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ON THE EIGENVALUE AND EIGENVECTOR DERIVATIVES OF A GENERAL MATRIX

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**ABSTRACT**

A novel approach is introduced to address the problem of existence of differentiable eigenvalues and eigenvectors for a general matrix which may have repeated eigenvalues. A method is shown using the singular value decomposition algorithm to compute the eigenspace which contains the differentiable eigenvectors for a general matrix associated with repeated eigenvalues. The solutions of eigenvalue and eigenvector derivatives for repeated eigenvalues are derived. Several examples are given to illustrate the validity of formulations developed in this paper.

## NOMENCLATURE

$\lambda_i, \psi_i, \phi_i$	i-th eigenvalue, right and left eigenvector of matrix A
$S_{\psi_i}, S_{\phi_i}$	right and left eigenvector subspaces of i-th eigenvalue
$\Psi_i, \Phi_i$	collection of right and left eigenvectors of i-th eigenvalue
$\hat{\psi}_i$	i-th differentiable eigenvector of matrix A
$\hat{\phi}_i$	collection of differentiable left eigenvectors corresponding to i-th eigenvalue
$\lambda_i^{(j)}, \psi_i^{(j)}, \phi_i^{(j)}$	i-th eigenvalue, right and left eigenvector of the matrix $\partial^j A / \partial \rho^j$
$r_i$	multiplicity of i-th eigenvalue
$S_{\psi_i}^{(j)}$	right eigenvector space corresponding to the eigenvalue $\lambda_i^{(j)}$
$r_i^{(j)}$	dimension of the i-th eigenvector space $S_{\psi_i}^{(j)}$ of the matrix $\partial^j A / \partial \rho^j$
$\hat{r}_{ik}^{(j)}$	dimension of the intersection subspace between $S_{\psi_i}^{(j)}$ and $S_{\psi_k}^{(j)}$
$\hat{r}_i$	number of differentiable eigenvectors for i-th eigenvalue
$x_i$	coefficients corresponding to i-th eigenvector derivative
$\rho$	scalar parameter of matrix A
$\Omega$	subspace intersection
$\oplus$	direct sum of linearly independent subspaces

## INTRODUCTION

Current space structures may be dynamically very complex. Situations frequently arise in which vibration modes are very close together in frequency. This condition may occur, for example, when structures are nearly symmetric in a plane or when loosely coupled branch systems have natural frequencies which are very close to those of the global system. Prediction of the dynamic behavior of structures under these conditions is difficult. The phasing of branch responses relative to the parent structure may be entirely reversed in certain modes depending upon the accuracy of a crucial stiffness parameter.

System performance or control stability may depend upon the ability to predict the structural behavior under these sensitive conditions. Thus understanding the sensitivity of the eigenvalues and eigenvectors with respect to some parameter such as a stiffness or mass value under conditions of close or repeated eigenvalues is essential. The eigenvalue and eigenvector derivatives for a matrix with distinct eigenvalues are well documented in references [1-4]. The general expressions for eigenvalue and eigenvector derivatives for non-self-adjoint systems appear to be first given by Rogers [1]. The formulation was correct but incomplete in the sense that only directional changes were included in the eigenvector sensitivity. Lim, et al [4], re-examined the eigenvector derivatives by including the contribution of eigenvector change in the nominal eigenvector direction. The existence of derivatives of eigenvalues and eigenvectors corresponding to a repeated eigenvalue, however, has not been addressed.

The purpose of this paper is to discuss the existence and some computational aspects of the derivatives of eigenvalues and eigenvectors of a general matrix which may have repeated eigenvalues. The singular value decomposition algorithm is used to compute the basis for an eigenspace which

determine the differentiable eigenvectors associated with repeated eigenvalues. Examples are presented to illustrate the concepts.

