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Approximation of Discrete-Time LQG Compensators for Distributed Systems with Boundary Input and Unbounded Measurement†

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ABSTRACT

We consider the approximation of optimal discrete-time linear quadratic Gaussian (LQG) compensators for distributed parameter control systems with boundary input and unbounded measurement. Our approach applies to a wide range of problems that can be formulated in a state space on which both the discrete-time input and output operators are continuous. Approximating compensators are obtained via application of the LQG theory and associated approximation results for infinite dimensional discrete-time control systems with bounded input and output. Numerical results for spline and modal based approximation schemes used to compute optimal compensators for a one dimensional heat equation with either Neumann or Dirichlet boundary control and pointwise measurement of temperature are presented and discussed.

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1. Introduction

In this paper we develop an approximation theory for the computation of optimal discrete-time linear quadratic Gaussian (LQG) compensators (combined feedback control law and state estimator) for distributed parameter systems with boundary input or control and unbounded output or measurement. In a continuous time setting, boundary input typically results in an unbounded input operator. That is, the system's input operator maps the control input into a space larger than the state space in which the open-loop system is usually formulated. In the discrete-time case, on the other hand, for a wide class of distributed systems, the resulting input operator is bounded on the usual underlying state space. By unbounded output or measurement is meant that the system output operator has domain in a space smaller than the usual open-loop state space.

For continuous time systems, Pritchard and Salamon (1987) have established an abstract semigroup theoretic framework for treating the linear quadratic regulator problem (control only) for infinite dimensional systems with unbounded input and output operators. Their approach is based upon a weak or distributional formulation of the Riccati equations which characterize the optimal feedback control laws in an appropriate dual space . Curtain (1984) provides a procedure for the design of finite dimensional compensators for parabolic systems with unbounded control and observation. In (Curtain and Salamon, 1986) a finite dimensional compensator design procedure for a wider class of infinite dimensional systems with unbounded input (but bounded output) including hereditary systems with control delays and partial differential systems with boundary control is developed. Lasiecka and Triggiani have looked at linear regulator problems for parabolic (1983a, 1987a) and hyperbolic (1983b, 1986) systems with boundary control and obtained, among other things, global and local regularity results for the optimal controls and state trajectories. In (Lasiecka and Traggiani, 1987b) Galerkin approximations and an associated convergence theory for closed-loop solutions to regulator problems for parabolic systems with Dirichlet boundary input are studied. A more complete survey of the boundary control literature including references to some of the poineering work in this area can be found in (Pritchard and Salamon, 1987).

In our treatment here, we consider the discrete-time problem (i.e. piecewise constant input and sampled output). Our interest in the discrete-time or digital formulation is motivated by 1) the fact that it represents a more accurate or realistic description of how the linear-quadratic theory for distributed systems would actually be applied in practice, and by 2) how the boundedness of the discrete-time input operator in the usual underlying state space facilitates the development of an approximation theory which can simultaneously handle both unbounded input and unbounded output. Our approach is based upon an application of the theory we developed earlier in (Gibson and Rosen, 1985 and 1986) for the approximation of optimal discrete-time LQG compensators for infinite dimensional systems with bounded input and output. Our results are applicable to a rather wide class of boundary control systems in which a restriction of the state transition operator and the discrete-time input operator are bounded on a space on which the output operator is bounded as well.

An outline of the remainder of the paper is as follows. In Section 2 we describe an abstract framework for the study of boundary control systems and their discrete-time formulation. In Section 3 we review the LQG theory for infinite dimensional discrete-time systems and associated abstract approximation results. In the fourth section, we illustrate the application of our technique on an example involving the spline and modal subspace based approximation of optimal compensators for a one dimensional heat equation with either Neumann or Dirichlet boundary control and pointwise measurement of temperature. Section 5 contains some concluding remarks.

2. The Boundary Control System and its Discrete-Time Formulation

We employ a semigroup theoretic formulation for a class of abstract boundary control systems which has appeared elsewhere in the literature. See, for example, (Curtain and Salamon, 1986). Let W,V and H be Hilbert spaces with W and V densely and continuously embedded in H. We consider boundary control systems of the form

(2.1) $\dot{w}(t) = \Delta w(t), \quad t > 0$

(2.2) $w(0) = w_0$

(2.3) $\Gamma w(t) = v(t), \quad t \geq 0$

(2.4) $y(t) = Cw(t), \quad t \geq 0$

where $\Delta \in \mathcal{L}(W,H)$, the boundary input operator Γ is an element in $\mathcal{L}(W,R^m)$ and the output operator C is an element in $\mathcal{L}(V,R^p)$.

We require the assumptions that 1) Γ is surjective and its null space, $\mathcal{N}(\Gamma) = \{\varphi \in W: \Gamma\varphi = 0\}$, is dense in H, 2) the operator $\mathcal{A}$, defined to be the restriction of the operator Δ to $\mathcal{N}(\Gamma)$, is a closed operator on H and has non-empty resolvent set and 3) for each $T > 0$, all $w_0 \in W$, and $v \in C^1(0,T; R^m)$ with $\Gamma w_0 = v(0)$, there exists a unique $w \in C([0,T]; W) \cap C^1([0,T]; H)$ which depends continuously on w_0 and v and which satisfies (2.1) - (2.3) for each $t \in [0,T]$. It then follows (see Hille and Phillips, 1957) that the operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset H \to H$ given by $\mathcal{A}\varphi = \Delta\varphi$ for $\varphi \in \mathrm{Dom}(\mathcal{A}) = \mathcal{N}(\Gamma)$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup, $\{\mathcal{T}(t) : t \geq 0\}$, of bounded linear operators on H.

Define the space Z as the dual of $\mathrm{Dom}(\mathcal{A}^*) \subset H$. Then H is densely and continuously embedded in Z, $\{\mathcal{T}(t) : t \geq 0\}$ can be uniquely extended to a $\mathcal{C}_0$ semigroup of bounded linear operators on Z, and the operator $\mathcal{A}$ can be uniquely extended to the operator in $\mathcal{L}(H,Z)$ given by $(\mathcal{A}\varphi)(\psi) = \langle\varphi, \mathcal{A}^*\psi\rangle_H$ for $\varphi \in H$, $\psi \in \mathrm{Dom}(\mathcal{A}^*)$.

For $v \in R^m$, since Γ is onto, there exists a $w \in W$ such that $\Gamma w = v$. With this w define $\mathcal{B}v = \Delta w - \mathcal{A}w$. It can be shown (see Curtain and Salamon, 1986) that $\mathcal{B}$ is a well defined element in $\mathcal{L}(R^m, Z)$ and that for each $w_0 \in H$ and $v \in L_2(0,T; R^m)$ there exists a unique $w \in C([0,T]; H) \cap H^1(0,T; Z)$ which depends continuously on w_0 and v and which satisfies

$$\dot{w}(t) = \mathcal{A}w(t) + \mathcal{B}v(t), \quad t > 0$$
$$w(0) = w_0$$

in Z. The function w is given by

(2.5) $$w(t) = \mathcal{T}(t)w_0 + \int_0^t \mathcal{T}(t-s)\,\mathcal{B}v(s)ds, \quad t \geq 0$$

and is referred to as a weak solution to the boundary control system (2.1) - (2.3).

The discrete-time formulation of (2.1) - (2.4) is found by considering piecewise-constant controls of the form

(2.6) $v(t) = u_k, \quad t \in [k\tau, (k+1)\tau), \quad k = 0,1,2,...$

where τ denotes the length of the sampling interval. Let $w_k = w(k\tau)$, $k = 0,1,2,...$ where $w(\cdot)$ is the unique weak solution to (2.1) - (2.3) given by (2.5) corresponding to $w_0 \in H$ and input v given by (2.6). (We note that with piecewise constant input of the form (2.6), the solution w is in fact a strong solution on each subinterval $[k\tau, (k+1)\tau]$.) Recalling that Γ is a surjection, let $\Gamma^+ \in \mathcal{L}(R^m, W)$ denote <u>any</u> right inverse of Γ and for each $k = 0,1,2,...$ define $z_k \in C([k\tau, (k+1)\tau]; H)$ by $z_k(t) = w(t) - \Gamma^+ u_k$, $t \in [k\tau, (k+1)\tau]$. Then

$$\begin{aligned}
\dot{z}_k(t) = \dot{w}(t) &= \mathcal{A}w(t) + \mathcal{B}u_k \\
&= \mathcal{A}z_k(t) + (\mathcal{A} + \mathcal{B}\Gamma)\Gamma^+ u_k \\
&= \mathcal{A}z_k(t) + \Delta\Gamma^+ u_k, \qquad t \in (k\tau, (k+1)\tau],
\end{aligned}$$

$$z_k(k\tau) = w_k - \Gamma^+ u_k .$$

Therefore

$$\begin{aligned}
w_{k+1} &= z_k((k+1)\tau) + \Gamma^+ u_k \\
&= \mathcal{T}(\tau)(w_k - \Gamma^+ u_k) + \int_0^\tau \mathcal{T}(s)\,\Delta\,\Gamma^+ u_k\, ds + \Gamma^+ u_k \\
&= \mathcal{T}(\tau)w_k + (I - \mathcal{T}(\tau))\,\Gamma^+ u_k + \int_0^\tau \mathcal{T}(s)\Delta\Gamma^+ u_k ds,
\end{aligned}$$

or

$$\begin{aligned}
w_{k+1} &= Tw_k + Bu_k, \quad k = 0,1,2,... \\
w_0 &\in H
\end{aligned}$$

where $T \in \mathcal{L}(H)$ and $B \in \mathcal{L}(R^m, H)$ are given by $T = \mathcal{T}(\tau)$ and $B = (I - \mathcal{T}(\tau))\,\Gamma^+ + \int_0^\tau \mathcal{T}(s)\Delta\Gamma^+ ds$ respectively.

