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The Nonlinear Interaction of Tollmien-Schlichting Waves and Taylor-Görtler Vortices in Curved Channel Flows.

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Abstract

It is known that a viscous fluid flow with curved streamlines can support both Tollmien-Schlichting and Taylor-Görtler instabilities. The question of which linear mode is dominant at finite values of the Reynolds numbers was discussed by Gibson and Cooke (1973). In a situation where both modes are possible on the basis of linear theory a nonlinear theory must be used to determine the effect of the interaction of the instabilities. The details of this interaction are of practical importance because of its possible catastrophic effects on mechanisms used for laminar flow control. Here this interaction is studied in the context of fully developed flows in curved channels. Apart from technical differences associated with boundary layer growth the structures of the instabilities in this flow are very similar to those in the practically more important external boundary layer situation. The interaction is shown to have two distinct phases depending on the size of the disturbances. At very low amplitudes two oblique Tollmien-Schlichting waves interact with a Görtler vortex in such a manner that the amplitudes become infinite at a finite time. This type of interaction is described by ordinary differential amplitude equations with quadratic nonlinearities. A stronger type of interaction occurs at larger disturbance amplitudes and leads to a much more complicated type of evolution equation. The solution of these equations now depends critically on the angle between the Tollmien-Schlichting wave and the Görtler vortex. Thus if this angle is greater than 41.6^o this interaction again terminates in a singularity at a finite time; otherwise the breakdown is exponential taking an infinite time. Moreover the strong interaction can take place in the absence of curvature, in which case the Görtler vortex is entirely driven by the Tollmien-Schlichting waves.

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1. Introduction

In recent years there has been renewed interest in the development of laminar flow wings. In particular the wing developed at NASA Langley, see for example the paper by Harvey and Pride (1982), has been designed so as to achieve laminar flow by a combination of suction and passive mechanisms. This particular wing has significant regions of concave curvature on its lower side so the possibility of transition through the Görtler vortex instability mechanism exists. However, the complicated three-dimensional flow over the wing can also support Tollmien-Schlichting waves and crossflow vortices so the transition process for the flow is likely to be dominated by an interaction involving all three instability mechanisms. Thus an understanding of the nonlinear interaction of these instabilities is necessary if they are to be successfully supressed.

Here the nonlinear interaction of Tollmien-Schlichting waves and Taylor-Görtler vortices in fully developed flows in curved channels is discussed. Hereafter we refer to these modes as "TS" waves and "TG" vortices respectively. In particular the interaction between lower branch TS waves and small wavenumber TG vortices is investigated. It is this type of interaction which will in general occur in external flows, but with the added complication of boundary layer growth. It is known from the work of Hall (1983) that there are significant differences between TG vortices in internal and external flows; the major difference is of course that non-parallel effects cannot be ignored in the external case. Such a difference does not exist between TS waves in external and internal flows and the reader is referred to the papers by Smith (1979a,b), Hall and Smith (1982, 1984) for a discussion of the linear and weakly nonlinear stages of the growth of these instabilities. More recently Smith and Burggraf (1985) and Smith and Stewart (1987) have given a description of the nonlinear development of TS waves well beyond the weakly nonlinear stage.

Though the primary motivation for the present calculation comes from the external boundary layer problem there are many engineering situations where fully developed flows in curved channels can support both types of instability . In addition further motivation for a study of this interaction problem comes from the experimental results of Nerem, Seed and Wood (1972) who found that blood flow in the curved part of the aorta can support turbulent bursts for part of the cardiac cycle. It is known from the work of Sobev (1978) and Papageorgiou (1987) that TS and TG instabilities can occur in unsteady flows in curved pipe flows so it is likely that the transition process there is again dominated by an interaction between these types of disturbance.Experiments of some relevance here , in curved channels or boundary layers ,are presented by Peerhossaini , Clement and Wesfried (1985) ,Kohama(1987) ,and Winoto and Crane (1980) .

The only previous investigation of the external interaction problem is that due to Nayfeh (1981). In that calculation Nayfeh arbitrarily assigned an amplitude to a Görtler vortex obtained by solving the parallel flow instability equations. The effect of this disturbance on a pair of linear oblique TS waves was then calculated. Nayfeh found that the vortex could have a large effect on the growth rate of the TS waves. However, more recently Malik (1986) repeated Nayfeh's calculation and found that the effect of the TG mode on a TS wave was negligible. The difference was apparently due to an inconsistency between Nayfeh's Orr-Sommerfeld boundary layer scaling and that for the vortex flow. Notwithstanding the doubts about the conclusions reached by Nayfeh on the basis of his numerical calculation there are other reasons why it is possible that his work is fatally flawed. The most obvious reason why Nayfeh's approach is not valid is that it is known from the work of Hall (1982a,b) and Hall (1983) that in the wavenumber regime appropriate to his calculation, a linear TG vortex is fully non-parallel and the parallel flow theory used by Nayfeh gives results having no connection with the non-parallel theory. Furthermore, the amplitude of the TG mode cannot be specified arbitrarily; it must be calculated by solving the Navier-Stokes equations numerically as in Hall (1987). The only situation where this is not necessary is when the amplitude of the TG mode is sufficiently small for linear theory to be applicable. However, in that situation the TS and TG modes become coupled so that again it is not possible to specify the amplitude of the Görtler vortex.

In internal flows it is possible to study the two types of interaction problem alluded to above. Firstly, it is reasonably straightforward to calculate numerically the fully nonlinear three-dimensional flow in a curved channel even at Görtler numbers much above the critical. The linear instability of this flow to linear TS waves can then be investigated in a self consistent manner. This type of calculation has already been carried out by Bennett and Hall (1987), who found that both two and three-dimensional TS waves can be greatly destabilized at relatively low Görtler numbers. Here the second type of interaction problem in which small TS and TG disturbances interact with each other is studied. In neither of these approaches is it possible for the amplitude of the TG vortex to be specified arbitrarily.

The interaction problem which occurs at very low levels of the disturbance amplitude can in the first place be most conveniently studied through the nonlinear equations governing fully nonlinear unsteady TG vortices in fully developed channel flows. This is surprising since these equations do not contain the interactive pressure gradient in the flow direction, yet it is well-known that this gradient is crucial in determining the structure of two-dimensional TS waves. However the interactions which are considered in this paper involve oblique TS waves and there is sufficient structure in the non-linear TG vortex equations to enable unstable three-dimensional TS waves to develop also. The oblique modes initially considered have their wavefronts almost parallel to the flow direction. Later this restriction is relaxed and TS waves inclined at an $O(1)$ angle to the flow direction are investigated.

The first weak interaction discussed above involves a TG vortex and two oblique TS waves inclined at an equal angle to the flow direction. The interaction is found to be characterised by three coupled quadratic amplitude equations typical of resonant triad interactions. In general it is found that this weak interaction leads to a finite time singularity in the solution of the amplitude equations. This type of interaction will be referred to as a type I interaction. The form of the solution when the singularity appears suggests a second type of stronger interaction, hereafter referred to as a type II interaction. In this situation the amplitudes of the TG and TS modes are much larger and the interaction is governed by a partial differential system for the TG mode coupled to an integro-differential equation for the TS wave. A major property of this type of interaction is that it can take place in a straight channel; in this case the longitudinal vortex is driven by the interacting oblique TS waves. In addition the longitudinal vortex is sufficiently strong so as to react back on the TS waves. Here again the ultimate effect of the interaction is to cause the growth of both disturbances to a more nonlinear state. However the nature of this breakdown now depends crucially on the orientation of the oblique TS waves. In particular it is shown that TS waves propagating at an angle of more than 41.6^o to the flow direction produce a breakdown at a finite time. TS waves inclined at an angle of less than 41.6 degrees also lead to a breakdown but this takes place over an infinite time interval and is therefore less catastrophic. Each type of breakdown for these type II interactions can occur with or without curvature of the channel; in the presence of curvature the route to breakdown can be more complicated but the final state has little dependence on the curvature.

The procedure adopted in the rest of this paper is as follows . In §2 the nonlinear equations governing the growth of nonlinear TG vortices in fully developed channel flows are formulated. In §3 these equations are solved for weak type I interactions, whilst in §4 the corresponding equations governing much stronger type II interactions are derived. In §5 the solution of the interaction equations found in §4 is discussed. Finally in §6 the results obtained in the previous sections are discussed and some conclusions drawn.

2. Formulation of the nonlinear disturbance equations

Consider the steady flow of a viscous incompressible fluid of density ρ^*, kinematic viscosity ν^* in a curved channel driven by an azimuthal pressure gradient. The channel walls are defined by $r' = R_1$, $r' = R_2$ with respect to cylindrical polar co-ordinates (r', θ', z') and it is assumed that the curvature parameter δ defined by

$$\delta = \frac{R_2 - R_1}{R_1}, \tag{2.1}$$

is small. The co-ordinates (x, y, z) are defined by

$$x = \frac{R_1\theta'}{R_2 - R_1}, \qquad y = \frac{r' - R_1}{R_2 - R_1}, \qquad z = \frac{z'}{R_2 - R_1}$$

and the corresponding velocity field (u, v, w) in the small gap limit $\delta << 1$ can be written

$$(u, v, w) = \mathbf{u}_B = U_0(\bar{u}(y), 0, 0) + O(\delta U_0) \tag{2.2}$$

where

$$\bar{u} = y(1 - y),$$

whilst $\frac{U_0}{4}$ is the maximum value of the zeroth order flow velocity. The instability of (2.2) to three- dimensional time-dependent Taylor-Görtler vortices and Tollmien- Schlichting waves will now be considered.

The appropriate non-dimensional time scale t for the disturbances is slow scaled on $(R_2 - R_1)/U_0$, whilst the perturbation velocities have the usual TG scales. The total velocity field is now written as

$$\mathbf{u} = \mathbf{u}_B + U_0(U, R_E^{-1}V, R_E^{-1}W), \tag{2.3}$$

where R_E is a Reynolds number defined by

$$R_E = \frac{U_0(R_2 - R_1)}{\nu^*}, \tag{2.4}$$

and U, V, W and the pressure perturbation P, scaled on $\rho^*\nu^{*2}(R_2 - R_1)^{-2}$, are all functions of x, y, z and t. The Görtler number G is defined by

$$G = 2R_E^2\delta \tag{2.5}$$

and is taken to be $O(1)$ in the limit $\delta \to 0$. The disturbance equations correct to $O(\delta^0)$ can then be written in the form

$$\mathcal{L}(u, v, w) = (0, P_y, P_z) + (\mathbf{u}.\nabla)\mathbf{u} + (\bar{v}\bar{u}', G\bar{u}u + G\frac{u^2}{2}, 0)$$
$$u_x + v_y + w_z = 0, \tag{2.6a, b, c, d}$$

where $$\mathcal{L} \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t}.$$

It should be noted that the pressure does not enter the x-component of the momentum equations; this is the most significant difference between the equations for TG modes and TS modes inclined at an $O(1)$ angle to the basic flow direction. However, the equations (2.6) have sufficient structure to support TS modes which have wavefronts inclined at a small angle to the flow direction (see Hall and Smith (1984)); it is the interaction of these modes and the TG mode which will be first investigated.

The equations (2.6) reduce in the small amplitude case with $\frac{\partial}{\partial x} \equiv 0$ to Dean's (1928) equations governing the linear instability of Poiseiulle flow in a curved channel. It is known that instability occurs for sufficiently large G and that β, the z-wavenumber of the vortices, is proportional to either $G^{-\frac{1}{2}}$ or $G^{\frac{1}{4}}$ where $G >> 1$. The manner in which nonlinear effects operate in these limits is quite different, thus for $\beta << 1$ the interactions are governed by a Stuart-Watson type of interaction whilst for $\beta >> 1$ a mean field interaction of the type discussed for external flows by Hall (1982b) applies. Here the most relevant case is when $\beta << 1$ so that there is a strong interaction between the TG and TS modes. The nature of the interaction depends on the size of the disturbance velocities; in the next section the simplest nonlinear interaction of the resonant triad type is considered. Later more complicated higher-amplitude interactions which lead to partial differential evolution equations are considered.

