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SPLITTING METHODS FOR LOW MACH NUMBER
EULER AND NAVIER-STOKES EQUATIONS

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ABSTRACT

In this paper, we examine some splitting techniques for low Mach number Euler flows. We point out shortcomings of some of the proposed methods and suggest an explanation for their inadequacy. We then present a symmetric splitting for both the Euler and Navier-Stokes equations which removes the stiffness of these equations when the Mach number is small. The splitting is shown to be stable.

INTRODUCTION

For many computational problems in low speed fluid-dynamics, it has been customary to use the incompressible Euler or Navier-Stokes equations. There are essentially two reasons for doing this: there is one less variable, since the density remains constant, and the stability limit is independent of the sound speed. Recently, however, there has been increased interest in studying compressibility effects even for low Mach number fluid flows. The compressible equations, unfortunately, have stiff coefficients due to the disparity in the magnitude of the flow velocity and the speed of sound. To overcome this difficulty various splitting methods have been proposed to remove the stiffness from the matrix coefficients of the equations, [3, 4, 7]. Some of these methods, however, have not performed as anticipated; in fact, often, for the the stipulated stability limits on the time step, the calculations diverged.

In this paper, we first propose an explanation for this behavior. We give examples in the first three sections which show that splittings resulting in matrices which are not simultaneously symmetrizable (such as in [7]) may be ill-posed at the p.d.e. level. Similar results are presented for some explicit numerical schemes, both finite difference and spectral. Thus, the intent of these sections is to caution against unrestrained use of splitting methods.

In Section IV, we present a transformation of variables which symmetrizes the Euler equations. Under the assumption of low Mach number flow, we are able to propose an efficient splitting technique for the compressible equations. The resulting algorithm, given both for the Euler and Navier-Stokes' equations, is unstiff for the nonlinear field, and the other split operators are linear and may therefore be solved implicitly with ease. (The implicit-

ness is necessary to overcome the stiffness which was transferred into the linear part.) The total scheme may be shown to be stable under the less restrictive time step of the nonlinear part. In a future paper, we intend to present computational results for our proposed algorithm.

1. A MODEL PROBLEM

Consider the initial value problem for the following symmetric hyperbolic system

$$w_t = \begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = A\, w_x . \tag{1.1}$$

β is a real number, $|\beta| > 1$. The eigenvalues of A are

$$\mu_1(A) = 1 + \beta, \qquad \mu_2(A) = 1 - \beta,$$

and therefore an explicit scheme will have the CFL condition

$$\Delta t \leq \frac{const}{1+|\beta|}\, \Delta x. \tag{1.2}$$

For example, the Lax-Wendroff scheme

$$w_j^{n+1} = w_j^n + \frac{\Delta t A}{2\Delta x}(w_{j+1}^n - w_{j-1}^n) + \frac{A^2(\Delta t)^2}{2(\Delta x)}(w_{j+1}^n - 2w_j^n + w_{j-1}^n) \tag{1.3}$$

is stable under the condition

$$\frac{\Delta t}{\Delta x}(1 + |\beta|) \leq 1.$$

Suppose now that one attempts to advance the solution of (1.1), equation by equation, rather than to use the form of the system as in (1.3). This amounts to splitting the matrix A into the sum of two matrices B and C

$$A = \begin{pmatrix} 1 & \beta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta & 1 \end{pmatrix} = B + C \tag{1.4}$$

and advancing the solution by using first the equation

$$w_t^{(1)} = B \, w_x^{(1)} \tag{1.5}$$

and then

$$w_t^{(2)} = C \, w_x^{(2)} \tag{1.6}$$

where the initial value of (1.6) at every time step is the value of $w^{(1)}$ obtained after advancing (1.5) one time step. This procedure yields a scheme which is first order in time and second order in space. We note that the systems defined in (1.5) and (1.6) are strictly hyperbolic and hence well-posed. The eigenvalues of B and C are 0 and 1, and therefore the Lax-Wendroff scheme for (1.5) and (1.6) separately will be stable under the condition

$$\frac{\Delta t}{\Delta x} \leq 1 \tag{1.7}$$

allowing a time step much larger than the one allowed by (1.2) if β is a large number. However, even if a numerical method is stable for (1.5) and (1.6) separately, it need not be stable for the combination of (1.5) and (1.6). In fact, consider the Lax-Wendroff scheme for (1.5) and (1.6). The amplification matrix G of the combined scheme is given by

$$G = \begin{pmatrix} g(\xi) + \beta^2(1 - g(\xi))^2 & -\beta g(\xi)(1 - g(\xi)) \\ -\beta(1 - g(\xi)) & g(\xi) \end{pmatrix} \qquad (1.7)$$

where

$$\xi = \sin \frac{k\Delta x}{2}$$

$$g(\xi) = 1 - 2\lambda^2\xi^2 + 2i\lambda\xi\sqrt{1 - \xi^2} \qquad \lambda = \frac{\Delta t}{\Delta x} .$$

