM_i})}{|\vec{R}_s^{(SUN)} - \vec{R}_s^{CM_i}|^3} \Bigg) + (\dot{\Omega}^T + \Omega^{T^2}) \vec{r}_i \Bigg) \Bigg] \Bigg\} = \mathcal{M}_{\varphi_i} ,
\end{aligned}
$$

(4-93)

where the symbol $\mathcal{M}_{\varphi_i}$ denotes the control torque applied to induce the appropriate change in φ_i.

It is important to note that equations (4-85)' and (4-93) were written for an appendage similar to that in Figure 1, but having two rotational degrees of freedom, and on the assumption that the solar panels designated "flexible appendage i" and "flexible appendage i + 1" in Figure 2 may move independently of each other, and further, that these equations hold whether both the θ_i motion and the φ_i motion are executed simultaneously or θ_i is held constant while φ_i changes and φ_i is held constant while θ_i changes. Obvious simplifications are possible if either θ_i or φ_i is constant while the other changes (in such a case, equation (4-85)' becomes (4-85)). If flexible appendages i and i + 1 of Figure 2 constitute a single unit and always move in unison (so that $\theta_{i+1} = -\theta_i$, $\varphi_{i+1} = -\varphi_i$), then equations (4-85)' and (4-93) apply to the combination (which should be relabeled "flexible appendage i") provided the symbols with subscript i or superscript i in those equations are pertinent to the combination.

Terms of (4-53)' and (4-73) which are impertinent to the configuration of Figure 2, such as those attributed to swiveled engines (not present on this station), must be deleted as should an entire equation corresponding to a non-existent system coordinate.

If a module, which may be regarded rigid but is joined to the central carrier by means of a flexible attachment, is not adequately modeled as a point mass with only one displacement d.o.f. relative to the carrier then, obviously, an equation such as (4-74) does not provide an adequate description of its motion. Permitting such a module to have three translational d.o.f's and three rotational d.o.f's relative to the carrier would require that six equations be appended to the system equations of motion, and further, that still more terms be added to equation (4-53), the system translational equation, and to (4-73), the system rotational equation. Nomenclature pertinent to this type of component follows.

N_{RB} = number of rigid components with 6 d.o.f.'s relative to carrier

m_{RB_i} = mass of the i^{th} rigid component with 6 d.o.f.'s relative to carrier

$\vec{r}_{RB_i}$ = position, referred to $\widetilde{B}$-frame, of the origin of $x_{RB_i}\ y_{RB_i}\ z_{RB_i}$ (RBi frame)

$x_{RB_i}\ y_{RB_i}\ z_{RB_i}$ – a right-handed rectangular frame, arbitrarily but sensibly oriented, at rest relative to the $\tilde{B}$-frame (and hence, at rest relative to the rigid central carrier), and with origin in the vicinity of the idealized attachment point (or points)

$x'_{RB_i}\ y'_{RB_i}\ z'_{RB_i}$ – the $(RB_i)'$-frame has origin at the CM of m_{RB_i} and is oriented as the RB_i-frame when m_{RB_i} has not undergone a rotation relative to the carrier. (The $(RB_i)'$ frame is at rest relative to m_{RB_i})

$\square^{RB_i}$ = the inertia matrix of m_{RB_i} referred to $x'_{RB_i}\ y'_{RB_i}\ z'_{RB_i}$ (context should make clear whether the symbol m_{RB_i} denotes the body whose mass is m_{RB_i} or the numerical value of that mass)

φ_{PRB_i}, φ_{YRB_i}, φ_{rRB_i} – Euler angles defining the orientation of the $(RB_i)'$ frame relative to the RB_i-frame

$T_{RB_i} = T\ (\tilde{B} \rightarrow RB_i)$ – a rotation matrix defining the transformation from the $\tilde{B}$ resolution to the RB_i resolution

$T'_{RB_i} = T(RBi \rightarrow (RB_i)')$ – the rotation matrix defining the transformation from the RBi resolution to the $(RB_i)'$ resolution

$T'_{RB_i} = [\varphi_{rRB_i}]_{(1)}\ [\varphi_{yRB_i}]_{(3)}\ [\varphi_{PRB_i}]_{(2)}$ for a 2,3,1 sequence through φ_{PRB_i}, φ_{yRB_i}, φ_{rRB_i} in passing from the RB_i orientation to the $(RB_i)'$ orientation

$\tilde{T}_{RB_i} = T'_{RB_i}\ T_{RB_i} = T\ (\tilde{B} \rightarrow (RB_i)')$

$\vec{\ell}^{(o)}_{RB_i}$ = position, vector referred to the RB_i frame, of the CM of m_{RB_i} (the origin of the $(RB_i)'$ frame) when m_{RB_i} has not experienced either translation or rotation relative to the central carrier

$\vec{\Delta}_{RB_i}$ = displacement, relative to the carrier, of the CM of m_{RB_i}. (This vector has the RB_i resolution)

$\vec{\omega}_{RB_i}$ = angular velocity of the $(RB_i)'$ frame relative to the RB_i frame (this vector has the $(RB_i)'$ resolution)

$$\vec{\omega}_{RB_i} \equiv \begin{bmatrix} \omega_{1RB_i} \\ \omega_{2RB_i} \\ \omega_{3RB_i} \end{bmatrix}_{(RB_i)'} = T_{\omega RB_i} \begin{bmatrix} \dot{\varphi}_{PRB_i} \\ \dot{\varphi}_{yRB_i} \\ \dot{\varphi}_{rRB_i} \end{bmatrix}$$

$$\dot{\vec{\omega}}_{RB_i} = \left(\frac{d}{dt}\right)_{(RB_i)'} \vec{\omega}_{RB_i}$$

$$T_{\omega RB_i} = \begin{bmatrix} \sin \varphi_{yRB_i} & , & 0 & , & 1. \\ \cos \varphi_{yRB_i} \cos \varphi_{rRB_i} & , & \sin \varphi_{rRB_i} & , & 0 \\ -\cos \varphi_{yRB_i} \sin \varphi_{rRB_i} & , & \cos \varphi_{rRB_i} & , & 0 \end{bmatrix}$$

(for a 2,3,1 sequence through φ_{PRB_i}, φ_{yRB_i}, φ_{rRB_i})

$$\Omega^T_{RB_i} = \begin{bmatrix} 0 & -\omega_{3RB_i} & \omega_{2RB_i} \\ \omega_{3RB_i} & 0 & -\omega_{1RB_i} \\ -\omega_{2RB_i} & \omega_{1RB_i} & 0 \end{bmatrix}$$

$$\dot{\tilde{T}}^T_{RB_i} = \frac{d}{dt} \tilde{T}^T_{RB_i} = \tilde{T}^T_{RB_i} \Omega^T_{RB_i}$$

$\Omega^T_{RB_i} \vec{v} \equiv \vec{\omega}_{RB_i} \times \vec{v}$ for any vector $\vec{v}$ having the $(RB_i)'$ resolution

$\vec{R}_S^{CMRB_i}$ = position vector of the CM of m_{RB_i} referred to the S-frame

$\vec{r}'_{RB_i}$ = position vector of a generic point of m_{RB_i} referred to x'_{RB_i} y'_{RB_i} z'_{RB_i}

A_{RB_i} = surface area of m_{RB_i}

$\vec{\omega}'_{RB_i} = \tilde{T}^T_{RB_i} \, \vec{\omega}_{RB_i} = \tilde{B}$ resolution of $\vec{\omega}_{RB_i}$ (also the B resolution)

$$\dot{\vec{\omega}}'_{RB_i} = \left(\frac{d}{dt}\right)_B \vec{\omega}'_{RB_i} = \tilde{T}^T_{RB_i} \, \dot{\vec{\omega}}_{RB_i}$$

$\vec{M}_{gRB_i}$ = torque on m_{RB_i} alone due to Earth's gravity field

$\vec{M}_{gRB_i}^{(SUN)}$ = torque on m_{RB_i} alone due to Sun's gravity field

$\vec{M}_{gRB_i}^{(MOON)}$ = torque on m_{RB_i} alone due to Moon's gravity field

$\vec{M}_{RB_i}^{(ATTACHMENT)}$ = sum of the damping moment and restoring moment resisting, through the attachment, any rotation of m_{RB_i} relative to the carrier .

The equation governing the translational motion (relative to the central carrier) of m_{RB_i} is found to be

$$m_{RBi} \, \ddot{\vec{\Delta}}_{RBi} + \begin{bmatrix} C_{xRBi} & 0 & 0 \\ 0 & C_{yRBi} & 0 \\ 0 & 0 & C_{zRBi} \end{bmatrix} \dot{\vec{\Delta}}_{RBi} + \begin{bmatrix} K_{xRBi} & 0 & 0 \\ 0 & K_{yRBi} & 0 \\ 0 & 0 & K_{zRBi} \end{bmatrix} \vec{\Delta}_{RBi}$$

$$+ m_{RBi} \, T_{RBi} \Bigg\{ 2 \, \Omega^T \, T^T_{RBi} \, \dot{\vec{\Delta}}_{RBi} + (\dot{\Omega}^T + \Omega^{T^2}) \, [\tilde{\vec{r}}_{RBi} + T^T_{RBi} \, (\vec{\ell}^{(o)}_{RBi} + \vec{\Delta}_{RBi})]$$

$$+ T \left[\ddot{\vec{R}}_s + \frac{\mu_M \, \vec{R}_s^{(MOON)}}{|\vec{R}_s^{(MOON)}|^3} - \frac{\mu_M \, (\vec{R}_s^{(MOON)} - \vec{R}_s^{CMRBi})}{|\vec{R}_s^{(MOON)} - \vec{R}_s^{CMRBi}|^3} + \frac{\mu_s \, \vec{R}_s^{(SUN)}}{|\vec{R}_s^{(SUN)}|^3} \right.$$

$$\left. - \frac{\mu_s \, (\vec{R}_s^{(SUN)} - \vec{R}_s^{CMRBi})}{|\vec{R}^{(SUN)} - \vec{R}_s^{CMRBi}|^3} \right] \Bigg\} - \vec{F}_{gRBi} - T_{RBi} \int_{(A_{RBi})} (d\vec{F}_{AERO})_{(B)}$$

$$- T_{RBi} \int_{(A_{RBi})} (d\vec{F}_{SOLAR})_{(B)} \approx \vec{0} \quad , \quad i = 1, \ldots N_{RB} \quad . \qquad (4\text{-}94)$$

Clearly evident in equation (4-94) is the assumption of viscous damping and linear restoring forces at the attachment. The symbol $\vec{F}_{gRBi}$ denotes the RBi resolution of the Earth's gravitational force on m_{RBi} and is obtainable by an equation analogous to (A-30). The procedure for finding $\vec{F}_{gRBi}$ should be clear after reading the paragraph containing equation (A-30). In fact, that portion of the paragraph starting with its second sentence can be made applicable to $\vec{F}_{gRBi}$ in the following way: replace i by RBi when i is used as a subscript or superscript; replace the expression "flexible appendage i" with the symbol m_{RBi} and delete the expression "in its deformed state;" delete also the remark within parentheses appearing as the last part of the final sentence of that paragraph; and ignore the references to Sections 2 and 3. In passing, one may observe that if m_{RBi} is regarded as a point mass (having a null inertia matrix and zero surface area), if $\vec{\ell}^{(o)}_{RBi} = \vec{0}$, if the two components Δ_{yRBi} and Δ_{zRBi} of the vector $\vec{\Delta}_{RBi} \equiv [\Delta_{xRBi}, \Delta_{yRBi}, \Delta_{zRBi}]^T$ are constrained to be zero, and if both members of equation (4-94) are premultiplied by $\vec{i}^{\,T}_{RBi} \equiv [1, 0, 0]_{RBi}$, the result is an equation completely equivalent to (4-74), in content, that is, which is as it should be.

The contribution of the relative motion of the m_{RBi}, $i = 1, \ldots N_{RB}$, to the "system" translational motion is realized by adding to the right member of equation (4-53) the expression (4-95) below, and the contribution to the "system" rotational motion is expressed by the addition of expression (4-96) to the right member of equation (4-73)

$$-\sum_{i=1}^{N_{RB}} m_{RBi}\, T^T_{RBi}\, \ddot{\vec{\Delta}}_{RBi} - 2\,\Omega^T \sum_{i=1}^{N_{RB}} m_{RBi}\, T^T_{RBi}\, \dot{\vec{\Delta}}_{RBi} \quad , \tag{4-95}$$

$$-\sum_{i=1}^{N_{RB}} m_{RBi} \{[\vec{\tilde{r}}_{RBi} - \vec{\tilde{r}}_{CM} + T^T_{RBi}(\vec{\ell}^{(o)}_{RBi} + \vec{\Delta}_{RBi})] \times [T^T_{RBi}\, \ddot{\vec{\Delta}}_{RBi}] + \Omega^T\, [(\vec{\tilde{r}}_{RBi} - \vec{\tilde{r}}_{CM}$$

$$+ T^T_{RBi}(\vec{\ell}^{(o)}_{RBi} + \vec{\Delta}_{RBi})) \times (T^T_{RBi}\, \dot{\vec{\Delta}}_{RBi})]\}$$

$$-\sum_{i=1}^{N_{RB}} \{\tilde{T}^T_{RBi}(\square^{RBi}\, \dot{\vec{\omega}}_{RBi} + \vec{\omega}_{RBi} \times \square^{RBi}\, \vec{\omega}_{RBi}) + \vec{\omega}_B \times (\tilde{T}^T_{RBi}\, \square^{RBi}\, \vec{\omega}_{RBi})\} \quad ,$$

$$\tag{4-96}$$

If equation (4-73.14) is designated the "system" rotational equation, then the expression (4-96) with $\vec{r}_{CM}$ deleted should be added to the right member of equation (4-73.14).

The second sum in expression (4-96) is equivalent to the expression (4-96A),

$$- \sum_{i=1}^{N_{RB}} \{\tilde{T}^{T}_{RBi} \square^{RBi} \tilde{T}_{RBi} \dot{\vec{\omega}}'_{RBi} + (\vec{\omega}_{B} + \vec{\omega}'_{RBi}) \times \tilde{T}^{T}_{RBi} \square^{RBi} \tilde{T}_{RBi} \vec{\omega}'_{RBi}\} \quad , \qquad (4\text{-}96A)$$

which expression one could rightly expect after an inspection of equation (4-73).