3. Type I interactions between Tollmien-Schlichting waves and Taylor-Görtler vortices.

Suppose that a TG mode of instability with cross stream wavenumber β is superimposed on the basic state (2.2). If β is small the disturbance equations (2.6) can support TS waves inclined at a real angle $O(\beta^7)$ to the flow direction. Thus the pair of skewed TS waved proportional to

$$\begin{aligned} E_1^{\pm 1} &= \exp \pm i \left[\frac{\alpha x}{\beta^6} + \frac{i\beta z}{2} \right] \\ E_2^{\pm 1} &= \exp \pm i \left[\frac{\alpha x}{\beta^6} - \frac{i\beta z}{2} \right] \end{aligned} \tag{3.1a,b}$$

will interact nonlinearly with a TG vortex proportional to

$$E_3^{\pm 1} = \exp\left(\pm i\beta z\right) \tag{3.2}$$

at second order, to reinforce themselves and the TG mode. This resonant triad type of interaction has been investigated elsewhere (see, for example, Craik (1971) or Smith (1987)) in the context of purely TS instabilities of boundary layers.

The crucial size of the disturbance associated with (3.1) and (3.2) is fixed from the following considerations. Firstly suppose that the TS and TG modes have amplitude $A(\beta)$ and $C(\beta)$ for the streamwise velocity components in the core. The relative size of A and C is chosen such that the amplitude equation for A driven by terms proportional to $A\bar{C}$ arises at the same order as the equation for C driven by terms proportional to $|A|^2$ so that nonlinear interactions occur over comparable time scales. Further limits away from this regime can then be taken to examine the linearized problems associated with $A/C \to 0$ or $C/A \to 0$. Before writing down the scalings deduced from the above considerations it is convenient to define a slow time scale τ by

$$\tau = \beta^{k-4} t$$

and expand the Görtler number G in the form

$$G = \frac{1}{\beta^4}\{g_0 + \beta^{k-4}g_1 + \cdots\} \tag{3.3}$$

where $k > 4$ is a constant associated with the elevation of the Görtler number G above the neutral value $g_0\beta^{-4}$. The expansion procedure given in this Section fails when $k = 4$, in which case the appropriate evolution equations are partial differential equations but can be derived by some minor modifications to the present procedure. It is now necessary to be more precise about the downstream wavenumber of the TS waves; thus E_1 and E_2 are now taken to be

$$\begin{aligned} E_1 &= \exp i\left\{\frac{\alpha x}{\beta^6} + \frac{i\beta z}{2} - \frac{i\gamma t}{\beta^4}\right\} \\ E_2 &= \exp i\left\{\frac{\alpha x}{\beta^6} - \frac{i\beta z}{2} - \frac{i\gamma t}{\beta^4}\right\} \end{aligned} \tag{3.4a, b}$$

where

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1\beta + \alpha_2\beta^2 \cdots + \tilde{\alpha}\beta^k + \cdots \\ \gamma &= \gamma_0 + \gamma_1\beta + \cdots \end{aligned} \tag{3.5a, b}$$

The ordering of the term β^k in (3.5a) depends on the size of k; we assume that all the quantities in (3.5a,b) are real so that

$$E_1\bar{E}_2 = E_3, \quad E_2E_3 = E_1, \quad E_1\bar{E}_3 = E_2.$$

In the absence of a Görtler vortex (3.5a,b) would of course reduce to the lower branch asymptotic expansions for the TS neutral wavenumber and frequency at a given value of β.

The coreflow solution

The above considerations about disturbance size lead us to write down the following expansions for the disturbance velocity field in the main part of the channel:

$$\begin{aligned} U &= \beta^k u_3 E_3 + \beta^{k+3} u_1 E_1 + \beta^{k+3} u_2 E_2 + \cdots + C.C., \\ V &= \beta^k v_3 E_3 + \beta^{k-3} v_1 E_1 + \beta^{k-3} v_2 E_2 + \cdots + C.C., \\ W &= \beta^{k-1} w_3 E_3 + \beta^{k-2} w_1 E_1 + \beta^{k-2} w_2 E_2 + \cdots + C.C., \\ P &= \beta^{k-2} p_3 E_3 + \beta^{k-9} p_1 E_1 + \beta^{k-9} p_2 E_2 + \cdots + C.C. \end{aligned} \tag{3.6a, b, c, d}$$

where

$$u_1 = u_{10} + \cdots + \beta^k u_{11} + \cdots,$$
$$u_2 = u_{20} + \cdots + \beta^k u_{21} + \cdots, \qquad (3.7a,b,c)$$
$$u_3 = u_{30} + \beta^k u_{31} + \cdots.$$

together with similar expansions for $v_1, v_2, v_3, w_1, w_2, w_3, p_1, p_2, p_3$. The terms in (3.7a,b) represented by $\cdots$ depend on the particular size chosen for k but do not enter the calculation given in this Section. The TS modes in (3.6) are valid only away from viscous layers of thickness β^2 at $y = 0, 1$ where the no slip condition must be satisfied. In contrast the Görtler expansions are such that viscosity enters the coreflow problem at zeroth order and hence the no-slip condition can be satisfied directly. The disturbance expansions (3.6),(3.7) are then substituted into (2.6) and like powers of β are equated. The zeroth order problems for $(u_{10}, v_{10}, w_{10}, p_{10})$ and (v_{30}, w_{30}, p_{30}) are found to be

$$\begin{aligned} i\alpha_0 \bar{u} u_{10} + v_{10} \bar{u}' &= 0, \\ i\alpha_0 \bar{u} v_{10} + p_{10_y} &= 0, \\ i\alpha_0 \bar{u} w_{10} &= -\frac{i}{2} p_{10}, \\ i\alpha_0 u_{10} + v_{10_y} &= 0. \end{aligned} \qquad (3.8)$$

for the TS part and

$$\begin{aligned} u_{30_{yy}} &= v_{30} \bar{u}_y, \\ p_{30_y} &= g_0 \bar{u} u_{30}, \\ w_{30_{yy}} &= i p_{30}, \\ v_{30_y} + i w_{30} &= 0. \end{aligned} \qquad (3.9)$$

for the Görtler part.

As expected the system (3.8) for the TS part has no solution with the no-slip condition at $y = 0, 1$ is satisfied. Thus if the boundary conditions on u_{10}, w_{10} at $y = 0, 1$ are dropped the appropriate solution is

$$u_{10} = a_0(\tau) \bar{u}', \qquad v_{10} = -i\alpha_0 a_0(\tau) \bar{u},$$
$$p_{10} = -\alpha_0^2 a_0(\tau) \int_0^y \bar{u}^2 \, dy, \qquad w_{10} = \frac{-p_{10}}{2\alpha_0 \bar{u}} \qquad (3.10)$$

where $a(\tau)$ is an as yet unknown amplitude function. A similar analysis shows that

$$u_{20} = b_0(\tau)\bar{u}', \qquad v_{20} = -i\alpha_0 b_0(\tau)\bar{u},$$
$$p_{20} = -\alpha_0^2 b_0(\tau)\int_0^y \bar{u}^2 dy, \qquad w_{20} = \frac{p_{20}}{2\alpha_0\bar{u}} \tag{3.11}$$

where b_0 is the second TS 'amplitude', or 'displacement' function to be found at higher order.

The zeroth order Görtler equations (3.9) retain the viscous derivatives in the y direction and therefore can be solved subject to $u_{30} = v_{30} = w_{30} = 0, \quad y = 0, 1$. However this system will have a solution for only certain values of g_0; the lowest one of which corresponds to the left hand branch of the lowest neutral curve for the Dean problem. The solution can be written

$$u_{30} = c_0(\tau)U_{30}, \quad v_{30} = c_0(\tau)V_{30}, \quad w_{30} = c_0(\tau)W_{30}, \quad p_{30} = c_0(\tau)P_{30} \tag{3.12a, b}$$

where $c_0(\tau)$ is another amplitude function and U_{30}, V_{30} are determined by the eigenvalue problem

$$U_{30}'' = V_{30}\bar{u}',$$
$$V_{30}'''' - g_0\bar{u}u_{30} = 0, \tag{3.13}$$
$$U_{30} = V_{30}' = V_{30} = 0, \qquad y = 0, 1.$$

The functions u_{11}, u_{21}, u_{31} etc. are found by proceeding to higher order in the coreflow expansions. In particular it is found that $(u_{11}, v_{11}, w_{11} p_{11})$ satisfies an inhomogeneous form of (3.8) forced by the interaction of the TS waves proportional to E_2 and the Görtler vortex. The appropriate solution of the system is

$$u_{11} = a_1(\tau)\bar{u}' + b_0 c_0 U_{30}'$$
$$v_{11} = -i\alpha_0 a_1(\tau)\bar{u} - i\alpha_0 b_0 c_0 U_{30}$$
$$p_{11} = -\alpha_0^2 a_1(\tau)\int_0^y \bar{u}^2\, dy - 2\alpha_0^2 b_0 c_0 \int_0^y U_{30}\bar{u}\, dy,$$

where $a_1(\tau)$ is another amplitude function to be determined at higher order. The solution for (u_{22}, v_{22}, p_{22}) is given by (3.1) with b_0 replaced by c_0 and a_1 by the function $b_1(\tau)$. Finally to complete the coreflow solution to this order the functions u_{31}, v_{31}, w_{31} and p_{31} satisfy

$$u_{31yy} - v_{31}\bar{u}' = u_{30\tau},$$

$$0 = -p_{31_y} + g_0\bar{u}u_{31} + \tilde{g}\bar{u}u_{30} - 4a_0\bar{b}_0\alpha_0^2\bar{u}\bar{u}',$$

$$w_{31_{yy}} = ip_{31} + w_{30_\tau} + ia_0\bar{b}_0\alpha_0^2\bar{u}^2,$$

$$v_{31_y} + iw_{31} = 0.$$

Here the forcing terms arise from the interaction of the two skewed TS waves in the core. An examination of the TS waves in the wall layers shows that the interactions in the wall layers have a negligible effect on the Görtler vortex in these regions so that the above sysytem is to be solved subject to the no-slip condition at $y = 0, 1$. In this case a suitable solution exists if and only if an appropriate orthogonality condition is satisfied; this condition can be written

$$\frac{dc_0}{dt} = \tilde{g}\theta c_0 + \gamma a_0\bar{b}_0 \tag{3.14}$$

where

$$\theta = \frac{-\int_0^1 U_{30}\bar{u}P(y)\,dy}{\int_0^1 (U_{30}Q(y) + V_{30}''P(y))\,dy},$$

and

$$\gamma = \frac{2\alpha_0^2\int_0^1 \bar{u}\bar{u}'P(y)\,dy}{\int_0^1 (V_{30}Q(y) + V_{30}''P(y))\,dy}.$$

Here P,Q are solutions of the adjoint problem

$$Q'' - g_0\bar{u}P = 0$$

$$P'''' = Q\bar{u}'$$

with
$$P = Q = P' = 0, \qquad y = 0, 1.$$

At this stage in the coreflow problem the TS amplitude functions are not determined and in fact are found by solving the equations in the wall layers.

The solution in the wall layers.

At the upper and lower boundaries there exist layers of thickness β^2 where the TS waves adjust their structure so as to satisfy the no-slip condition. Thus near $y = 0$ it is necessary to define a stretched variable Y by

$$Y = \beta^{-2}y$$

and (3.6), (3.7) are modified by writing

$$
\begin{aligned}
u &= \beta^{2+k} Y E_3 c_0 U_3'(0) + \beta^{k+3}\{U_1E_1 + U_2E_2\} + \cdots + C.C. \\
v &= \beta^{4+k} Y^2 E_3 c_0 V_3''(0) + \beta^{k-1}\{V_1E_1 + V_2E_2\} + \cdots + C.C. \\
w &= \beta^{k+1} Y E_3 c_0 W_3'(0) + \beta^{k-4}\{W_1E_1 + W_2E_2\} + \cdots + C.C. \\
p &= \beta^{k} E_3 c_0 P_3(0) + \beta^{k-9}\{P_1E_1 + P_2E_2\} + \ldots + C.C.
\end{aligned}
\qquad (3.15a,b,c,d)
$$

where

$$
\begin{aligned}
U_1 &= U_{10} + \cdots + \beta^k U_{11} + \cdots, \\
V_2 &= U_{20} + \cdots + \beta^k U_{21} + \cdots \; .
\end{aligned}
\qquad (3.16a,b)
$$

Here the first terms in the expansions (3.15a,b,c,d) are just the first terms in the small y expansions of the zeroth order Gortler vortex in the core. Again the ordering of the correction terms of order β^k in (3.16a,b) depends on the precise value of k. For brevity the wall layer solution for the TS wave proportional to E_1 will be given, the details for E_2 being identical apart from some sign and notational changes. The expansions (3.15), (3.16) are then substituted into (2.6) and like powers of β are equated. At zeroth order it is found that $(U_{10}, V_{10}, W_{10}, P_{10})$ satisfies

$$
\begin{aligned}
\{\frac{d^2}{dY^2} - i\alpha_0 Y + i\gamma_0\}U_{10} &= V_{10} \\
\frac{dP_{10}}{dY} &= 0 \\
\{\frac{d^2}{dY^2} - i\alpha_0 Y + i\gamma_0\}W_{10} &= \frac{i}{2}P_{10} \\
i\alpha_0 U_{10} + V_{10} + \frac{i}{2}W_{10} &= 0
\end{aligned}
\qquad (3.17a,b,c,d)
$$

and these must be solved subject to the conditions $U_{10} = V_{10} = W_{10} = 0, \quad Y = 0$.