We will show that the eigenvalues of G are greater than one in modulus for any λ, and thus the combined scheme is unconditionally unstable. To do that we look at the mode $\xi = 1$:

$$G(\xi = 1) = \begin{pmatrix} 1 - 2\lambda^2 + 4\beta^2\lambda^4 & -\beta(1 - 2\lambda^2)2\lambda^2 \\ -2\beta\lambda^2 & 1 - 2\lambda^2 \end{pmatrix} .$$

The eigenvalues of G $\mu\pm$ are given by

$$\mu\pm = 1 - 2\lambda^2 + 2\beta^2\lambda^4 \pm \sqrt{\beta^2\lambda^4 + 1 - 2\lambda^2} \; 2\beta\lambda^2 .$$

The scheme is clearly unstable for

$$\lambda^2\beta^2 \geq 1$$

since in this case $\mu+ > 1$ for $\beta > 1$ and $\mu_- > 1$ for $\beta < 1$. It is also easily verified that $\mu+ > 1$ for $\beta > 1$ for any λ. Thus, the splitting (1.5) - (1.6) is the wrong way of splitting.

Perhaps a deeper insight is obtained if we Fourier transform (1.1), (1.5), and (1.6). The solution operator for (1.1) in Fourier space is

$$\hat{E}(\omega, \Delta t) = e^{iA\omega t}$$

where ω is the dual Fourier variable.

The solution operator for the split scheme (1.5), (1.6) over one time step is

$$\hat{S}(\omega, \Delta t) = e^{iB\omega\Delta t} e^{iC\omega\Delta t}. \tag{1.8}$$

For every fixed ω

$$\hat{S}(\omega, \Delta t)^{t/\Delta t} \rightarrow \hat{E}(t).$$

However, since $C^2 = C$, $B^2 = B$, an expansion of the right hand side of (1.8) shows that

$$\hat{S}(\omega, \Delta t) = [I + B(e^{i\omega\Delta t} - 1)][I + C(e^{i\omega\Delta t} - 1)].$$

If we put $\Delta t\omega = \pi$, we get

$$\hat{S}(\omega, \Delta t) = (I - 2B)(I - 2C) = \begin{pmatrix} 4\beta^2 - 1 & 2\beta \\ -2\beta & -1 \end{pmatrix}$$

and for any $|\beta| > 1$ $\hat{S}(\omega, \Delta t)$ has eigenvalues larger than 1. This illustrates the instability.

2. THE ISENTROPIC EULER EQUATIONS

The isentropic Euler equations in one space dimension may be written as

$$w_t = \begin{bmatrix} u \\ p \end{bmatrix}_t = -\begin{bmatrix} u & 1/\rho \\ \gamma p & u \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}_x = A\, w_x, \tag{2.1}$$

where u is the velocity, p is the pressure, ρ is the density, and γ is the adiabatic constant of the fluid. The normalized equation of state for the fluid is

$$p = \rho^\gamma. \tag{2.2}$$

The eigenvalues of the matrix A in (2.1) are $u - c$ and $u + c$, where $c = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed. Thus, if we were to solve (2.1) by an explicit difference scheme, we would have to impose a CFL condition of the form

$$\frac{\Delta t}{\Delta x} \leq \frac{const}{|u|+c}. \tag{2.3}$$

We wish to study (2.1) in the low mach number regime so that $\rho \simeq \rho_0$, where ρ_0 is the base flow density. We define

$$\epsilon = \frac{\rho_0 - \rho}{\rho}.$$

Then $|\epsilon| \ll 1$. Using (2.2) we conclude

$$p - p_0 = -\gamma \epsilon \rho_0^\gamma + 0(\epsilon^2),$$

where p_0 is the base flow pressure.