We note that the discrete-time input operator B is well defined and does not depend upon a particular choice for Γ^+. Indeed if B_1 and B_2 are the input operators which correspond to the choices Γ_1^+ and Γ_2^+ then for $u \in R^m$ we have

$$(B_1 - B_2)u = (I - \mathcal{T}(\tau))(\Gamma_1^+ - \Gamma_2^+)u + \int_0^\tau \mathcal{T}(s)\Delta(\Gamma_1^+ - \Gamma_2^+)u ds.$$

But $(\Gamma_1^+ - \Gamma_2^+)u \in \mathcal{N}(\Gamma) = \mathrm{Dom}(\mathcal{A})$ and therefore

$$\int_0^\tau \mathcal{T}(s)\Delta(\Gamma_1^+ - \Gamma_2^+)u ds = \int_0^\tau \mathcal{T}(s)\,\mathcal{A}(\Gamma_1^+ - \Gamma_2^+)\,u ds$$

$$= \int_0^\tau \frac{d}{ds}\,\mathcal{T}(s)(\Gamma_1^+ - \Gamma_2^+)u ds = (\mathcal{T}(\tau) - I)(\Gamma_1^+ - \Gamma_2^+)\,u\,.$$

In addition, if Γ^+ is chosen so that $\mathcal{R}(\Gamma^+) \subset \mathcal{N}(\Delta)$, B takes on the particularly simple form $B = (I - \mathcal{T}(\tau))\Gamma^+$. It is also worth noting that a simple calculation reveals that

$$B = \int_0^\tau \mathcal{T}(s)\,\mathcal{B} ds$$

in agreement with the standard technique for obtaining the discrete or sampled time formulation of a continuous time system in either a finite dimensional or bounded input setting.

It is our intention here to apply the approximation theory we developed earlier in (Gibson and Rosen, 1986) for the design of optimal discrete-time LQG compensators for infinite dimensional systems with bounded input and output operators. We therefore require the additional assumptions that 4) $T = \mathcal{T}(\tau) \in \mathcal{L}(V)$ and 5) $\mathcal{R}(\Gamma^+) \subset V$. Although not all boundary control systems we might formulate would satisfy these conditions, as will become evident in Section 4 below, a wide class of interesting and important systems do. In this case, the control system (2.1) - (2.4) takes the form

$$(2.7)\qquad w_{k+1} = Tw_k + Bu_k, \quad k = 0,1,2,\ldots$$

$$(2.8)\qquad w_0 \in V$$

$$(2.9)\qquad y_k = Cw_k, \qquad k = 0,1,2,\ldots .$$

3. LQG Theory for Infinite Dimensional Discrete-Time Systems and Finite Dimensional Approximation

The discrete-time linear quadratic control or regulator problem for the boundary control system (2.1) - (2.3) takes the form

Find $u^* = \{u_k^*\}_{k=0}^{*} \in \ell_2(0,\infty; R^m)$ which minimizes the quadratic performance index

$$J(u) = \sum_{k=0}^{\infty} \langle Qw_k, w_k \rangle_V + u_k^T Ru_k$$

where $Q \in \mathcal{L}(V)$ is self-adjoint and nonnegative, $R \in \mathcal{L}(R^m)$ is a symmetric positive definite $m\times m$ matrix and the state $w = \{w_k\}_{k=0}^{\infty}$ evolves according to the recurrence (2.7), (2.8).

The optimal control is given in closed-loop, linear state feedback form by $u_k^* = -Fw_k^*$, $k = 0,1,2,\ldots$ where $F = (R + B^*\Pi B)^{-1}B^*\Pi T$ and Π is the minimal nonnegative, self-adjoint solution (if it exists) to the operator algebraic Riccati equation

$$(3.1)\qquad \Pi = T^*(\Pi - \Pi B(R + B^*\Pi B)^{-1}B^*\Pi)T + Q .$$

A control u is said to be admissible for the initial data w_0 if the resulting state trajectory $w_k = w_k(w_0,u)$, $k = 0,1,2,\ldots$ is such that $J(u) < \infty$. If for each $w_0 \in V$ there exists an admissible control u, then the Riccati equation (3.1) admits a self-adjoint nonnegative solution Π. If, in addition, an admissible control drives the state w_k to zero, asymptotically as $k \to \infty$, then this solution is unique (see (Gibson and Rosen, 1985)). The optimal state trajectory $w^* = \{w_k^*\}_{k=0}^{*}$

evolves according to $w_k^* = S^k w_0$, $k = 0,1,2,...$ where the closed loop state transition operator $S \in \mathcal{L}(V)$ is given by $S = T - BF$. If Q is coercive, then S has spectral radius less than one and is uniformly exponentially stable. From the finite dimensionality of the control space we obtain

$$(3.2) \qquad u_k^* = - \langle f, w_k^* \rangle_V, \quad k = 0,1,2,...$$

where $f = (f_1, f_2, ..., f_m)^T \in \times_{j=1}^{m} V$ is referred to as the optimal functional feedback control gains.

When only a finite dimensional measurement $y = \{y_k\}_{k=0}^{\infty}$ of the infinite dimensional state w is available (i.e. equation (2.9)) a state estimator or observer is required. For a given input sequence u and corresponding output sequence y the optimal LQG estimator is given by

$$(3.3) \qquad \hat{w}_{k+1} = T\hat{w}_k + Bu_k + \hat{F}\{y_k - C\hat{w}_k\}, \quad k = 0,1,2,...$$

$$(3.4) \qquad \hat{w}_0 \in V$$

where the optimal estimator or observer gains $\hat{F} \in \mathcal{L}(R^p, V)$ are given by $\hat{F} = T\hat{\Pi}C^*(\hat{R} + C\hat{\Pi}C^*)^{-1}$ with $\hat{\Pi} \in \mathcal{L}(V)$ the minimal self-adjoint, nonnegative solution (if one exists) to the operator algebraic Riccati equation

$$(3.5) \qquad \hat{\Pi} = T(\hat{\Pi} - \hat{\Pi}C^*(\hat{R} + C\hat{\Pi}C^*)^{-1}C\hat{\Pi})T^* + \hat{Q}.$$

The operator $\hat{Q} \in \mathcal{L}(V)$ is assumed to be self-adjoint, nonnegative and the $p \times p$ matrix $\hat{R}$ is assumed to be symmetric, positive definite. In a stochastic setting, the operator $\hat{Q}$ and the matrix $\hat{R}$ are assumed to be respectively the covariance operator and matrix for uncorrelated, zero-mean, stationary, Gaussian white noise processes which corrupt the state and measurement. In this case, if $\hat{Q}$ is trace class, (3.3), (3.4) is the infinite dimensional analog of the discrete-time Kalman-Bucy filter. In a strictly deterministic setting, $\hat{Q}$ and $\hat{R}$ are assumed to be determined via engineering design criteria such as stability margins, robustness of the closed-loop system, etc.

Sufficient conditions for existence and uniqueness of self-adjoint, nonnegative solutions $\hat{\Pi}$ to the algebraic Riccati equation (3.5) are of course analogous to the ones given earlier for the control problem (i.e. with regard to the usual duality which exists between the optimal LQG control and estimator problems).

Since $\hat{F} \varepsilon (R^p,V)$, it has a representation of the form

$$\hat{F}y = \hat{f}^T y \quad , \quad y \ \varepsilon \ R^p$$

where $\hat{f} = (f_1, f_2, \ldots, f_p)^T \ \varepsilon \ \overset{p}{\underset{j=1}{\times}} V$ is referred to as the optimal functional observer gains.