The appropriate solution is obtained in terms of the variable ξ defined by

$$
\xi = \Delta\{Y - \frac{\gamma_0}{\alpha_0}\}, \quad \Delta = (i\alpha_0)^{\frac{1}{3}}
$$

and is given by

$$\alpha_0 U_{10} + i/2 W_{10} = A_0 \int_{\xi_0}^{\xi} A_i(\eta)\, d\eta, \qquad (3.18a,b)$$

$$W_{10} = \frac{i}{2} P_{10} \Delta^{-2} \mathcal{L}(\xi).$$

Here A_i is the Airy function whilst $\mathcal{L}(\xi)$ satisfies an inhomogeneous Airy equation with unity on the right hand side and zero boundary conditions at $0, \infty$.

The amplitude function A_0 can be related to the pressure P_{10} using the x and z momentum equations evaluated at $Y = 0$ to give

$$\frac{iP_{10}}{4} = (i\alpha_0)^{\frac{2}{3}} A_i'(\xi_0) A_0. \qquad (3.19)$$

The zeroth order eigenrelation for the Tollmien-Schlichting wave follows by matching (3.18) and the corresponding upper layer solution with the coreflow. If we denote the functions in the upper layer corresponding to U_{10}, A_0 etc. by $\tilde{U}_{10}, \tilde{A}_0$ etc. it follows from (3.11) that

$$\tilde{P}_{10} - P_{10} = \frac{-\alpha_0^2}{30} a_0(t), \qquad (3.20a,b)$$

$$a_0 = \frac{A_0}{\alpha_0} \int_{\xi_0}^{\infty} A_i\, dy = \frac{-\tilde{A}_0}{\alpha_0} \int_{\xi_0}^{\infty} A_i\, d\eta$$

and using (3.19) and the corresponding upper layer relationship it is found that

$$\Delta^2 A_i'(\xi_0) = \frac{i\alpha_0}{240} \int_{\xi_0}^{\infty} A_i(\eta)\, d\eta \qquad (3.21)$$

and α_0, γ_0 must be chosen to satisfy the above eigenrelationship.

A similar analysis applies to the second TS mode which has amplitude B_0 and clearly this mode leads to the same eigenrelation (3.21). The amplitude equations for $a_0(\tau)$ and $b_0(\tau)$ will now be obtained by considering the $O(\beta^k)$ disturbance equations in the wall layers. The equations to determine $(U_{11}, V_{11}, W_{11}, P_{11})$ in the layer at $y = 0$ are:

$$\{\frac{d^2}{dY^2} - i\alpha_0 Y + i\gamma_0\} U_{11} - V_{11} = U_3'(0) Y c_0 \{U_{20} i\alpha_0 + iW_{20}\} + V_{20} c_0 U_3'(0) + i\tilde{\alpha} Y U_{10} + U_{10}$$

$$\{\frac{d^2}{dY^2} - i\alpha_0 Y + i\gamma_0\}W_{11} - \frac{i}{2}P_{11} = U_3'(0)Y c_0 W_{20} + i\tilde{\alpha} Y W_{10} + W_{10\tau}$$

$$i\alpha_0 U_{11} + V_{11Y} + \frac{i}{2}W_{11} = -i\tilde{\alpha} U_{10}$$

$$U_{11} = V_{11} = W_{11} = 0, Y = 0. \tag{3.22}$$

The appropriate solution of (3.22) is given by

$$\begin{aligned} i\alpha_0\{\alpha_0 U_{11} + \frac{1}{2}W_{11}\} = {} & i\tilde{\alpha}_0 A_0\{\frac{1}{3}\xi A_i(\xi) - \xi_0 A_i(\xi)\} \\ & + \Delta\frac{dA_0}{dt}\{A_i(\xi) - A_i(\xi_0)\} + i\alpha_0 A_1 \int_{\xi_0}^{\xi} A_i d\xi \\ & - c_0 B_0 U_3'(0) i\alpha_0 A_i'(\xi_0)\{\mathcal{L}(\xi) + (\xi - 2\xi_0)\mathcal{L}'(\xi) + \xi_0\mathcal{L}'(\xi_0)\} \\ & + U_3'(0) i\alpha_0 c_0 B_0\{\frac{1}{3}\xi A_i(\xi) - \xi_0 A_i(\xi) + \frac{2}{3}\xi_0 A_i(\xi_0)\} \end{aligned} \tag{3.23}$$

$$\begin{aligned} \frac{iP_{11}}{4} = \Delta^{-1}\{ & \frac{2i\tilde{\alpha}A_0}{3}[A_i'(\xi_0) - \xi_0^2 A_i(\xi_0)] + \Delta\frac{dA_0}{dt}\xi_0 A_i(\xi_0) \\ & + i\alpha_0 A_1 A_i'(\xi_0) + \frac{2i\alpha_0}{3}c_0 B_0 U_3'(0)[-\xi_0^2 A_i(\xi_0) + A_i'(\xi_0)] \\ & - A_i'(\xi_0) c_0 i\alpha_0 B_0 U_3'(0)\left[3 + \frac{\xi_0^2\chi}{A_i(\xi_0)}\right]\} \end{aligned} \tag{3.24}$$

Here A_1 is another amplitude function to be found and $\chi = \int_{\xi_0}^{\infty} A_i(\eta)\, d\eta$. A similar analysis applies in the upper wall layer and the matching between the velocity and pressure fields where the wall layers merge with the coreflow produces an equation to determine $a_0(\tau)$ the amplitude of the first TS mode in the core. The equation, after some manipulation, can be written

$$\frac{da_0}{d\tau} = \tilde{\alpha}\nu a_0 + \mu b_0 c_0, \tag{3.25}$$

where

$$\nu = \frac{\frac{4}{3}i\left\{\frac{\xi_0 A_i(\xi_0)}{\chi} + \frac{1}{2} + \xi_0^2\frac{A_i(\xi_0)}{A_i'(\xi_0)}\right\}}{2\Delta Ai(\xi_0)\left[\frac{\xi_0}{A_i'(\xi_0)} + \frac{1}{\chi}\right]}$$

and

$$\mu = \{\Delta i\alpha_0 J\chi + i\alpha_0[U_3'(0) - U_3'(1)]\Big\{\frac{1}{\chi}\Big[\frac{2}{3}\xi_0 A_i(\xi_0)$$
$$+ \frac{\xi_0 A_i'(\xi_0)\chi}{A_i(\xi_0)}\Big] - 1 - \frac{2}{3}\Big(1 - \xi_0^2\frac{A_i(\xi_0)}{A_i'(\xi_0)}\Big)$$
$$+ \Big(3 + \frac{\chi\xi_0^2}{A_i(\xi_0)}\Big)\Big\}2\Delta A_i(\xi_0)\Big[\frac{\xi_0}{A_i'(\xi_0)} + \frac{1}{\chi}\Big]\}$$

with

$$J = \int_0^1 U_{30}(y)\bar{u}(y)\,dy. \qquad (3.26a,b,c,)$$

A similar analysis for the second TS mode produces the amplitude equation

$$\frac{db_0}{d\tau} = \tilde{\alpha}\nu b_0 + \mu a_0 \bar{c}_0. \qquad (3.27)$$

Thus a_0, b_0, and c_0 satisfy the resonant triad equations (3.14), (3.25), (3.27). These equations can be put in a more convenient form by writing

$$a_0 = ah, \quad b_0 = bh, \quad c_0 = ck,$$
$$h^2 = \frac{|\theta\tilde{g}|^2}{|\mu\gamma|}, \quad \mu\gamma = |\mu\gamma| e^{i\phi}, \quad k = \frac{\gamma |h|^2}{|\theta\tilde{g}|}, \qquad \nu = |\nu| e^{i\omega}, \qquad (3.28)$$
$$t = |\theta\tilde{g}|\tau, \quad \tilde{\alpha} = \frac{\alpha}{|\nu|}|\theta\tilde{g}|$$

to give the equations governing the nonlinear interaction of TS and TG instability modes:

$$\frac{da}{dt} = \alpha e^{i\omega}a + e^{i\phi}bc$$
$$\frac{db}{dt} = \alpha e^{i\omega}b + e^{i\phi}a\bar{c} \qquad (3.29a,b,c)$$
$$\frac{dc}{dt} = \pm c + a\bar{b}.$$

Here the $\pm$ signs in (3.29c) are taken depending on whether $\tilde{g} > 0$ or $\tilde{g} < 0$. The constants ω and ϕ can be found by evaluating the coefficients in (3.14), (3.25), (3.27) numerically and using (3.20). It was found that

$$e^{i\phi} = (0.998, 0.049) \qquad (3.30a)$$

corresponds to $g_0 = 4.27 \times 10^5$, the smallest neutral value of the scaled Görtler number. The second mode for the Görtler problem has $g_0 = 4.94 \times 10^7$ and in that situation

$$e^{i\phi} = (0.79, 0.61). \tag{3.30b}$$

The equations (3.29) permit the particular solution $a = b$ with c real and the following discussion is restricted to this situation. In this case it is convenient to write $\rho = \mid a \mid^2$ so that (3.29) can be replaced by

$$\begin{aligned} \frac{1}{2}\frac{d\rho}{dt} &= \{\alpha \cos\omega + c\cos\phi\}\rho, \\ \frac{dc}{dt} &= \pm c + \rho. \end{aligned} \tag{3.31}$$

Apart from the trivial solution $\rho = c = 0$ an equilibrium solution of (3.31) corresponding to a stationary Görtler vortex and a time-periodic TS wave is

$$\rho = \mp c = \pm \frac{\alpha \cos\omega}{\cos\phi} \tag{3.32}$$

It will be seen shortly that (3.31) apply to another interaction problem involving TS waves and TG vortices so it is convenient to discuss (3.31) in general and later discuss the special case corresponding to (3.30). However in all the situations which arise in this paper $\cos\omega > 0$ so that the properties of the limit cycle solution depend only on the sign of $\cos\phi$. A linear instability analysis of the limit cycle solution leads to the following conclusions:

Case a $\cos\phi > 0$

Here the limit cycle exists if the TS and TG modes are both linearly stable or both linearly unstable. In either situation the limit cycle is unstable.

Case b $\cos\phi < 0$

Here the limit cycle exists if the TS mode is unstable and the TG mode is stable or vice versa. The former situation is stable and the latter situation is unstable.

Another property of (3.31) which has some relevance is the question of whether the system has a solution which breaks down at a finite value of the time $t = \bar{t}$. In order to investigate this possibility ρ and c are expanded in the form

$$\begin{aligned} \rho &= \frac{\rho_0}{(t-\bar{t})^2} + \cdots, \\ c &= \frac{c_0}{(t-\bar{t})} + \cdots, \end{aligned} \tag{3.33a, b}$$

and substitution of these expansions into (3.31) yields

$$1 = -c_0 \cos\phi$$

$$\rho_0 = -c_0 = \frac{1}{\cos\phi}$$

so that a finite time breakdown occurs if $\cos\phi > 0$.