One possible splitting for (2.1) [7], is to write A as the sum of two matrices A_1 and A_2 as follows

$$A = -\begin{bmatrix} 0 & 1/\rho_0 \\ \gamma p_0 & 0 \end{bmatrix} - \begin{bmatrix} u & 1/\rho - 1/\rho_0 \\ \gamma(p-p_0) & u \end{bmatrix} = A_1 + A_2. \tag{2.4}$$

We then advance the solution of (2.1) by first using the equation

$$w_t^{(1)} = A_1 w_x^{(1)},$$

(2.5)

and then the equation

$$w_t^{(2)} = A_2 w_x^{(2)}.$$

(2.6)

Since A_1 is a constant matrix, we could solve (2.5) analytically thus doing away with any CFL restriction. The eigenvalues of the matrix A_2, however, are $-u \pm i\sqrt{\gamma}\, c_0 \epsilon + O(\epsilon^2)$. Thus, the splitting (2.5 - 2.6) is not a hyperbolic splitting.

To examine the stability of the split scheme, we examine the Fourier transform of the solution operator, $S(\Delta t)$, over one time step. The Fourier transform of S is $\hat{S}$:

$$\hat{S}(\omega, \Delta t) = e^{iA_2 \omega \Delta t}\, e^{iA_1 \omega \Delta t}.$$

(2.7)

Let $\alpha = c_0\, \omega \Delta t$, and $\beta = \sqrt{\gamma}\, c_0\, \epsilon\, \omega \Delta t$.

After some computation, we obtain

$$e^{iA_1 \omega \Delta t} = \begin{bmatrix} \cos\alpha & -i\sin\alpha / c_0 \rho_0 \\ -i c_0 \rho_0 \sin\alpha & \cos\alpha \end{bmatrix}.$$

To first order in ϵ, we may write A_2 as

$$A_2 = \begin{bmatrix} u & \epsilon/\rho_0 \\ -\gamma^2 \rho_0^\gamma \epsilon & u \end{bmatrix}.$$

We then have

$$e^{iA_2\omega\Delta t} = e^{iu\omega\Delta t}\begin{bmatrix} \cosh\beta & -i\sinh\beta/\sqrt{\gamma}\ c_0\rho_0 \\ -i\sqrt{\gamma}\ c_0\rho_0\sinh\beta & \cosh\beta \end{bmatrix}.$$

Hence,

$$\hat{S}(\omega,\Delta t) = e^{iu\omega\Delta t}\begin{bmatrix} \cosh\beta\cos\alpha - \dfrac{\sinh\beta\sin\alpha}{\sqrt{\gamma}} & \dfrac{-1}{c_0\rho_0}(\cosh\beta\sin\alpha + \sinh\beta\cos\alpha) \\ i\rho_0 c_0(\sqrt{\gamma}\ \sinh\beta\cos\alpha - \cosh\beta\sin\alpha) & \sqrt{\gamma}\ \sinh\beta\sin\alpha + \cosh\beta\cos\alpha \end{bmatrix}.$$

The eigenvalues of $\hat{S}(\omega,\Delta t)$ are roots of the polynomial

$$p(\lambda) = \lambda^2 - (2\cosh\beta\cos\alpha + \frac{(\gamma-1)}{\sqrt{\gamma}}\sinh\beta\sin\alpha)\lambda + 1 = 0.$$

By Miller's criterion [8] the roots of $p(\lambda)$ are inside the unit disc if and only if

$$\theta = \left|\cosh\beta\cos\alpha + \frac{\gamma-1}{2\sqrt{\gamma}}\sinh\beta\sin\alpha\right| \leq 1.$$

If we let $\alpha = c_0\ \omega\Delta t = \pi$, then $\theta > 1$, whenever $\beta > 0$. Hence, at least one of the eigenvalues of $\hat{S}(\omega,\Delta t)$ lies outside the unit disc in a neighborhood of $\alpha = \pi$.

If we were to solve (2.5) - (2.6) using a pseudospectral difference scheme, we would have to impose a CFL restriction of

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{c_0}$$

to ensure the stability of our split scheme. For if we were to use the Fourier modes $\{e^{ikx}\}$, $k = -N, \cdots, N,$ as a basis for our numerical scheme the mesh width Δx for the grid of collocation points would be given by

$$\Delta x = \frac{2\pi}{2(N+1)} \ .$$

Since our stability condition dictates that

$$c_0 N \Delta t < \pi,$$

the CFL condition for our scheme assumes the form

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{c_0} \ .$$

Nothing has been gained, therefore, from this splitting technique.