### EXISTENCE OF THE EIGENVALUE AND EIGENVECTOR DERIVATIVES

Consider the right eigenvalue problem

$$A\psi_i = \lambda_i \psi_i \quad (i=1, \dots, n) \quad \text{or} \quad A\Psi = \Psi\Lambda \quad (1)$$

and the left eigenvalue problem

$$\phi_i^T A = \lambda_i \phi_i^T \quad (i=1, \dots, n) \quad \text{or} \quad \Phi^T A = \Lambda \Phi^T \quad (2)$$

where  $\Phi = [\phi_1, \dots, \phi_n]$ ,  $\Psi = [\psi_1, \dots, \psi_n]$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ . For the class of matrices which are non-defective, a full linearly independent set of eigenvectors always exist and the right and left eigenvectors can be normalized such that

$$\Phi^T \Psi = I_n \quad (3)$$

where  $I_n$  is an identity matrix of order  $n$ .

Assume that the derivatives of the scalar  $\lambda_i$ , vector  $\psi_i$ , and matrix  $A$  with respect to a parameter  $\rho$  exist. Taking the partial derivative of Eq. (1) with respect to the parameter  $\rho$  and premultiplying the resulting equation by  $\phi_i^T$  yields

$$\phi_i^T [\partial A / \partial \rho] \psi_i - (\partial \lambda_i / \partial \rho) e_i + \Lambda \phi_i^T [\partial \psi_i / \partial \rho] - \phi_i^T [\partial \psi_i / \partial \rho] \lambda_i = 0 \quad (4)$$

where  $e_i^T = [0, \dots, 1, \dots, 0]$  is a null vector except that the  $i$ -th element is unity. It is obvious that the coexistence of an eigenvalue and the corresponding eigenvector derivatives are required for Eq. (4) to exist. In other words, the eigenvalue derivative exists if and only if the corresponding eigenvector derivative exists. Let

$$y_i = \Phi^T [\partial A / \partial \rho] \psi_i - (\partial \lambda_i / \partial \rho) e_i \quad (5)$$

and

$$x_i = \Phi^T [\partial \psi_i / \partial \rho] \quad (6)$$

Introduction of these two variables into Eq.(4) gives the following linear vector equation

$$y_i + \Lambda x_i - x_i \lambda_i = 0 \quad (7)$$

or

$$[y_{ji}] + [\Lambda - \lambda_i I][x_{ji}] = 0 \quad (8)$$

in terms of each individual element.

Assuming that the eigenvalue  $\lambda_i$  has a multiplicity  $r_i$ , i.e.  $\lambda_k$  ( $k=i, i+1, \dots, i+r_i-1$ ) =  $\lambda_i$ , Eq. (8) is satisfied if and only if

$$y_{ki} = 0 \quad \text{for } k = i, i+1, \dots, i+r_i-1 \quad (9)$$

with arbitrary  $x_{ki}$  to be determined. Let  $\phi_i = [\phi_i, \phi_{i+1}, \dots, \phi_{i+r_i-1}]$ , and

$\psi_i = [\psi_i, \psi_{i+1}, \dots, \psi_{i+r_i-1}]$  respectively represent the collection of the

left and right eigenvectors associated with the repeated eigenvalues  $\lambda_i$ .

Since any nonzero combination of eigenvectors associated with a repeated eigenvalue is also an eigenvector of the eigenvalue, the column vectors of

the matrices  $\phi_i$  and  $\psi_i$  respectively span a left eigenspace  $S_{\phi_i}$  and a right

eigenspace  $S_{\psi_i}$  for the eigenvalue  $\lambda_i$ . To satisfy Eq. (9), a subset of  $r_i$

independent vectors collected in the  $n$  by  $r_i$  matrix  $\hat{\phi}_i$  should be chosen from

the eigenspace  $S_{\phi_i}$  such that

$$\hat{\phi}_i^T [\partial A / \partial \rho] \hat{\psi}_i = (\partial \lambda_i / \partial \rho) \hat{e}_i \quad (10)$$

where  $\hat{\psi}_i$  is a vector chosen from the eigenspace  $S_{\psi_i}$  and  $\hat{e}_i$  is an  $r_i$  by 1 null vector except the first element is equal to unity. For a distinct eigenvalue  $r_i = 1$ , Eq. (10) is automatically satisfied by which the well-known equation of the eigenvalue derivative is obtained.