When the optimal control law is used together with the optimal observer, that is

$$\text{(3.6)} \qquad \hat{u}_k^* = -F\,\hat{w}_k^* \ , \ k = 0,1,2,\ldots$$

where $\hat{w}^* = \{\hat{w}_k^*\}_{k=0}^{\infty}$ is given by (3.3), (3.4) with input $u = \hat{u}^*$ and corresponding output $y = y^*$, we obtain the optimal LQG compensator. The resulting closed-loop system is given by

$$\mathcal{W}_k = \mathcal{A}^k \mathcal{W}_0, \quad k = 0,1,2,\ldots$$

where $\mathcal{W}_k = (w_k, \hat{w}_k^*)^T$ with $\{w_k\}_{k=0}^{*}$ the state trajectory which results from the input (3.6) and $\mathcal{A} \ \varepsilon \ \mathcal{L}(V\times V)$ is given by

$$\mathcal{A} = \begin{bmatrix} T & -BF \\ \hat{F}C & T-BF-\hat{F}C \end{bmatrix} .$$

If we define e_k to be the difference between w_k and $\hat{w}^*_k$; $e_k = w_k - \hat{w}^*_k$, we find $e_k = \hat{S}^k e_0$, $k = 0,1,2,...$ where $\hat{S} = T - \hat{F}C$. It can be shown that if S and $\hat{S}$ are uniformly exponentially stable, so too is $\mathcal{S}$ and its spectrum, $\sigma(\mathcal{S})$, is given by $\sigma(\mathcal{S}) = \sigma(S) \cup \sigma(\hat{S})$.

For each N = 1,2,... let V_N denote a finite dimensional subspace of V and let P_N be a bounded linear mapping from V onto V_N (for example, the orthogonal projection with respect to either the V or H inner product). Let T_N, Q_N, $\hat{Q}_N \in \mathcal{L}(V_N)$, $B_N \in \mathcal{L}(R^m, V_N)$ and $C_N \in \mathcal{L}(V_N, R^p)$ and set

$$F_N = (R + B^*_N \Pi_N B_N)^{-1} B^*_N \Pi_N T_N$$

and

$$\hat{F} = T_N \hat{\Pi}_N C^*_N (\hat{R} + C_N \hat{\Pi}_N C^*_N)^{-1}$$

where Π_N and $\hat{\Pi}_N$ are the minimal, self-adjoint, nonnegative solutions (once again if they exist) to the finite dimensional operator algebraic Riccati equations

(3.7) $$\Pi_N = T^*_N(\Pi_N - \Pi_N B_N(R + B^*_N \Pi_N B_N)^{-1} B^*_N \Pi_N) T_N + Q_N$$

and

(3.8) $$\hat{\Pi}_N = T_N(\hat{\Pi}_N - \hat{\Pi}_N C^*_N (\hat{R} + C_N \hat{\Pi}_N C^*_N)^{-1} C_N \hat{\Pi}_N) T^*_N + \hat{Q}_N$$

respectively. The approximating optimal compensator is given by $\hat{u}^*_{N,k} = -F_N \hat{w}^*_{N,k}$ $k = 0,1,2,..$ where $\hat{w}^*_N = \{w^*_{N,k}\}^{\infty}_{k=0}$ is determined according to the approximating observer

$$\hat{w}^*_{N,k+1} = T_N \hat{w}^*_{N,k} + B_N \hat{u}^*_{N,k} + \hat{F}_N\{y^*_{N,k} - C_N w^*_{N,k}\}, \quad k = 0,1,2\ldots$$
$$w^*_{N,0} = P_N \hat{w}_0 \,\varepsilon\, V_N .$$

The measurements y^*_N are given by $y^*_{N,k} = Cw_{N,k}$, $k = 0,1,2,\ldots$ where

$$w_{N,k+1} = Tw_{N,k} - B\hat{u}^*_{N,k}, \qquad k = 0,1,2,\ldots$$
$$w_{N,0} = w_0 .$$

The resulting closed-loop system is given by $\mathcal{W}_{N,k} = \mathcal{S}^k_N \mathcal{W}_{N,0}$, $k = 0,1,2,\ldots$ where $\mathcal{W}_{N,k} = (w_{N,k}, \hat{w}^*_{N,k})^T$ and $\mathcal{S}_N \,\varepsilon\, \mathcal{L}(V \times V_N)$ is given by

$$\text{(3.9)} \qquad \mathcal{S}_N = \begin{bmatrix} T & -BF_N \\ \hat{F}_N C & T_N - B_N F_N - \hat{F}_N C \end{bmatrix} .$$

Let $S_N = T_N - B_N F_N$ and $\hat{S}_N = T_N - \hat{F}_N C_N$ and assume that the spaces V_N are V-approximating in the sense that $P_N \to I$ strongly on V as $N \to \infty$. Assume further that $T_N P_N \to T$, $T^*_N P_N \to T^*$, $Q_N P_N \to Q$ and $\hat{Q}_N P_N \to \hat{Q}$ strongly on V and that $B_N \to B$ and $C_N P_N \to C$ in norm as $N \to \infty$. If the pairs (T_N, B_N) and (T^*_N, C^*_N) are uniformly exponentially stabilizable and the pairs (T_N, Q_N) and $(T^*_N, \hat{Q}_N)$ are detectable (see Kwakernaak and Sivan, 1972) then there exist unique, self-adjoint, nonnegative solutions Π_N and $\hat{\Pi}_N$ to the algebraic Riccati equations (3.1) and (3.5). If Π_N and $\hat{\Pi}_N$ are bounded from above uniformly in N, then $\Pi_N P_N$ and $\hat{\Pi}_N P_N$ converge weakly to Π and $\hat{\Pi}$ respectively as $N \to \infty$.

If, in addition, S_N and $\hat{S}_N$ are uniformly exponentially stable, uniformly with respect to N, then $\Pi_N P_N$ and $\hat{\Pi}_N P_N$ converge strongly. Weak convergence of $\Pi_N P_N$ to Π yields strong convergence of $F_N P_N$ to F and $S_N P_N$ to S. If $\Pi_N P_N$ converges strongly then $F_N P_N \to F$ in norm. Weak convergence of $\hat{\Pi}_N P_N$ to $\hat{\Pi}$ yields weak convergence of $\hat{F}_N$ to $\hat{F}$ and $\hat{S}_N P_N$ to

$\hat{S}$. When $\hat{\Pi}_N P_N \to \hat{\Pi}$ strongly, then $\hat{F}_N \to \hat{F}$ in norm and $\hat{S}_N P_N \to \hat{S}$ strongly in V as $N \to \infty$. Finally, if we let $\mathcal{P}_N$ denote the mapping of $V \times V$ onto $V \times V_N$ given by $\mathcal{P}_N(w_1, w_2) = (w_1, P_N w_2)$ then $\Pi_N P_N \to \Pi$ weakly or strongly is sufficient to conclude that $\mathcal{S}_N \mathcal{P}_N \to \mathcal{S}$ weakly or strongly depending only upon whether $\hat{\Pi}_N P_N \to \hat{\Pi}$ weakly or strongly as $N \to \infty$. Under appropriate additional hypotheses on the spectral properties of the open-loop system, the nature of the approximation spaces V_N and the mappings P_N, it is possible to obtain a result regarding the norm convergence of $\mathcal{S}_N \mathcal{P}_N$ to $\mathcal{S}$. Norm convergence of the closed-loop state transition operators is sufficient to conclude that uniform exponential stability of $\mathcal{S}$ implies uniform exponential stability of $\mathcal{S}_N$ for all N sufficiently large (see Gibson and Rosen 1986).

In practice, the finite dimensional approximating subspaces V_N are often constructed using any one of a number of common finite element bases, e.g. polynomial and hermite spline functions, mode shapes, orthogonal polynomials, etc. For some of the discrete-time boundary control systems of particular interest to us here, the approximations to T and B, T_N and B_N, are typically obtained by approximating the continuous time semigroup, $\{\mathcal{T}(t) : t \geq 0\}$, by a semigroup of bounded linear operators on V_N, $\{\mathcal{T}_N(t): t \geq 0\}$. (In actual fact it is the infinitesimal generator $\mathcal{A}$ of the semigroup $\{\mathcal{T}(t): t \geq 0\}$ which is approximated by a bounded linear operator $\mathcal{A}_N$ on V_N with $\{\mathcal{T}_N(t): t \geq 0\}$ then being defined by $\mathcal{T}_N(t) = \exp(\mathcal{A}_N t), t \geq 0)$. With $T_N = \mathcal{T}_N(\tau)$ and $B_N = (I - \mathcal{T}_N(\tau))P_N \Gamma^+ + \int_0^\tau \mathcal{T}_N(s) P_N \Delta \Gamma^+ ds$, the necessary conditions for convergence are argued using the well known Trotter-Kato semigroup approximation result (see (Kato, 1966)). The approximations to Q, $\hat{Q}$ and C, Q_N, $\hat{Q}_N$ and C_N respectively, are typically taken to be $Q_N = P_N Q$, $\hat{Q}_N = P_N \hat{Q}$ and

$C_N = CP_N$.

Let $\{\varphi_j^N\}_{j=1}^{n_N}$ denote a basis for V_N and set $\Phi^N = (\varphi_1^N, \varphi_2^N, \ldots, \varphi_{n_N}^N)^T \varepsilon \overset{n_N}{\underset{j=1}{\times}} V_N$.