The rotational motion of m_{RBi} relative to the carrier is determined by

$$\begin{aligned} &\square^{RBi} (\dot{\vec{\omega}}_{RBi} + \tilde{T}_{RBi} \dot{\vec{\omega}}_{B} + \Omega_{RBi} \tilde{T}_{RBi} \vec{\omega}_{B}) + (\vec{\omega}_{RBi} + \tilde{T}_{RB_i} \vec{\omega}_{B}) \\ &\qquad \times \square^{RBi} (\vec{\omega}_{RBi} + \tilde{T}_{RBi} \vec{\omega}_{B}) - \vec{M}_{gRBi} - \int_{(A_{RBi})} \vec{r}'_{RBi} \\ &\qquad \times \tilde{T}_{RBi} (d\vec{F}_{SOLAR})_{B} - \int_{(A_{RBi})} \vec{r}'_{RBi} \times \tilde{T}_{RBi} (d\vec{F}_{AERO})_{B} \\ &\qquad - \vec{M}_{gRBi}^{(SUN)} - \vec{M}_{gRBi}^{(MOON)} - \vec{M}_{RBi}^{(ATTACHMENT)} = \vec{0} \quad , \\ &\qquad\qquad i = 1, \ldots N_{RB} \quad . \end{aligned} \qquad (4\text{-}97)$$

If the dimensions of m_{RBi} are not sufficiently large to warrant retention of the gravity terms, solar terms, and aerodynamic terms in equation (4-97), as is likely to be the case, then they may be ignored along with terms arising from certain other sources of excitation mentioned near the beginning of this section (see the paragraph following equation (4-1G)). Even if deemed important, the gravity terms due to Sun and Moon should be adequately approximated via equations similar to (4-55) and (4-56). Equations (4-55) and (4-56) are made directly applicable to the computation of $\vec{M}_{gRBI}^{(SUN)}$ and $\vec{M}_{gRBi}^{(MOON)}$ by merely replacing the subscript B by RBi, the symbol $\square$ by $\square^{RBi}$, and the superscript CM in the definitions of $\vec{R}_{SUN}$ and $\vec{R}_{MOON}$ by CMRBi. Just as $\vec{F}_{gRBi}$ may be approximated by an equation similar to (A-30), as explained above, so can $\vec{M}_{gRBi}$ be approximated by an equation similar to (A-31).

Not only would the presence of the m_{RBi}, $i = 1, \ldots N_{RB}$, require the additions to equations (4-53) and (4-73) as described above, but would result in slight change in the expressions for $\vec{\tilde{r}}_{CM}$, $\vec{\tilde{r}}_{CM}^{(o)}$, and $\square$ as well. To the expression within braces in equation (D-1), one should add

$$\sum_{i=1}^{N_{RB}} m_{RBi} \, T_{RBi}^{T} \, \vec{\Delta}_{RBi} \quad , \tag{4-98}$$

while to that within braces in equation (D-2), one should add

$$\sum_{i=1}^{N_{RB}} m_{RBi} \, (\vec{\tilde{r}}_{RBi} + T_{RBi}^{T} \, \vec{\ell}_{RBi}^{(o)}) \quad .$$

Obviously, the necessary modifications to (D-3) and (D-4) are made by adding the first time derivative of expression (4-98) to the expression within braces in equation (D-3) and the second time derivative of expression (4-98) to the expression within braces in equation (D-4). Equation (E-4), defining the system inertia matrix $\square$, would be modified by the addition to its right member of the expression (4-99).

$$\sum_{i=1}^{N_{RB}} \{\tilde{T}_{RBi}^{T} \, \square^{RBi} \, \tilde{T}_{RBi} + m_{RBi} \, \mathcal{S} \, (\vec{q}_{RBi}) \, \mathcal{S} \, (-\vec{q}_{RBi})\} \quad . \tag{4-99}$$

The vector $\vec{q}_{RBi}$, not defined in Appendix E, denotes the instantaneous position of the CM of m_{RBi} relative to the B-frame and is given by

$$\vec{q}_{RBi} = \vec{\tilde{r}}_{RBi} - \vec{\tilde{r}}_{CM} + T_{RBi}^{T} \, (\vec{\ell}_{RBi}^{(o)} + \vec{\Delta}_{RBi}) \quad .$$

The subsequent modification of equation (E-5) is effected by simply adding to its right member the first time derivative of expression (4-99).

No attempt will be made in this paper to assess the effect of liquid sloshing on vehicle motion since it is hardly possible that the author could add anything to the extensive literature on that subject. However, with regard to slosh, Reference 21 should be cited as one providing a collection of formulae pertinent to a variety of tank configurations plus an exhaustive list of technical papers on sloshing in both high-g and low-g environments. Due consideration cannot be given to slosh until tank geometry and mechanical analog of the liquid have been decided. Only then can one write the sloshing equations and add the appropriate terms to equations (4-53), (4-73), and (D-1). Introduction of the mechanical analog requires that the contribution of the fluid to the system inertia matrix $\square$ (see equation (E-4)) is understood to be the "effective" inertia matrix of the fluid. The structure of some slosh models is such that certain of the equations (4-74) or (4-76) may be designated as slosh mass equations.

V. REMARKS ON SIMULATION

To digitally simulate vehicle motion, such a simulation being nothing more than the numerical solution of the system of differential equations descriptive of that motion, one must first recast the underlying equations in a form suitable to programming, that is, a form amenable to the direct application of the selected numerical integration scheme. Use of the popular fourth order Runge-Kutta technique, for example, would require that the system of equations be put in its equivalent first order form.

Also necessary to the construction of a simulation program, as clearly indicated by the equations in Section IV, are many subsidiary relations, not the least important of which are the control equations. Linked to the control equations are expressions for approximating measurements made by onboard sensors. Equations for determining the output of the filter networks designed to filter those measurements should be implemented. It is hardly necessary to point out that the equations in the previous sections and the appendices are indicative of program input with the exception of such data as the time interval of interest, integration step size, etc.

The reader who is concerned about the existence and uniqueness of solutions is hereby advised to consult the literature on ordinary differential equations.

A proposed sequel to this paper is one wherein the equations of this paper are applied to one or more specific station configurations.

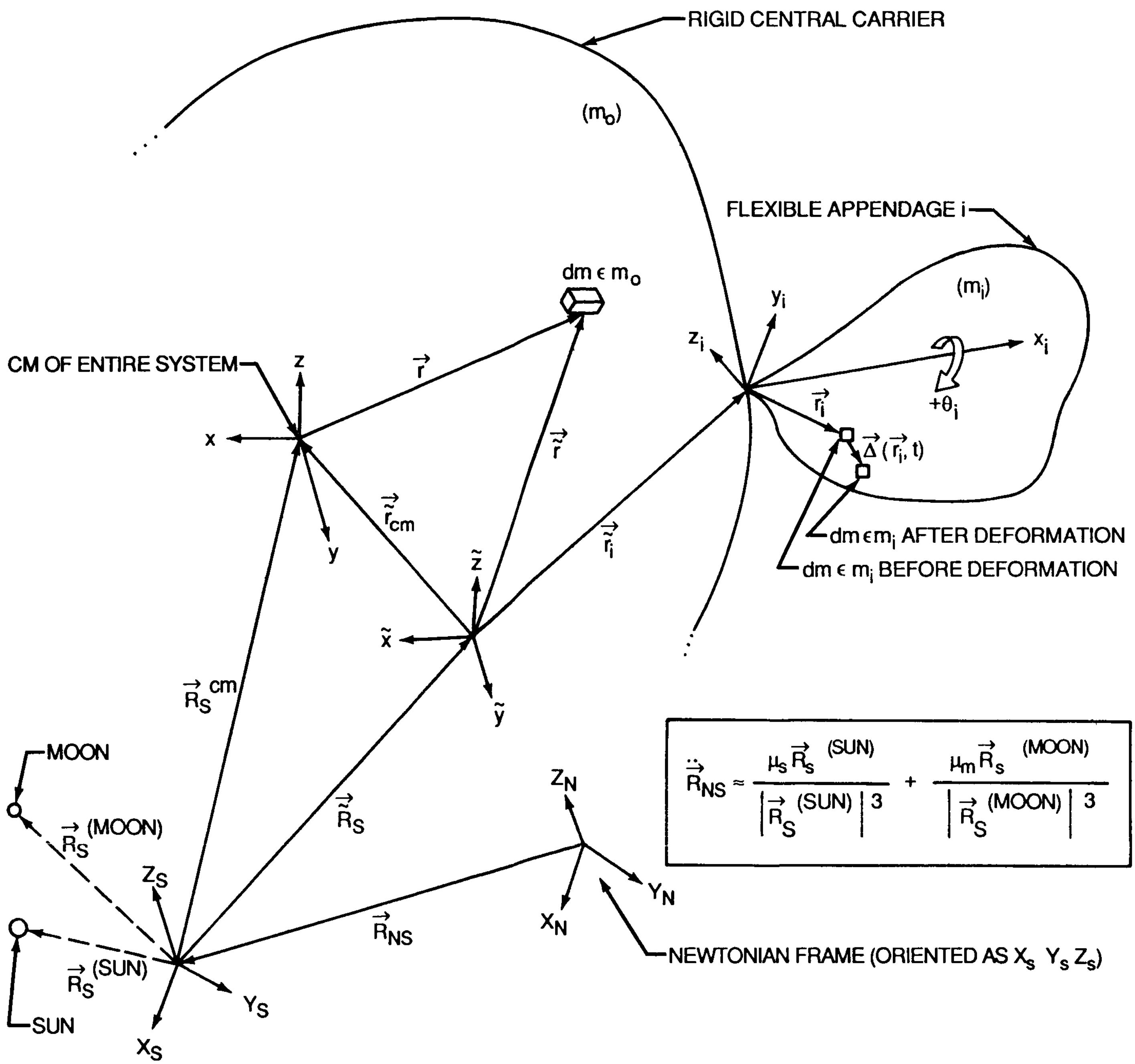
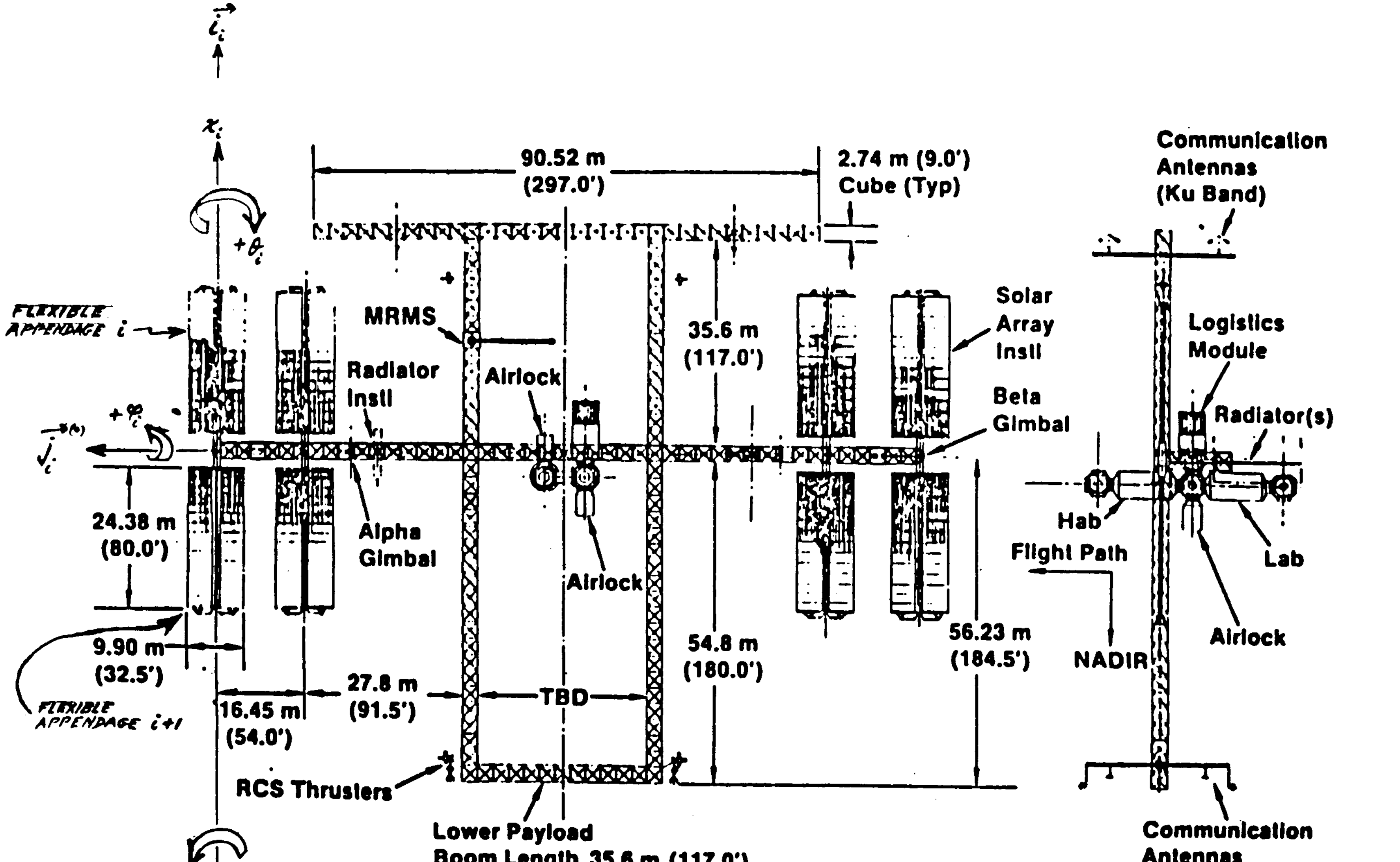

RIGID CENTRAL CARRIER
(m_o)
FLEXIBLE APPENDAGE i
(m_i)
$dm \in m_o$
CM OF ENTIRE SYSTEM
$\vec{r}$
$\vec{\tilde{r}}$
$\vec{r}_{cm}$
$\vec{\tilde{r}}_i$
$\vec{r}_i$
$+\theta_i$
$\vec{\Delta}(\vec{r}_i, t)$
$dm \in m_i$ AFTER DEFORMATION
$dm \in m_i$ BEFORE DEFORMATION
$\vec{R}_S^{\,cm}$
$\vec{\tilde{R}}_S$
$\vec{R}_{NS}$
MOON
$\vec{R}_S$ (MOON)
SUN
$\vec{R}_S$ (SUN)
Z_S Y_S X_S
Z_N Y_N X_N
NEWTONIAN FRAME (ORIENTED AS X_s Y_s Z_s)
$\ddot{\vec{R}}_{NS} \approx \frac{\mu_s \vec{R}_s^{\,(SUN)}}{\left|\vec{R}_S^{\,(SUN)}\right|^3} + \frac{\mu_m \vec{R}_s^{\,(MOON)}}{\left|\vec{R}_S^{\,(MOON)}\right|^3}$

Figure 1.



SIGN CONVENTION FOR θ_i AND φ_i IS IN ACCORD WITH THE RIGHT HAND RULE

$$\begin{bmatrix} \vec{i}_i \\ \vec{j}_i \\ \vec{k}_i \end{bmatrix} = \left[\theta_i\right]_{(1)} \left[\varphi_i\right]_{(2)} \begin{bmatrix} \vec{i}_i^{(0)} \\ \vec{j}_i^{(0)} \\ \vec{k}_i^{(0)} \end{bmatrix}$$

WHERE THE UNIT VECTORS $\vec{i}_i^{(0)}$, $\vec{j}_i^{(0)}$ AND $\vec{k}_i^{(0)}$ DEFINE THE "NULL" ORIENTATION OF THE i-FRAME.

Figure 2. Manned Core Space Station.