Observe that, if the vector  $\hat{\psi}_i$  is also a right eigenvector of the matrix  $[\partial A / \partial \rho]$ , then Eq.(10) is satisfied with the aid of the normalization condition in Eq. (3). Let  $\psi_k^{(1)}$  be the right eigenvector of the matrix  $[\partial A / \partial \rho]$  associated with the eigenvalue  $\lambda_k^{(1)}$  such that

$$[\partial A / \partial \rho] \psi_k^{(1)} = \psi_k^{(1)} \lambda_k^{(1)} ; \quad k = 1, \dots, m \quad (11)$$

The superscript (i) (i=1,2,...) indicates that the corresponding variable is associated with the eigenvalue problem of the i-th partial derivative of the matrix A with respect to the parameter  $\rho$ . Let  $r_k^{(1)}$  be the number of independent eigenvectors  $\psi_k^{(1)}$  associated with the eigenvalue  $\lambda_k^{(1)}$ , and  $S_{\psi_k}^{(1)}$  be the right eigenspace generated by the  $r_k^{(1)}$  independent eigenvectors  $\psi_k^{(1)}$ . The eigenvector  $\hat{\psi}_i$  in Eq. (10) can then be determined from the direct sum of intersection subspaces

$$\hat{\psi}_i \in \text{span} [S_i] \quad (12)$$

$$S_i \triangleq (S_{\psi_i} \cap S_{\psi_1}^{(1)}) \oplus (S_{\psi_i} \cap S_{\psi_2}^{(1)}) \oplus \dots \oplus (S_{\psi_i} \cap S_{\psi_m}^{(1)})$$

where  $\cap$  and  $\oplus$  represent the intersection and direct sum of subspaces respectively [5]. Note that the direct sum for  $S_i$  results from the linear

independence of eigenvector subspaces,  $S_{\psi_i}^{(1)}, \dots, S_{\psi_m}^{(1)}$ . It follows that the

dimension,  $\hat{r}_i$ , of the subspace  $S_i$  can be written as

$$\hat{r}_i = \sum_{k=1}^m \dim (S_{\psi_i} \cap S_{\psi_k}^{(1)}) \triangleq \sum_{k=1}^m \hat{r}_{ik}^{(1)} \quad (13)$$

The procedures for determining the differentiable eigenvectors associated with a repeated eigenvalue can be summarized as follows.

- (1) Compute the eigenvalues  $\lambda_i$ , left eigenvectors  $\phi_i$ , and right eigenvectors  $\psi_i$  of the state matrix A (see Eqs. (1) and (2)).
- (2) Determine the number,  $r_i$ , of independent eigenvectors associated with a repeated eigenvalue  $\lambda_i$ , and form the eigenspace  $S_{\psi_i}$  spanned by the eigenvectors.
- (3) Compute the eigenvalues  $\lambda_i^{(1)}$ , left eigenvectors  $\phi_i^{(1)}$ , and right eigenvectors  $\psi_i^{(1)}$  of the sensitivity matrix  $[\partial A / \partial \rho]$  (see Eq.(11)).
- (4) Determine the number,  $r_i^{(1)}$ , of independent eigenvectors associated with each eigenvalue  $\lambda_i^{(1)}$  and form the eigenspace  $S_{\psi_i}^{(1)}$  spanned by the eigenvectors. Note that the dimension  $r_i^{(1)}$  of the eigenspace may be unity.
- (5) Apply the concept of intersection between two subspaces to determine the differentiable eigenvectors associated with the repeated eigenvalue  $\lambda_i$  (see Eq.(12)).