3. THE EULER EQUATIONS

We write the Euler equations in one space dimension as

$$w_t = \begin{bmatrix} \rho \\ m \\ p \end{bmatrix}_t = - \begin{bmatrix} 0 & 1 & 0 \\ -u^2 & 2u & 1 \\ -c^2 u & c^2 & u \end{bmatrix} \begin{bmatrix} \rho \\ m \\ p \end{bmatrix}_x = A w_x. \tag{3.1}$$

Here ρ, m, p, and u denote the density, the momentum, the pressure, velocity, and c is the sound speed of the fluid. We analyze (3.1) in the low mach number regime.

The eigenvalues of A are $u - c$, u and $u + c$. Therefore, an explicit

difference scheme for (3.1) would have a CFL restriction of the form

$$\frac{\Delta t}{\Delta x} \leq \frac{const}{|u|+c} \, . \tag{3.2}$$

One possible splitting for (3.1) could be obtained by writing A as

$$A = - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & c_0^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ -u^2 & 2u & 0 \\ -c^2 u & c^2 - c_0^2 & u \end{bmatrix} = A_1 + A_2 \tag{3.3}$$

where c_0 is the sound speed of the base flow. We then solve (3.1) by using
first the equation

$$w_t^{(1)} = A_1 w_x^{(1)} \tag{3.4}$$

and then

$$w_t^{(2)} = A_2 w_x^{(2)}. \tag{3.5}$$

The advantage of such a splitting, it would seem, is that since A_1 is a
constant matrix we can obtain an analytical solution of (3.4) without any
restriction on the time step. Further, since the eigenvalues of A_2 are 0,
$-u$, and $-2u$, we can solve (3.5) by a difference scheme with a large CFL con-
dition of the form

$$\frac{\Delta t}{\Delta x} \leq \frac{const}{2|u|} \, . \tag{3.6}$$

We examine the Fourier transform of the solution operator $S(\Delta t)$ over one
time step. Then $\hat{S}(\omega, \Delta t) = e^{iA_2 \omega \Delta t} e^{iA_1 \omega \Delta t}$.

Let $$\alpha = c_0 \omega \Delta t$$

$$\eta = \frac{c^2 - c_0^2}{c_0^2} .$$

We choose $u = 0$ and $\eta > 0$. Then

$$e^{iA_1\omega\Delta t} = \begin{bmatrix} 1 & -\dfrac{i\sin\alpha}{c_0} & -\dfrac{1+\cos\alpha}{c_0^2} \\ 0 & \cos\alpha & -\dfrac{i\sin\alpha}{c_0} \\ 0 & -ic_0\sin\alpha & \cos\alpha \end{bmatrix}$$

and

$$e^{iA_2\omega\Delta t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -i\eta c_0\alpha & 1 \end{bmatrix} .$$

Hence

$$\hat{S}(\omega,\Delta t) = \begin{bmatrix} 1 & \dfrac{i\sin\alpha}{c_0} & -\dfrac{1+\cos\alpha}{c_0^2} \\ 0 & \cos\alpha & -\dfrac{i\sin\alpha}{c_0} \\ 0 & -i(\eta c_0\alpha \cos\alpha + c_0\sin\alpha) & (-\eta\alpha\sin\alpha + \cos\alpha) \end{bmatrix} .$$

The eigenvalues of $\hat{S}(\omega,\Delta t)$ are 1, and the roots of the polynomial

$$p(\lambda) = \lambda^2 - (2\cos\alpha - \eta\alpha\sin\alpha) + 1 = 0.$$

By Miller's criterion the roots of $p(\lambda)$ are inside the unit disc if and only if

$$\theta = \left|\frac{2\cos\alpha - \eta\alpha\sin\alpha}{2}\right| \leq 1 .$$

Let $\alpha = -\pi + \delta$.

Then $\theta = 1 + \eta\pi\delta + O(\delta^2)$.

Hence $\hat{S}(\omega,\Delta t)$ has at least one root outside the unit disc near $\alpha = c_0\omega\Delta t = \pi$. Thus, this proposed splitting, once more, has undesirable properties. If we were to solve (3.4) — (3.5) using a pseudospectral difference scheme, we would have to impose a very restrictive CFL condition of the form $\frac{\Delta t}{\Delta x} \leq \frac{const}{c_0}$.

Gustafsson and Guerra [3] showed how to split (2.1) in a way that avoids the pitfalls pointed out above. The main idea in their work was to obtain two symmetric split operators. This, of course, is harder to do for more complicated systems. In the following sections, we generalize this approach to the problem of obtaining split operators which are simultaneously symmetrizable in the case of the full Euler and Navier-Stokes equations.