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APPENDIX A

ON GRAVITY FORCE AND GRAVITY TORQUE

The manipulations leading to the expressions below for the force and moment exerted by the Earth's external gravitational field upon a body of arbitrary shape are based upon the gravitational potential function of Reference 6 which includes spherical harmonics through fourth order and reads, in the notation of this paper, as follows:

$$U_g = \frac{\mu_E}{R} \left\{ 1 - \sum_{n=2}^{4} \sum_{m=0}^{n} J_{nm} \left(\frac{a}{R}\right)^n P_n^m (\sin \delta) \cos m (\lambda - \lambda_{nm}) \right\} , \qquad \text{(A-1)}$$

wherein μ_E denotes the product of the mass of the Earth and the universal gravitational constant; R, λ, and δ are the spherical coordinates of the field point relative to $X_E Y_E Z_E$ (see Sections 2 and 3); λ_{nm} is the longitude (positive east of the prime meridian) of the principal meridian of symmetry for the nm harmonic; the J_{nm}, n = 2, 3, 4, m = 0, ... n, are dimensionless coefficients peculiar to the planet Earth with $J_{21} = 0$ by virtue of the assumption that the Z_E axis (and also the Z_s axis) is a principal axis of inertia for the Earth; a is the mean equatorial radius; and $P_n^m (\sin \delta)$ is the associated Legendre function of the first kind of degree n and order m defined by

$$P_n^m (\sin \delta) = \cos^m \delta \, \frac{d^m P_n (\sin \delta)}{d (\sin \delta)^m} ,$$

the function $P_n(\sin\delta)$ being the Legendre polynomial of degree n in the argument $\sin \delta$.

As seen in equation (A-1) the dimension of U_g is that of work per unit mass or potential energy per unit mass, the equivalent of the product of unit of force and unit of length divided by the unit of mass. From dimensional considerations alone it is evident that a differentiation of U_g with respect to distance in a given direction (the directional derivative, that is) gives the gravitational force per unit mass, more frequently called the gravitational acceleration, in that direction. It can also be argued (Reference 7) that a differentiation with respect to arc length along the curvilinear coordinate curves whose unit tangent vectors are the $\vec{u}_R$, $\vec{u}_\lambda$ and $\vec{u}_\delta$ defined in Section 3 gives the components in the directions of those unit vectors of $\vec{A}_g$, the symbol $\vec{A}_g$ here denoting the acceleration due to the Earth's gravity at the field point with spherical coordinates (R, λ, δ) referred to the E-frame. Thus,

$$\vec{A}_g \equiv A_{gR}\,\vec{u}_R + A_{g\lambda}\,\vec{u}_\lambda + A_{g\delta}\,\vec{u}_\delta = \left(\frac{\partial U_g}{\partial R}\right)\vec{u}_R + \left(\frac{1}{R\cos\delta}\,\frac{\partial U_g}{\partial \lambda}\right)\vec{u}_\lambda + \left(\frac{1}{R}\,\frac{\partial U_g}{\partial \delta}\right)\vec{u}_\delta \quad . \tag{A-2}$$

The results of performing the indicated differentiations in equation (A-2) are given in Reference 6 and are repeated here (in the notation of this paper) for ready reference. (The author of this paper has verified, through tedius scratchwork not shown, that the results appearing in Reference 6 are, apart from some obvious typographical errors, correct. Minor typographical errors are also present in the expression for the potential function in the reference cited.)

$$\begin{aligned} A_{gR} = \frac{\partial U_g}{\partial R} = \frac{\mu_E}{R^2}\Bigg\{ -1 + \left(\frac{a}{R}\right)^2 \Bigg[& \frac{3\,J_{20}}{2}\,(3\sin^2\delta - 1) + 9\,J_{22}\cos^2\delta\,\cos 2\,(\lambda - \lambda_{22}) \\ & + 2\left(\frac{a}{R}\right) J_{30}\,(5\sin^2\delta - 3)\sin\delta + 6\left(\frac{a}{R}\right) J_{31}\,(5\sin^2\delta - 1)\cos\delta\,\cos\,(\lambda - \lambda_{31}) \\ & + 60\left(\frac{a}{R}\right) J_{32}\cos^2\delta\,\sin\delta\,\cos 2\,(\lambda - \lambda_{32}) \\ & + 60\left(\frac{a}{R}\right) J_{33}\cos^3\delta\,\cos 3\,(\lambda - \lambda_{33}) + \frac{5}{8}\left(\frac{a}{R}\right)^2 J_{40}\,(35\sin^4\delta - 30\sin^2\delta + 3) \\ & + \frac{25}{2}\left(\frac{a}{R}\right)^2 J_{41}\,(7\sin^2\delta - 3)\cos\delta\,\sin\delta\,\cos\,(\lambda - \lambda_{41}) \\ & + \frac{75}{2}\left(\frac{a}{R}\right)^2 J_{42}\,(7\sin^2\delta - 1)\cos^2\delta\,\cos 2\,(\lambda - \lambda_{42}) \\ & + 525\left(\frac{a}{R}\right)^2 J_{43}\cos^3\delta\,\sin\delta\,\cos 3\,(\lambda - \lambda_{43}) \\ & + 525\left(\frac{a}{R}\right)^2 J_{44}\cos^4\delta\,\cos 4\,(\lambda - \lambda_{44})\Bigg]\Bigg\} \quad , \end{aligned} \tag{A-3}$$

$$A_{g\lambda} = \frac{1}{R \cos \delta} \frac{\partial U_g}{\partial \lambda} = \frac{\mu_E}{R^2} \left(\frac{a}{R}\right)^2 \Big\{ 6 J_{22} \cos \delta \sin 2 (\lambda - \lambda_{22})$$

$$+ \frac{3}{2} \left(\frac{a}{R}\right) J_{31} (5 \sin^2 \delta - 1) \sin (\lambda - \lambda_{31})$$

$$+ 30 \left(\frac{a}{R}\right) J_{32} \cos \delta \sin \delta \sin 2 (\lambda - \lambda_{32})$$

$$+ 45 \left(\frac{a}{R}\right) J_{33} \cos^2 \delta \sin 3 (\lambda - \lambda_{33})$$

$$+ \frac{5}{2} \left(\frac{a}{R}\right)^2 J_{41} (7 \sin^2 \delta - 3) \sin \delta \sin (\lambda - \lambda_{41})$$

$$+ 15 \left(\frac{a}{R}\right)^2 J_{42} (7 \sin^2 \delta - 1) \cos \delta \sin 2 (\lambda - \lambda_{42})$$

$$+ 315 \left(\frac{a}{R}\right)^2 J_{43} \cos^2 \delta \sin \delta \sin 3 (\lambda - \lambda_{43})$$

$$+ 420 \left(\frac{a}{R}\right)^2 J_{44} \cos^3 \delta \sin 4 (\lambda - \lambda_{44}) \Big\} , \qquad \text{(A-4)}$$

$$A_{g\delta} = \frac{1}{R} \frac{\partial U_g}{\partial \delta} = \frac{\mu_E}{R^2} \left(\frac{a}{R}\right)^2 \Big\{ - 3 J_{20} \sin \delta \cos \delta + 6 J_{22} \cos \delta \sin \delta \cos 2 (\lambda - \lambda_{22})$$

$$- \frac{3}{2} \left(\frac{a}{R}\right) J_{30} (5 \sin^2 \delta - 1) \cos \delta$$

$$+ \frac{3}{2} \left(\frac{a}{R}\right) J_{31} (15 \sin^2 \delta - 11) \sin \delta \cos (\lambda - \lambda_{31})$$

$$+ 15 \left(\frac{a}{R}\right) J_{32} (3 \sin^2 \delta - 1) \cos \delta \cos 2 (\lambda - \lambda_{32})$$

$$+ 45 \left(\frac{a}{R}\right) J_{33} \cos^2 \delta \sin \delta \cos 3 (\lambda - \lambda_{33})$$

$$- \frac{5}{2} \left(\frac{a}{R}\right)^2 J_{40} (7 \sin^2 \delta - 3) \sin \delta \cos \delta$$

(A-5)
(Continued)

$$+ \frac{5}{2} \left(\frac{a}{R}\right)^2 J_{41} \, (28 \sin^4 \delta - 27 \sin^2 \delta + 3) \cos (\lambda - \lambda_{41})$$

$$+ 30 \left(\frac{a}{R}\right)^2 J_{42} \, (7 \sin^2 \delta - 4) \cos \delta \sin \delta \cos 2 (\lambda - \lambda_{42})$$

$$+ 105 \left(\frac{a}{R}\right)^2 J_{43} \, (4 \sin^2 \delta - 1) \cos^2 \delta \cos 3 (\lambda - \lambda_{43})$$

$$+ 420 \left(\frac{a}{R}\right)^2 J_{44} \cos^3 \delta \sin \delta \cos 4 (\lambda - \lambda_{44}) \Big\} \quad , \qquad \text{(A-5) (Concluded)}$$

The force and torque* (attributed to the Earth's gravity field) experienced by a space vehicle** will here be denoted by $\vec{F}_{gB}$ and $\vec{M}_{gB}$, respectively, the second subscript B being indicative of the B-resolution.

Definitions of symbols incident to the development of expressions for $\vec{F}_{gB}$ and $\vec{M}_{gB}$ are in order now (some being repeated elsewhere).

$\vec{R}_s^{CM}$ = position referred to the S-frame (see Section 2) of the CM of the entire vehicle system.

$\vec{r} = x \, \vec{i} + y \, \vec{j} + z \, \vec{k}$ = position referred to the B-frame (see Section 2) of a generic point of the vehicle.

$\vec{R}_s = \vec{R}_s^{CM} + T^T \, \vec{r}$ = position of a generic point of the vehicle relative to the S-frame ($T = T(S \rightarrow B)$ defined in Section 3).

$\vec{R}_o = T \, \vec{R}_s^{CM}$ = B-resolution of $\vec{R}_s^{CM}$, $R_o = |\vec{R}_o| = |\vec{R}_s^{CM}|$

$\vec{R} = T \, \vec{R}_s = \vec{R}_o + \vec{r}$ = B-resolution of $\vec{R}_s$, $R = |\vec{R}| = |\vec{R}_o + \vec{r}|$

$(\vec{u}_R)_B = \vec{R}/R$ = B-resolution of $\vec{u}_R$ (see Section 3)

$\vec{u}_{Ro} = \vec{R}_o/R_o$ = B-resolution of $\vec{R}_s^{CM}/|\vec{R}_s^{CM}|$

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* The moment reference point is the vehicle center of mass (CM).

** The body need not be a space vehicle but may be any body of arbitrary shape and finite dimensions occupying a position in the Earth's external field.

$\vec{n} = T\ \vec{K}_s$ = B-resolution of $\vec{K}_s$

$$\square \equiv \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$ = Inertia matrix of entire system referred to xyz (the B-frame with origin at the system CM).

$$I_{xx} = \int_{(m)} (y^2 + z^2)\, dm \quad , \quad I_{yy} = \int_{(m)} (x^2 + z^2)\, dm \quad , \quad I_{zz} = \int_{(m)} (x^2 + y^2)\, dm$$

$$I_{\xi\eta} = - \int_{(m)} \xi\eta\, dm \quad , \quad \xi \neq \eta \quad , \quad \xi, \eta = x, y, z$$

$$\int_{(m)} (\ldots)\, dm = \iiint_{(m)} (\ldots)\ \sigma\,(x, y, z)\, dxdydz \quad , \quad (\sigma(x, y, z) = \text{mass density})$$

(as used immediately above the subscript (m) indicates that the integration is to be extended over the volume occupied by the entire vehicle system. Elsewhere the symbol m will denote the numerical value of the mass of the entire system.)

$$Tr(\square) = \text{trace of } \square = I_{xx} + I_{yy} + I_{zz} \quad .$$

In approximating $\vec{M}_{gB}$ the contribution of the longitude dependent terms (the tesseral and sectorial harmonics) was completely ignored while in approximating $\vec{F}_{gB}$ their effect was accounted for "in part," that is, their variation over the vehicle was ignored, it being assumed that their values at any point of the vehicle differ negligibly from those at the vehicle CM.

The contribution to $\vec{F}_{gB}$ of the dominant term in the expression for A_{gR}, here denoted by $\vec{F}_{gR}^{\ (o,o)}$, is given by

$$\vec{F}_{gR}^{\ (o,o)} = \int_{(m)} d\vec{F}_{gR}^{\ (o,o)} = - \mu_E \int_{(m)} \frac{(\vec{R}_o + \vec{r})}{|\vec{R}_o + \vec{r}|^3}\, dm$$

$$\approx - \frac{m\ \mu_E}{R_o^{\ 2}}\ \vec{u}_{Ro} - \frac{3\ \mu_E}{R_o^{\ 4}}\ \square \vec{u}_{Ro} - \frac{3\ \mu_E}{2\ R_o^{\ 4}} \left[Tr(\square) - 5\ \vec{u}_{Ro}^{\,T}\ \square \vec{u}_{Ro} \right] \vec{u}_{Ro}\ , \quad \text{(A-6)}$$

while its contribution to $\vec{M}_{gB}$ is

$$\vec{M}_{gR}{}^{(o,o)} = \int_{(m)} \vec{r} \times d\vec{F}_{gR}{}^{(o,o)} \approx \frac{3\,\mu_E}{R_o^{\;3}}\, \vec{u}_{Ro} \times (\square\, \vec{u}_{Ro}) \quad . \tag{A-7}$$

In arriving at the approximations (A-6) and (A-7), use was made of the approximate expansion

$$R^{-K} \equiv |\vec{R}_o + \vec{r}|^{-K} \approx R_o^{\;-K} - K\,R_o^{\;-K-1}\,\vec{u}_{Ro}^{\;T}\,\vec{r} - \frac{K\,R_o^{\;-K-2}}{2}\,\vec{r}^{\,T}\,\vec{r}$$

$$+ \frac{K\,(K+2)}{2}\,R_o^{\;-K-2}\,\vec{u}_{Ro}^{\;T}\,\vec{r}\,\vec{r}^{\,T}\,\vec{u}_{Ro} \quad , \tag{A-8}$$

and the relations

$$\int_{(m)} \vec{r}\,dm = \vec{0} \qquad \text{(by definition of the mass center)} \quad , \tag{A-9}$$

$$\int_{(m)} \vec{r}^{\,T}\,\vec{r}\,dm = \tfrac{1}{2}\,\mathcal{T}r(\square) \tag{A-10}$$

$$\int_{(m)} \vec{r}\,\vec{r}^{\,T}\,dm = -\square + \tfrac{1}{2}\,\mathcal{T}r(\square)\,I_{(3x3)} \quad , \quad I_{(3x3)} \equiv \begin{bmatrix} 1. & 0 & 0 \\ 0 & 1. & 0 \\ 0 & 0 & 1. \end{bmatrix} , \tag{A-11}$$

in addition to further simplifications made by discarding all terms in $x^i y^j z^k$ when $i + j + k \geq 3$ (such terms have denominators of the order of $R_o^{\;5}$ and greater).

It should be remarked that the axes xyz (the B-axes) have not been restricted to be principal axes of inertia and further that the inertia matrix $\square$ must be interpreted as that of the instantaneous deformed configuration referred to the xyz axes which have origin at the instantaneous position of the CM.