Adopting the convention that [L] denotes the matrix representation with respect to the basis $\{\varphi_j^N\}_{j=1}^{n_N}$ for a linear operator L with domain and/or range in V_N, we find that $[F_N] = (R + [B_N]^T \Theta^N [B_N])^{-1} [B_N]^T \Theta^N [T_N]$ and $[\hat{F}_N] = [T_N] \hat{\Theta}^N [C_N]^T (\hat{R} + [C_N] \hat{\Theta}^N [C_N]^T)^{-1}$ where Θ^N and $\hat{\Theta}^N$ are the unique, symmetric, nonnegative solutions to the $n_N \times n_N$ matrix algebraic Riccati equations

(3.10) $$\Theta^N = [T_N]^T(\Theta^N - \Theta^N[B_N](R + [B_N]^T\Theta^N[B_N])^{-1}[B_N]^T\Theta^N)[T_N] + M^N[Q_N]$$

and

(3.11) $$\hat{\Theta}_N = [T_N](\hat{\Theta}^N - \hat{\Theta}^N[C_N]^T(\hat{R} + [C_N]\Theta^N[C_N]^T)^{-1}[C_N]\Theta^N)[T_N]^T + [\hat{Q}_N](M^N)^{-1} .$$

The matrix M^N is the $n_N \times n_N$ Gramian matrix $<\Phi^N, (\Phi^N)^T>_V$.

If we write $\hat{w}^*_{N,k} = (\Phi^N)^T \hat{W}^*_{N,k}$ with $\hat{W}^*_{N,k} \in R^{n_N}$, we obtain $\hat{u}^*_{N,k} = -[F_N]\, W_{N,k}$, $k = 0,1,2,...$ with

$$\hat{W}^*_{N,k+1} = [T_N]\,\hat{W}^*_{N,k} + [B_N]\,\hat{u}^*_{N,k} + [\hat{F}_N]\{\hat{y}^*_{N,k} - [C_N]\,\hat{W}^*_{N,k}\},\ k = 0,1,2,...$$

$$\hat{W}^*_{N,0} = (M^N)^{-1} < \Phi^N, \hat{w}_0>_V .$$

The approximating optimal functional feedback control gains, $f^N = (f^N_1, f^N_2,..., f^N_m)^T \in \times_{j=1}^{m} V_N$ are given by $f^N = [F_N](M^N)^{-1}\Phi^N$ and the approximating optimal functional observer gains $\hat{f}^N = (\hat{f}^N_1, \hat{f}^N_2,..., \hat{f}^N_p)^T \in \times_{j=1}^{p} V_N$ by $\hat{f} = [\hat{F}_N]^T\Phi^N$. If $\Pi_N P_N \to \Pi$ weakly (strongly) then $f^N_i \to f_i$, $i = 1,2,..., m$ weakly (strongly) in V. If $\hat{\Pi}_N P_N \to \hat{\Pi}$ weakly (strongly) then $\hat{f}^N_i \to \hat{f}_i$, $i = 1,2,..., p$ weakly (strongly) in V. If the injection $V \subset H$ is compact, then in either case we have $f^N_i \to f_i$, $i = 1,2,..., m$ and $\hat{f}^N_i \to \hat{f}_i$, $i = 1,2,...,p$ strongly in H.

4. Examples and Numerical Result

We consider the one dimensional heat equation

$$\text{(4.1)} \qquad \frac{\partial w}{\partial t}(t,x) = a^2 \frac{\partial^2 w}{\partial x^2}(t,x), \quad 0 < x < 1, t > 0,$$

where $a \varepsilon R^1$, with a homogeneous Dirichlet boundary condition at $x = 0$,

$$\text{(4.2)} \qquad w(t,0) = 0, \quad t > 0,$$

and either Neumann,

$$\text{(4.3)} \qquad \frac{\partial w}{\partial x}(t,1) = v(t), t > 0,$$

or Dirichlet,

$$\text{(4.4)} \qquad w(t, 1) = v(t), \quad t > 0,$$

boundary control or input at $x = 1$ where $v \varepsilon L_2(0, \infty)$. For output we take a temperature measurement

$$\text{(4.5)} \qquad y(t) = w(t, \zeta), \qquad t \geq 0,$$

at some fixed point $\zeta \varepsilon (0, 1)$. Initial conditions for these systems are assumed to be of the form

$$\text{(4.6)} \qquad w(0, x) = w_0(x), \qquad 0 \leq x \leq 1$$

where $w_0 \varepsilon L_2(0, 1)$.

Although the two control systems above appear to be similar, they are, in fact, quite different and must be treated separately. We begin with the more straight forward of the two - the case of

Neumann boundary control. Let $H = L_2(0, 1)$, $V = H^1_L(0, 1) = \{\varphi \in H^1(0, 1) : \varphi(0) = 0\}$ and $W = H^2(0, 1) \cap H^1_L(0, 1)$. With H endowed with the usual L_2 inner product, V with the inner product $<\varphi, \psi>_V = \int_0^1 D\varphi\, D\psi$ and W with the inner product $<\varphi, \psi>_W = \int_0^1 D^2\varphi\, D^2\psi$ we have the continuous and dense embeddings $W \subset V \subset H \subset V' \subset W'$. Define $\Delta \in \mathcal{L}(W, H)$, $\Gamma \in \mathcal{L}(W, R^1)$ by $\Delta\varphi = a^2 D^2\varphi$, $\Gamma\varphi = D\varphi(1)$ and $C\varphi = \varphi(\zeta)$ respectively. With these definitions it is immediately clear that the boundary control system (4.1) - (4.3), (4.5), (4.6) is of the general form (2.1) - (2.4). The operator $\mathcal{Q}$: $\mathrm{Dom}(\mathcal{Q}) \subset H \rightarrow H$ is given by $\mathcal{Q}\varphi = a^2 D^2\varphi$ for $\varphi \in \{\varphi \in H^2(0, 1): \varphi(0) = D\varphi(1)) = 0\}$. It is densely defined, negative definite, self-adjoint and it is the infintesimal generator of a uniformly exponentially stable analytic semigroup $\{\mathcal{T}(t): t \geq 0\}$ of bounded, self-adjoint linear operators on H. In addition, $\{\mathcal{T}(t): t \geq 0\}$ is also a uniformly exponentially stable, analytic semigroup of bounded, self-adjoint operators on V with generator $\tilde{\mathcal{Q}}$ given by $\tilde{\mathcal{Q}}\varphi = \mathcal{Q}\varphi$ for $\varphi \in \{\varphi \in H^3(0, 1): \varphi(0) = D\varphi(1) = D^2\varphi(0) = 0\}$. Choosing $\Gamma^+ \in \mathcal{L}(R^1, W)$ as $(\Gamma^+ u)(x) = xu$ for $x \in [0, 1]$, we have $\mathcal{R}(\Gamma^+) \subset V$, $\mathcal{R}(\Gamma^+) \subset \mathcal{N}(\Delta)$ and that conditions 1) -5) given in Section 2 are satisfied. For the optimal control and estimator problems, we take $Q = qI$, $\hat{Q} = \hat{q}\, I$, $R = r$ and $\hat{R} = \hat{r}$ where I is the identity operator on V, q, $\hat{q} \geq 0$ and r, $\hat{r} > 0$. The uniform exponential stability of the semigroup $\{\mathcal{T}(t): t \geq 0\}$ on V implies that the algebraic Riccati equations (3.1) and (3.5) admit unique, nonnegative self-adjoint solutions Π and $\hat{\Pi}$ respectively. The optimal control (3.2) takes the form

$$(4.7) \qquad u_k^* = - \int_0^1 Df\, Dw_k^*, \qquad k = 0,1,2,...$$

where the optimal functional feedback control gain f along with the optimal functional observer gain $\hat{f}$ are elements in $H^1_L(0,1)$.

We construct an approximation scheme using a linear spline based Ritz-Galerkin approach. For each N = 1,2,... let $\{\varphi_j^N\}_{j=0}^N$ denote the usual linear spline or "hat" functions defined on the interval [0,1] with respect to the uniform mesh {0, 1/N, 2/N,...,1}. We discard the element centered at x = 0, φ_0^N, set $V_N = \{\varphi_j^N\}_{j=1}^N$ and choose P_N to be the orthogonal projection of V onto V_N with respect to the V inner product. It is clear that V_N is an N dimensional subspace of V.

For $\varphi \in \mathrm{Dom}(\mathcal{A})$, $|\mathcal{A}\varphi|_H \geq a^2|\varphi|_V \geq a^2|\varphi|_H$ and therefore $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}: H \to \mathrm{Dom}(\mathcal{A})$ satisfies $|\mathcal{A}^{-1}\varphi|_V \leq a^{-2}|\varphi|_H$ for $\varphi \in H$. We define $\mathcal{A}_N : V_N \to V_N$ as the inverse of the operator $\mathcal{A}_N^{-1} = P_N\mathcal{A}^{-1}$ restricted to V_N. That the operator $\mathcal{A}_N$ is well defined follows from the fact that $< \mathcal{A}_N^{-1}\varphi_N, \varphi_N>_V = -a^{-2}|\varphi_N|_H^2$ for $\varphi_N \in V_N$ and that it is self-adjoint from $<\mathcal{A}_N\varphi_N, \psi_N>_V = -a^{-2}<\mathcal{A}_N\varphi_N, \mathcal{A}_N\psi_N>_H$. For $\varphi_N \in V_N$ and $\psi_N = \mathcal{A}_N\varphi_N$, the estimate

$$
\begin{aligned}
< \mathcal{A}_N\varphi_N, \varphi_N>_V &= < \psi_N, \mathcal{A}_N^{-1}\psi_N>_V = -a^{-2}| \psi_N|_H^2 \\
&\leq -a^2|\mathcal{A}^{-1}\psi_N|_V^2 \leq -a^2 |P_N\mathcal{A}^{-1} \psi_N|_V^2 = -a^2| \mathcal{A}_N^{-1} \psi_N|_V^2 \\
&= -a^2 | \varphi_N|_V^2
\end{aligned}
$$

implies that $\mathcal{A}_N$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $\{\mathcal{T}_N(t) : t \geq 0\}$ of bounded, self-adjoint linear operators on V_N satisfying $|\mathcal{T}_N(t)| \leq e^{-a^2 t}$, $t \geq 0$.