4. CORRECT SPLITTING FOR THE EULER EQUATIONS

In the preceding sections, we gave examples of "natural" splitting procedures which led either to instabilities or to stability conditions which at best did not represent an improvement over the original ones. A common feature of those split operators was that they were not simultaneously symmetrizable.

To avoid the dangers pointed out by these examples, we propose to remove the stiffness of a given stable symmetric operator by instituting a splitting procedure such that all the split-off operators are simultaneously symmetrizable. If each of these new operators is discretized in a stable manner, then the overall scheme will remain stable.

A prescription for a general operator achieving this goal is not known to us. We would like, however, to suggest such a procedure for compressible, low Mach number flows governed by either the Euler or the Navier-Stokes equations. These systems are chosen in view of the "counter-examples" given in Section 3. The Euler equations may be symmetrized nonlinearly by using "entropy-variables" [5, 6]. The system thus obtained is of the form

$$Pq_t + \sum_{i=1}^{3} A_i q_{x_i} = 0 \qquad (4.1)$$

where P and the A_i's are symmetric matrix functions of the vector q. The premultiplying matrix P is usually non-sparse, and hence it is not clear how to remove the stiffness (if there is any) from the A_i's. In the Euler equations, it is well known that the eigenvalues of A_1 are u, $u + c$, $u - c$ where u is the x-component of velocity and c is the speed of sound. At low Mach number flows, $u \ll c$ everywhere; hence, a Von-Neumann like stability condition

$$\Delta t \leq \frac{\Delta x}{|u|+c} \qquad (4.2)$$

gives an over-restricted condition. In this sense, the system is stiff (see Sections 2 and 3).

Our approach is motivated by previous results [1] valid for the linearized frozen coefficient case.

Consider the Euler equations for a gas in their nonconservative form in two-space dimensions (the three-dimensional case follows directly from the results of this section):

$$\frac{\partial \hat{V}}{\partial t} + \hat{A} \frac{\partial \hat{V}}{\partial x} + \hat{B} \frac{\partial \hat{V}}{\partial y} = 0 \qquad (4.3)$$

where $\hat{V}$ is the column vector $(\hat{\rho}, \hat{u}, \hat{v}, \hat{p})$ and the coefficient matrices are given by

$$\hat{A} = \begin{bmatrix} \hat{u} & \hat{\rho} & 0 & 0 \\ 0 & \hat{u} & 0 & 1/\hat{\rho} \\ 0 & 0 & \hat{u} & 0 \\ 0 & \gamma \hat{p} & 0 & \hat{u} \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} \hat{v} & 0 & \hat{\rho} & 0 \\ 0 & \hat{v} & 0 & 0 \\ 0 & 0 & \hat{v} & 1/\hat{\rho} \\ 0 & 0 & \gamma \hat{p} & 0 \end{bmatrix}, \quad (4.4)$$

where $\hat{\rho}, \hat{u}, \hat{v}, \hat{p}$ and γ are, respectively, the density, the velocity components in the $\hat{x}$ and $\hat{y}$ directions, the pressure and the ratio of specific heats at constant pressure and volume $(\gamma = C_p/C_v)$. Next, nondimensionalize the quantities in (4.3) as follows:

$$t = \frac{\hat{t} U_\infty}{\ell}, \quad x = \frac{\hat{x}}{\ell}, \quad y = \frac{\hat{y}}{\ell}, \quad \rho = \frac{\hat{\rho}}{\rho_\infty}$$

$$\qquad (4.6)$$

$$u = \frac{\hat{u}}{u_\infty}, \quad v = \frac{\hat{v}}{v_\infty}, \quad p = \frac{\hat{p}}{\rho_\infty u_\infty^2}$$

where the subscript ∞ indicates free stream condition and ℓ is a reference length. Equations (4.3) and (4.4) then retain the same form exactly with the superscript $\wedge$ removed. In particular, the dimensionless speed of sound retains its functional form, i.e.,

$$c = \frac{\hat{c}}{u_\infty} = \sqrt{\gamma p/\rho} \ . \qquad (4.7)$$

We now propose the following change of variables:

$$V = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix} \qquad \rightarrow \qquad q = \begin{bmatrix} \left(c_1/\sqrt{\gamma}\right)\ell n\ \rho \\ u \\ v \\ \dfrac{2c}{\sqrt{\gamma(\gamma-1)}} \end{bmatrix} \tag{4.8}$$