(6) Determine the number,  $r_{ik}^{(1)}$ , of independent differentiable eigenvectors associated with the repeated eigenvalue  $\lambda_i$  (see Eq. (13)).

The three numbers  $r_i$ ,  $r_i^{(1)}$ , and  $r_{ik}^{(1)}$  are related by

$$r_i \geq r_{ik}^{(1)} \text{ and } r_i^{(1)} \geq r_{ik}^{(1)} \quad (14)$$

Assume that the differentiable vector  $\hat{\psi}_i$  is in the intersection space between  $S_{\psi_i}^{(1)}$  and  $S_{\psi_k}^{(1)}$ . The vector  $\hat{\psi}_i$  will then be an eigenvector of the state matrix A as well as the sensitivity matrix  $[\partial A / \partial \rho]$ . From the fact that explicit formula for eigenvector derivative exists for simple eigenvalues, it is suspected that the quantity  $[\partial \hat{\psi}_i / \partial \rho]$  may perhaps be computed by taking partial derivative of Eq. (11) which is the eigenvalue equation for the sensitivity matrix  $[\partial A / \partial \rho]$ .

Based on the above hypothesis, let the matrix A in Eqs. (1) and (2) be replaced by  $[\partial A / \partial \rho]$  and the sensitivity matrix  $[\partial A / \partial \rho]$  in Eqs. (4), (5) and (11) be replaced by  $[\partial^2 A / \partial \rho^2]$ . Of course, it is assumed that the above derivatives exist. If  $r_i^{(1)} > 1$ , perform the above-stated procedures (1) - (6), using these new matrices  $[\partial A / \partial \rho]$  and  $[\partial^2 A / \partial \rho^2]$ . We will obtain a subspace

$$S_i^{(2)} = (S_{\psi_i} \cap S_{\psi_k}^{(1)} \cap S_{\psi_l}^{(2)}) \quad (15)$$

of dimension  $r_i^{(2)}$  which contains the differentiable eigenvector  $\hat{\psi}_i$ . In Eq. (15), the subscripts "k" and "l" range over the corresponding number of invariant eigenvector subspaces so that any other pair of "k" and "l" may also be chosen to define  $S_i^{(2)}$ . If the dimension  $r_i^{(2)}$  for the subspace  $S_{\psi_i}^{(2)}$

is still larger than 1, replace the matrix  $[\partial A/\partial \rho]$  by the matrix  $[\partial^2 A/\partial \rho^2]$  and the matrix  $[\partial^2 A/\partial \rho^2]$  by  $[\partial^3 A/\partial \rho^3]$ . The above-stated procedures (1) to (6) can be repeated until a subspace with unit dimension

$$S_i^{(k)} = (S_{\psi_i} \Omega S_{\psi_\kappa}^{(1)} \Omega S_{\psi_\ell}^{(2)} \Omega \dots \Omega S_{\psi_\omega}^{(k)}) \quad (16)$$

is obtained such that the dimension  $r_i^{(k)}$  of the subspace  $S_{\psi_i}^{(k)}$  is also equal to unity. Note that there may not exist a solution which satisfies Eq. (16). Assume that the solution exists. The differentiable eigenvector  $\hat{\psi}_i$  thus determined will then be the eigenvector of the matrices  $[\partial^j A/\partial \rho^j]$  ( $j=0,1,2,\dots,k$ ) where  $[\partial^0 A/\partial \rho^0] = A$ . The derivative  $[\partial \hat{\psi}_i/\partial \rho]$  can then be computed by taking partial derivative of the following eigenvalue problem

$$[\partial^k A/\partial \rho^k] \hat{\psi}_i = \hat{\psi}_i \lambda_i^{(k)} \quad (17)$$

where the eigenvalue  $\lambda_i^{(k)}$  is now assumed non-repeated.