In finding the contributions of the zonal harmonics to $\vec{F}_{gB}$ and $\vec{M}_{gB}$ (these contributions being identified herein as $\vec{F}_{gR}{}^{(k,o)}$, $\vec{M}_{gR}{}^{(k,o)}$, $\vec{F}_{g\delta}{}^{(k,o)}$, $\vec{M}_{g\delta}{}^{(k,o)}$, $k = 2, 3, 4$) the relations (A-8), (A-9), (A-10), and (A-11) again find repeated application as do the following:

C-2

$$\sin\delta_o = \vec{u}_{Ro}^T\,\vec{n} \quad , \quad (\delta_o = \text{geocentric latitude of vehicle CM}) \quad , \tag{A-12}$$

$$\sin\delta = \vec{n}^T\,(\vec{u}_R)_B = \frac{R_o}{R}\left(\sin\delta_o + \frac{\vec{n}^T\,\vec{r}}{R_o}\right) \tag{A-13}$$

$$\cos\delta\,(\vec{u}_\delta)_B = \vec{n} - \sin\delta\,(\vec{u}_R)_B \tag{A-14}$$

the relation (A-14) being invoked only in the development of expressions for the $\vec{F}_{g\delta}^{(k,o)}$ and $\vec{M}_{g\delta}^{(k,o)}$, $k = 2, 3, 4$. Notice that the use of (A-14) circumvents the need for an approximate expansion for $\cos\delta$.

No exposition of the detailed manipulations leading to the approximations below will be given, it being presumed that the reader can, with the time and inclination, supply all the manipulations omitted.

$$\vec{F}_{gR}^{(2,o)} = \int_{(m)} d\vec{F}_{gR}^{(2,o)} = \frac{3\,\mu_E\,a^2\,J_{20}}{2}\int_{(m)}\frac{(3\sin^2\delta - 1)\,(\vec{R}_o + \vec{r})}{R^5}\,dm$$

$$\approx \frac{3\,\mu_E\,a^2\,J_{20}}{2}\left\{\frac{m}{R_o^4}(3\sin^2\delta_o - 1) + \frac{3}{R_o^6}(7\sin^2\delta_o - 2)\,Tr(\square)\right.$$

$$+ \frac{1}{2R_o^6}(35 - 189\sin^2\delta_o)\,\vec{u}_{Ro}^T\,\square\,\vec{u}_{Ro}$$

$$\left. - \frac{3}{R_o^6}\,\vec{n}^T\,\square\,\vec{n} + \frac{42\sin\delta_o}{R_o^6}\,\vec{n}^T\,\square\,\vec{u}_{Ro}\right\}\vec{u}_{Ro}$$

$$+ \frac{3\,\mu_E\,a^2\,J_{20}}{2\,R_o^6}\left\{(21\sin^2\delta_o - 5)\left[\square - \tfrac{1}{2}\,Tr(\square)\,I_{(3x3)}\right]\vec{u}_{Ro}\right.$$

$$\left. - 6\sin\delta_o\left[\square - \tfrac{1}{2}\,Tr(\square)\,I_{(3x3)}\right]\vec{n}\right\} \quad , \tag{A-15}$$

$$\vec{M}_{gR}^{(2,o)} = \int_{(m)} \vec{r} \times d\vec{F}_{gR}^{(2,o)} = -\frac{3\,\mu_E\, a^2\, J_{20}}{2} \left\{ \vec{R}_o \times \int_{(m)} \frac{(3 \sin^2 \delta - 1)\vec{r}}{R^5}\, dm \right\}$$

$$\approx -\frac{3\,\mu_E\, a^2\, J_{20}}{2\, R_o^5} \{(21 \sin^2 \delta_o - 5)\, \vec{u}_{Ro} \times (\Box\, \vec{u}_{Ro}) - 6 \sin \delta_o\, \vec{u}_{Ro} \times (\Box\, \vec{n})$$

$$+ 3 \sin \delta_o\, \mathcal{T}r(\Box)\, (\vec{u}_{Ro} \times \vec{n})\} \quad , \qquad \text{(A-16)}$$

$$\vec{F}_{gR}^{(3,o)} = \int_{(m)} d\vec{F}_{gR}^{(3,o)} = 2\,\mu_E\, a^3\, J_{30} \int_{(m)} \frac{(5 \sin^2 \delta - 3) \sin \delta\, (\vec{R}_o + \vec{r})}{R^6}\, dm$$

$$\approx 2\,\mu_E\, a^3\, J_{30} \left\{ \frac{m}{R_o^5} (5 \sin^3 \delta_o - 3 \sin \delta_o) \right.$$

$$+ \frac{3}{R_o^7} (15 \sin^3 \delta_o - 8 \sin \delta_o)\, \mathcal{T}r(\Box)$$

$$+ \frac{9}{2R_o^7} (21 \sin \delta_o - 55 \sin^3 \delta_o)\, \vec{u}_{Ro}^T\, \Box\, \vec{u}_{Ro}$$

$$\left. - \frac{15 \sin \delta_o}{R_o^7}\, \vec{n}^T\, \Box\, \vec{n} + \frac{1}{R_o^7} (135 \sin^2 \delta_o - 21)\, \vec{n}^T\, \Box\, \vec{u}_{Ro} \right\} \vec{u}_{Ro}$$

$$+ \frac{2\,\mu_E\, a^3\, J_{30}}{R_o^7} \left\{ 3 \sin \delta_o\, (15 \sin^2 \delta_o - 7) \left[\Box - \frac{1}{2}\, \mathcal{T}r(\Box)\, I_{(3x3)} \right] \vec{u}_{Ro} \right.$$

$$\left. + 3\,(1 - 5 \sin^2 \delta_o) \left[\Box - \frac{1}{2}\, \mathcal{T}r(\Box)\, I_{(3x3)} \right] \vec{n} \right\} \quad , \qquad \text{(A-17)}$$

$$\vec{M}_{gR}^{(3,o)} = \int_{(m)} \vec{r} \times d\vec{F}_{gR}^{(3,o)} = -2\,\mu_E\, a^3\, J_{30} \left\{ \vec{R}_o \times \int_{(m)} \frac{(5 \sin^3 \delta - 3 \sin \delta)\vec{r}}{R^6}\, dm \right\}$$

$$\approx \frac{-2\,\mu_E\, a^3\, J_{30}}{R_o^{\,6}} \left\{ 3 \sin \delta_o\, (15 \sin^2 \delta_o - 7)\, \vec{u}_{Ro} \times (\square\, \vec{u}_{Ro}) \right.$$

$$\left. + 3\,(1-5 \sin^2 \delta_o)\, \vec{u}_{Ro} \times (\square\, \vec{n}) - \frac{3}{2}\,(1-5 \sin^2 \delta_o)\, \mathcal{T}r(\square)\, (\vec{u}_{Ro} \times \vec{n}) \right\} , \qquad \text{(A-18)}$$

$$\vec{F}_{gR}^{(4,o)} = \int_{(m)} d\vec{F}_{gR}^{(4,o)} = \frac{5\,\mu_E\, a^4\, J_{40}}{8} \int_{(m)} \frac{1}{R^7}\, (35 \sin^4 \delta - 30 \sin^2 \delta + 3)\, (\vec{R}_o + \vec{r})\, dm$$

$$\approx \frac{5\,\mu_E\, a^4\, J_{40}}{8} \left\{ \frac{m}{R_o^{\,6}}\, (35 \sin^4 \delta_o - 30 \sin^2 \delta_o + 3) \right.$$

$$+ \frac{1}{R_o^{\,8}}\, (385 \sin^4 \delta_o - 300 \sin^2 \delta_o + 27)\, \mathcal{T}r(\square)$$

$$- \frac{1}{2R_o^{\,8}}\, (5005 \sin^4 \delta_o - 2970 \sin^2 \delta_o + 189)\, \vec{u}_{Ro}^{\,T}\, \square\, \vec{u}_{Ro}$$

$$\left. + \frac{30}{R_o^{\,8}}\, (1-7 \sin^2 \delta_o)\, \vec{n}^T \square\, \vec{n} + \frac{20 \sin \delta_o}{R_o^{\,8}}\, (77 \sin^2 \delta_o - 27)\, \vec{n}^T\, \square\, \vec{u}_{Ro} \right\} \vec{u}_{Ro}$$

$$+ \frac{5\,\mu_E\, a^4\, J_{40}}{8R_o^{\,8}} \left\{ (385 \sin^4 \delta_o - 270 \sin^2 \delta_o + 21)\, \square\, \vec{u}_{Ro} \right.$$

$$- 20 \sin \delta_o\, (7 \sin^2 \delta_o - 3)\, \square\, \vec{n} - \frac{1}{2}\, (385 \sin^4 \delta_o - 270 \sin^2 \delta_o + 21)\, \mathcal{T}r(\square)\, \vec{u}_{Ro}$$

$$\left. + 10 \sin \delta_o\, (7 \sin^2 \delta_o - 3)\, \mathcal{T}r(\square)\, \vec{n} \right\} , \qquad \text{(A-19)}$$

$$\vec{M}_{gR}^{(4,o)} = \int_{(m)} \vec{r} \times d\vec{F}_{gR}^{(4,o)} = \frac{-5\,\mu_E\, a^4 J_{40}}{8} \left\{ \vec{R}_o \times \int_{(m)} \frac{1}{R^7} (35 \sin^4 \delta - 30 \sin^2 \delta + 3)\, \vec{r}\, dm \right\}$$

$$\approx \frac{-5\,\mu_E\, a^4 J_{40}}{8 R_o^{\,7}} \{ (385 \sin^4 \delta_o - 270 \sin^2 \delta_o + 21)\, \vec{u}_{Ro} \times (\Box\, \vec{u}_{Ro})$$

$$- 20 (7 \sin^3 \delta_o - 3 \sin \delta_o)\, \vec{u}_{Ro} \times (\Box\, \vec{n})$$

$$+ 10 (7 \sin^3 \delta_o - 3 \sin \delta_o)\, Tr(\Box)\, (\vec{u}_{Ro} \times \vec{n}) \} \quad , \qquad \text{(A-20)}$$

$$\vec{F}_{g\delta}^{(2,o)} = \int_{(m)} d\vec{F}_{g\delta}^{(2,o)} = -3\,\mu_E\, a^2 J_{20} \int_{(m)} \frac{1}{R^4} \sin \delta \cos \delta\, (\vec{u}_\delta)_B\, dm$$

$$= -3\,\mu_E\, a^2 J_{20} \left\{ \int_{(m)} \frac{\sin \delta}{R^4}\, dm \right\} \vec{n} + 3\,\mu_E\, a^2 J_{20} \int_{(m)} \frac{\sin^2 \delta}{R^5} (\vec{R}_o + \vec{r})\, dm$$

$$\approx \frac{3\,\mu_E\, a^2 J_{20}}{R_o^{\,4}} \left\{ \sin^2 \delta_o \left[m + \frac{7\, Tr(\Box)}{2R_o^{\,2}} - \frac{63}{2R_o^{\,2}}\, \vec{u}_{Ro}^{\,T}\, \Box\, \vec{u}_{Ro} \right] \right.$$

$$\left. + \frac{14 \sin \delta_o}{R_o^{\,2}}\, \vec{n}^T \Box\, \vec{u}_{Ro} - \frac{1}{R_o^{\,2}}\, \vec{n}^T \Box\, \vec{n} + \frac{Tr(\Box)}{2R_o^{\,2}} \right\} \vec{u}_{Ro}$$

$$- \frac{3\,\mu_E\, a^2 J_{20}}{R_o^{\,4}} \left\{ \sin \delta_o \left[m + \frac{5\, Tr(\Box)}{R_o^{\,2}} - \frac{35}{2R_o^{\,2}}\, \vec{u}_{Ro}^{\,T}\, \Box\, \vec{u}_{Ro} \right] + \frac{5}{R_o^{\,2}}\, \vec{n}^T \Box\, \vec{u}_{Ro} \right\} \vec{n}$$

$$+ \frac{3\,\mu_E\, a^2 J_{20}}{R_o^{\,6}} \left\{ 7 \sin^2 \delta_o\, \Box\, \vec{u}_{Ro} - 2 \sin \delta_o \left[\Box\, \vec{n} - \frac{Tr(\Box)}{2}\, \vec{n} \right] \right\} \quad , \qquad \text{(A-21)}$$

$$\vec{M}_{g\delta}^{(2,o)} = \int_{(m)} \vec{r} \times d\vec{F}_{g\delta}^{(2,o)} = 3\,\mu_E\, a^2 J_{20}\, \vec{n} \times \left\{ \int_{(m)} \frac{\sin\,\delta}{R^4}\, \vec{r}\, dm \right\}$$

$$- 3\,\mu_E\, a^2 J_{20}\, \vec{R}_o \times \left\{ \int_{(m)} \frac{\sin^2\,\delta}{R^5}\, \vec{r}\, dm \right\}$$

$$\approx \frac{-3\,\mu_E\, a^2 J_{20}}{R_o^{\;5}} \Big\{ \vec{n} \times (\square\vec{n}) + \sin\,\delta_o\, [7 \sin\,\delta_o\, \vec{u}_{Ro} \times (\square\vec{u}_{Ro})$$

$$-5\,\vec{n} \times (\square\vec{u}_{Ro}) - 2\,\vec{u}_{Ro} \times (\square\vec{n})$$

$$+ \frac{3}{2}\, \mathcal{T}r(\square)\, (\vec{n} \times \vec{u}_{Ro})] \Big\}\;, \qquad \text{(A-22)}$$

$$\vec{F}_{g\delta}^{(3,o)} = \int_{(m)} d\vec{F}_{g\delta}^{(3,o)} = \frac{-3\,\mu_E\, a^3 J_{30}}{2} \int_{(m)} \frac{1}{R^5}\, (5 \sin^2\,\delta - 1) \cos\,\delta\, (\vec{u}_\delta)_B\, dm$$

$$= \left\{ \frac{-3\,\mu_E\, a^3 J_{30}}{2} \int_{(m)} \frac{1}{R^5}\, (5 \sin^2\,\delta - 1)\, dm \right\} \vec{n}$$

$$+ \frac{3\,\mu_E\, a^3 J_{30}}{2} \int_{(m)} \frac{1}{R^6}\, (5 \sin^3\,\delta - \sin\,\delta)\, (\vec{R}_o + \vec{r})\, dm$$

$$\approx \frac{-3\,\mu_E\, a^3 J_{30}}{2R_o^{\;5}} \{m\, (5 \sin^2\,\delta_o - 1)\, (\vec{n} - \sin\,\delta_o\, \vec{u}_{Ro})\}$$

$$+ \frac{3\,\mu_E\, a^3 J_{30}}{2R_o^{\;7}} \Big\{ [5(1-7 \sin^2\,\delta_o)\, \mathcal{T}r(\square) + \frac{35}{2}\, (9 \sin^2\,\delta_o - 1)\, \vec{u}_{Ro}^{\,T}\, \square\vec{u}_{Ro}$$

$$+ 5\, \vec{n}^T \square\vec{n} - 70 \sin\,\delta_o\, \vec{n}^T\, \square\vec{u}_{Ro}]\vec{n} + [3 \sin\,\delta_o\, (15 \sin^2\,\delta_o - 1)\, \mathcal{T}r(\square)$$

$$+ \frac{9}{2} \sin\,\delta_o\, (7-55 \sin^2\,\delta_o)\, \vec{u}_{Ro}^{\,T}\, \square\vec{u}_{Ro} - 15 \sin\,\delta_o\, \vec{n}^T \square\vec{n}$$

$$+ (135 \sin^2\,\delta_o - 7)\vec{n}^T \square\,\vec{u}_{Ro}]\, \vec{u}_{Ro} + \sin\,\delta_o\, (45 \sin^2\,\delta_o - 7)\, \square\vec{u}_{Ro} \qquad \text{(A-23) (Cont.)}$$

$$+ (1-15 \sin^2 \delta_o) [\ ] \vec{n} + \frac{1}{2} [\sin \delta_o (7-45 \sin^2 \delta_o) \vec{u}_{Ro} + (15 \sin^2 \delta_o - 1)\vec{n}] \mathcal{T}r([\ ]) \Big\} ,$$