The discussion leading to the amplitude equations (3.31) is easily modified to account for a related resonant triad interaction problem. It should be recalled that lower branch Tollmien-Schlichting waves inclined at an $O(1)$ angle to the flow direction have x and z wavenumbers scaled on $R_E^{-1/7}$. Since the z variation in Section 2 was taken to be on a length scale $O(1)$ it follows that the above situation will occur when $\beta = O(R_E^{-1/7})$ so that $\alpha = O(R_E^{-6/7})$. The expansion procedure leading to (3.31) is essentially unchanged in this region, the only significant alteration being that the pressure gradient in the x-momentum equation is no longer negligible in the description of the two skewed TS waves. The amplitude equation (3.14) for the TG mode remains unchanged whilst the nonlinear term in (3.25) has coefficient μ given by

$$\begin{aligned} 2\Delta\Big(\xi_0 + \frac{A_i(\xi_0)}{\chi}\Big)A_i(\xi_0)\frac{\mu}{\chi} = {} & \frac{-2\alpha_0^2 J}{\Delta^2} - i\alpha_0\frac{A_i'(\xi_0)}{\chi}(U_3'(0) - U_3'(1)) \\ & + \frac{i\alpha_0 A_i(\xi_0)}{(\alpha_0^2 + \beta_0^2/4)\chi}(U_3'(0) - U_3'(1))\Big[\Big(\frac{2\xi_0^2}{3} - \frac{2A_i'(\xi_0)}{3A_i(\xi_0)} + \frac{2\xi_0}{3\chi}A_i'(\xi_0)\Big)(\alpha_0^2 + \frac{\beta_0^2}{4}) \\ & + \Big(\frac{3A_i'(\xi_0)}{A_i(\xi_0)} + \xi_0^2\frac{\chi A_i'(\xi_0)}{A_i^2(\xi_0)} + \xi_0\frac{A_i'(\xi_0)}{A_i(\xi_0)}\Big)\frac{\beta_0^2}{4}\Big] \end{aligned} \tag{3.34}$$

where $\beta_0 = \beta R_E^{-1/7}$ and the eigenrelation corresponding to (3.21) is

$$\Delta^2 A_i'(\xi_0) = \frac{i\alpha_0}{60}\left(\alpha_0^2 + \frac{\beta_0^2}{4}\right)\chi \tag{3.35}$$

The amplitude equations can again be rescaled into the form(3.31) and the coefficients $\cos\omega$ and $\cos\phi$ are now functions of β_0 with α_0 , γ_0 determined by the eigenrelation (3.35). In Figure 1 the dependence of α_0 on β_0 is shown. For large β_0, α_0 asymptotes to zero like β_0^{-6} which is consistent with the earlier analysis of §3. The constant $\cos\omega$ is positive in all cases whilst $e^{i\phi}$ is shown in Figures 2,3 for the first and second Görtler modes. The lowest mode always has $\cos\phi > 0$ so that the finite time breakdown discussed above will always occur. The second mode has $\cos\omega < 0$ and $cos\phi < 0$ for $\beta_0/\alpha_0 < 1.29$ so that a stable finite amplitude equilibrium state is possible outside this region. However the first Görtler mode is unstable in this situation and might possibly destabilize this state.

4. Larger amplitude Type II interactions governed by an integro-differential system.

It is now shown that by increasing the disturbance size the coupled triad interaction equations derived in the previous section alter their form. The choice of k in (3.3) effectively fixes the size of the TS and TG modes in the core. If $k > 4$ the zeroth order TG mode has as its zeroth order problem (3.9). At this stage the interaction of the two TS modes in the core does not force the TG mode. However, if k is now taken to be 4 then this is no longer the case and the forcing terms which appear in (3.13) now appear on the right hand sides of the zeroth order problem corresponding to (3.9). In contrast the forcing of the TS modes by their interaction with the TG mode remains a higher order effect and does not enter the zeroth order eigenrelation problem involving γ_0 and α_0. Thus if the analysis of the previous section is repeated with $k = 4$ throughout then, after eliminating w_{30} and p_{30}, it is found that u_{30} and v_{30} satisfy

$$\begin{aligned}
&\left\{\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right\} u_{30} = v_{30}\bar{u}', \\
&\left\{\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right\} \frac{\partial^2}{\partial y^2} v_{30} = g_0 \bar{u} u_{30} - 2\rho\bar{u}\bar{u}', \\
&u_{30} = v_{30} = \frac{\partial v_{30}}{\partial y} = 0, \qquad y = 0, 1.
\end{aligned} \tag{4.1}$$

Here attention has been restricted to the situation where the skewed TS modes have identical amplitudes and again $\rho = | a_0 |^2$. Thus (4.1) now replaces (3.9) and (3.13) and the outcome is that the zeroth order Görtler problem is now forced by the TS modes so that g_0 is no longer required to be an eigenvalue of the Dean problem. Indeed the structure that is now developing occurs in the absence of curvature in which case the streamwise vortices present in the flow are a direct result of the interaction of the TS modes.

The derivation of the amplitude equation corresponding to (3.25) is essentially identical to that given in Section 3. However the appearance of U_3 in (3.26) means that the corresponding amplitude equation for $\rho = \mid a_0 \mid^2$ now depends on u_{30}. In fact the appropriate equation is

$$\frac{1}{2}\frac{d}{dt}\rho = \rho\{\tilde{\alpha}\nu_r + \tilde{\gamma}\int_0^1 \bar{u}u_{30}\,dy + \tilde{\theta}\{u_{30y}(0,t) - u_{30y}(1,t)\}\} \tag{4.2}$$

where ν is defined by (3.26a) whilst

$$\tilde{\gamma} = \text{Real}\left\{i\alpha_0\chi/(2A_i(\xi_0)[\frac{\xi_0}{A_i'(\xi_0)} + \frac{1}{\chi}])\right\}, \tag{4.3a, b}$$

$$\tilde{\theta} = \text{Real}\left[\frac{i\alpha_0[\frac{1}{\chi}(\frac{2}{3}\xi_0 A_i(\xi_0)) + \frac{\xi_0 A_i'(\xi_0)\chi}{A_i(\xi_0)}] - 1 - \frac{2}{3}(1 - \frac{\xi_0^2 A_i(\xi_0)}{A_i'(\xi_0)}) + (3 + \frac{\chi\xi_0^2}{A_i(\xi_0)})}{2\Delta A_i(\xi_0)[\xi_0/A_i'(\xi_0) + \frac{1}{\chi}]}\right].$$

If β is reduced to be $O(R^{-1/7})$ then the three dimensional effects again modify the co-efficients in the amplitude equation (4.2) for the TS mode. For the purposes of this paper it is sufficient to note that when $\alpha = \alpha_0 R^{-6/7}$, $\beta = \beta_0 R^{-1/7}$ the eigenrelation (3.35) applies but with $\tilde{\gamma}, \tilde{\theta}$ now defined by

$$\tilde{\gamma} = \text{Real}\left\{\frac{-\alpha_0^2\chi(\alpha_0^2 + \frac{\beta_0^2}{4})}{\Delta^3(\xi_0 + \frac{A_1'(\xi_0)}{\chi})A_i(\xi_0)}\right\}, \tag{4.4a, b}$$

$$\tilde{\theta} = \text{Real}\Big[\Big\{\frac{-i\alpha_0 A_i'(\xi_0)}{\chi} + \frac{i\alpha_0 A_i(\xi_0)}{(\alpha_0^2 + \beta_0^2/4)\chi}\Big((\alpha_0^2 + \beta_0^2/4)\Big(\frac{2}{3}\xi_0^2 - \frac{2A_i'(\xi_0)}{3A_i(\xi_0)} + \frac{2}{3}\frac{\xi_0 A_i'(\xi_0)}{3\chi}\Big) + \Big(\frac{3A_i'(\xi_0)}{A_i(\xi_0)} + \frac{\xi_0^2\chi A_i'(\xi_0)}{A_i^2(\xi_0)} + \frac{\xi_0 A_i'(\xi_0)}{A_i^2(\xi_0)}\Big)\frac{\beta_0^2}{4}\Big)\Big\}/2\Delta(\xi_0 + \frac{A_i'(\xi_0)}{\chi})\frac{A_i(\xi_0)}{\chi}\Big].$$

For convenience (4.2) is rescaled by writing $\alpha = 2\tilde{\alpha}\nu_r$, $\theta = 2\tilde{\theta}$, $\gamma = 2\tilde{\gamma}$ so that ρ now satisfies

$$\frac{d\rho}{dt} = \rho\{\alpha + \gamma \int_0^1 \bar{u}u_{30}\, dy + \theta[u_{30y}(0,t) - u_{30y}(1,t)]\} \tag{4.5}$$

which must be solved together with (4.1). It should be stressed at this stage that setting $g_0 = 0$ in (4.1) does not force $u_{30} = v_{30} = 0$ so that in the absence of curvature the longitudinal vortex is driven by the interaction of the skewed TS waves.

The solution of the coupled system (4.1), (4.5) is discussed in the following section. It will be seen that the nature of the solution depends crucially on the ratio θ/γ which must be calculated numerically from (4.3) or (4.4b) depending on the wavenumber under investigation. In Figures 4a,b the values of $\tilde{\gamma}$ and $\theta/\tilde{\gamma}$ calculated from (4.4) are shown as functions of β_0/α_0. The ratio θ/γ is seen to increase monotonically from $-.0138$ at $\beta_0/\alpha_0 = 0$ to $.028$ when $\beta_0/\alpha_0 \to \infty$. This limiting value corresponds to the value of θ/γ which is obtained from (4.3). The discussion of the following section shows that there is a significant difference in the solution of (4.1), (4.5) from $\theta/\gamma < -1/140$ and $\theta/\gamma > -1/140$. For future reference it should be noted that $\theta/\gamma = -1/140$ when $\beta_0/\alpha_0 = .8887$, so that there is a change in the type of nonlinear interaction when the angle between the TG vortices and the TS waves falls below 41.6°.

5. Solution of the evolution equations for Type II waves.

Initially it is assumed in the following discussion that θ and γ in (4.5) are arbitrary real constants; later solutions of (4.1), (4.5) for values of θ and γ appropriate to TS waves in channels will be considered. Firstly it is noted that an equilibrium solution of (4.1), (4.5) is

$$\rho = u_3 = v_3 = 0, \tag{5.1}$$

but an instability analysis of this equation shows that this solution is unstable if g_0 or α is positive. In addition a limit cycle solution exists with a_0 periodic in time and ρ, u_3 and v_3 stationary. This solution takes the form

$$u_3 = U(y), \quad v_3 = V(y), \quad \rho = \rho_e \tag{5.2}$$

where

$$U'' = V\bar{u}', \quad V'''' = -2\rho_e \bar{u}\bar{u}' + g_0 \bar{u} U, \quad U = V = V' = 0, \qquad y = 0, 1, \tag{5.3}$$

and

$$\alpha + \gamma \int_0^1 \bar{u} U \, dy + \theta [U'(0) - U'(1)] = 0 \tag{5.4}$$

For a given value of g_0 it follows from (5.4) that, since ρ_e is necessarily positive, the limit cycle solution exists for either $\alpha < 0$ or $\alpha > 0$. In fact the choice of sign required for the existence of the limit cycle solution is determined by θ/γ. A numerical solution of (5.3) for different values of g_0 shows that

$$\gamma \int_0^1 \bar{u} U \, dy + \theta [U'(0) - U'(1)] = 0 \tag{5.5}$$

on the discontinuous curve C shown in Figure 5. The integral $J = \int_0^1 \bar{u} U \, dy$ and the shear difference $\mu = U(0) - U'(1)$ change sign along C ; the different combinations of signs possible for $0 < g_0 < 10^9$ are indicated in that Figure. The Görtler numbers g_{01} and g_{02} correspond to the first two neutral curves at low wavenumbers in the absence of TS waves. At these positions μ and J necessarily change signs but these quantities are found to also change sign independently at different values of g_0. These sign changes mean that the limit cycle exists for positive values of α when $(\theta/\gamma, g_0)$ lies below the continuous parts and above the dotted parts of C. Elsewhere the limit cycle solution exists for negative values of α. The stability properties of the limit cycle solution can be found from a small perturbation

$$u_3 = U(y) + u(y)e^{st}, \quad v_3 = V(y) + v(y)e^{st}, \quad \rho = \rho_e + \tilde{\rho} e^{st}$$

in which case the eigenvalue s is determined by

$$\left(\frac{d^2}{dy^2} - s\right) v = v\bar{u}',$$

$$\left(\frac{d^2}{dy^2} - s\right) \frac{d^2 v}{dy^2} = g_0 \bar{u} u - 2\tilde{\rho}\bar{u}\bar{u}', \tag{5.6}$$

$$\tilde{\rho} s = \left\{ \gamma \int_0^1 \bar{u} u \, dy + \theta \{u'(0) - u'(1)\} \right\} \rho_e$$

It can be shown from (5.6) by expanding for $| \alpha | << 1$ that the limit cycle solution is unstable whenever it exists for $\alpha < 0, | \alpha | << 1$. Numerical investigations of (5.6) confirmed this result for α in this region and showed that the limit cycle remains stable for $O(1)$ negative values of α. These investigations also showed that the limit cycle solution is unstable whenever $g_0 > g_{01}$ This is entirely consistent with the results obtained for the ordinary differential equations discussed in §3. This is to be expected since these equations can be retrieved from (4.1), (4.5) by taking the limit $g_0 \to g_{01}$.