Let  $\Phi^{(k)} = [\hat{\phi}_1, \dots, \hat{\phi}_n]$ ,  $\Psi^{(k)} = [\hat{\psi}_1, \dots, \hat{\psi}_n]$  and  $\Lambda^{(k)} = \text{diag}(\hat{\lambda}_1, \dots, \hat{\lambda}_n)$  be respectively the left eigenvectors, right eigenvectors, and eigenvalues of the sensitivity matrix  $[\partial^k A/\partial \rho^k]$ . Assuming that a full linearly independent set of eigenvectors exist, the right and left eigenvectors can be normalized such that

$$\Phi^{(k)T} \Psi^{(k)} = I_n \quad (18)$$

where  $I_n$  is an identity matrix of order  $n$ .

Taking the partial derivative of Eq. (17) with respect to the parameter  $\rho$  and premultiplying the resulting equation by  $\Phi^{(k)T}$  yields

$$\Phi^{(k)T} [\partial^{k+1} A/\partial \rho^{k+1}] \hat{\psi}_i - (\partial \hat{\lambda}_i/\partial \rho) e_i + \Lambda^{(k)} \Phi^{(k)T} [\partial \hat{\psi}_i/\partial \rho] - \Phi^{(k)T} [\partial \hat{\psi}_i/\partial \rho] \hat{\lambda}_i = 0 \quad (19)$$

where  $e_i^T = [0, \dots, 1, \dots, 0]$  is a null vector except that the  $i$ -th element is unity. Let

$$\hat{y}_i = \phi^{(k)T} [\partial^{(k+1)} A / \partial \rho^{(k+1)}] \hat{\psi}_i - (\partial \hat{\lambda}_i / \partial \rho) e_i \quad (20)$$

and

$$\hat{x}_i = \phi^{(k)T} [\partial \hat{\psi}_i / \partial \rho] \quad (21)$$

Introduction of these two variables into Eq.(19) gives the following linear vector equation

$$\hat{y}_i + \Lambda^{(k)} \hat{x}_i - \hat{x}_i \hat{\lambda}_i = 0 \quad (22)$$

or

$$[\hat{y}_{ji}] + [\Lambda^{(k)} - \hat{\lambda}_i I][\hat{x}_{ji}] = 0 \quad (23)$$

in terms of each individual element.

Since the eigenvalue  $\hat{\lambda}_i$  is non-repeated, Eq. (23) yields

$$\hat{x}_{ji} = \hat{y}_{ji} / (\hat{\lambda}_j - \hat{\lambda}_i) \quad \text{for } j \neq i; \quad j = 1, \dots, n \quad (24)$$

with arbitrary  $x_{ii}$  to be determined, as in reference [4], from the

normalization of the eigenvector  $\hat{\psi}_i$  in Eq. (17) such that

$$\hat{\psi}_i^T \hat{\psi}_i = 1 \quad (25)$$

Taking partial derivative of Eq. (25) relative to the parameter  $\rho$  yields

$$[\partial \hat{\psi}_i^T / \partial \rho] \hat{\psi}_i + \hat{\psi}_i^T [\partial \hat{\psi}_i / \partial \rho] = 0 \quad (26)$$

From Eqs. (18) and (21), one obtains

$$[\partial \hat{\psi}_i / \partial \rho] = \Psi^{(k)} \hat{x}_i \quad (27)$$

Substitution of Eq.(27) into Eq.(26) thus yields

$$\hat{x}_i^T [\Psi^{(k)T} \hat{\psi}_i] + [\hat{\psi}_i^T \Psi^{(k)}] \hat{x}_i = 0 \quad (28)$$

where the elements of  $x_i$  associated with the eigenvalues  $\hat{\lambda}_j \neq \hat{\lambda}_i$  are computed from Eq.(23). The only unknown  $\hat{x}_{ii}$  in Eq. (28) can thus be determined.