(A-23)
(Conc.)

$$\vec{M}_{g\delta}^{(3,o)} = \int_{(m)} \vec{r} \times d\vec{F}_{g\delta}^{(3,o)} = \vec{n} \times \left\{ \frac{3 \mu_E a^3 J_{30}}{2} \int_{(m)} \frac{1}{R^5} (5 \sin^2 \delta - 1)\vec{r} \, dm \right\}$$

$$- \vec{R}_o \times \left\{ \frac{3 \mu_E a^3 J_{30}}{2} \int_{(m)} \frac{1}{R^6} (5 \sin^3 \delta - \sin \delta)\vec{r} \, dm \right\}$$

$$\approx \frac{15 \mu_E a^3 J_{30}}{2R_o^6} \Big\{ (7 \sin^2 \delta_o - 1) \left[\vec{n} \times ([\ ]\vec{u}_{RO}) - \frac{1}{2} \mathcal{T}r([\ ]) (\vec{n} \times \vec{u}_{Ro}) \right]$$

$$- 2 \sin \delta_o \, \vec{n} \times ([\ ]\vec{n}) \Big\}$$

$$- \frac{3 \mu_E a^3 J_{30}}{2R_o^6} \Big\{ \sin \delta_o (45 \sin^2 \delta_o - 7) \vec{u}_{Ro} \times ([\ ]\vec{u}_{Ro})$$

$$+ (1-15 \sin^2 \delta_o) \left[\vec{u}_{Ro} \times ([\ ]\vec{n}) - \frac{1}{2} \mathcal{T}r([\ ]) (\vec{u}_{Ro} \times \vec{n}) \right] \Big\} , \tag{A-24}$$

$$\vec{F}_{g\delta}^{(4,o)} = \int_{(m)} d\vec{F}_{g\delta}^{(4,o)} = \frac{-5 \mu_E a^4 J_{40}}{2} \int_{(m)} \frac{1}{R^6} (7 \sin^2 \delta - 3) \sin \delta \cos \delta \, (\vec{u}_\delta)_B \, dm$$

$$= \left\{ \frac{-5 \mu_E a^4 J_{40}}{2} \int_{(m)} \frac{1}{R^6} (7 \sin^2 \delta - 3) \sin \delta \, dm \right\} \vec{n}$$

$$+ \frac{5 \mu_E a^4 J_{40}}{2} \int_{(m)} \frac{1}{R^7} (7 \sin^2 \delta - 3) \sin^2 \delta \, (\vec{R}_o + \vec{r}) \, dm$$

$$\approx \frac{-5 \mu_E a^4 J_{40}}{2R_o^6} \Big\{ m (7 \sin^3 \delta_o - 3 \sin \delta_o) + \frac{21 \sin \delta_o}{R_o^2} (3 \sin^2 \delta_o - 1) \mathcal{T}r([\ ])$$

(A-25)
(Cont.)

$$+ \frac{63 \sin \delta_o}{2R_o^{\,2}} (3-11 \sin^2 \delta_o)\, \vec{u}_{Ro}^{\,T}\, \Box \vec{u}_{Ro} - \frac{21 \sin \delta_o}{R_o^{\,2}}\, \vec{n}^T \Box\, \vec{n}$$

$$+ \frac{21}{R_o^{\,2}} (9 \sin^2 \delta_o - 1)\, \vec{n}^T \Box \vec{u}_{Ro} \Big\}\, \vec{n}$$

$$+ \frac{5\, \mu_E\, a^4\, J_{40}}{2R_o^{\,6}} \Big\{ m\, (7 \sin^4 \delta_o - 3 \sin^2 \delta_o)$$

$$+ \frac{1}{2R_o^{\,2}} (154 \sin^4 \delta_o - 39 \sin^2 \delta_o - 3)\, \mathcal{T}r(\Box)$$

$$+ \frac{\sin^2 \delta_o}{2R_o^{\,2}} (297-1001 \sin^2 \delta_o)\, \vec{u}_{Ro}^{\,T}\, \Box \vec{u}_{Ro} + \frac{3}{R_o^{\,2}} (1-14 \sin^2 \delta_o)\, \vec{n}^T \Box \vec{n}$$

$$+ \frac{2 \sin \delta_o}{R_o^{\,2}} (154 \sin^2 \delta_o - 27)\, \vec{n}^T\, \Box\, \vec{u}_{Ro} \Big\}\, \vec{u}_{Ro}$$

$$+ \frac{5\, \mu_E\, a^4\, J_{40}}{2R_o^{\,8}} \Big\{ \sin^2 \delta_o\, (77 \sin^2 \delta_o - 27)\, [\Box \vec{u}_{Ro} - \tfrac{1}{2}\, \mathcal{T}r(\Box)\, \vec{u}_{Ro}]$$

$$+ 2 \sin \delta_o\, (3-14 \sin^2 \delta_o)\, [\Box \vec{n} - \tfrac{1}{2}\, \mathcal{T}r(\Box)\, \vec{n}] \Big\} \,, \qquad \text{(A-25)} \atop \text{(Conc.)}$$

$$\vec{M}_{g\delta}^{(4,o)} = \int_{(m)} \vec{r} \times d\vec{F}_{g\delta}^{(4,o)} = \vec{n} \times \left\{ \frac{5\,\mu_E\, a^4\, J_{40}}{2} \int_{(m)} \frac{1}{R^6}\,(7 \sin^2 \delta - 3) \sin \delta\, \vec{r}\, dm \right\}$$

$$- \vec{R}_o \times \left\{ \frac{5\,\mu_E\, a^4\, J_{40}}{2} \int_{(m)} \frac{1}{R^7}\,(7 \sin^2 \delta - 3) \sin^2 \delta\, \vec{r}\, dm \right\}$$

$$\approx \frac{5\,\mu_E\, a^4\, J_{40}}{2R_o^{\,7}} \Big\{ 21 \sin \delta_o\, (3 \sin^2 \delta_o - 1)\, \vec{n} \times [\Box \vec{u}_{Ro} - \tfrac{1}{2}\, \mathcal{T}r(\Box)\, \vec{u}_{Ro}]$$

$$-2 \sin \delta_o\, (3 - 14 \sin^2 \delta_o)\, \vec{u}_{Ro} \times [\Box \vec{n} - \tfrac{1}{2}\, \mathcal{T}r(\Box)\, \vec{n}]$$

$$+ 3\,(1 - 7 \sin^2 \delta_o)\, \vec{n} \times (\Box \vec{n}) - \sin^2 \delta_o\, (77 \sin^2 \delta_o - 27)\, \vec{u}_{Ro} \times (\Box \vec{u}_{Ro}) \Big\}\,, \tag{A-26}$$

As remarked in a previous paragraph, the variation of the longitude dependent terms (in the expressions for the components of $\vec{A}_g$) over the vehicle will be ignored, it being assumed that their values at any point of the vehicle are equal to their values at the vehicle CM. An immediate consequence of this assumption is that the approximation to $\vec{M}_{gB}$ is devoid of any contribution from the tesseral harmonics whose contribution to $\vec{F}_{gB}$ must then be

$$\vec{F}^{\,*}_{gB} = mT\, [\alpha_o]^T_{(3)}\, [-\delta_o]^T_{(2)}\, (\vec{A}^*_g)_{(o)} \;. \tag{A-27}$$

In equation (A-27), m denotes the mass of the entire vehicle system; $T \equiv T(S \rightarrow B)$ is the rotation matrix defined in Section 3; the angles α_o and δ_o are, respectively, the right ascension and declination (geocentric latitude) of the vehicle CM, these being given by (among other expressions)

$$\alpha_o = \tan^{-1} (Y_s^{CM}/X_s^{CM}) \quad , \quad 0 \leq \alpha < 2\pi \quad ,$$

$$\delta_o = \sin^{-1} (Z_s^{CM}/R_s^{CM}) \quad , \quad -\pi/2 \leq \delta_o \leq \pi/2 \quad ,$$

where X_s^{CM}, Y_s^{CM} and Z_s^{CM} are the components of $\vec{R}_s^{CM}$ and $R_s^{CM} = |\vec{R}_s^{CM}|$; and $\vec{A}_g^*$ denotes the "longitude dependent part" of $\vec{A}_g$ with $\vec{u}_R$, $\vec{u}_\lambda$ and $\vec{u}_\delta$ components defined by

$A_{gR}^* = A_{gR}$ minus the terms in J_{20}, J_{30}, J_{40} and minus the term $-\mu_E/R^2$

$A_{g\lambda}^* = A_{g\lambda}$

$A_{g\delta}^* = A_{g\delta}$ minus the terms in J_{20}, J_{30}, and J_{40} ,

the subscript (o) on $\vec{A}_g^*$ indicating evaluation at $R = R_o = |\vec{R}_s^{CM}|$, $\delta = \delta_o$, $\lambda = \lambda_o$, where λ_o, the east longitude of the vehicle CM, is given by

$$\lambda_o = \alpha_o - \alpha_P - \omega_\ell(t - t_o) \quad , \quad 0 \leq \lambda_o \leq 2\pi \quad .$$

From the foregoing, the approximations to $\vec{F}_{gB}$ and $\vec{M}_{gB}$, in the notation of this paper, read as follows

$$\vec{F}_{gB} \approx \vec{F}_{gB}^* + \sum_{K=0,2,3,4} \vec{F}_{gR}^{(K,o)} + \sum_{K=2,3,4} \vec{F}_{g\delta}^{(K,o)} \quad , \tag{A-28}$$

$$\vec{M}_{gB} \approx \sum_{K=0,2,3,4} \vec{M}_{gR}^{(K,o)} + \sum_{\vec{K}=2,3,4} \vec{M}_{g\delta}^{(K,o)} \quad . \tag{A-29}$$

The reader should be aware that the dependence of $\vec{F}_{gB}$ and $\vec{M}_{gB}$ upon vehicle attitude enters through the dependence of the unit vectors $\vec{u}_{Ro}$ and $\vec{n}$ upon the Euler angles specifying the orientation of the B-frame relative to the S-frame. One should notice also that it has not been necessary to introduce an additional reference frame (an "orbital" frame whose attitude relative to the S-frame also changes); and therefore, no additional attitude reference angles, thereby circumventing the need to transform the inertia matrix. The more prominent writers on the subject of gravity torque have used an orbital reference system (such as that alluded to) in defining "attitude deviation" angles in terms of which they expressed the potential function (via the transformed inertia matrix) and subsequently differentiated the potential function with respect to these angles to get the components of gravity torque.

Though the expressions for $\vec{F}_{gB}$ and $\vec{M}_{gB}$ were derived with the entire vehicle system in mind, they are applicable to any component (as if the component were up there alone) by a proper interpretation of the symbols, that is, by an appropriate alteration and/or change in the meaning of the symbols. In particular, they can be made to apply to the i^{th} flexible appendage of the space station model described in Section 1, and use will be made of them in writing the contribution of gravity to the equation corresponding to the coordinate θ_i (associated with the i^{th} flexible appendage and denoting the rotation of that appendage, as a whole, relative to the rigid central carrier).

The symbol $\vec{F}_{gi}$, denoting the i-resolution (see Section 2 for the definition of the $x_i y_i z_i$ frame) of the Earth's gravitational force on appendage i alone, presents** itself in the equation corresponding to the coordinate θ_i. On introducing the symbols

m_i = mass of flexible appendage i

$(\vec{A}^*_{gi})_{(o)} = \vec{A}^*_g$ evaluated at $R = R_{oi}$, $\lambda = \lambda_{oi}$, $\delta = \delta_{oi}$

$\vec{R}_s^{CMi} = [X_s^{CMi}, Y_s^{CMi}, Z_s^{CMi}]^T$ = position referred to the S-frame of the CM of flexible appendage i (in its deformed state)

$\alpha_{oi} = \tan^{-1}(Y_s^{CMi}/X_s^{CMi})$ = right ascension of the CM of the flexible appendage i ($0 \le \alpha_{oi} < 2\pi$)

$\delta_{oi} = \sin^{-1}(Z_s^{CMi}/|\vec{R}_s^{CMi}|) = \sin^{-1}(\vec{n}_i^T \vec{u}_{Roi})$ = declination or geocentric latitude of the CM of appendage i ($-\pi/2 \le \delta_{oi} \le \pi/2$)

$\lambda_{oi} = \alpha_{oi} - \alpha_P - \omega_\ell(t - t_o)$ = east longitude of the CM of appendage i ($0 \le \lambda_{oi} < 2\pi$)

$\vec{R}_{oi} = \tilde{T}_i T \vec{R}_s^{CMi}$ = i - resolution of $\vec{R}_s^{CMi}$ (see Section 3)

$R_{oi} = |\vec{R}_{oi}| = |\vec{R}_s^{CMi}|$

$\vec{u}_{Roi} = \vec{R}_{oi}/R_{oi}$

$\vec{n}_i = \tilde{T}_i T \vec{K}_s$ = i-resolution of $\vec{K}_s$ (see Sections 2 and 3),

**As one factor of a scalar triple product.

it should be obvious, without recourse to the method of deriving (A-28), that

$$\vec{F}_{gi} \approx \tilde{T}_i\ T\ [\alpha_{oi}]^T_{(3)}\ [-\delta_{oi}]^T_{(2)} \left\{m_i\ (\vec{A}^*_{gi})_{(o)}\right\} + \sum_{K=0,2,3,4} \vec{F}_{gRi}{}^{(K,o)}$$

$$+ \sum_{K=2,3,4} \vec{F}_{g\delta i}{}^{(K,o)} \quad , \tag{A-30}$$

the $\vec{F}_{gRi}{}^{(k,o)}$ and $\vec{F}_{g\delta i}{}^{(k,o)}$ in equation (A-30) being found by simply replacing m, R_o, $\vec{u}_{Ro}$, $\vec{n}$, sin δ_o, $\square$, and $Tr(\square)$ in the expressions for $\vec{F}_{gR}{}^{(k,o)}$ and $\vec{F}_{g\delta}{}^{(k,o)}$ by m_i, R_{oi}, $\vec{u}_{Roi}$, $\vec{n}_i$, sin δ_{oi}, $\square^i$, and $Tr(\square^i)$, respectively. The symbol $\square^i$ is to be interpreted as the inertia matrix of flexible appendage i (in its deformed state) referred to the axes $x_i'y_i'z_i'$ defined in Section 2 (recall that the axes $x_i'y_i'z_i'$, though oriented as $x_iy_iz_i$, have origin at the instantaneous CM of flexible appendage i).

Also, appearing (as one factor of the scalar product of two vectors) in the equation corresponding to the coordinate θ_i is the symbol $\vec{M}_{gi}$ representing the i-resolution of the gravity torque on flexible appendage i alone about its instantaneous CM. By a direct application of equation (A-29), one has immediately the approximation

$$\vec{M}_{gi} \approx \sum_{K=0,2,3,4} \vec{M}_{gRi}{}^{(K,o)} + \sum_{K=2,3,4} \vec{M}_{g\delta i}{}^{(K,o)} \quad , \tag{A-31}$$

the expressions for the $\vec{M}_{gRi}{}^{(k,o)}$ and $\vec{M}_{g\delta i}{}^{(k,o)}$ being obtained from those for $\vec{M}_{gR}{}^{(k,o)}$ and $\vec{M}_{g\delta}{}^{(k,o)}$ by merely replacing R_o, $\vec{u}_{Ro}$, $\vec{n}$, sin δ_o, $\square$ and $Tr(\square)$ by R_{oi}, $\vec{u}_{Roi}$, $\vec{n}_i$, sin δ_{oi}, $\square^i$ and $Tr(\square^i)$, respectively.