where c_1 is a constant to be specified later. One may then cast (4.3) in the form of (4.1) where:

$$P = \begin{bmatrix} \dfrac{c^2}{c_1^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.9}$$

$$A_1 = \begin{bmatrix} \dfrac{c^2}{c_1^2}u & \dfrac{c^2}{c_1\sqrt{\gamma}} & 0 & 0 \\ \dfrac{c^2}{c_1\sqrt{\gamma}} & u & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\,c \\ 0 & 0 & u & 0 \\ 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\,c & 0 & u \end{bmatrix}, \ A_2 = \begin{bmatrix} \dfrac{c^2}{c_1^2}v & 0 & \dfrac{c^2}{c_1\sqrt{\gamma}} & 0 \\ 0 & v & 0 & 0 \\ \dfrac{c^2}{c_1\sqrt{\gamma}} & 0 & v & \sqrt{\dfrac{\gamma-1}{\gamma}}\,c \\ 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\,c & v \end{bmatrix}. \tag{4.10}$$

With these definitions of P, A_1, and A_2 the Euler equations

$$Pq_t + A_1 q_x + A_2 q_y = 0 \tag{4.11}$$

are symmetric hyperbolic.

We now wish to motivate the way in which the operators in (4.11) ($A_1 \frac{\partial}{\partial x}$ and $A_2 \frac{\partial}{\partial y}$) are split. The starting point is the fact that we are here interested in low Mach number flow. Such flows are characterized by two facts: the first is that $c^2 \gg u^2 + v^2$ everywhere; secondly, for reasonable initial conditions

$$\frac{c^2(x,y,t) - c_\infty^2}{c_\infty^2} \ll 1. \tag{4.12}$$

For example, at steady state

$$\frac{c^2(x,y) - c_\infty^2}{c_\infty^2} \leq \frac{T_{st} - T_\infty}{T_\infty} = \frac{\gamma-1}{2} M_\infty^2 \tag{4.13}$$

where T_{st} is the stagnation temperature, M_∞ is the free stream Mach number; hence, for low Mach numbers (4.12) is valid. In view of the above, we choose $c_1 = c_\infty$, and we rewrite (4.11) as follows

$$Pq_t + (R_1 + S_1)q_x + (R_2 + S_2)q_y = 0 \tag{4.14}$$

where

$$R_1 = \begin{bmatrix} \dfrac{c^2}{c_\infty^2} & \dfrac{c^2-c_\infty^2}{c_\infty\sqrt{\gamma}} & 0 & 0 \\[2ex] \dfrac{c^2-c_\infty^2}{c_\infty\sqrt{\gamma}} & u & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}(c-c)_\infty \\[2ex] 0 & 0 & u & 0 \\[2ex] 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}(c-c_\infty) & 0 & u \end{bmatrix} \quad R_2 = \begin{bmatrix} \dfrac{c^2}{c_\infty^2}\,v & 0 & \dfrac{c^2-c_\infty^2}{c_\infty\sqrt{\gamma}} & 0 \\[2ex] 0 & v & 0 & 0 \\[2ex] \dfrac{c^2-c_\infty^2}{c_\infty\sqrt{\gamma}} & 0 & v & \sqrt{\dfrac{\gamma-1}{\gamma}}(c-c_\infty) \\[2ex] 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}(c-c_\infty) & v \end{bmatrix}$$

and

$$S_1 = \begin{bmatrix} 0 & \dfrac{c_\infty}{\sqrt{\gamma}} & 0 & 0 \\[2ex] \dfrac{c_\infty}{\sqrt{\gamma}} & 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\,c_\infty \\[2ex] 0 & 0 & 0 & 0 \\[2ex] 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\,c_\infty & 0 & 0 \end{bmatrix} \quad S_2 = \begin{bmatrix} 0 & 0 & \dfrac{c_\infty}{\sqrt{\gamma}} & 0 \\[2ex] 0 & 0 & 0 & 0 \\[2ex] \dfrac{c_\infty}{\sqrt{\gamma}} & 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\,c_\infty \\[2ex] 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\,c_\infty & 0 \end{bmatrix}.$$

The four eigenvalues of $P^{-1}R_1$ are

$$\lambda = u,\ u,\ u \pm (c - c_\infty)\left[1 + \frac{2}{\sqrt{\gamma}}\left(\frac{c_\infty}{c}\right) + \frac{1}{\sqrt{\gamma}}\left(\frac{c_\infty}{c}\right)^2\right]^{1/2}.$$

It is clear from (4.11) that

$$c - c_\infty \leq (\gamma-1)M_\infty$$

while

$$\frac{c_\infty}{c} \approx O(1).$$

Thus, none of the eigenvalues gets to be large unlike the original unsplit scheme which had eigenvalues u, u, $u \pm c$ (recall that in our case $u \approx O(1)$ while $c \approx 1/M_\infty$.)