After the vector  $\hat{\psi}_i$  is determined, the first element of Eq. (10) produces the following known equality

$$(\partial \lambda_i / \partial \rho) = \hat{\phi}_i^T [\partial A / \partial \rho] \hat{\psi}_i = \hat{\phi}_i^T \hat{\psi}_i \lambda_{\rho i} = \lambda_{\rho i} \quad (29)$$

if the normalization condition (3) is imposed, where  $\hat{\phi}_i$  is the right eigenvector defined in Eq. (10) which can also be determined in the same manner as the left eigenvector  $\hat{\psi}_i$ .

Summary: The derivatives of differentiable eigenvectors associated with repeated eigenvalues for a general matrix can be computed using the higher order derivatives of the matrix if they exist, whereas the corresponding eigenvalue derivatives are equal to the eigenvalues of the sensitivity matrix  $[\partial A / \partial \rho]$ .

#### Computation of Differentiable Eigenvectors:

To compute the intersection of subspaces, the principal angles of the subspace pairs in Eq. (12) should be determined (see Ref. 5). Assume, for example, that  $\theta_1, \dots, \theta_{r_1}^{(1)}$  represent the principal angles between subspaces

$S_{\psi_i}$  and  $S_{\psi_1}^{(1)}$  whose dimensions satisfy

$$r_i = \dim(S_{\psi_i}) \geq \dim(S_{\psi_1}^{(1)}) = r_1^{(1)} \geq 1 \quad (30)$$

Compute the singular value decompositions (SVD)

$$\psi_i = U_i \Sigma_i V_i^T \quad (31)$$

$$\Psi_1^{(1)} = U_1^{(1)} \Sigma_1^{(1)} V_1^{(1)T} \quad (32)$$

Then form the matrix  $C = U_i^T U_1^{(1)}$  and compute the SVD such that

$$U_i^T C V = \text{diag}(\cos\theta_k) ; k = 1, 2, \dots, r_1^{(1)} \quad (33)$$

and

$$U_i U = [u_1, \dots, u_{r_i}] \quad (34)$$

$$U_1^{(1)} V = [v_1, \dots, v_{r_1^{(1)}}] \quad (35)$$

where  $u_k$  and  $v_k$  are the principal vectors of the subspace pair  $(S_{\psi_i}, S_{\psi_1}^{(1)})$ .

By taking the vector  $u_k$  corresponding to  $\cos\theta_k = 1$ , an unitary basis is

formed to generate the intersection subspace  $S_{\psi_i} \cap S_{\psi_1}^{(1)}$ . Note that the

intersection subspace  $S_{\psi_i} \cap S_{\psi_1}^{(1)}$  may be null. All the other intersection

subspaces in Eq. (12) can be computed similarly. Assume that the dimension

of the intersection space  $S_{\psi_i} \cap S_{\psi_j}^{(1)}$  ( $j=1, \dots, m$ ) is  $\pi_j$ . Note that  $\hat{r}_i$  as

defined in Eq.(13) is the total number of differentiable eigenvectors.

### EXAMPLE

Consider a series of three springs and two masses in series as shown in Fig. 1. Let  $k_1$ ,  $k_2$  and  $k_m$  respectively be the first, second, and middle

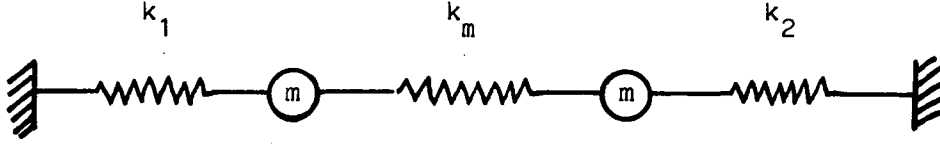


Fig. 1 Three-Spring System

spring constants. The stiffness matrix for the system is simply

$$\begin{bmatrix} k_1+k_m & -k_m \\ -k_m & k_2+k_m \end{bmatrix} = (k_1+k_m) \begin{bmatrix} 1 & -\epsilon \\ -\epsilon & 1+2\eta \end{bmatrix} \quad (36)$$

$$\text{where } \epsilon = k_m/(k_1+k_m) \text{ and } \eta = (k_2-k_1)/2(k_1+k_2) \quad (37)$$