Thus the limit cycle solution is only stable for $g_0 < g_{01}$ and values θ/γ below C in Figure 5. For ordinary differential equations it is known that unstable limit cycles often lead to a finite time singularity beyond which the solution cannot be continued. Though the evolution equations (4.1), (4.5) are not ordinary differential equations the possibility of such a 'blow-up' is now investigated.

In fact the nonlinear partial differential system (4.1)-(4.5) leads to a stronger singularity than that found in the ordinary differential lower amplitude system of §3. It follows from (4.1) and (4.5) that if a singularity develops then the dominant balances in these equations have

$$\frac{\partial u_3}{\partial t} \simeq v_3, \qquad \rho \simeq \frac{\partial v_3}{\partial t}, \qquad \frac{d\rho}{dt} \simeq u_3 \rho$$

which suggests that in the neighbourhood of the singularity at $t = t_0, u_3, v_3$ and ρ expand as

$$u_3 = \frac{u_{30}}{(t_0 - t)} + \cdots, \qquad v_3 = \frac{v_{30}}{(t_0 - t)^2}, \qquad \rho = \frac{\rho_0}{(t_0 - t)^3} + \cdots \quad (5.7a, b, c)$$

Substitution of the above expansions into (4.1), (4.5) shows that

$$u_{30} = -v_{30}\bar{u}', \qquad v_{30}'' = \rho_0 \bar{u}\bar{u}',$$

and the solution of the above system which satisfies $u_3 = v_{30} = 0$ at $y = 0, 1$ is

$$\begin{aligned} v_{30} &= \frac{\rho_0}{2}\left[\int_0^y \bar{u}^2\, dy - y\int_0^1 \bar{u}^2\, dy,\right] \\ u_{30} &= -\frac{\bar{u}'}{2}\rho_0\left[\int_0^y \bar{u}^2\, dy - y\int_0^1 \bar{u}^2\, dy\right]. \end{aligned} \qquad (5.8a, b)$$

where $\bar{u} = y(1-y)$.

Thus it is not possible to satisfy $v'_{30} = 0$ at the boundaries at this stage. This is of course a consequence of the fact that at $O(1)$ values of y the second derivative $\frac{\partial^2}{\partial y^2}$ in (4.1) is always negligible compared to $\frac{\partial}{\partial t}$. The remedy is to allow for extra sub-layers at $y = 0, 1$ and, as is usually the case for the heat equation operator $\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}$, the layers are of thickness $(t_0 - t)^{\frac{1}{2}}$. Thus near $y = 0$ the variable ξ is defined by

$$\xi = \frac{y}{[t_0 - t]^{\frac{1}{2}}} \tag{5.9}$$

and (5.7) is then modified by writing

$$u_3 = \frac{U_{30}(\xi)}{\mid t_0 - t \mid^{1/2}} + \cdots, \quad v_3 = \frac{V_{30}(\xi)}{\mid t_0 - t \mid^{3/2}} + \cdots, \quad \rho = \frac{\rho_0}{\mid t_0 - t \mid^3} + \cdots \tag{5.10a, b, c,}$$

The function V_{30} now satisfies

$$\left\{ \frac{d^2}{d\xi^2} - \frac{\xi}{2}\frac{d}{d\xi} - \frac{3}{2} \right\} V_{30\xi\xi} = 0, \tag{5.11}$$

and the appropriate boundary conditions are

$$V_{30} = V'_{30} = 0, \qquad \xi = 0.$$

The condition that (5.10b) matches with (5.8a) when $\xi \to \infty$ requires that

$$V'_{30} \to -\frac{\rho_0}{60}, \qquad \xi \to \infty$$

so that the appropriate solution of (5.11) is

$$\frac{60 V_{30}}{\rho_0} = \xi - \sqrt{2} U\left(\tfrac{5}{2}, \tfrac{\xi}{\sqrt{2}}\right) \frac{e^{\xi^2/8}}{U'(\frac{5}{2}, 0)} + \frac{U(\frac{5}{2}, 0)\sqrt{2}}{U'(\frac{5}{2}, 0)}.$$

Here $U(a, x)$ denotes the parabolic cylinder function which decays when $x \to \infty$. The function $U_{30}(\xi)$ then satisfies

$$\frac{60}{\rho_0} \left\{ \frac{d^2}{d\xi^2} - \frac{\xi}{2}\frac{d}{d\xi} - \frac{1}{2} \right\} U_{30} = -\xi + \frac{\sqrt{2} U(\frac{5}{2}, \frac{\xi}{\sqrt{2}}) e^{\frac{\xi^2}{8}}}{U'(\frac{5}{2}, 0)} - \frac{U(\frac{5}{2}, 0)\sqrt{2}}{U'(\frac{5}{2}, 0)}$$

$$U_{30} = 0, \qquad \xi = 0,$$

whilst matching with (5.8b) requires that

$$U_{30}' \to \frac{\rho_0}{60}, \qquad \xi \to \infty.$$

The appropriate solution for U_{30} is

$$\frac{60}{\rho_0} U_{30} = \frac{e^{\xi^2/8} 2\sqrt{2}}{U'(\frac{5}{2},0)} U(\frac{1}{2}, \frac{\xi}{\sqrt{2}}) \int_0^{\frac{\xi}{\sqrt{2}}} U^{-2}(\frac{1}{2},\eta)\, d\eta \int_\infty^\eta U(\frac{5}{2},\tilde{\eta}) U(\frac{1}{2},\tilde{\eta})\, d\tilde{\eta}$$
$$+ \xi + \frac{2\sqrt{2} U(\frac{5}{2},0)}{U'(\frac{5}{2},0)} \left[1 - \frac{e^{\frac{\xi^2}{8}} U(\frac{1}{2}, \frac{\xi}{\sqrt{2}})}{U(\frac{1}{2},0)} \right]. \tag{5.12}$$

After some manipulation (5.12) yields

$$U_{30}'(0) = \frac{\rho_0}{120}$$

so that the gradient in u_{30} at the edge of the boundary layer is exactly twice the value at the wall. An identical analysis to the above applies at the upper wall $y = 1$ and then (4.5) yields at zeroth order

$$\frac{3}{2} = \gamma \int_0^1 \bar{u} u_{30}\, dy + 2\theta U_{30}'$$

so that ρ_0 is determined by

$$\frac{3}{2} = \frac{\rho_0}{60} \left[\frac{\gamma}{140} + \theta \right]. \tag{5.13}$$

Therefore this type of finite time singularity can only occur whenever

$$\frac{\theta}{\gamma} > -\frac{1}{140} \tag{5.14}$$

Hence in the half plane $\theta/\gamma > -1/140$, u_3 , v_3 and ρ_3 can become infinite at a finite time. This singularity is associated with either the unstable zero solution or an unstable limit cycle solution.

Some clarification of the regions (i) and (ii) defined by
(i) $g_0 > g_{01}$, $\theta/\gamma < -1/140$
(ii) the area bounded by $\theta/\gamma = -1/140$, the curve C and $g_0 = 0$
is required since here an unstable limit cycle solution exists but a finite time breakdown of the type discussed above cannot occur. The possibility of the existence of an alternative form for the algebraic singularity was investigated but none was found. However it can be shown that an alternative breakdown on an infinite time scale can occur. Here u_3 , v_3 and ρ are written

$$\begin{aligned}(u_3, v_3) &= (\tilde{u}_{30}(y), \tilde{v}_{30}(y))e^{st} + (\tilde{u}_{31}(y), \tilde{v}_{31}(y)) + O(e^{-st}) \\ \rho &= \rho_0 e^{st} + \rho_1 + O(e^{-st})\end{aligned} \tag{5.15a, b}$$

where $s > 0$ so that u_3 , v_3 and ρ go to infinity exponentially at the same rate. This exponential rate of growth satisfies the nonlinear eigenvalue problem:

$$\left(\frac{d^2}{dy^2} - s\right)\tilde{u}_{30} = \tilde{v}_{30}\bar{u}'$$

$$\left(\frac{d^2}{dy^2} - s\right)\frac{d^2}{dy^2}\tilde{v}_{30} = g_0\bar{u}\tilde{u}_{30} - 2\rho_0\bar{u}\bar{u}'$$

$$\tilde{u}_{30} = \tilde{v}_{30} = \tilde{v}'_{30} = 0, \qquad y = 0, 1.$$

$$\gamma\int_0^1 \bar{u}\tilde{u}_{30}\,dy + \theta\{\tilde{u}'_{30}(0) - \tilde{u}'_{30}(1)\} = 0 \tag{5.16}$$

The latter condition ensures that the terms proportional to e^{2st} on the right hand side of (4.2) vanish identically. Otherwise ρ would go to infinity like e^{2st} and the structure (5.15) would not be possible. The eigenvalue s must of course be positive for (5.15) to be valid. It turns out that the eigenvalue s is closely related to the limit cycle solution and the existence of the finite time 'blow-up' solution (5.7).

Firstly it should be noted that when θ/γ tends to any value on the curve C of Figure 5 then $(\tilde{u}_{30}, \tilde{v}_{30})$ tends to $(\rho_0/\rho)(U, V)$ where (U, V) is the limit cycle solution determined by (5.2)-(5.4). The eigenvalue s of (5.16) goes through zero on C so solutions of the form (5.15) exist in a neighbourhood either above or below C. Secondly it can be shown that eigenvalues of (5.16) with $s >> 1$ exist either immediately above or below the line $\theta/\gamma = -1/140$. This result is not immediately obvious but follows by seeking a solution of (5.16) with $s >> 1$. The functions $\tilde{u}_{30}, \tilde{v}_{30}$ for $O(1)$ values of y satisfy the same equations as the blow-up solution (u_{30}, v_{30}). Again it is found that sub-layers at $y = 0, 1$ are required in order to satisfy the no-slip condition there. The solution in these layers is in fact different from that for the 'blow-up' solution but the outcome is essentially the same. Thus it is found that eigenvalues of (5.16) exist for θ/γ sufficiently close to the line $\theta/\gamma = -1/140$. This region is above or below $\theta/\gamma = -1/140$ depending on the sign of the difference in the shear $\tilde{u}'_{30}(0) - u'_{30}(1)$ at the lower and upper boundaries.The eigenvalues of (5.16) appear or disappear on the curve C and the line $\theta/\gamma = -1/140$. A numerical investigation of (5.14) shows that for $g_0 < g_{01}$ these eigenvalues are connected so that the exponential blow-up can occur in region (ii). Beyond $g_0 = g_{01}$ the situation is more complex and the eigenvalues emanating from C and $\theta/\gamma = -1/140$ are not always connected. In Figure 5 the regions where eigenvalues of (5.14) were found numerically to exist are shaded.

Thus if it is not possible for u_3, v_3 and ρ to become infinite in a finite time there exists the possibility of a breakdown in an infinite time. The only region of Figure 5 where neither kind of breakdown can occur is the area below C between $g_0 = 0$ and $g_0 = g_{01}$. In this region the stable zero or limit cycle solutions are presumably set up as a result of any initial disturbances

It should also be noted that another type of exponential blow-up is always available for $g_0 > g_{01}$. This type of breakdown has $\rho = 0, \quad (u_3, v_3) = e^{st}(u^{+}_{30}, v^{+}_{30})$ where s is the eigenvalue for a Görtler vortex. This type of breakdown is not always stable in the sense that a small perturbation of ρ will be forced by the exponentially growing Görtler vortex and will possibly grow. Thus it is likely that this kind of breakdown cannot occur in regions where either the finite time breakdown or the exponential breakdown (5.15) exists.