Next we notice that S_1 and S_2 are constant matrices. A difficulty remains however in the nonlinear element of P. We shall deal with this as the method of solution is presented.

Step I in the solution of (4.12) is to numerically advance

$$Pq_t + R_1 q_x + R_2 q_y = 0 \qquad (4.15)$$

by one time step. We have just demonstrated that stability criterion for (4.15) is not stiffly restricted. In fact, for most explicit schemes, to within a constant of order unity,

$$\Delta t \leq \min_{x,y} \left\{ \frac{\Delta x}{|u|+|c-c_\infty|} , \frac{\Delta y}{|v|+|c-c_\infty|} \right\},$$

as compared with (4.2). The gain is obvious. Step II in the procedure is to solve

$$Pq_t + S_1 q_x + S_2 q_y = 0. \qquad (4.16)$$

The initial conditions for (4.16) are given by the solution to (4.15) at $t = \Delta t$. Notice that while S_1 and S_2 are constant matrices, P has the

nonlinear element c^2/c_∞^2. This means that removal of the stricter time-step due to S_1 and S_2 ($\Delta t \leq$ const. $\Delta x/c_0$) cannot be done easily via implicit method implementation of (4.16). To overcome this difficulty, we split S_1 and S_2 as follows:

$$S_1 = S_1^I + S_1^{II} \qquad S_2 = S_2^I + S_2^{II}$$

where

$$S_1^I = \begin{bmatrix} 0 & \dfrac{c_\infty}{\sqrt{\gamma}} & 0 & 0 \\ \dfrac{c_\infty}{\sqrt{\gamma}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad S_1^{II} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\, c_\infty \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{\dfrac{r-1}{\gamma}}\, c_\infty & 0 & 0 \end{bmatrix} \qquad (4.17)$$

and

$$S_2^I = \begin{bmatrix} 0 & 0 & \dfrac{c_\infty}{\sqrt{\gamma}} & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{c_\infty}{\sqrt{\gamma}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad S_2^{II} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\, c_\infty \\ 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\, c_\infty & 0 \end{bmatrix}. \qquad (4.18)$$

Thus, we replace (4.16) with the sequence

$$Pq_t + S_1^I q_x + S_2^I q_y = 0 \qquad (4.19)$$

$$Pq_t + S_1^{II} q_x + S_2^{II} q_y = 0. \qquad (4.20)$$

Note that in (4.19), c_t is identically zero, and so over that time step we take $c = c(x,y)$ from (4.15). The rest of (4.19) is therefore linear (because $c^2 = c^2(x,y)$ is known) and can be solved implicitly with relative ease. Alternatively, the first three equations in (4.19) may be combined into a variable coefficient wave equation for $\ell n \, \rho$, namely:

$$\frac{\partial^2}{\partial t^2} (\ell n \, \rho) = \frac{c_\infty^4}{c^2(x,y)} \nabla^2 (\ell n \, \rho); \qquad (4.21)$$

u and v are then obtained directly from the middle two equations of (4.19). In (4.20) it is $\ell n \, \rho$ that does not change over the time step. The rest of the system is linear with constant coefficients and may also be cast into a wave equation for c:

$$\frac{\partial^2 c}{\partial t^2} = \frac{\gamma-1}{\gamma} \, c_\infty^2 \, \nabla^2 c, \qquad (4.22)$$

and again u and v are found directly from the middle two equations of (4.20).

This completes the splitting method for the Euler equation. The temporal and spatial discretization depends on the particular problem. Straight splitting as described here will result in only first order accuracy in time. Alternating the order of solving between (4.15) → (4.19) → (4.20) to (4.20) → (4.19) → (4.15) will yield second order in time, see [2].