There are two parameters, namely  $\epsilon$  and  $\eta$ , involved in Eq.(29). For convenience in the subsequent analysis, the following notions are introduced

$$\begin{aligned} A &= \begin{bmatrix} 1 & -\epsilon \\ -\epsilon & 1+2\eta \end{bmatrix}, \quad [\partial A/\partial \epsilon] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ [\partial A/\partial \eta] &= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and } A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (38)$$

The matrix A can then be written by

$$A = A_0 + [\partial A/\partial \epsilon]\epsilon + [\partial A/\partial \eta]\eta \quad (39)$$

Instead of directly studying the sensitivity problems of the stiffness matrix as shown in Eq. (36), let us examine the three matrices  $A_0$ ,  $[\partial A/\partial \epsilon]$ ,

and  $[\partial A/\partial \eta]$ . The matrix  $A_0$  has repeated eigenvalues  $\lambda_{01} = \lambda_{02} = 1$  and an eigenspace spanned by  $\psi_1^T = [1, 0]$  and  $\psi_2^T = [0, 1]$ .

Case 1:

The eigenvalues and eigenvectors of the sensitivity matrix  $[\partial A/\partial \epsilon]$  are

$$\lambda_{\epsilon 1} = -1, \quad \psi_{\epsilon 1}^T = [1, 1]/\sqrt{2} \text{ and } \lambda_{\epsilon 2} = 1, \quad \psi_{\epsilon 2}^T = [1, -1]/\sqrt{2} \quad (40)$$

which are obviously in the eigenspace  $S_\psi$  and thus differentiable for the perturbed matrix defined by

$$A_1 = A_0 + [\partial A/\partial \epsilon]\epsilon = \begin{bmatrix} 1 & -\epsilon \\ -\epsilon & 1 \end{bmatrix} \quad (41)$$

In fact,  $\psi_{\epsilon 1}$  and  $\psi_{\epsilon 2}$  are eigenvectors of the matrix  $A_1$  corresponding to two distinct eigenvalues  $\lambda_{11} = 1+\epsilon$  and  $\lambda_{12} = 1-\epsilon$ . This indicates that the eigenvectors  $\psi_{\epsilon 1}$  and  $\psi_{\epsilon 2}$  are not function of the parameter  $\epsilon$ . The eigenvector derivatives for the matrix  $A_1$  are thus zero, which is consistent with the theoretical results developed in the preceding section, by simply applying  $[\partial^2 A/\partial \epsilon^2]=0$  into Eqs. (20) and (24).

Case 2:

The eigenvalues and eigenvectors of the sensitivity matrix  $[\partial A/\partial \eta]$  are

$$\lambda_{\eta 1} = 0, \quad \psi_{\eta 1}^T = [1, 0] \text{ and } \lambda_{\eta 2} = 2, \quad \psi_{\eta 2}^T = [0, 1] \quad (42)$$

which are the basis vectors for the eigenspace  $S_\psi$  and thus are also differentiable for the perturbed matrix defined by

$$A_2 = A_0 + [\partial A/\partial \eta]\eta = \begin{bmatrix} 1 & 0 \\ 0 & 1+2\eta \end{bmatrix} \quad (43)$$

Obviously,  $\psi_{\eta 1}$  and  $\psi_{\eta 2}$  are also the eigenvectors associated with the eigenvalues  $\lambda_{21} = 1$  and  $\lambda_{22} = 1 + 2\eta$  of the matrix  $A_2$  no matter what the value  $\eta$  is. The eigenvector derivatives are thus zero for the perturbed matrix  $A_2$  which again verifies the preceding theoretical studies.