There are regions in Figure 4 where the finite time blow up or either or both of the exponentially growing solutions can occur. It is not clear which of these would be the ultimate state for an arbitrary initial perturbation. However one would expect that a sufficiently complicated initial perturbation would, in general, lead to the finite time singularity. This was certainly found to be the case for the numerical integrations of (4.1), (4.5) to be reported later in this section.

To determine the relevance of the different regions of Figure 5 to the interaction problem of §4 it is necessary to compute the constant θ/γ appearing in (4.5). This is shown in Figure 4 as a function of β_0 for oblique waves inclined at an $O(1)$ angle to the flow direction. It can be seen that θ/γ increases monotonically with β_0 and $\theta/\gamma = -1/140$ when $\beta_0/\alpha_0 \approx .8887$. The finite time breakdown discussed above can therefore only occur for waves inclined at angles of more than 41.6 degrees to the flow direction. TS waves inclined at a lesser angle break down on an infinite time scale.

Finally in this section the results of a numerical integration of (4.1), (4.5) are reported. These calculations were carried out in order to check on the breakdown structures described above. The integrations were performed by use of finite differences with the spatial derivatives on the left hand side of (4.1) evaluated implicitly. The terms on the right hand sides of (4.1) and (4.5) were treated explicitly. This scheme is stable for time step Δt, the same order of magnitude as the step length Δy.

Firstly consider the situation when $g_0 < g_{01}$ so that the limit cycle is stable below C in Figure 4. Here we found that any nonzero initial disturbance decayed to zero if $\alpha < 0$ and approached the limit cycle solution otherwise. For values of $\theta/\gamma > -1/140$ we found that, dependent of α, a suffiently large initial disturbance terminated in a finite time singularity. If α was negative however a sufficiently small initial disturbance was found to decay to zero. For values of θ/γ between C and $\theta/\gamma = -1/140$ a similar result was found except that the amplitudes were found to grow exponentially when breakdown occured.

The above results are entirely consistent with our previous discussion. Similarly the numerical calculations for $g_0 > g_{01}$ were consistent with the prediction given above. Clearly a numerical investigation of (4.1) –(4.5) can describe only a finite number of initial disturbances but, in all the cases investigated, we found that if both the algebraic and exponential 'blow–ups' were possible then except in special circumstances the algebraic type took place. The exception corresponded in all cases to an initial disturbance with the TS amplitude set equal to zero.

6. Conclusion

It has been shown in the previous sections that there are at least two types of nonlinear interaction possible where small amplitude TG longitudinal vortices and TS disturbances interact in curved channel flows. The weakest 'Type I' interaction is governed by the coupled quadratic amplitude equation typical of resonant triad interactions. This interaction can only take place in the presence of curvature and breaks down at a finite time with the amplitudes of both modes becoming infinite.

At larger amplitudes the 'Type *II*' interaction occurs and leads to the evolution equations (4.1), (4.5). The outcome of this interaction is again a singularity at a finite time if the wavefronts of the TS waves make an angle with the vortices which is greater than 41.6°. Perhaps the most significant property of this type of interaction is that it can also take place in a straight channel. In this situation the Görtler vortex would decay to zero in the absence of the oblique TS waves, thus the longitudinal vortex structure is then a direct consequence of these waves. Thus, in a straight channel, sufficiently large TS waves are capable of driving longitudinal vortices with precisely the same structure as those which would occur if the channel were curved. This suggests that if transition in channel flows is caused by interacting oblique waves, then waves making an angle of more than 41.6° with the flow direction will lead to a catastrophic change in the flow structure.

In external flows it is well known that, in the initial stages of transition, longitudinal vortex structures can develop in the flowfield. (See, for example, Klebanoff, Tidstrom and Sargent (1962)). In addition, turbulent bursts in boundary layers are thought to originate from the spanwise locations where these vortex structures are most unstable. It follows from the present work that if boundary layer growth does not destroy the interaction mechanism discussed in this paper then there exists a mechanism for producing the longitudinal vortices observed in the initial stages of of boundary layer transition.

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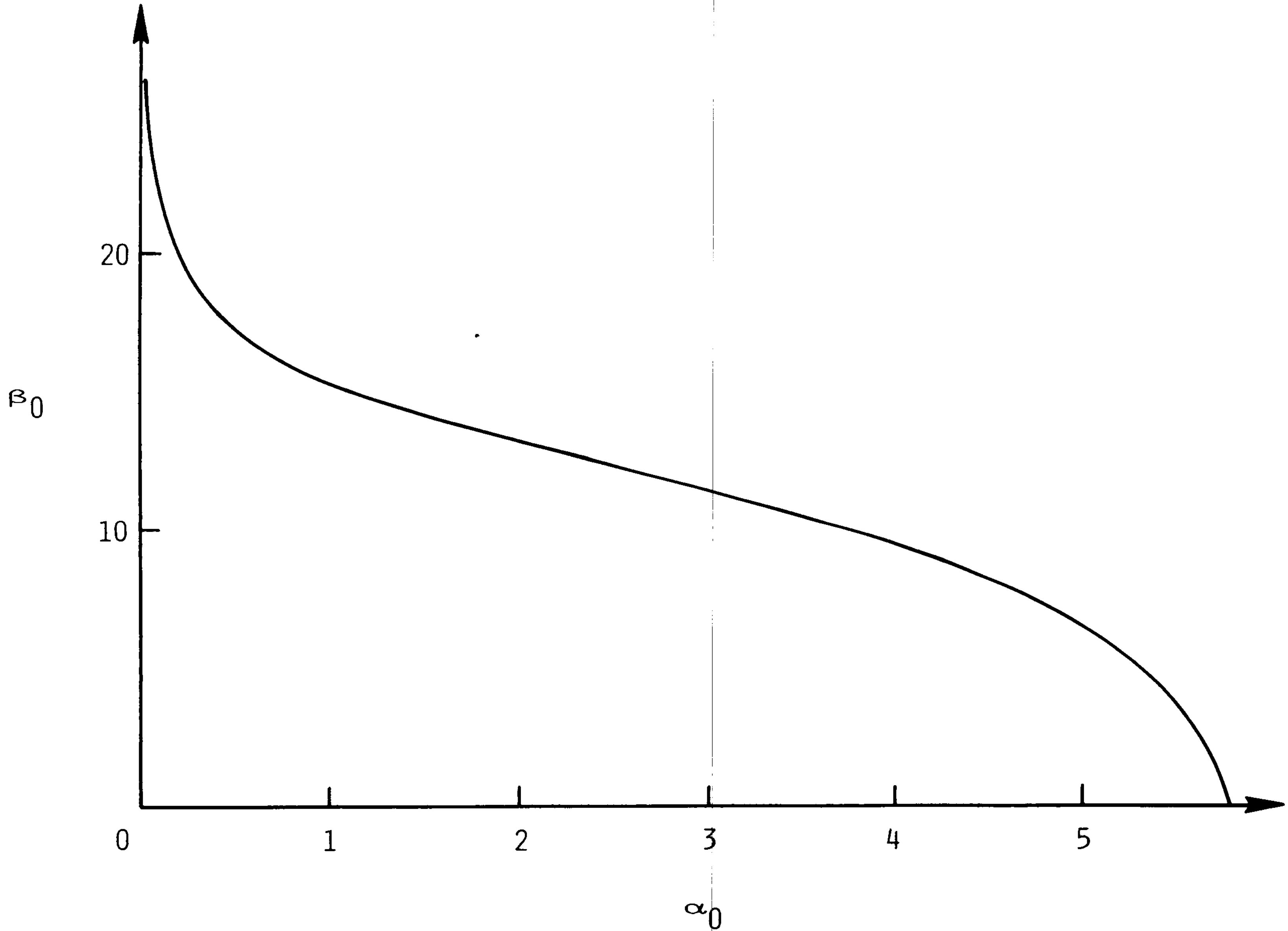

Figure 1. β_0 as a function of α_0 calculated from the eigenrelation (3.35).

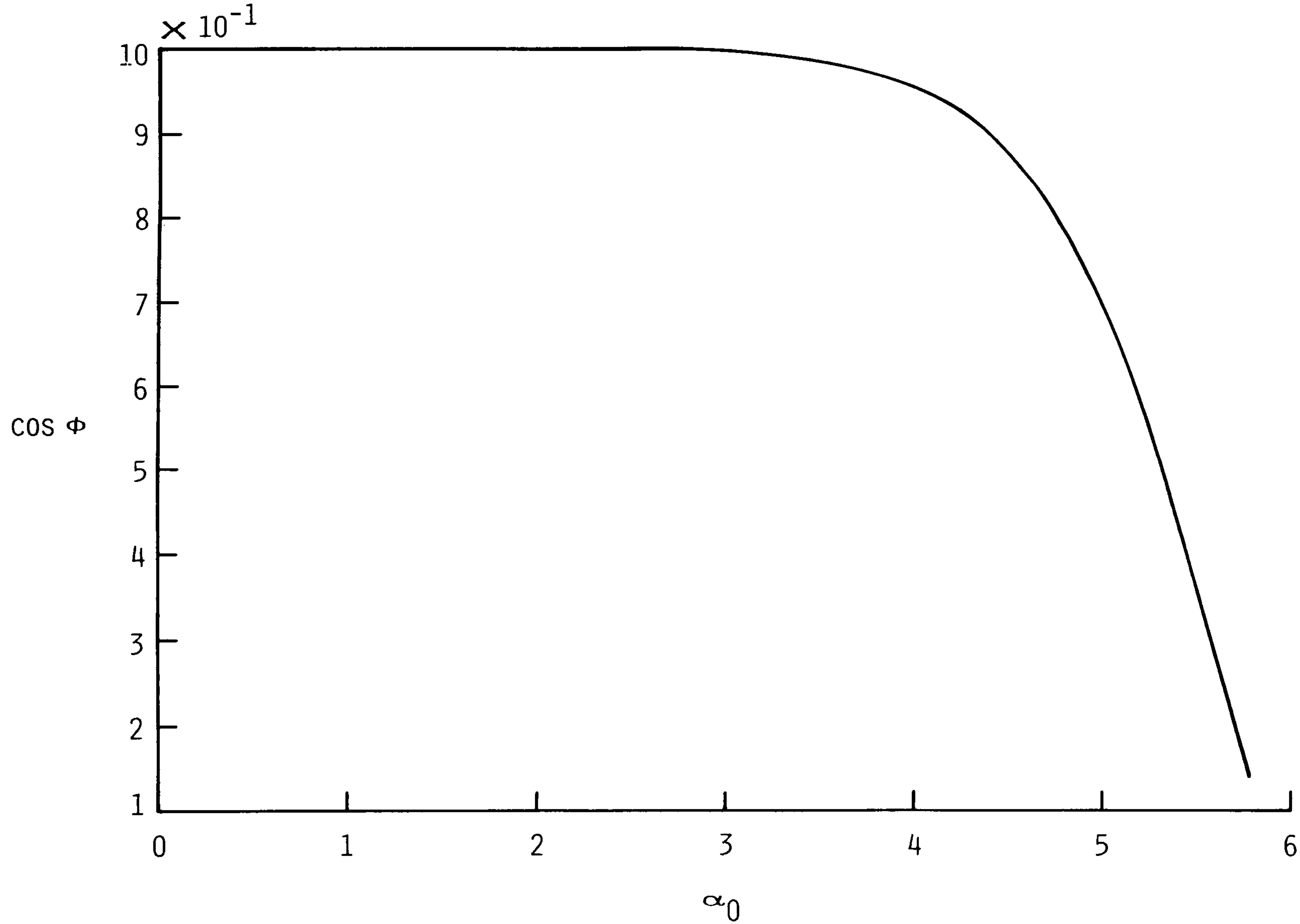

Figure 2a. Cos ϕ as a function of α_0 for the first Görtler mode.