In the equation corresponding to the coordinate θ_i the terms in $\vec{F}_{gi}$ and $\vec{M}_{gi}$ can be combined to give the expression

$$\vec{i}_i^{\,T}\ \{\vec{M}_{gi} + (\vec{\ell}_i{}^{(o)} + \psi^{(i)}\ \vec{\eta}^{(i)})\ \times\ \vec{F}_{gi}\} \quad , \tag{A-32}$$

which, after some reflection on the matter, one should recognize as the x_i component of the gravity torque on flexible appendage i alone about the origin of $x_iy_iz_i$, (see again the coordinate system definitions of Section 2). In the expression (A-32), the vector $\vec{\ell}_i^{(o)}$ is the position vector, referred to $x_iy_iz_i$, of the CM of appendage i when $\vec{\eta}^{(i)} = \vec{0}$, that is, when appendage i is undeformed; $\vec{\eta}^{(i)}$ is the N_i x 1 column matrix of generalized bending displacement coordinates η_1^i, η_2^i, ... η_{Ni}^i (functions of time to be determined); and the 3 x N_i matrix $\Psi^{(i)}$ has for its j^{th} column the column matrix $\vec{\psi}_j^{(i)}$ defined by

$$\vec{\psi}_j^{(i)} = \frac{1}{m_i} \int_{(m_i)} \vec{\varphi}_j^{(i)} \, dm \quad , \quad j = 1, \ldots N_i \quad , \tag{A-33}$$

the 3 x 1 column matrix $\vec{\varphi}_j^{(i)}$ being the j^{th} column of the undamped modal matrix $\Phi^{(i)}$ associated with flexible appendage i, i = 1, ... NA. The problem of deciding the number N_i, that is, an adequate number of mode shape functions $\vec{\varphi}_j^{(i)}$ to include in the truncated modal matrix $\Phi^{(i)}$, will not be addressed in this paper since suitable criteria for selecting the modes to be accounted for have been the subject of investigation by more learned men whose work will be found in the literature. It should be remarked (if not already clear to the reader) that it is herein supposed that $\vec{\varphi}_j^{(i)}$ is a function of position referred to the i-frame so that one could write (in the usual functional notation)

$$\vec{\varphi}_j^{(i)} \equiv \vec{\varphi}_j(\vec{r}_i) = \begin{bmatrix} \varphi_{jx}(x_i, y_i, z_i) \\ \varphi_{jy}(x_i, y_i, z_i) \\ \varphi_{jz}(x_i, y_i, z_i) \end{bmatrix} \quad , \quad j = 1, \ldots N_i \quad .$$

APPENDIX B

ON AERODYNAMIC FORCE AND AERODYNAMIC TORQUE

At space station altitudes, the flow regime is, presumably, free molecular flow. Following the development in Reference 10 pertinent to such a regime, the aerodynamic force, in the notation of this paper, is given by

$$\vec{F}_{AERO} = \int_{(A_T)} \begin{bmatrix} dF_{AERO}^{(x)} \\ dF_{AERO}^{(y)} \\ dF_{AERO}^{(z)} \end{bmatrix} \tag{B-1}$$

where the subscript A_T on the integral sign indicates that the integration extends over the vehicle surface* and

$$dF_{AERO}^{(\xi)} = (1/2)\,\rho_a V_R^2 \Bigg\{ [\sigma_t\,(\varepsilon_1\,\gamma_{1\xi} + \varepsilon_3\,\gamma_{3\xi}) + (2 - \sigma_n)\,\varepsilon_2\,\gamma_{2\xi}]$$

$$\left[\varepsilon_2\,(1 + \text{erf}\,\varepsilon_2 S) + \frac{1}{S\sqrt{\pi}}\,\exp\,(-\varepsilon_2^2\,S^2)\right] + \frac{\gamma_{2\xi}}{2S^2}\,(2 - \sigma_n)\,(1 + \text{erf}\,\varepsilon_2 S)$$

$$+ \frac{\sigma_n\,\gamma_{2\xi}}{2}\sqrt{\frac{T'_w}{T'_i}}\,\left[\frac{\varepsilon_2\,\sqrt{\pi}}{S}\,(1 + \text{erf}\,\varepsilon_2 S)\right.$$

$$\left.\left. + \frac{1}{S^2}\,\exp\,(-\varepsilon_2^2\,S^2)\right]\right\}\,dA\quad ,\quad \xi = x, y, z\quad . \tag{B-2}$$

The symbol ρ_a denotes local atmospheric density. V_R is the magnitude of the relative velocity vector, $\vec{V}_R$, herein defined** by

*The vehicle surface is here presumed a convex surface, thereby ruling out the effect of molecules reflected from other parts of the body upon the force on a differential element of surface area.

**Note the tacit assumption that the velocity of all points of the surface is the same as that of the vehicle CM.

$$\vec{V}_R = T\,(-\dot{\vec{R}}_S^{\,CM}) \quad , \quad T \equiv T\,(S\rightarrow B) \quad , \tag{B-3}$$

The ε_1, i = 1, 2, 3, are the direction cosines of the relative velocity vector in the local coordinate system, the local y-axis being directed as the inward normal to the surface and the local x and z axes tangent to the surface with arbitrary (but sensibly chosen) directions. If $T_{BL} \equiv T\,(B\rightarrow LOCAL)$ denotes the rotation matrix defining the transformation from the B resolution of a vector to its resolution along the local axes, then the ε_i, i = 1, 2, 3, are determined by

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = T_{BL} \left(\frac{\vec{V}_R}{V_R} \right) \quad . \tag{B-4}$$

The triples $\gamma_{1\xi}$, $\gamma_{2\xi}$, $\gamma_{3\xi}$, ξ = x, y, z, are, respectively, the direction cosines of the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ in the local coordinate system, that is,

$$\begin{bmatrix} \gamma_{1x} \\ \gamma_{2x} \\ \gamma_{3x} \end{bmatrix} = T_{BL}\,\vec{i} = T_{BL} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{B-5}$$

$$\begin{bmatrix} \gamma_{1y} \\ \gamma_{2y} \\ \gamma_{3y} \end{bmatrix} = T_{BL}\,\vec{j} = T_{BL} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{B-6}$$

$$\begin{bmatrix} \gamma_{1z} \\ \gamma_{2z} \\ \gamma_{3z} \end{bmatrix} = T_{BL}\,\vec{k} = T_{BL} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{B-7}$$

The symbols T_w' and T_i' denote, respectively, the surface temperature and temperature of the incident molecules. The dimensionless quantity S, called the molecular speed ratio, is defined by

$$S = V_R \;/\; \sqrt{2\,RT_i'} \quad ,$$

the symbol R denoting the appropriate gas constant. Current best estimates of the tangential and normal reflection coefficients, σ_t and σ_n, respectively, restrict both to the range 0.8 to 1.0, completely diffuse reflection corresponding to $\sigma_t = \sigma_n = 1$. These dimensionless coefficients are also called momentum accommodation coefficients and sometimes momentum exchange coefficients.

The symbol erf $(\varepsilon_2 S)$ denotes the error function with argument $\varepsilon_2 S$, that is,

$$\operatorname{erf}\,(\varepsilon_2 S) = \frac{2}{\sqrt{\pi}} \int_o^{\varepsilon_2 S} e^{-x^2}\, dx \quad .$$

In a simulation program, the parameters ρ_a and T_i' should be available from atmosphere tables based upon an acceptable atmosphere model while the assumption of a constant T_w' would not be unlikely.

From equations (B-1) through (B-7), it should be obvious that $\vec{F}_{AERO}$ has the B-resolution as does the aerodynamic moment, $\vec{M}_{AERO}$, given by equation (B-8).

$$\vec{M}_{AERO} = \int_{(A_T)} \vec{r} \times d\vec{F}_{AERO} = \int_{(A_T)} \vec{r} \times \begin{bmatrix} dF_{AERO}^{(x)} \\ dF_{AERO}^{(y)} \\ dF_{AERO}^{(z)} \end{bmatrix} \tag{B-8}$$

wherein $\vec{r}$ denotes the position referred to the B-frame of the point of application of $d\vec{F}_{AERO}$.

Except for plane and cylindrical surfaces, the surface integrals in equations (B-1) and (B-8) defy closed form evaluation. In the case of conical surfaces and spherical segments, certain simplifications are possible, but even after such simplifications are made, there remain integrals which yield only to numerical methods.

As pointed out in Reference 10, a body in free molecular flow does not alter the flow, thereby permitting one to subdivide the body into a finite number of "simple" bodies (flat plates, cylinders, cones, etc.), determine the contribution of each part to $\vec{F}_{AERO}$ and $\vec{M}_{AERO}$, and then combine the individual contributions by simple addition.

In the reference cited above (Reference 10), the development of a general expression for the differential of force coefficient corresponding to a prescribed direction is followed by the development of expressions for the normal force coefficient, axial force coefficient, and moment coefficient pertinent to a flat plate, a circular cylinder, a right circular cone frustum, and a spherical segment, the developments being valid for complete diffuse reflection. In the notation of this paper, the differential of the force coefficient C_ξ corresponding to the spacecraft* ξ-axis direction (ξ = x, y, z) is obtainable from equation (B-2) by dividing both of its members by $\frac{1}{2}\, \rho_a\, V^2_R\, A_{REF}$ (A_{REF} denoting a reference area), setting $\sigma_t = \sigma_n = 1$, and replacing T_w' by T_r' (the symbol T_r' denoting the temperature of the reflected molecules), the result being

$$dC_\xi = \frac{1}{A_{REF}} \left\{ (\varepsilon_1\, \gamma_{1\xi} + \varepsilon_2\, \gamma_{2\xi} + \varepsilon_3\, \gamma_{3\xi}) \left[\varepsilon_2\, (1 + \text{erf}\ \varepsilon_2 S) + \frac{1}{S\sqrt{\pi}} \exp(-\varepsilon_2^2\, S^2) \right] + \frac{\gamma_{2\xi}}{2S^2}\, (1 + \text{erf}\ \varepsilon_2 S) + \frac{\gamma_{2\xi}}{2} \sqrt{\frac{T_r'}{T_i'}} \left[\frac{\varepsilon_2\sqrt{\pi}}{S}\, (1 + \text{erf}\ \varepsilon_2 S) + \frac{1}{S^2} \exp(-\varepsilon_2^2\, S^2) \right] \right\} dA \quad , \quad \text{(B-9)}$$

$$\xi = x, y, z \quad .$$

*One should be aware that the spacecraft ξ-axis direction does not, in general, coincide with the "local" ξ-axis direction.

In Reference 10, the development of equation (B-9) precedes that of (B-2) which follows from (B-9) via introduction of the reflection coefficients σ_t and σ_n, the purpose of the reflection coefficients being to admit both specular and diffuse reflection (complete specular reflection is realized when $\sigma_t = \sigma_n = 0$).

Once the dimensionless coefficient C_ξ is known, the force component $F^{(\xi)}_{AERO}$ is found via the familiar equation

$$F^{(\xi)}_{AERO} = (1/2)\; C_\xi \; \rho_a V^2_R \; A_{REF} \quad , \qquad \xi = x, y, z \quad . \tag{B-10}$$

If C_ξ and C_η, $\xi \neq \eta$, denote, respectively, the force coefficients pertinent to the B-frame ξ-axis and η-axis directions, then the differential, $d\tilde{C}_\zeta$, of the moment coefficient $\tilde{C}_\zeta$ pertinent to the B-frame ζ-axis, the ζ-axis being orthogonal to both the ξ-axis and the η-axis, is expressible in terms of the differentials dC_ξ and dC_η. Subsequent integration then gives $\tilde{C}_\zeta$, which determines the ζ-component of $\vec{M}_{AERO}$ in accordance with

$$M^{(\zeta)}_{AERO} = (1/2)\; \rho_a \; V^2_R \; \tilde{C}_\zeta \; A_{REF} \; L_{REF} \quad , \tag{B-11}$$

the symbol L_{REF} denoting a reference length.

An alternate expression (see References 9 and 12) for $\vec{F}_{AERO}$, somewhat more tractable than (and deemed "inferior" to) that above is the following:

$$\vec{F}_{AERO} \approx -\rho_a \; V^2_R \int_{(A_T)} [(2 - \sigma_n - \sigma_t)\; (\vec{\ell}_v \cdot \vec{\ell}_n)^2 \; \vec{\ell}_n + \sigma_t \; (\vec{\ell}_v \cdot \vec{\ell}_n)\; \vec{\ell}_v]\; dA \quad . \tag{B-12}$$

In equation (B-12), both the unit vectors $\vec{\ell}_n$ and $\vec{\ell}_v$ are presumed expressed on the B-vector basis, the vector $\vec{\ell}_n$ being the unit outward normal to the differential element of surface area dA and $\vec{\ell}_v$ defined by

$$\vec{\ell}_v = \vec{V}_R / V_R \quad , \tag{B-13}$$

the $\vec{V}_R$ in equation (B-13) being the negative of that defined by (B-3).

APPENDIX C

ON THE VECTORS $\vec{F}_{SOLAR}$ AND $\vec{M}_{SOLAR}$

References 11 and 13 (among others) provide the following expression for the force attributed to direct solar radiation

$$\vec{F}_{SOLAR} = P_{SOLAR} \int_{(A_T)} \{- [(1 + C_{rs}) \cos \hat{\theta} + (2/3)\, C_{rd}]\, \vec{n}$$

$$+ [(1 - C_{rs}) \sin \hat{\theta}]\, \vec{\tau}\} \cos \hat{\theta}\, dA \quad . \tag{C-1}$$

In equation (C-1), the unit vectors $\vec{n}$ and $\vec{\tau}$ define, respectively, the outward normal and tangential directions to the differential element of surface area dA; the angle $\hat{\theta}$ is the angle between the incident ray and $\vec{n}$; the symbol C_{rs} denotes the coefficient of specular reflection (the fraction of incident radiation reflected specularly) and C_{rd} is the coefficient of diffuse reflection (the fraction of incident radiation reflected diffusely). The unit vectors $\vec{n}$ and $\vec{\tau}$ are here presumed to have the B-resolution so that $\vec{F}_{SOLAR}$ too has the B-resolution. The value assigned to P_{SOLAR}, the solar radiation pressure, depends upon the vehicle's position. If the vehicle lies within the Earth's umbra $P_{SOLAR} = 0$; if the vehicle is within the penumbra, $0 < P_{SOLAR} <$ (MAXIMUM SOLAR RADIATION PRESSURE IN VICINITY OF EARTH); and if the vehicle lies in neither umbra nor penumbra, the value assigned to P_{SOLAR} in a simulation program is likely to be its value at 1 AU (one astronomical unit).

Regarding equation (C-1), it should be remarked that it does not account for the effect of radiation reflected from one vehicle part upon another part or shading of one part by another. Furthermore, it should be obvious that the integration need not extend over the entire surface area but only over that part exposed to direct solar radiation.