Let $\mathcal{I}_N$ denote the interpolation operator from V onto V_N. Then for $\varphi \in W$, elementary approximation properties of linear interpolatory spline functions (see (Schultz, 1971)) imply

$$
|(P_N - I)\varphi|_V \leq |(\mathcal{I}_N - I)\varphi|_V \leq \frac{1}{N\pi} |D^2\varphi|_H
$$

and therefore, since W is dense in V, that $P_N \to I$ strongly on V as $N \to \infty$. Also, it follows that $\mathcal{A}_N^{-1} = P_N\mathcal{A}^{-1} \to \mathcal{A}^{-1}$ strongly on V as $N \to \infty$. If we define $T_N = \mathcal{T}_N(\tau)$, then the Trotter-Kato approximation theorem (see (Pazy, 1983)) yields $T_NP_N \to T$ (and therefore of course that $T_N^*P_N \to T^*$) strongly on V as $N \to \infty$ where $T = T^* = \mathcal{T}(\tau)$.

Since $\mathcal{R}(\Gamma^+) \subset V_N$, we define the approximating input operators B_N by $B_N = (I - \mathcal{T}_N(\tau))\Gamma^+$ and set $Q_N = qI$, $\hat{Q}_N = \hat{q}\, I$ and $C_N = C$. The strong convergence of P_N to the identity and T_NP_N to T together with the finite dimensionality of the domain of B and the range of C are sufficient to conclude that $Q_NP_N \to Q$, $\hat{Q}_NP_N \to \hat{Q}$ strongly on V and that $B_N \to B$ and $C_NP_N \to C$ in norm as $N \to \infty$.

The uniform exponential stability of the semigroups $\{ \mathcal{T}_N(t) : t \geq 0\}$ implies

(4.8) $\qquad |T^k_N|_V = |(T^*_N)^k|_V \le r^k, \quad k = 0,1,2,...$

with $r = e^{-a^2\tau} < 1$. Consequently the pairs (T_N, B_N) and (T_N, C^*_N) are uniformly exponentially stabilizable and the pairs (T_N, Q_N) and $(T_N, \hat{Q}_N)$ are detectable. it follows therefore that there exist unique self-adjoint, nonnegative solutions Π_N and $\hat{\Pi}_N$ to the finite dimensional algebraic Riccati equations (3.7) and (3.8) respectively. The uniform exponential bound (4.8) with $r < 1$ on the approximating open-loop state transition operators T_N imples that the zero control yields a uniform upper bound for Π_N and $\hat{\Pi}_N$ and the uniform exponential stability of $S_N = T_N - B_N F_N$ and $\hat{S}_N = T_N - \hat{F}_N C_N$. We can conclude therefore that $\Pi_N P_N$ and $\hat{\Pi}_N P_N$ converge strongly in V to Π_N and $\hat{\Pi}_N$ respectively and that $F_N P_N$ and $\hat{F}_N$ converge to F and $\hat{F}$ in norm as $N \to \infty$. The approximating optimal functional feedback control and observer gains, f_N and $\hat{f}_N$, converge respectively to f and $\hat{f}$ in the H^1 norm as $N \to \infty$.

In implementing the scheme outlined above, eigenvector decomposition of the associated Hamiltonian matrix was used to solve the matrix algebraic Riccati equations (3.10) and (3.11) (see Pappas, et. al., 1980). Matrix exponentials, where required, were also computed using eigenvalue/eigenvector decomposition. All calculations were carried out via codes written in Fortran and run on an IBM PC AT. We set $a^2 = .1$, $q = \hat{q} = r = \hat{r} = 1.0$, $\xi = \sqrt{2}/2$ and $\tau = .01$ and obtained the functional gains plotted in Figs. 4.1 and 4.2 below. We plot f_N and $\hat{f}_N$ as well as Df_N and $D\hat{f}_N$ so as to exhibit the H^1 convergence and since it is Df (or Df_N) which actually appears as the feedback kernel in the optimal control law (4.7).

We also simulated the operation of the closed-loop system with an approximating compensator. Using a 20 mode model for the infinite dimensional system and N = 12, we computed the closed-loop spectrum of the approximating compensator (i.e. the eigenvalues of the operator $\mathcal{S}_N$ given by (3.9) with N = 12). These eigenvalues along with the first 20 open-loop eigenvalues (i.e. the first 20 eigenvalues of the operator $T = \mathcal{T}(\tau)$) and the approximating closed-loop control and observer eigenvalues are tabulated in Table 4.1 below. Table 4.1 reveals that the last seven open-loop eigenvalues remain essentially unchanged in the closed-loop system-i.e. these modes are neither controlled nor observed by the finite dimensional compensator. Also, as one would expect, $\sigma(\mathcal{S}_N)$

consists essentially of the union of $\sigma(S_N)$, $\sigma(\hat{S}_N)$ and the eigenvalues corresponding to the uncontrolled/unobserved modes of the open-loop system.

It is worth noting that the scheme we have outlined above for the Neumann boundary control problem is the same scheme that one would ordinarily use if the problem were formulated in the space H - i.e. if the output operator C was bounded on $L_2(0,1)$ (see Gibson and Rosen, 1986). This is possible primarily because the space $V = H^1_L(0,1)$ is the natural energy space for the underlying homogeneous or open-loop system. Consequently, the inherent self-adjointness and coercivity in the problem is preserved when it is formulated in the stronger space. In the case of Dirichlet boundary control, the situation is quite different.

For the Dirichlet boundary control system (4.1), (4.2), (4.4) - (4.6), we choose the spaces H, V and W and their corresponding inner products to be the same as they were in the Neumann case. The operators $\Delta \varepsilon \mathcal{L}(W,H)$ and $C \varepsilon \mathcal{L}(V,R^1)$ also remain unchanged, however now we have $\Gamma \varepsilon \mathcal{L}(W,R^1)$ given by $\Gamma\varphi = \varphi(1)$. It then follows that the operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset H \to H$ is given by $\mathcal{A}\varphi = a^2D^2\varphi$ for $\varphi \varepsilon H^2(0,1) \cap H^1_0(0,1)$. It is well known that $\mathcal{A}$ is densely defined, negative definite and self-adjoint and that it is the infinitesimal generator of the uniformly exponentially stable analytic semigroup $\{\mathcal{T}(t) : t \geq 0\}$ of bounded, self-adjoint linear operators on H. However this time the operators $\mathcal{T}(t)$ for $t > 0$ are neither self-adjoint nor a semigroup on V. Indeed, since $\mathcal{R}(\mathcal{T}(t)) \subset H^1_0(0,1)$ for all $t > 0$ and since $H^1_0(0,1)$ is a closed proper subspace of $H^1_L(0,1)$, $\mathcal{T}(t)$ is not strongly continuous in the V-norm at $t = 0$. (The fact that our general framework requires $\Gamma\Gamma^+ = 1$ and $\mathcal{R}(\Gamma^+) \subset V$ precludes our choosing V to be $H^1_0(0,1)$.) On the other hand, $\{\mathcal{T}(t) : t \geq 0\}$ an analytic semigroup implies (see Pazy, 1983) that there exists a constant $\mu > 0$ for which $|\mathcal{A}\mathcal{T}(t)|_H \leq \mu t^{-1}$ for $t > 0$. Consequently, if we define $T = \mathcal{T}(\tau)$, then it follows that $T \varepsilon \mathcal{L}(V)$ and moreover, that

$$|T^k\varphi|^2_V = - <\mathcal{A}\mathcal{T}(k\tau)\varphi, \mathcal{T}(k\tau)\varphi> \leq |\mathcal{A}\mathcal{T}(k\tau)\varphi|_H \, |\mathcal{T}(k\tau)\varphi|_H$$

$$\leq \frac{\mu e^{-a^2 k\tau}}{k\tau} |\varphi|^2_H \leq \frac{\mu e^{-a^2 k\tau}}{k\tau} |\varphi|^2_V$$

for $k = 1,2,...$ and $\varphi \varepsilon V$. We have therefore

(4.9) $|T^k|_V = |(T^*)^k|_V \le Mr^k,\ k = 0,1,2,\ldots$

where $M > 0$ and $r < 1$.