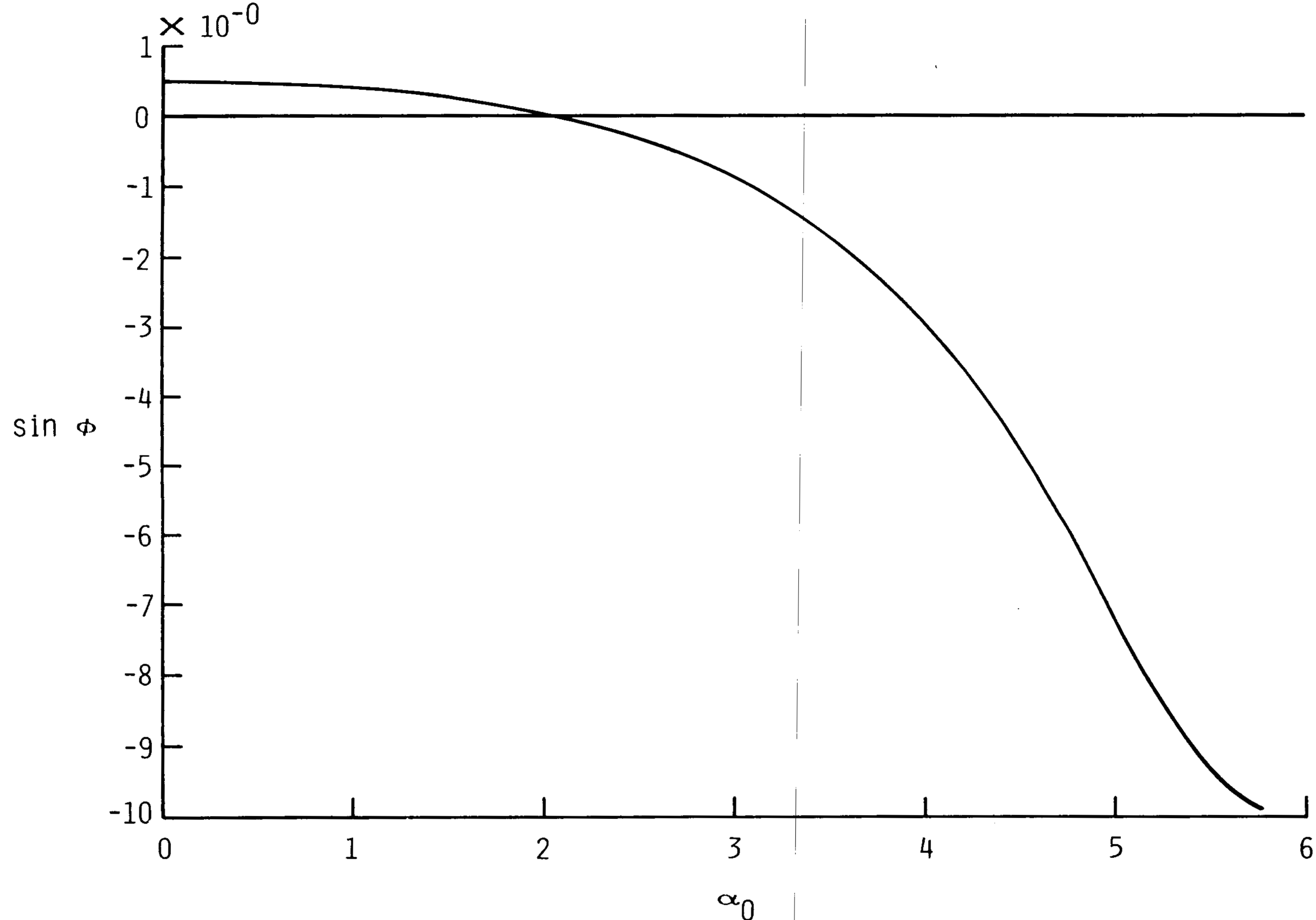

Figure 2b. Sin ϕ as a function of α_0 for the first Görtler mode.

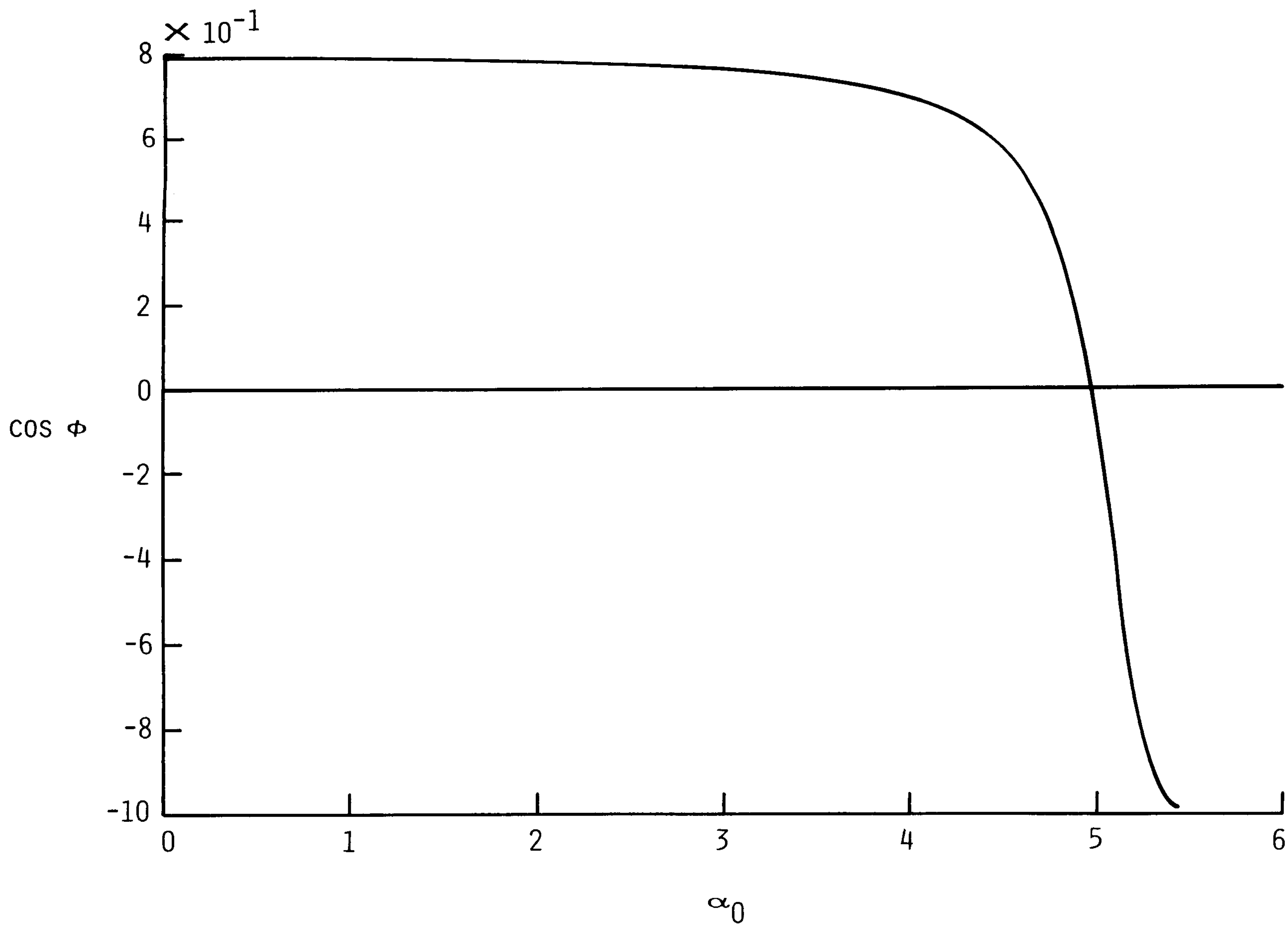

Figure 3a. Cos ϕ as a function of α_0 for the second Görtler mode.

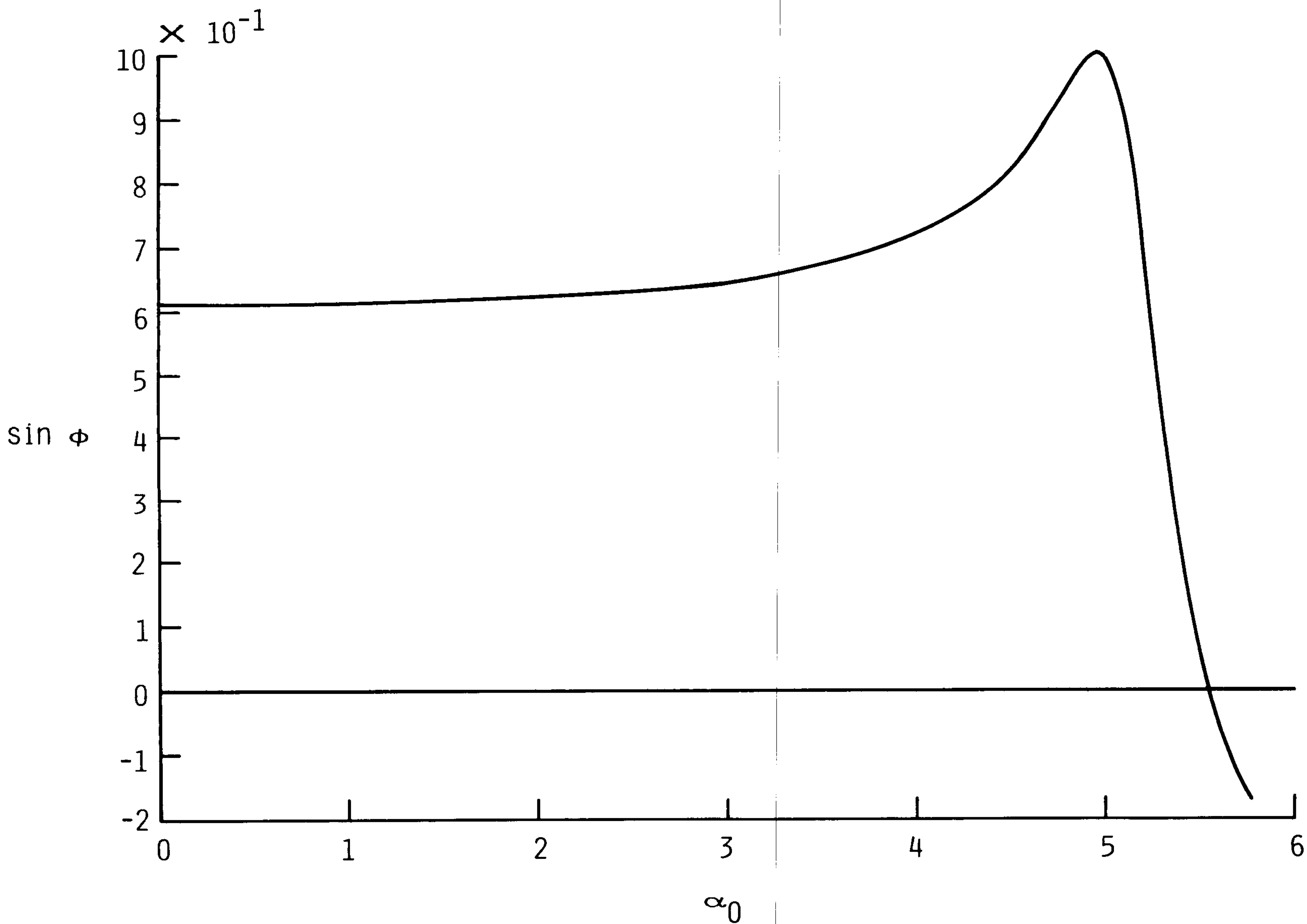

Figure 3b. Sin ϕ as a function of α_0 for the second Görtler mode.