Case 3:

Now examine the matrix equation

$$A = A_1 + [\partial A / \partial \eta] \eta \quad (44)$$

where the matrix  $A$  has eigenvectors  $\psi_{\epsilon 1}^T = [1, 1]/\sqrt{2}$  and  $\psi_{\epsilon 2}^T = [1, -1]/\sqrt{2}$  associated with the eigenvalues  $\lambda_{11} = 1 - \epsilon$  and  $\lambda_{12} = 1 + \epsilon$  respectively. Since eigenvalues  $\lambda_{11}$  and  $\lambda_{12}$  are distinct, application of Eqs. (5)-(8) yields

$$[\partial \psi_{\epsilon 1} / \partial \eta]^T = [1, -1]/(2\epsilon\sqrt{2}) \quad \text{and} \quad [\partial \psi_{\epsilon 2} / \partial \eta]^T = [1, 1]/(2\epsilon\sqrt{2}) \quad (45)$$

It is seen that  $[\partial \psi_{\epsilon 1} / \partial \eta]$  and  $[\partial \psi_{\epsilon 2} / \partial \eta]$  are inversely proportional to the parameter  $\epsilon$  and thus very sensitive in the neighborhood of  $\eta = 0$ , particularly when  $\epsilon$  is sufficiently small. This is due to the fact that, when  $\epsilon = 0$ , the eigenvectors become  $\psi_{\eta 1}^T = [1, 0]$  and  $\psi_{\eta 2}^T = [0, 2]$ , corresponding to the eigenvalues  $\lambda_{21} = 1$  and  $\lambda_{22} = 1 + 2\eta$  of the matrix  $A_2$  shown in Eq. (36). The eigenvectors of  $A$  suffer radical changes with only small perturbations in the entries of the matrix  $A$ . The eigenvectors  $\psi_{\epsilon 1}$  and  $\psi_{\epsilon 2}$  for all  $\epsilon > 0$  are not approximately parallel to either of the two vectors  $\psi_{\eta 1}$  and  $\psi_{\eta 2}$  no matter how small the parameter  $\epsilon$  is. The transition of the eigenvectors  $\psi_{\epsilon 1}$  and  $\psi_{\epsilon 2}$  to the vectors  $\psi_{\eta 1}$  and  $\psi_{\eta 2}$  is possible only when they are all in the same eigenspace associated with a repeated eigenvalue for the case where  $\epsilon = 0$ . In other words, for any nonzero  $\epsilon$  no matter how



small it is, the transition from the vectors  $\psi_{\epsilon 1}$  and  $\psi_{\epsilon 2}$  to the vectors  $\psi_{\eta 1}$  and  $\psi_{\eta 2}$  becomes impossible.

### CONCLUSIONS

The existence of differentiable eigenvalues and eigenvectors for a general matrix is addressed. The eigenspace which contains the differentiable eigenvectors are determined and computed by using the concept of subspace intersection in conjunction with the singular value decomposition algorithm. The differentiable eigenvectors associated with repeated eigenvalues for a general matrix should simultaneously be the eigenvectors of the corresponding sensitivity matrix. Furthermore, the derivatives for differentiable eigenvectors associated with repeated eigenvalues can be computed from higher order derivatives of the matrix, whereas the corresponding eigenvalue derivatives are the eigenvalues of the sensitivity matrix.

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16. Abstract  The existence of differentiable eigenvalues and eigenvectors for a general matrix is addressed. The eigenspace which contains the differentiable eigenvectors are determined and computed by using the concept of subspace intersection in conjunction with the singular value decomposition algorithm. The differentiable eigenvectors associated with repeated eigenvalues should be simultaneously the eigenvectors of the general matrix and its corresponding sensitivity matrix. Furthermore, the derivatives for differentiable eigenvectors associated with repeated eigenvalues can be computed using higher order derivatives of the matrix, whereas the corresponding eigenvalue derivatives are the eigenvalues of the sensitivity matrix.			
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