A GENERAL METHOD FOR DYNAMIC ANALYSIS OF STRUCTURES

OVERVIEW

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I. Definition of the Term "Element"

An element is defined as any structural unit whose degrees of freedom (DOFS) can be categorized as either interface DOFS or non-interface (internal) DOFS.

Interface DOFS are those DOFS through which the element is connected to one or more neighboring elements. Non-interface DOFS are internal to the element and do not directly couple to neighboring elements.

The term "element" then, has a rather broad meaning. An element could be a fundamental structural unit such as a rod, a beam, a plate, etc. or it could be an entire structural component. Furthermore, the parameters of the element could be distributed or lumped. Figure 1 schematically illustrates the element concept.

Any structural system can now be thought of as a composite of n such elements. The choice of elements is totally arbitrary and is a matter of user convenience. In particular, the user does not have to worry about the "size" of the element as is the case for example when using a standard finite element approach. This means that in general n is relatively small and little bookkeeping is necessary.

Spatial periodicity of structures can be taken into account in a natural manner. These important advantages will be further clarified in the next few sections.

II. Modeling of an Element

Each element will be modeled using a set of assumed modes. In particular, a combination of interface constraint modes (ICM) and a set of interface restrained normal modes (IRNM) can be employed. Note that other types of assumed modes can be selected and should be investigated. The above choice is motivated by the Craig/Bampton approach to component modes synthesis and has several important advantages.

1. Interface Constraint Modes (ICM)

Let us assume the element has $\mathscr{I}$ interface DOFS x_{Ij} $(j=1,2,\cdots,\mathscr{I})$. The ICM corresponding to DOF x_{Ij} is defined as the static deformation pattern of the element for $x_{Ij} = 1$ and $x_{Ii} = 0$ (for all $i \neq j$). Note that many "shape functions" used in the finite element method are actually ICM.

2. Interface Restrained Normal Modes (IRNM)

IRNM are the regular mass normalized mode shapes for the element with fixed interface DOFS.

3. Displacement of an Element

The displacement vector of a discrete element can be partitioned as:

$$\underset{\sim}{X} = \left\{ \begin{array}{c} \underset{\sim}{X}_I \\ ---- \\ \underset{\sim}{X}_N \end{array} \right\} \tag{1}$$

The non-interface displacement can be written as

$$\underset{\sim}{X}_N = S_{NI} \, \underset{\sim}{X}_I + \underset{\sim}{\bar{X}}_N \tag{2}$$

where the first term $S_{NI}\underset{\sim}{X}_I$ represents a static deformation due to the interface displacements. The term $\underset{\sim}{\bar{X}}_N$ is best described as the difference

$$\underset{\sim}{\bar{X}}_N = \underset{\sim}{X}_N - S_{NI} \, \underset{\sim}{X}_I \tag{3}$$

The displacement vector $\underset{\sim}{\bar{X}}_N$ is now written as a linear combination of the IRNM

$$\underset{\sim}{\bar{X}}_N = \Phi_N \, \underset{\sim}{\bar{q}}_N \tag{4}$$

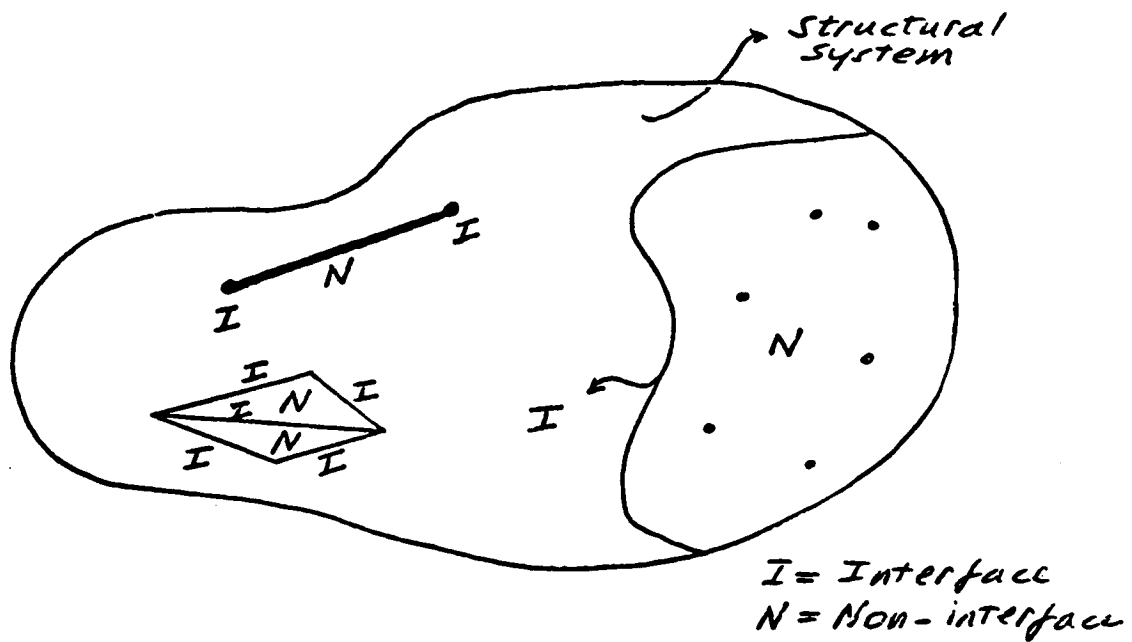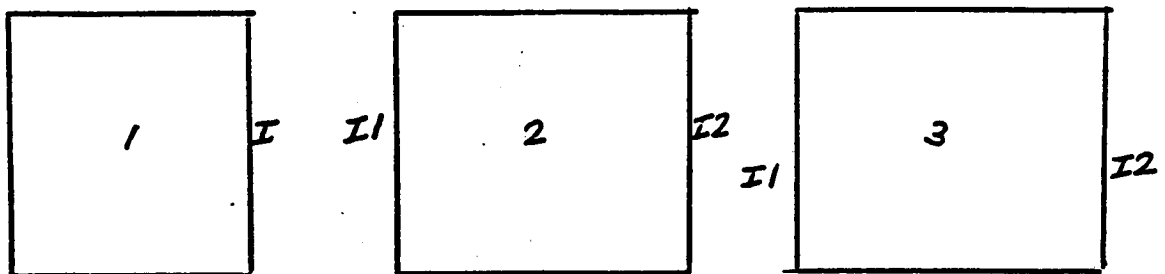**Figure 1**: A Schematic Representation of the Element Concept.