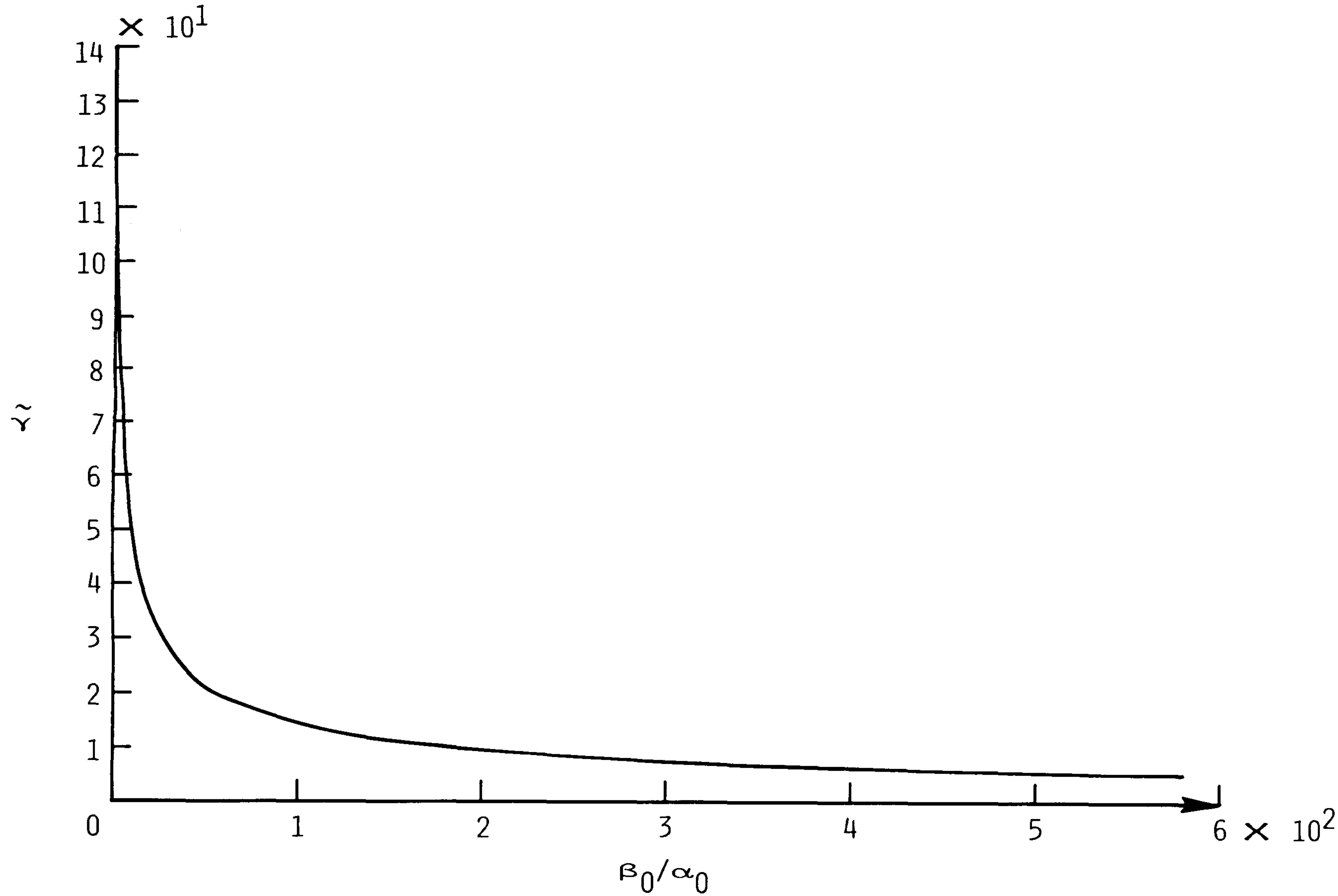

Figure 4a. $\tilde{\gamma}$ as a function of α_0/β_0.

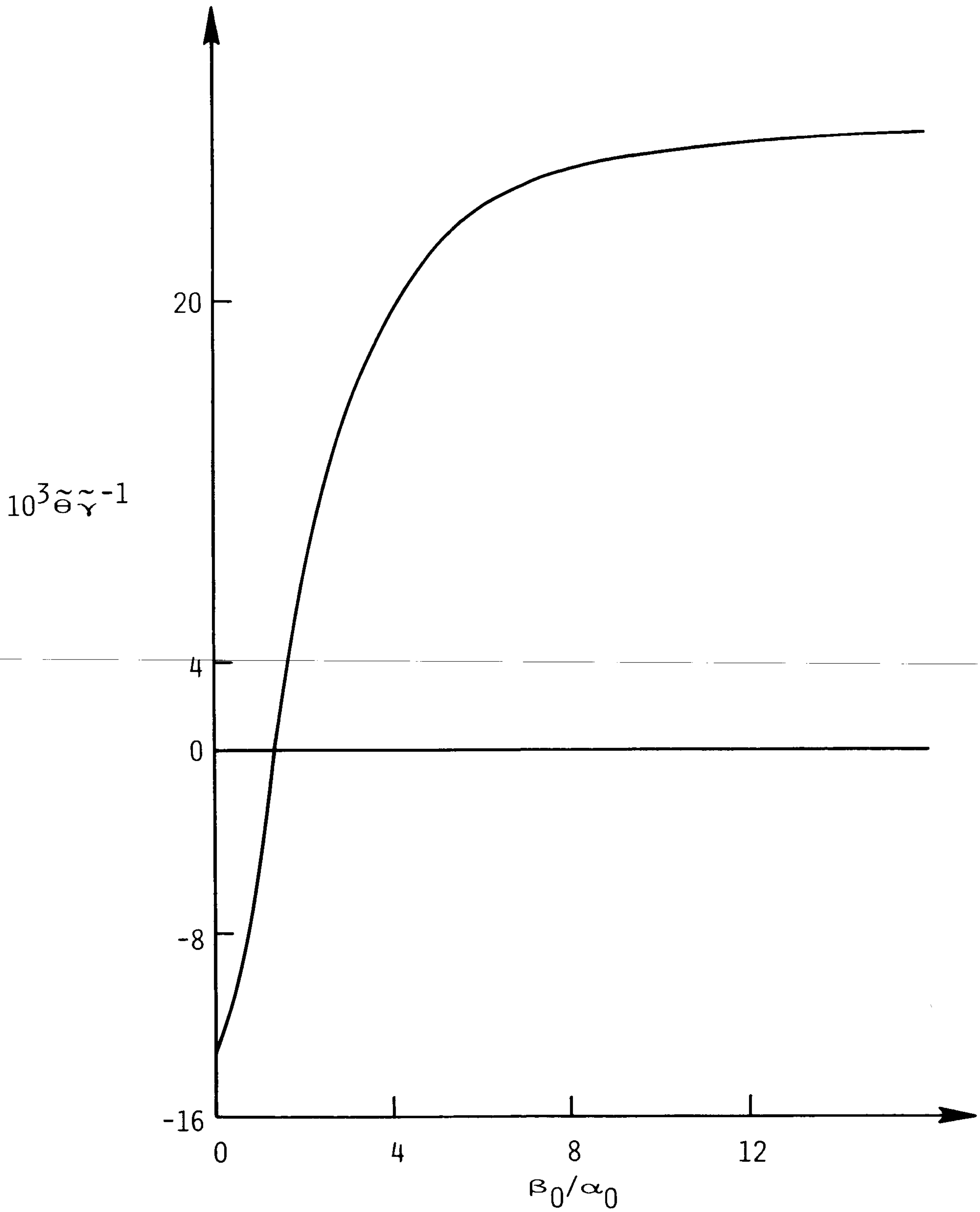

Figure 4b. $\tilde{\theta}\tilde{\gamma}^{-1}$ as a function of α_0/β_0.

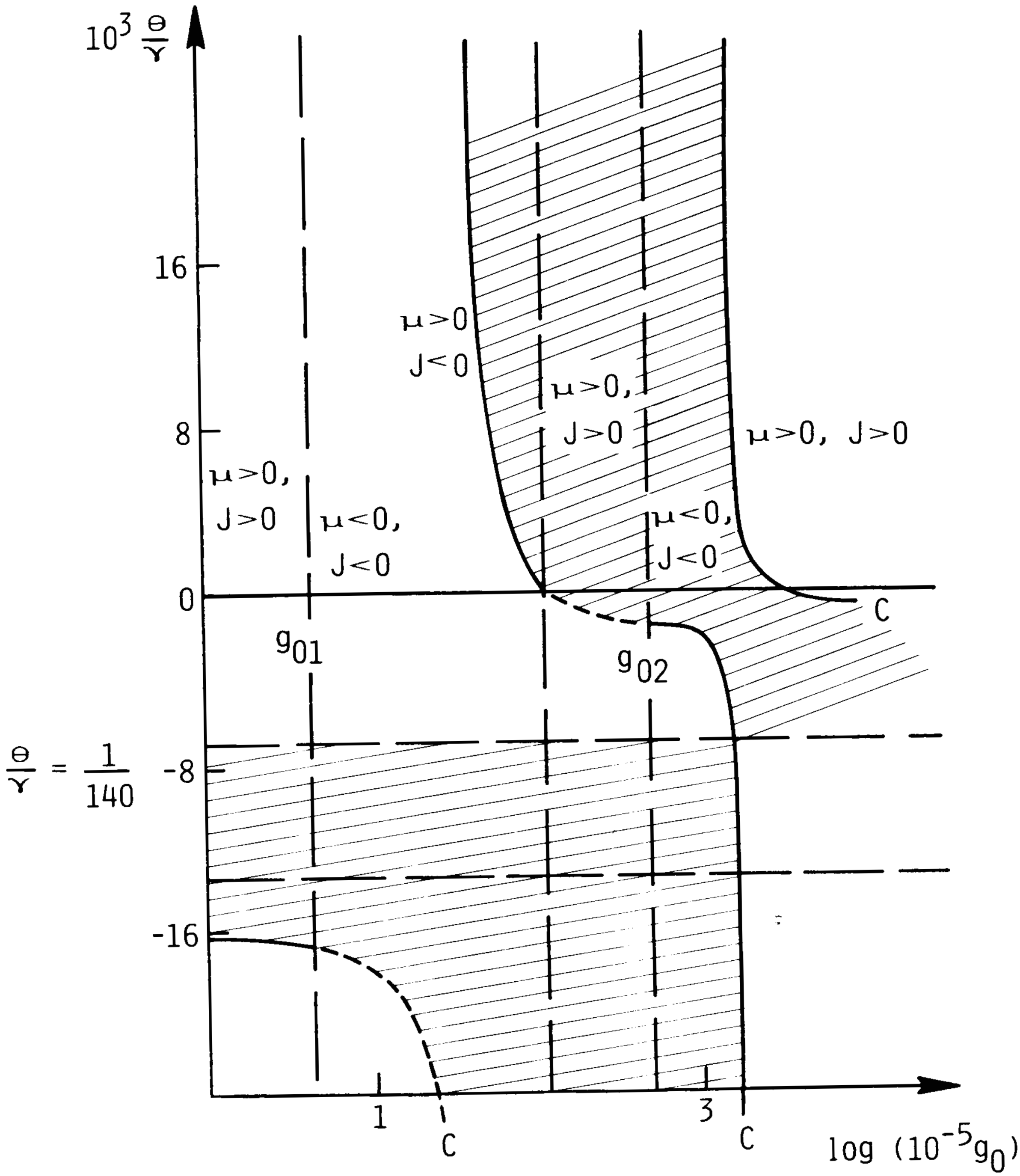

Figure 5. The curve C and the signs of μ, J in the $(\theta/\gamma - g_0)$ plane.

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16. Abstract

~~It is known that a viscous fluid flow with curved streamlines can support both~~ Tollmien-Schlichting and Taylor-Görtler instabilities. The question of which linear mode is dominant at finite values of the Reynolds numbers was discussed by Gibson and Cooke (1973). In a situation where both modes are possible on the basis of linear theory a nonlinear theory must be used to determine the effect of the interaction of the instabilities. The details of this interaction are of practical importance because of its possible catastrophic effects on mechanisms used for laminar flow control. Here this interaction is studied in the context of fully developed flows in curved channels. Apart from technical differences associated with boundary layer growth the structures of the instabilities in this flow are very similar to those in the practically more important external boundary layer situation. The interaction is shown to have two distinct phases depending on the size of the disturbances. At very low amplitudes two oblique Tollmien-Schlichting waves interact with a Görtler vortex in such a manner that the amplitudes become infinite at a finite time. This type of interaction is described by ordinary differential amplitude equations with quadratic nonlinearities. A stronger type of interaction occurs at larger disturbance amplitudes and leads to a much more complicated type of evolution equation. The solution of these equations now depends critically on the angle between the Tollmien-Schlichting wave and the Görtler vortex. Thus if this angle is greater than 41.6° this interaction again terminates in a singularity at a finite time; otherwise the breakdown is exponential taking an infinite time. Moreover the strong interaction can take place in the absence of curvature, in which case the Görtler vortex is entirely driven by the Tollmien-Schlichting waves.

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