Except for very special configurations, not likely to be encountered in practice, a closed form expression for the surface integral in (C-1) is out of the question. However simple the surface configuration may be, the evaluation of $\vec{F}_{SOLAR}$ will probably always rest upon numerical methods for evaluating integrals.

With $\vec{r}$ denoting (as usual) position referred to the B-frame, having origin at the vehicle CM (see Section 2), one can define the solar radiation torque about the system CM by (C-2).

$$\vec{M}_{SOLAR} = P_{SOLAR} \int_{(A_T)} \vec{r} \times \{-[(1 + C_{rs}) \cos \hat{\theta} + (2/3)\, C_{rd}]\, \vec{n}$$

$$+ [(1 - C_{rs}) \sin \hat{\theta}]\, \vec{\tau}\} \cos \hat{\theta}\, dA \quad . \tag{C-2}$$

Assuming it possible to subdivide the surface area into a finite number of "sub-areas" on the i^{th} of which is exerted the resultant solar radiation force $\vec{F}^{(i)}_{SOLAR}$ at the point with position vector $\vec{r}_{cpi}$ relative to the B-frame, one can write $\vec{M}_{SOLAR}$ as the sum of vector products

$$\vec{M}_{SOLAR} = \sum_i \vec{r}_{cpi} \times \vec{F}^{(i)}_{SOLAR} \quad . \tag{C-3}$$

In general, one can go no farther than the integral expressions of equations (C-1) and (C-2).

APPENDIX D

ON THE VECTOR $\vec{\tilde{r}}_{cm}$

To find an expression for $\vec{\tilde{r}}_{cm}$, one has only to invoke the definition of the center of mass to write

$$\int_{(m)} \vec{r}\, dm = \int_{(m)} (\vec{\tilde{r}} - \vec{\tilde{r}}_{cm})\, dm = \vec{0} \quad ,$$

subdivide m as indicated by the subscripts on the integral signs in equation (4-16), and integrate after substituting from (4-25), (4-28), (4-31), (4-34), (4-35), (4-37), (4-39), (4-41), (4-42), and (4-43), attention being paid to (4-52.1) and (4-52.2) in integrating over m_i. The result is expressible as

$$\vec{\tilde{r}}_{CM} = \vec{\tilde{r}}_{CM}^{(o)} + \frac{1}{m}\left\{ \sum_{i=1}^{NA} m_i \left[(\tilde{T}_i^T - \tilde{T}_{io}^T)\, \vec{\ell}_i^{(o)} + \tilde{T}_i^T\, \Psi^{(i)}\, \vec{\eta}^{(i)} \right] + \sum_{i=1}^{NP} m_{Pi}\, \xi_{Pi}\, \vec{\Lambda}_{Pi} - \sum_{i=1}^{NSE} m_{Ei}\, \ell_{Ei}\, (\vec{\Lambda}_{Ei} - \vec{\Lambda}_{Eio}) \right\} \tag{D-1}$$

where $\vec{\tilde{r}}_{CM}^{(o)}$, $\tilde{T}_{io}^T$ and $\vec{\Lambda}_{Eio}$ are given by

$$\vec{\tilde{r}}_{CM}^{(o)} = \frac{1}{m}\left\{ \int_{(m_o+m_f)} \vec{\tilde{r}}\, dm + \sum_{i=1}^{NSE} m_{Ei}\, (\vec{\tilde{r}}_{Ei} - \ell_{Ei}\, \vec{\Lambda}_{Eio}) + \sum_{i=1}^{NP} m_{Pi}\, \vec{\tilde{r}}_{Pi} + \sum_{i=1}^{NR} m_{Ri}\, \vec{\tilde{r}}_{Ri} \right.$$

$$+ \sum_{i=1}^{NA} m_i\, (\vec{\tilde{r}}_i + \tilde{T}_{io}^T\, \vec{\ell}_i^{(o)}) + \sum_{i=1}^{NSDOF} (m'_{Gi}\, \vec{\tilde{r}}_{Gi} + m'_{gi}\, \vec{\tilde{r}}_{gi})$$

$$\left. + \sum_{i=1}^{N2DOF} (m_{OG}\, \vec{\tilde{r}}_{OG} + m_{IG}\, \vec{\tilde{r}}_{IG} + m_g\, \vec{\tilde{r}}_g)_{(i)} \right\} \tag{D-2}$$

$$\tilde{T}_{io}^{T} = (\tilde{T}_i^{T})_{\theta_i=0} = T_i^{T}$$

$$\vec{\Lambda}_{Eio} = (\vec{\Lambda}_{Ei})_{\beta_{yi}=\beta_{pi}=0} \quad .$$

The integral in equation (D-2) is herein supposed a known function of time so that $\vec{\tilde{r}}_{cm}^{(o)}$ is presumed a known function of time.

Assuming the time derivatives of m negligible leads to the approximations

$$\dot{\vec{\tilde{r}}}_{CM} \approx \dot{\vec{\tilde{r}}}_{CM}^{(o)} + \frac{1}{m}\left\{ \sum_{i=1}^{NA} m_i \left[\dot{\tilde{T}}_i^{T} \left(\vec{\ell}_i^{(o)} + \psi^{(i)} \vec{\eta}^{(i)}\right) + \tilde{T}_i^{T} \psi^{(i)} \dot{\vec{\eta}}^{(i)}\right] \right.$$

$$\left. + \sum_{i=1}^{NP} m_{Pi} \dot{\xi}_{Pi} \vec{\Lambda}_{Pi} - \sum_{i=1}^{NSE} m_{Ei} \ell_{Ei} \dot{\vec{\Lambda}}_{Ei} \right\} \qquad \text{(D-3)}$$

$$\ddot{\vec{\tilde{r}}}_{CM} \approx \ddot{\vec{\tilde{r}}}_{CM}^{(o)} + \frac{1}{m}\left\{ \sum_{i=1}^{NA} m_i \left[\ddot{\tilde{T}}_i^{T} \left(\vec{\ell}_i^{(o)} + \psi^{(i)} \vec{\eta}^{(i)}\right) + 2\, \dot{\tilde{T}}_i^{T} \psi^{(i)} \dot{\vec{\eta}}^{(i)} \right.\right.$$

$$\left.\left. + \tilde{T}_i^{T} \psi^{(i)} \ddot{\vec{\eta}}^{(i)}\right] + \sum_{i=1}^{NP} m_{Pi} \ddot{\xi}_{Pi} \vec{\Lambda}_{Pi} - \sum_{i=1}^{NSE} m_{Ei} \ell_{Ei} \ddot{\vec{\Lambda}}_{Ei} \right\} \quad . \qquad \text{(D-4)}$$

APPENDIX E

ON THE MATRICES $\square$, $\square^i$ AND I^i

Among the symbols necessary to a "working" expression* for the inertia matrix $\square$, there are some which have not yet been defined. These include the following (with the exception of $\vec{q}_{Ei}$ whose definition will be repeated for ready reference): $\vec{\tilde{q}}_o$, the position, relative to the $\tilde{B}$-frame, of the CM of the rigid central carrier; $\vec{q}_o$, the position vector, relative to the B-frame, of the CM of the rigid central carrier; $\tilde{I}^o_{CM}$, the inertia matrix of the rigid central carrier referred to axes (oriented as both the B and $\tilde{B}$ frames) with origin at its CM; I^f, the inertia matrix of the fluid referred to the B-frame (further remarks regarding I^f will be made later); and the vectors $\vec{q}_i$, $\vec{q}_{Ei}$, $\vec{q}_{Ri}$, $\vec{q}_{Pi}$, $\vec{q}_{Gi}$, $\vec{q}_{gi}$, $\vec{q}_{OG}$, $\vec{q}_{IG}$, and $\vec{q}_g$, denoting, respectively, the instantaneous positions relative to the B-frame of the CM of flexible appendage i, the CM of swivel engine i, the CM of rotor i, the i^{th} point mass, the CM of the gimbal of the i^{th} SDOF CMG, the CM of the gyro element of the i^{th} SDOF CMG, the CM of the outer gimbal of the 2 DOF CMG in question, the CM of the inner gimbal of the 2 DOF CMG in question, and the CM of the gyro element of the 2 DOF CMG in question (the words "in question" being used to adhere to a previous agreement that an additional subscript i, or superscript i, would be suppressed on all symbols relating to a 2 DOF CMG). For a specific vehicle configuration, $\vec{\tilde{q}}_o$ will be a known constant vector and $\tilde{I}^o_{CM}$ a known matrix of constants, while I^f, though variable, "can be determined." By simply stating that I^f is "susceptible to being determined," the author has tacitly avoided the construction of a mechanical system whose motion duplicates the response of the fluid. Introduction of a mechanical analog would, of course, increase the number of system coordinates and require that more terms be added to both the translational and rotational equations. Knowing the rate at which mass is depleted through propellant consumption and assuming negligible sloshing, one might define I^f adequately via tables with time as argument.

In terms of symbols already defined, the position vectors (referred to the B-frame) of the previous paragraph must be given by

---

*By "working" expression is meant one which can be put to practical use such as serving as a guide to a programmer in coding a subroutine (of a simulation program) whose function is the construction of $\square$.

$$\vec{q}_o = \vec{\tilde{q}}_o - \vec{\tilde{r}}_{CM}$$

$$\vec{q}_i = \vec{\tilde{r}}_i - \vec{\tilde{r}}_{CM} + \tilde{T}_i^T \left(\vec{\ell}_i^{(o)} + \psi^{(i)} \vec{\eta}^{(i)} \right) \quad , \quad i = 1, \ldots NA \quad ,$$

$$\vec{q}_{Ei} = \vec{\tilde{r}}_{Ei} - \vec{\tilde{r}}_{CM} - \ell_{Ei} \vec{\Lambda}_{Ei} \quad , \quad i = 1, \ldots NSE \quad ,$$

$$\vec{q}_{Ri} = \vec{\tilde{r}}_{Ri} - \vec{\tilde{r}}_{CM} \quad , \quad i = 1, \ldots NR \quad ,$$

$$\vec{q}_{Pi} = \vec{\tilde{r}}_{Pi} - \vec{\tilde{r}}_{CM} + \xi_{Pi} \vec{\Lambda}_{Pi} \quad , \quad i = 1, \ldots NP \quad ,$$

$$\vec{q}_{Gi} = \vec{\tilde{r}}_{Gi} - \vec{\tilde{r}}_{CM} \quad , \quad i = 1, \ldots NSDOF \quad ,$$

$$\vec{q}_{gi} = \vec{\tilde{r}}_{gi} - \vec{\tilde{r}}_{CM} \quad , \quad i = 1, \ldots NSDOF \quad ,$$

$$\left. \begin{aligned} \vec{q}_{OG} &= \vec{\tilde{r}}_{OG} - \vec{\tilde{r}}_{CM} \quad , \\ \vec{q}_{IG} &= \vec{\tilde{r}}_{IG} - \vec{\tilde{r}}_{CM} \quad , \\ \vec{q}_{g} &= \vec{\tilde{r}}_{g} - \vec{\tilde{r}}_{CM} \quad . \end{aligned} \right\} \quad \text{CORRESPONDING TO EACH 2DOF CMG}$$

In deriving an equation for the computation of the inertia matrix I^i (defined in Section 4 in the paragraph containing equation (4-69)), the author has exploited certain properties of the operator $\mathcal{S}$ introduced in Section 4 (see equations (4-69.3) and (4-69.4) and the definition following). The properties alluded to are among those expressed by the relations

$$\mathcal{S}\,(-\vec{A}) = -\mathcal{S}\,(\vec{A}) = [\mathcal{S}(\vec{A})]^T$$

$$\mathcal{S}\,(\vec{A} \pm \vec{B}) = \mathcal{S}\,(\vec{A}) \pm \mathcal{S}\,(\vec{B})$$

$$\mathcal{S}\,(\vec{A})\, \mathcal{S}\,(\vec{B}) = [\mathcal{S}\,(\vec{B})\, \mathcal{S}\,(\vec{A})]^T$$

$$\mathcal{S}\,(\vec{A}) \sum_{i=1}^{N} a_i\, \mathcal{S}\,(\vec{B}_i) = \sum_{i=1}^{N} a_i\, \mathcal{S}\,(\vec{A})\, \mathcal{S}\,(\vec{B}_i)$$

$$\frac{d}{dt} \{\mathcal{S}\,[(\vec{A})_{(B)}]\} = \mathcal{S}\left\{\left(\frac{d}{dt}\right)_{(B)} (\vec{A})_{(B)}\right\}$$

$$\mathcal{S}\int \vec{A}\ dm = \int \mathcal{S}\ (\vec{A})\ dm$$

$$\frac{\partial}{\partial \eta^i_j} \{\mathcal{S}\,[\vec{\Delta}\ (\vec{r}_i,t)]\} = \mathcal{S}\left\{\left(\frac{\partial}{\partial \eta^i_j}\right)\vec{\Delta}\ (\vec{r}_i,t)\right\} \quad .$$

It is not difficult to show that

$$I^i \equiv (I^i)_{\vec{\eta}^{(i)}\neq\vec{0}} = (I^i)_{\vec{\eta}^{(i)}=\vec{0}} - \sum_{j=1}^{N_i} \eta^i_j\,(\mathcal{J}^i_{rj} + \mathcal{J}^{i\,T}_{rj}) - \sum_{j=1}^{N_i}\sum_{K=1}^{N_i} \eta^i_j\,\eta^i_k\,\mathcal{J}^i_{jK} \quad , \tag{E-1}$$

$$i = 1, \ldots NA,$$

use being made of equations (4-69.3) and (4-69.4). The notation in equation (E-1) should be self explanatory, that is, one should strongly suspect that the symbol $(I^i)_{\vec{\eta}^{(i)}=\vec{0}}$ simply denotes the inertia matrix of appendage i, referred to $x_iy_iz_i$, when the appendage is in its undeformed state which it assumes when $\vec{\eta}^{(i)}=\vec{0}$. The matrix $(I^i)_{\vec{\eta}^{(i)}=\vec{0}}$ will, for each i, be a known matrix of constants (provided by the manufacturer of the appendage).

Having found I^i via equation (E-1), the matrix $\square^i$ is determined by

$$\square^i = I^i - m_i\,\mathcal{S}\,(\vec{\ell}_i^{(o)} + \psi^{(i)}\,\vec{\eta}^{(i)})\,\mathcal{S}\,(-\vec{\ell}_i^{(o)} - \psi^{(i)}\,\vec{\eta}^{(i)}) \quad , \tag{E-2}$$

$\square^i$ being the inertia matrix of appendage i referred to axes $x_i'y_i'z_i'$ which have origin at the instantaneous CM of the appendage and are oriented as $x_iy_iz_i$.

In writing equation (E-2), a direct application was made of what might be called the "generalized transfer theorem for inertia matrices" which is expressible as

$$\square_{xyz} = \mathcal{J}^T\,\square_{\xi\eta\zeta}\,\mathcal{J} + M\,\mathcal{S}(\vec{q})\,\mathcal{S}(-\vec{q}) \tag{E-3}$$

wherein $\square_{\xi\eta\zeta}$ is the inertia matrix of an arbitrarily shaped body of mass M referred to right-handed rectangular axes $\xi\eta\zeta$ with origin at the CM of the body which has position vector $\vec{q}$ relative to the right-handed rectangular axes xyz; $\mathcal{J} \equiv \mathcal{J}(xyz \rightarrow \xi\eta\zeta)$ denotes the rotation matrix defining the transformation <u>from</u> the xyz vector basis <u>to</u> the $\xi\eta\zeta$ vector basis; and $\square_{xyz}$ is the inertia matrix of the body referred to the axes xyz. The expression $\mathcal{S}(\vec{q})\ \mathcal{S}(-\vec{q})$ in equation (E-3) replaces its equivalent, $(\vec{q}^T\ \vec{q})\ I_{(3x3)} - \vec{q}\ \vec{q}^T$, of the author's previous work (Reference 1), the symbol $I_{(3x3)}$ denoting the 3x3 identity matrix. Admittedly, use of the "$\mathcal{S}$" expression is the more convenient.