5. EXTENSION TO THE NAVIER-STOKES EQUATIONS (N. S. CASE)

The Navier-Stokes equations may be written as:

$$Pq_t + A_1 q_x + A_2 q_y = B_1 q_{xx} + B_2 q_{yy} + Cq_{xy} + F_1 + F_2 \qquad (5.1)$$

where q, P, A_1, and A_2 are as defined in the preceding section. The quantities on the right hand side are given by:

$$B_1 = \frac{1}{Re}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\gamma}{2Pr\rho} \end{bmatrix} \qquad B_2 = \frac{1}{Re}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{\gamma}{2Pr\rho} \end{bmatrix} \qquad \begin{matrix}(5.2)\\ \\ (5.3)\end{matrix}$$

$$C = \frac{1}{Re}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5.4)$$

$$F_1 = \frac{\sqrt{\gamma}}{\sqrt{\gamma-1}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{P_r R_e} \cdot \frac{1}{\rho}\left(\frac{c_x^2+c_y^2}{c}\right) \end{bmatrix} (5.5) \qquad F_2 = \frac{1}{\sqrt{\gamma}(\gamma-1)}\frac{1}{Re}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\Phi}{\rho c} \end{bmatrix} (5.6)$$

where the dimensionless viscous production function Φ is given by

$$\Phi = -\frac{2}{3}(u_x + v_y)^2 + 2[u_x^2 + v_y^2] + [u_y + v_x]^2. \qquad (5.7)$$

We can now describe the solution method: after obtaining the "hyperbolic"
solution (see equations (4.15) to (4.22)), we go through the following steps:

1) From (5.2) to (5.6), it follows that $q_{1_t} = 0$, i.e., during the "viscous"
integration $\rho = \rho(x,y)$.

2) multiply equation (5.1) by the matrix

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \cdot
$$

The resulting two "viscosity split" equations for u and v are

$$
u_t = \frac{1}{Re} \left[(\frac{4}{3} u_{xx} + u_{yy}) + \frac{1}{3} u_{xy} \right]
\tag{5.8}
$$

$$
v_t = \frac{1}{Re} \left[v_{xx} + \frac{4}{3} v_{xy} + \frac{1}{3} v_{xy} \right].
\tag{5.9}
$$

These may be easily solved implicitly since they are linear p.d.e.'s with con-
stant coefficients.

(3) The last step is to solve the viscous part of the energy equation which
may be cast in the form:

$$
\frac{\partial}{\partial t} (c^2) = \frac{\gamma}{2PrRe\rho} \left[\nabla^2 (c^2) \right] + \frac{1}{Re} \frac{\Phi}{\rho}.
\tag{5.10}
$$

Note that Φ/ρ is a function of the squares of u_x, u_y, v_x, v_y, and $\rho(x,y)$ only and may therefore be taken as known from the previous step. Equation (5.10) is then a scalar linear inhomogeneous heat equation which again may be easily solved by implicit methods.

Note that all the operators in (5.8) and (5.9) may be taken to be stable (i.e., $\|\cdot\| < 1$) in L_2. In addition, because $c^2 > 0$, $F_2 > 0$ (5.10) is also stabilizable under the L_1 norm for c^2; this assures the L_2 stability for q_4.

Notice the total algorithm (4.15) $\rightarrow$ (4.22) $\rightarrow$ (5.8) $\rightarrow$ (5.10) may be run partly in parallel thus enhancing its efficiency beyond the removal of the stiffness. Schematically, the tree of calculation may be shown as follows:
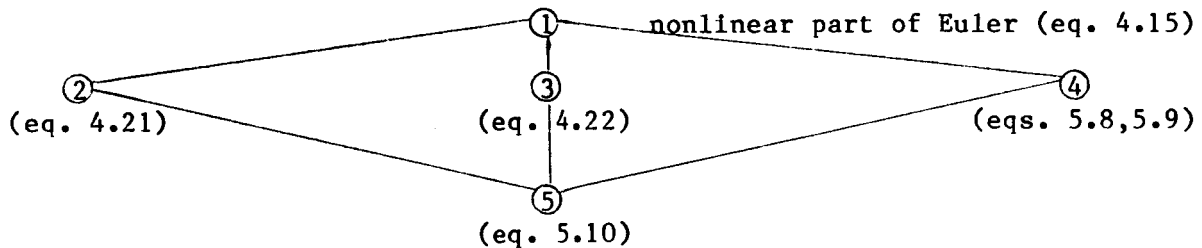
Thus, if parallel processors are available, we run only three computations instead of five.

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16. Abstract

In this paper, we examine some splitting techniques for low Mach number Euler flows. We point out shortcomings of some of the proposed methods and suggest an explanation for their inadequacy. We then present a symmetric splitting for both the Euler and Navier-Stokes equations which removes the stiffness of these equations when the Mach number is small. The splitting is shown to stable.

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