We again choose $\Gamma^+ \varepsilon\ \mathcal{L}(R^1,W)$ as $(\Gamma^+u)(x) = xu$ for $x\ \varepsilon\ [0,1]$. Then $\mathcal{R}(\Gamma^+) \subset \eta(\Delta)$ and we have reformulated the boundary control system (4.1), (4.2), (4.4) - (4.6) in the general form of (2.1) - (2.4) and conditions 1) - 5) are satisfied.

We formulate the optimal control problem with the performance index

$$J(u) = \sum_{k=0}^{\infty} q < w_k , w_k >_H + r u_k^2$$

where $q \ge 0$ and $r > 0$. That is, we take Q to be the bounded, self-adjoint nonnegative operator on $H^1_L(0,1)$ given by $(Q\varphi)(x) = q \int_0^x \int_y^1 \varphi(z)dzdy$ and R to be r. For the estimator problem we set $\hat{Q} = \hat{q}\, I$ and $\hat{R} = \hat{r}$ with $\hat{q} \ge 0$ and $\hat{r} > 0$.

The uniform exponential bound (4.9) implies the existence of unique, nonnegative, self-adjoint solutions Π and $\hat{\Pi}$ to the algebraic Riccati equations (3.1) and (3.5). The optimal control is again of the form (4.7) with the optimal functional gains f and $\hat{f}$ in H^1_L

The fact that $\{\mathcal{T}(t) : t \ge 0\}$ is not a semigroup on V precludes the use of a semigroup - theoretic approach to approximation. We therefore employ modal subspaces and approximate the open-loop state transition operator T directly as a bounded linear operator on V.

For each $N = 1,2,\ldots$ let $V_N = \text{span}\ \{\varphi_j\}_{j=0}^N$ where for $x\ \varepsilon\ [0,1]$, $\varphi_0(x) = x$ and $\varphi_j(x) = \sin j\pi x$, $j = 1,2,\ldots,N$. Let p_N denote the orthogonal projection of $H = L_2(0,1)$ onto span $\{\varphi_j\}_{j=1}^N$ and let P_N denote the orthogonal projection of V onto V_N. Using the fact that $V = H^1_0(0,1) \oplus \varphi_0$, it is not difficult to see that $P_N\varphi = \varphi(1)\varphi_0 + p_N(\varphi - \varphi(1)\varphi_0)$ for $\varphi\ \varepsilon$ V and hence, via elementary properties of Fourier series (see Tolstov, 1962), that $|(P_N - I)\varphi|_V = |(p_N - I)(\varphi - \varphi(1)\varphi_0)|_V \to 0$ as $N \to \infty$ for each $\varphi\ \varepsilon$ V.

If for $\psi_N = \sum_{j=0}^{N} \psi_N^j \varphi_j\ \varepsilon\ V_N$ we define $T_N\ \varepsilon\ \mathcal{L}(V_N)$ by

$$T_N \psi_N = P_N T \psi_N = P_N \mathcal{T}(\tau)\psi_N = \sum_{j=1}^{N} \{ \frac{2(-1)^j}{j\pi} \psi_N^0 + \psi_N^j \} e^{-a^2 j^2 \pi^2 \tau} \varphi_j ,$$

then $T^*_N = P_N T^*$, $|T^k_N| = |(T^*_N)^k|_V \le Mr^k$, $k = 0,1,2,\ldots$ with $M > 0$ and $r < 1$ independent of N,

$$|(T_N P_N - T)\varphi|_V \le |(P_N T P_N - P_N T)\varphi|_V + |(P_N - I)T\varphi|_V$$
$$\le Mr|(P_N - I)\varphi|_V + |(P_N - I)T\varphi|_V \to 0$$

as $N \to \infty$ for $\varphi \in V$. Similarly, $T^*_N P_N \to T^*$ strongly on V as $N \to \infty$.

The approximating input, output and state penalization operators B_N, C_N, Q_N and $\hat{Q}_N$ take the form

$$B_N u = (I - T_N)\Gamma^+ u = \varphi_0 u + \sum_{j=1}^{N} \frac{2(-1)^j}{j\pi} e^{-a^2 j^2 \pi^2 \tau} \varphi_j u \ ,$$

$C_N = C, Q_N = qP_N Q$ and $\hat{Q}_N = \hat{q} I$. Reasoning as we did in the Neumann case, the approximating algebraic Riccati equations (3.7) and (3.8) admit unique, nonnegative, self-adjoint solutions Π_N and $\hat{\Pi}_N$ respectively, $\Pi_N P_N \to \Pi$ and $\hat{\Pi}_N P_N \to \hat{\Pi}$ strongly on V and $F_N P_N \to F$ and $\hat{F}_N \to \hat{F}$ in norm as $N \to \infty$. The approximating functional feedback control and observer gains f_N and $\hat{f}_N$ converge to f and $\hat{f}$ respectively, strongly in H^1 as $N \to \infty$.

With $a^2 = 1.0$, $q = \hat{q} = r = 1.0$, $\hat{r} = 5.0$, $\xi = \sqrt{2}/2$ and $\tau = .01$ and the scheme outlined above we obtained the approximating optimal functional feedback control and observer gains plotted in Figs. 4.3 and 4.4 below. The first 12 open-loop and the approximating closed-loop control and observer eigenvalues for N = 12 are tabulated in Table 4.2.

Table 4.2 reveals an interlacing of the closed-loop control and open-loop eigenvalues. That is, the closed-loop control eigenvalues (i.e. the elements in the spectrum of S) are alternately more and less stable than the corresponding open-loop eigenvalues. We also have observed this phenomenon in other numerical studies we are carrying out involving LQG boundary control for flexible structures. In additon, in the Dirichlet boundary control system discussed above, if Q is chosen as the identity operator on $V = H^1_L(0,1)$, virtually all of the closed-loop control eigenvalues are less stable than the corresponding open-loop eigenvalues. It is clear that this non-standard behavior results from the presence of the one dimensional subspace represented by $\mathcal{R}(\Gamma^+)$. Indeed, the behavior of the closed-loop spectrum in the case of Neumann boundary control is as would be expected. We feel that what we are seeing can most likely be explained via infinite dimensional analogs of existing results relating the asymptotic properties of the closed-loop spectrum of a linear regulator and the zeros of the corresponding open-loop transfer function (see Kwakernaak and Sivan, 1972 and Harvey and Stein, 1978). However, as of yet, we have been unable to establish this conjecture satisfactorily and we consider it to be beyond the scope of this paper, which is primarily concerned with approximation. We leave it as an interesting open question worthy of further investigation.

5. Concluding Remarks

We have developed a framework for the finite dimensional approximation of optimal discrete-time LQG compensators for distributed parameter systems with boundary input and unbounded measurement. Our theory applies to the class of boundary control problems which can be formulated in a state space in which both the discrete-time input and output operators are continuous. We have used a functional analytic treatment to develop a convergence theory and have demonstrated the feasibility of our approach via examples involving either the Neumann or Dirichlet boundary control of a one dimensional heat equation with point measurement of temperature. We have shown that while both problems outwardly appear to be quite similar, they in fact require very different approaches to approximation. Also in the Dirichlet case the observed behavior of the resulting closed-loop spectrum is, in some ways unexpected and its explanation remains open.

Finally, we have been looking at the application of our schemes to LQG problems for flexible structures with boundary inputs and unbounded measurement and systems with control and/or observations delays. We have been considering vibration suppression for cantilevered beams via shear or moment inputs at the free end and pointwise observation of strain or acceleration. These studies are currently underway with the results to be reported elsewhere.