Repeated application of equation (E-3) leads to the following equation for the system inertia matrix $\square$ (referred to the B-frame)

$$
\begin{aligned}
\square = {} & I^f + \tilde{I}^o_{CM} + m_o\, \mathcal{S}(\vec{q}_o)\, \mathcal{S}(-\vec{q}_o) + \sum_{i=1}^{NA} \{\tilde{T}_i^T\, \square^i\, \tilde{T}_i + m_i\, \mathcal{S}(\vec{q}_i)\, \mathcal{S}(-\vec{q}_i)\} \\
& + \sum_{i=1}^{NSE} \{T^T_{Ei}\, I^{Ei}\, T_{Ei} + m_{Ei}\, \mathcal{S}(\vec{q}_{Ei})\, \mathcal{S}(-\vec{q}_{Ei})\} + \sum_{i=1}^{NP} m_{Pi}\, \mathcal{S}(\vec{q}_{Pi})\, \mathcal{S}(-\vec{q}_{Pi}) \\
& + \sum_{i=1}^{NR} \{T^T_{Ri}\, I^{Ri}\, T_{Ri} + m_{Ri}\, \mathcal{S}(\vec{q}_{Ri})\, \mathcal{S}(-\vec{q}_{Ri})\} + \sum_{i=1}^{NSDOF} \{T^T_{Gi}\, I^{Gi}\, T_{Gi} \\
& + m'_{Gi}\, \mathcal{S}(\vec{q}_{Gi})\, \mathcal{S}(-\vec{q}_{Gi}) + T^T_{gi}\, I^{gi}\, T_{gi} + m'_{gi}\, \mathcal{S}(\vec{q}_{gi})\, \mathcal{S}(-\vec{q}_{gi})\} \\
& + \sum_{i=1}^{N2DOF} \{T^T_{BOG}\, I^{OG}\, T_{BOG} + m_{OG}\, \mathcal{S}(\vec{q}_{OG})\, \mathcal{S}(-\vec{q}_{OG}) + T^T_{BIG}\, I^{IG}\, T_{BIG} \\
& + m_{IG}\, \mathcal{S}(\vec{q}_{IG})\, \mathcal{S}(-\vec{q}_{IG}) + T^T_{Bg}\, I^{g}\, T_{Bg} + m_g\, \mathcal{S}(\vec{q}_g)\, \mathcal{S}(-\vec{q}_g)\}_{(i)} \quad .
\end{aligned}
\tag{E-4}
$$

By direct differentiation of equation (E-4), there follows, after further manipulation (giving due regard to the definitions and relations in Section 3 and to the relevant properties of the operator $\mathcal{S}$),

$$\dot{\Box} = \dot{I}^f + m_o \{\mathcal{S}(\dot{\vec{q}}_o)\, \mathcal{S}(-\vec{q}_o) + \mathcal{S}(\vec{q}_o)\, \mathcal{S}(-\dot{\vec{q}}_o)\}$$

$$+ \sum_{i=1}^{NA} \{\tilde{T}_i^T\, \dot{\Box}^i\, \tilde{T}_i + \tilde{T}_i^T\, (\tilde{\Omega}_i^T\, \Box^i - \Box^i\, \tilde{\Omega}_i^T)\, \tilde{T}_i + m_i\, [\mathcal{S}(\dot{\vec{q}}_i)\, \mathcal{S}(-\vec{q}_i) + \mathcal{S}(\vec{q}_i)\, \mathcal{S}(-\dot{\vec{q}}_i)]\}$$

$$+ \sum_{i=1}^{NSE} \{T_{Ei}^T\, (\tilde{\Omega}_{Ei}^T\, I^{Ei} - I^{Ei}\, \tilde{\Omega}_{Ei}^T)\, T_{Ei} + m_{Ei}\, [\mathcal{S}(\dot{\vec{q}}_{Ei})\, \mathcal{S}(-\vec{q}_{Ei}) + \mathcal{S}(\vec{q}_{Ei})\, \mathcal{S}(-\dot{\vec{q}}_{Ei})]\}$$

$$+ \sum_{i=1}^{NP} m_{Pi}\, \{\mathcal{S}(\dot{\vec{q}}_{Pi})\, \mathcal{S}(-\vec{q}_{Pi}) + \mathcal{S}(\vec{q}_{Pi})\, \mathcal{S}(-\dot{\vec{q}}_{PI})\} + \sum_{i=1}^{NR} m_{Ri}\, \{\mathcal{S}(\dot{\vec{q}}_{Ri})\, \mathcal{S}(-\vec{q}_{Ri})$$

$$+ \mathcal{S}(\vec{q}_{Ri})\, \mathcal{S}(-\dot{\vec{q}}_{Ri})\} + \sum_{i=1}^{NSDOF} \{T_{Gi}^T\, (\Omega_{Gi}^T\, I^{Gi} - I^{Gi}\, \Omega_{Gi}^T)\, T_{Gi}$$

$$+ m'_{Gi}\, [\mathcal{S}(\dot{\vec{q}}_{Gi})\, \mathcal{S}(-\vec{q}_{Gi}) + \mathcal{S}(\vec{q}_{Gi})\, \mathcal{S}(-\dot{\vec{q}}_{Gi})] + T_{gi}^T\, (\Omega_{gi}^T\, I^{gi} - I^{gi}\, \Omega_{gi}^T)\, T_{gi}$$

$$+ T_{Gi}^T\, (\Omega_{Gi}^T\, I^{gi} - I^{gi}\, \Omega_{Gi}^T)\, T_{Gi} + m'_{gi}\, [\mathcal{S}(\dot{\vec{q}}_{gi})\, \mathcal{S}(-\vec{q}_{gi}) + \mathcal{S}(\vec{q}_{gi})\, \mathcal{S}(-\dot{\vec{q}}_{gi})]\}$$

$$+ \sum_{i=1}^{N2DOF} \{T_{BOG}^T\, [\Omega_{OG}^T\, (I^{OG} + I^{IG} + T_{OGIG}^T\, I^g\, T_{OGIG})$$

$$- (I^{OG} + I^{IG} + T_{OGIG}^T\, I^g\, T_{OGIG})\, \Omega_{OG}^T]\, T_{BOG} + T_{BIG}^T\, [\Omega_{IG}^T\, (I^{IG} + I^g)$$

$$- (I^{IG} + I^g)\, \Omega_{IG}^T]\, T_{BIG} + T_{Bg}^T\, (\Omega_g^T\, I^g - I^g\, \Omega_g^T)\, T_{Bg}$$

$$+ m_{OG}\, [\mathcal{S}(\dot{\vec{q}}_{OG})\, \mathcal{S}(-\vec{q}_{OG}) + \mathcal{S}(\vec{q}_{OG})\, \mathcal{S}(-\dot{\vec{q}}_{OG})]$$

$$+ m_{IG}\, [\mathcal{S}(\dot{\vec{q}}_{IG})\, \mathcal{S}(-\vec{q}_{IG}) + \mathcal{S}(\vec{q}_{IG})\, \mathcal{S}(-\dot{\vec{q}}_{IG})]$$

$$+ m_g\, [\mathcal{S}(\dot{\vec{q}}_g)\, \mathcal{S}(-\vec{q}_g) + \mathcal{S}(\vec{q}_g)\, \mathcal{S}(-\dot{\vec{q}}_g)]\}_{(i)} \quad . \qquad \text{(E-5)}$$

Regarding the matrix $\dot{I}^f$, all that will be said is that it is supposed here that a routine method exists for its computation.* As for the time derivatives of the position vectors, it is easily seen that they are given by

$$\dot{\vec{q}}_i = -\dot{\vec{\tilde{r}}}_{CM} + \tilde{T}_i^T \{\Psi^{(i)} \dot{\vec{\eta}}^{(i)} + \tilde{\Omega}_i^T [\vec{\ell}_i^{(o)} + \Psi^{(i)} \vec{\eta}^{(i)}]\}$$

$$\dot{\vec{q}}_{Ei} = -\dot{\vec{\tilde{r}}}_{CM} - \ell_{Ei} \dot{\vec{\Lambda}}_{Ei}$$

$$\dot{\vec{q}}_{Pi} = -\dot{\vec{\tilde{r}}}_{CM} + \dot{\xi}_{Pi} \vec{\Lambda}_{Pi}$$

$$\dot{\vec{q}}_{\xi} = -\dot{\vec{\tilde{r}}}_{CM} \quad , \qquad \xi = 0,\ Ri,\ Gi,\ gi,\ OG,\ IG,\ g,$$

while from equations (E-1) and (E-2), one has

$$\dot{I}^i = -\sum_{j=1}^{N_i} \dot{\eta}_j^i (\mathcal{J}_{rj}^i + \mathcal{J}_{rj}^{i^T}) - \sum_{j=1}^{N_i} \sum_{K=1}^{N_i} (\dot{\eta}_j^i \eta_K^i + \eta_j^i \dot{\eta}_K^i) \mathcal{J}_{jK}^i$$

$$\dot{\Box}^i = \dot{I}^i + m_i \{\mathcal{S}(\Psi^{(i)} \dot{\vec{\eta}}^{(i)}) \mathcal{S}(\vec{\ell}_i^{(o)} + \Psi^{(i)} \vec{\eta}^{(i)})$$

$$+ \mathcal{S}(\vec{\ell}_i^{(o)} + \Psi^{(i)} \vec{\eta}^{(i)}) \mathcal{S}(\Psi^{(i)} \dot{\vec{\eta}}^{(i)})\} \quad .$$

---

* Among the engineers engaged in digital simulation, few, if any, will be inclined to retain the term $\dot{I}^f$ in the expression for $\dot{\Box}$, not to mention certain of the other terms. In fact, most will modify the moment equation by deleting the term $\dot{\Box}\,\vec{\omega}_B$. One who is reluctant to retain all the terms belonging to $\dot{\Box}\vec{\omega}_B$ should be cautioned not to discard terms which could be of major importance, in particular the CMG terms.

APPENDIX F

ON TIME INDEPENDENT INTEGRALS

Appearing in the moment equation, the bending equations, and the equation corresponding to the system coordinate θ_i are terms having as a factor one of the time independent integrals defined below, their independence of time being a consequence of the assumption that the mode shape functions $\vec{\varphi}_j^{(i)}$, $j = 1, \ldots N_i$, are invariant under a rotation of flexible appendage i relative to the rigid central carrier. In a simulation program based upon the equations of this paper (and pertinent to a specific vehicle), those integrals would need to be evaluated but once.

$$\vec{\psi}_j^{(i)} = \frac{1}{m_i} \int_{(m_i)} \vec{\varphi}_j^{(i)} \, dm \quad , \quad j = 1, \ldots N_i \quad , \quad i = 1, \ldots NA$$

$$\vec{C}^i_{oj} = \int_{(m_i)} \vec{r}_i \times \vec{\varphi}_j^{(i)} \, dm \quad , \quad j = 1, \ldots N_i \quad , \quad i = 1, \ldots NA$$

$$\vec{C}^i_{Kj} = \int_{(m_i)} \vec{\varphi}_K^{(i)} \times \vec{\varphi}_j^{(i)} \, dm = - \vec{C}^i_{jK} \quad , \quad j,K = 1, \ldots N_i \quad , \quad i = 1, \ldots NA$$

$$\mathscr{I}^i_{rj} = \int_{(m_i)} \mathcal{S}(\vec{r}_i) \, \mathcal{S}(\vec{\varphi}_j^{(i)}) \, dm = \mathscr{I}^{i\,T}_{jr} \quad , \quad \text{(see equation (4-69.5))}$$

$$j = 1, \ldots N_i \quad , \quad i = 1, \ldots NA$$

$$\mathscr{I}^i_{jK} = \int_{(m_i)} \mathcal{S}(\vec{\varphi}_j^{(i)}) \, \mathcal{S}(\vec{\varphi}_K^{(i)}) \, dm = \mathscr{I}^{i\,T}_{Kj} \quad , \quad j,K = 1, \ldots N_i \quad , \quad i = 1, \ldots NA$$

$$D^i_{oj} = \int_{(m_i)} \vec{r}_i^{\,T} \, \vec{\varphi}_j^{(i)} \, dm$$

$$D^i_{rj} = \int_{(m_i)} \vec{r}_i \; \vec{\varphi}_j^{(i)^T} \, dm = D^{i^T}_{jr}$$

$$D^i_{jK} = \int_{(m_i)} \vec{\varphi}_j^{(i)} \; \vec{\varphi}_K^{(i)^T} \, dm = D^{i^T}_{Kj}$$

$$D^i_{jrr} = \int_{(m_i)} \vec{\varphi}_j^{(i)^T} \; \vec{r}_i \; \vec{r}_i^{\,T} \, dm$$

$$j = 1, \ldots N_i \quad , \quad i = 1, \ldots NA$$

$$D^i_{rrj} = \int_{(m_i)} \vec{r}_i^{\,T} \; \vec{r}_i \; \vec{\varphi}_j^{(i)^T} \, dm \neq D^i_{jrr}$$

$$D^i_{jrK} = \int_{(m_i)} \vec{\varphi}_j^{(i)^T} \; \vec{r}_i \; \vec{\varphi}_K^{(i)^T} \, dm$$

$$D^i_{jKr} = \int_{(m_i)} \vec{\varphi}_j^{(i)^T} \; \vec{\varphi}_K^{(i)} \; \vec{r}_i^{\,T} \, dm$$

$$D^i_{rKj} = \int_{(m_i)} \vec{r}_i^{\,T} \; \vec{\varphi}_k^{(i)} \; \vec{\varphi}_j^{(i)^T} \, dm \neq D^{i^T}_{jKr}$$

$$D^i_{jK\ell} = \int_{(m_i)} \vec{\varphi}_j^{(i)^T} \; \vec{\varphi}_K^{(i)} \; \vec{\varphi}_\ell^{(i)^T} \, dm \neq D^{i^T}_{\ell Kj} \quad , \quad j, K, \ell = 1, \ldots N_i \quad ,$$

$$i = 1, \ldots NA \quad .$$

APPROVAL

EQUATIONS OF MOTION OF A SPACE STATION WITH EMPHASIS ON THE EFFECTS OF THE GRAVITY GRADIENT

By L. P. Tuell

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

G. F. McDONOUGH
Director, Structures and Dynamics Laboratory

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16. ABSTRACT

The derivation of the equations of motion is based upon the principle of virtual work. As developed, these equations apply only to a space vehicle whose physical model consists of a rigid central carrier supporting several flexible appendages (not interconnected), smaller rigid bodies, and point masses. Clearly evident in the equations is the respect paid to the influence of the Earth's gravity field, considerably more than has been the custom in simulating vehicle motion. The effect of unpredictable crew motion is ignored.

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