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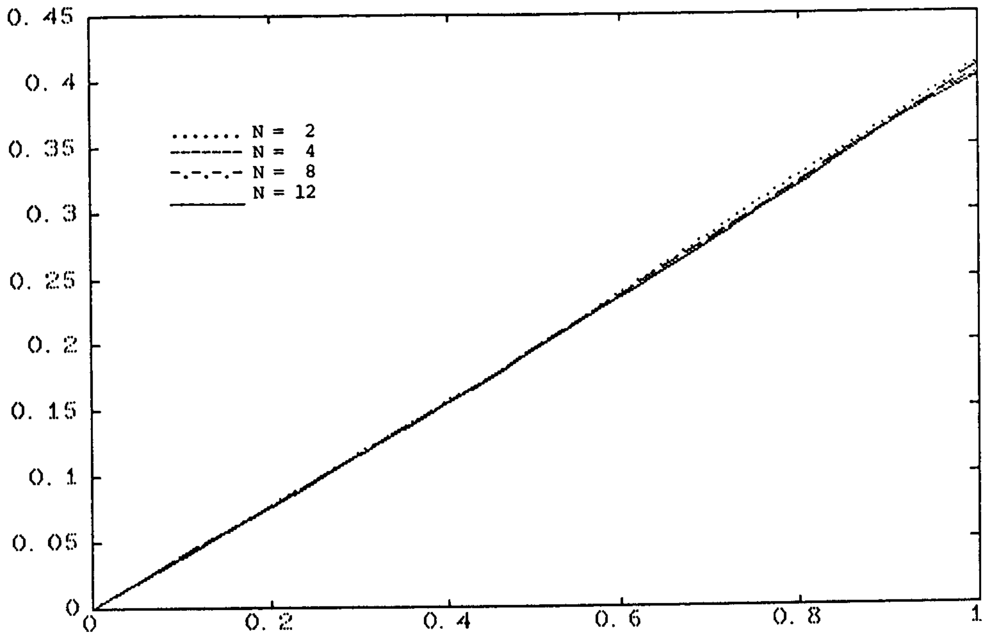
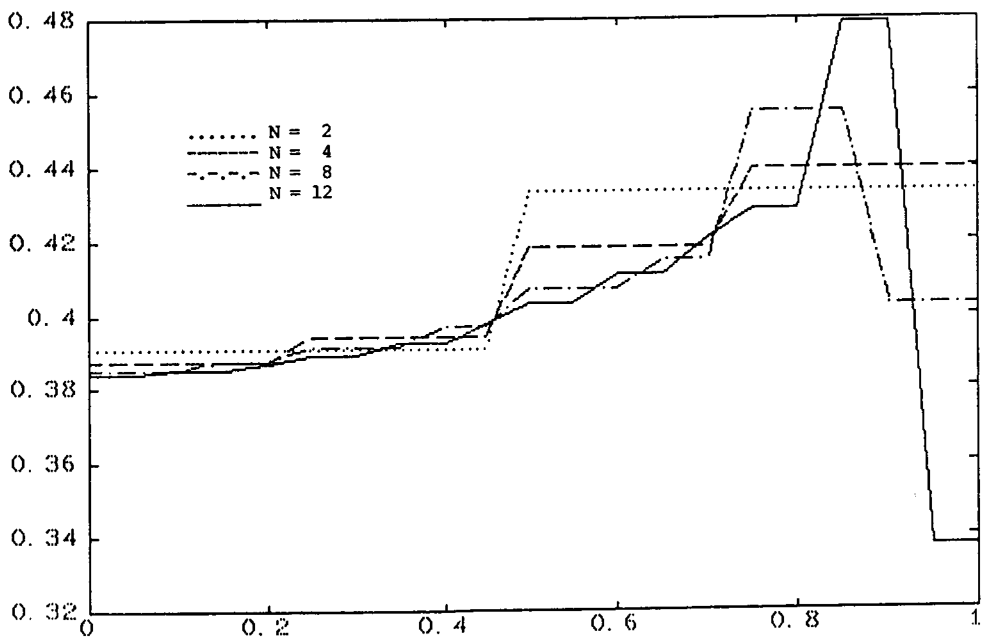
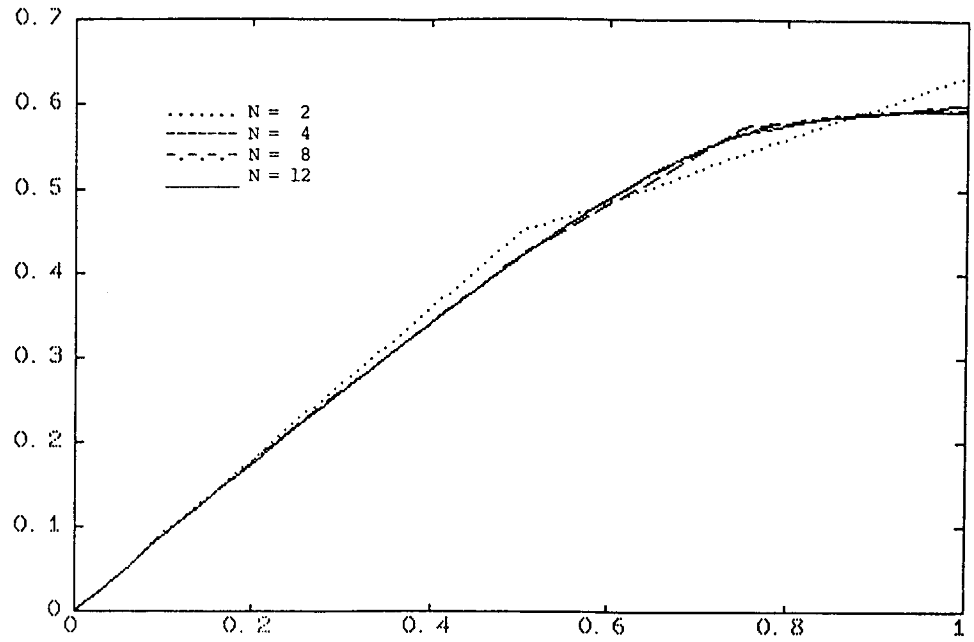


Neumann boundary control; approximating optimal fuctional feedback control gains, f_N.

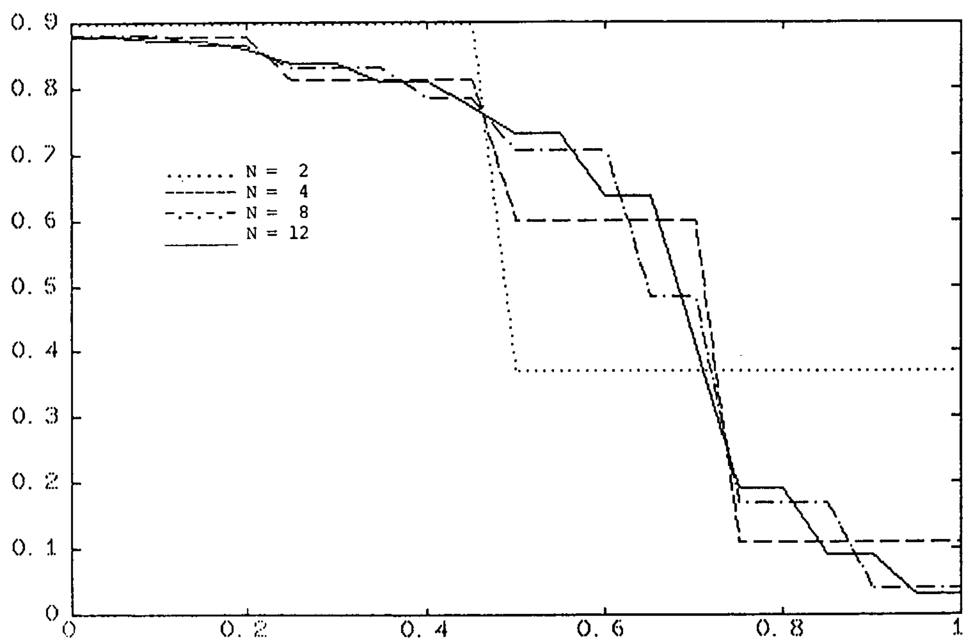


Neumann boundary control; first derivative of approximating optimal functional feedback control gains, Df_N.

Figure 4.1



Neumann boundary control; approximating optimal functional observer gains, $\hat{f}_N$.



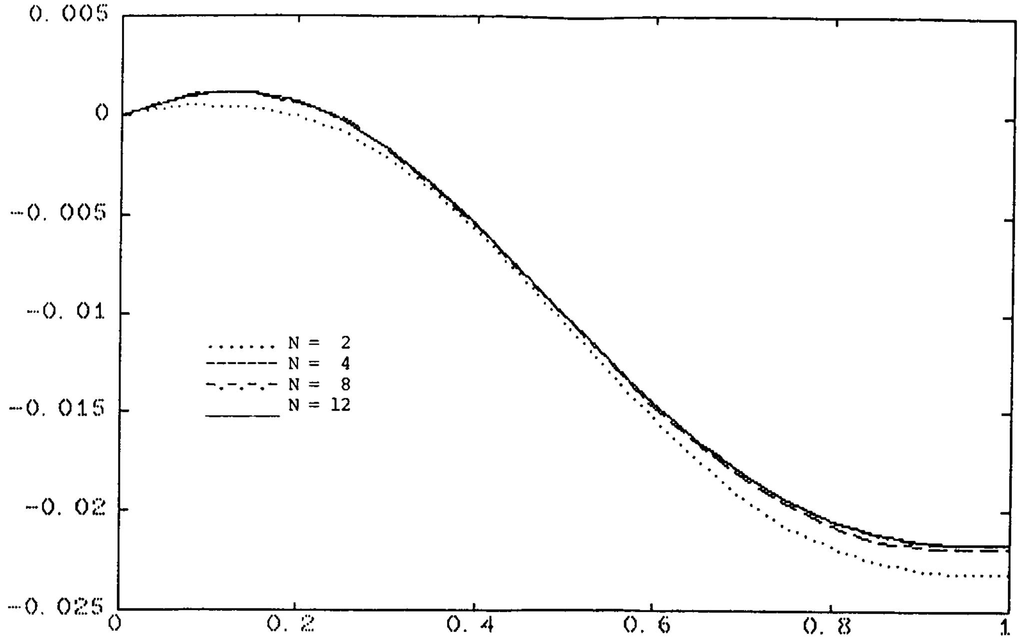
Neumann boundary control; first derivative of approximating optimal functional observer gains, $D\hat{f}_N$.

Figure 4.2

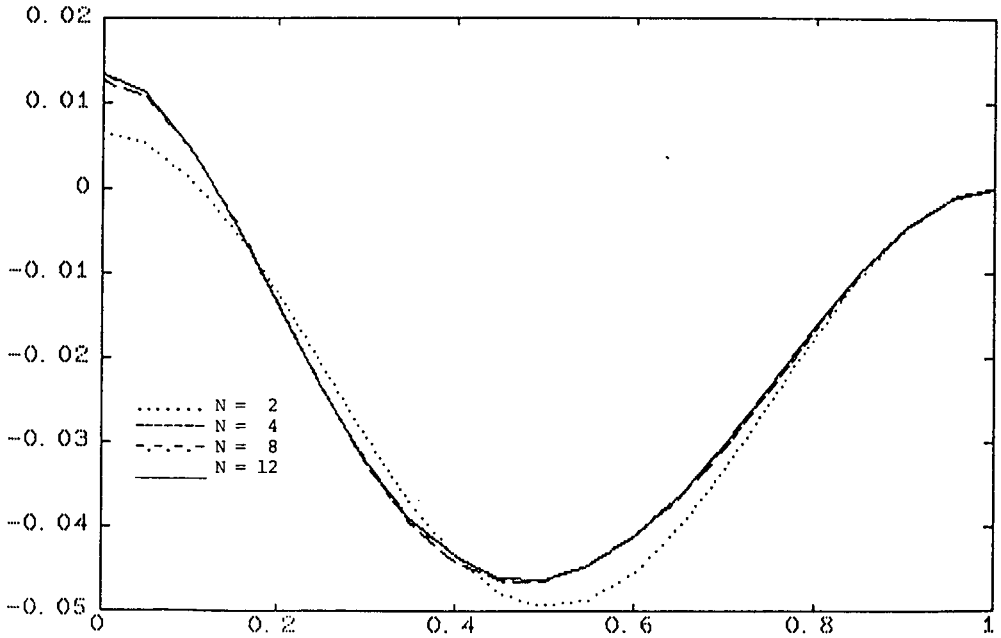
	Open-Loop	$\sigma(\mathcal{S}_{12})$	$\sigma(S_{12})$	$\sigma(\hat{S}_{12})$
1	.9975	.9968	.9968	
2	.9780	.9780		.9778
		.9769	.9768	
3	.9402	.9408		.9387
		.9371	.9371	
4	.8861	.8872		
		.8778	.8775	.8798
5	.8188	.8194		
		.7982	.7985	.7998
6	.7419	.7414		
		.7026	.7019	.7030
7	.6590	.6573		
		.5960	.5921	.5946
8	.5740	.5718		
		.4891	.4769	.4804
9	.4901	.4875		
		.4433		.4412
10	.4104	.4041		
		.3675	.3675	.3682
11	.3368	.3341		
		.2772	.2763	.2768
12	.2711	.2705		
		.2145	.2129	.2133
13	.2139	.2134		
		.1811	.1811	.1816
14	.1655	.1663		
15	.1255	.1260		
16	.0934	.0933		
17	.0681	.0677		
18	.0482	.0483		
19	.0341	.0340		
20	.0235	.0236		

Neumann boundary control; simulation results

Table 4.1

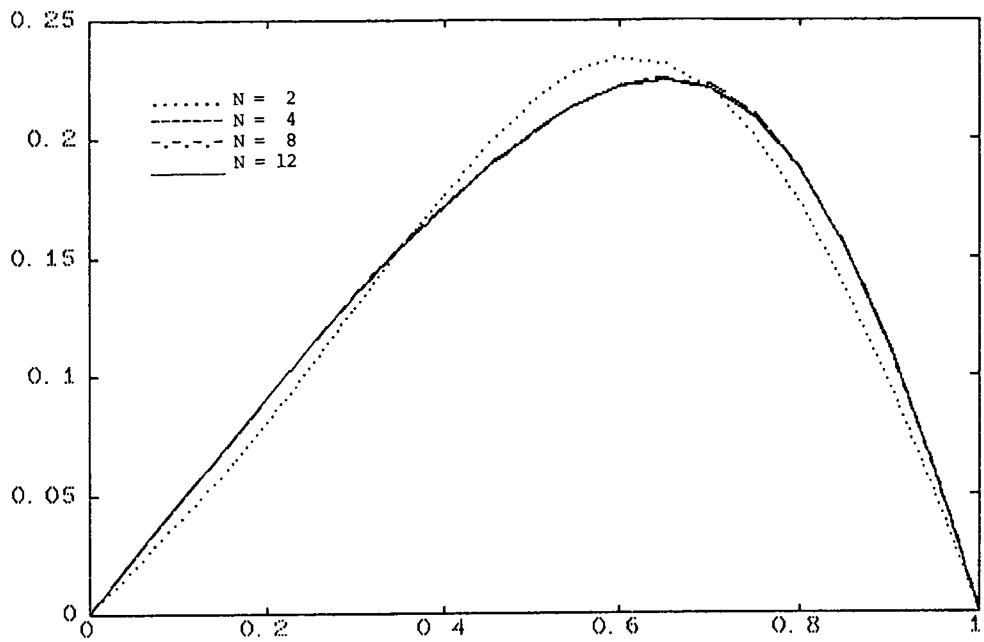


Dirichlet boundary control; approximating optimal functional feedback control gains, f_N.

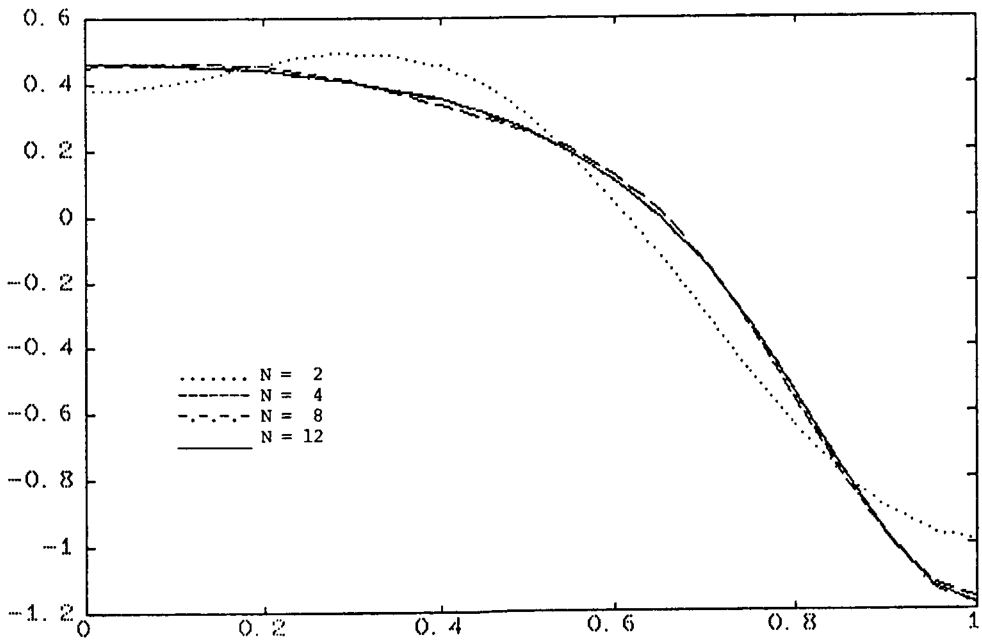


Dirichlet boundary control; first derivative of approximating optimal functional feedback control gains, Df_N.

Figure 4.3



Dirichlet boundary control; approximating optimal functional observer gains, $\hat{f}_N$.



Dirichlet boundary control; first derivative of approximating optimal functional observer gains, Df_N.

Figure 4.4

	Open-Loop	$\sigma(S_{12})$	$\sigma(\hat{S}_{12})$
1	.90601806	.90569591	.78573771
2	.67382545	.68243047	.57981918
3	.41136911	.40961171	.40082268
4	.20615299	.20758391	.20936323
5	.08480497	.08447005	.08636884
6	.02863695	.02873534	.02892353
7	.00793790	.00791793	.00792193
8	.00180617	.00180978	.00178763
9	.00033753	.00033682	.00033414
10	.00005172	.00005179	.00005162
11	.00000651	.00000650	.00000654
12	.00000067	.00000067	.00000068
13	——	.00000000	.00000000

Dirichlet boundary control; open and closed-loop spectrum

Table 4.2

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16. Abstract

We consider the approximation of optimal discrete-time linear quadratic Gaussian (LQG) compensators for distributed parameter control systems with boundary input and unbounded measurement. Our approach applies to a wide range of problems that can be formulated in a state space on which both the discrete-time input and output operators are continuous. Approximating compensators are obtained via application of the LQG theory and associated approximation results for infinite dimensional discrete-time control systems with bounded input and output. Numerical results for spline and modal based approximation schemes used to compute optimal compensators for a one dimensional heat equation with either Neumann or Dirichlet boundary control and pointwise measurement of temperature are presented and discussed.

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