Figure 2: Schematic Representation of
Three Elements

where $\bar{q}_N$ is a set of modal coordinates.

The displacement vector $\underset{\sim}{x}$ can now be written as

$$\underset{\sim}{x} = \begin{bmatrix} t & | & I^e \Phi_N \end{bmatrix} \left\{ \frac{\underset{\sim}{x}_I}{\bar{q}_N} \right\} \tag{5}$$

with

$$t \triangleq \begin{bmatrix} I \\ S_{NI} \end{bmatrix} \quad , \quad I^e \triangleq \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{6}$$

The columns of matrix t clearly represent the ICM of the element. The actual general form of the matrix S_{NI} will be given shortly.

The elements of the vectors $\underset{\sim}{x}_I$ and $\bar{q}_N$ will be the generalized coordinates used to model a structural element. Equation (5) is also valid for distributed parameter elements.

Equation (5) indicates why the elements can be chosen with a large degree of freedom. If the matrices t and $I^e \Phi_N$ can be determined with enough accuracy then a legitimate element is found regardless of its size. For example entire beams, rods and plates can be considered as one element. Even large components could be considered as single elements as long as Eq. (5) can be adequately written. As a consequence, a considerable amount of bookkeeping can be avoided, thereby reducing the cost of the analysis.

In the next section we will introduce the element equations of motion.

III. The Element Equations of Motion

From here on we will work with discrete systems because relationships can be shown more explicitly. It should be kept in mind however, that all results are equally valid for distributed-parameter systems.

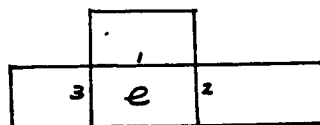The most general form of the element displacement vector $\underset{\sim}{x}$ is:

$$\underset{\sim}{x} = \left\{ \begin{array}{c} \underset{\sim}{x}_{I1} \\ \underset{\sim}{x}_{I2} \\ \vdots \\ \underset{\sim}{x}_{I\Delta} \\ \hline \underset{\sim}{x}_N \end{array} \right\} \tag{7}$$

Δ = number of distinct interfaces of the element
For Example:
$\Delta = 3$ for element e

with

$$\underset{\sim}{x}_I \triangleq \left\{ \begin{array}{c} \underset{\sim}{x}_{I1} \\ \underset{\sim}{x}_{I2} \\ \vdots \\ \underset{\sim}{x}_{I\Delta} \end{array} \right\} \tag{8}$$

$\underset{\sim}{x}$ can be written as:

$$\underset{\sim}{x} = \left\{ \begin{array}{c} \underset{\sim}{x}_I \\ \hline \underset{\sim}{x}_N \end{array} \right\} \tag{9}$$

Because distinct interfaces by definition do not connect directly to each other, the element stiffness matrix corresponding to vector $\underset{\sim}{x}$ in Eq. (7) is:

$$K = \begin{bmatrix} K_{I1I1} & 0 & \cdots & 0 & | & K_{I1N} \\ 0 & K_{I2I2} & \cdots & 0 & | & K_{I2N} \\ \vdots & \vdots & & \vdots & | & \vdots \\ 0 & 0 & \cdots & K_{I\Delta I\Delta} & | & K_{I\Delta N} \\ \hline K_{NI1} & K_{NI2} & \cdots & K_{NI\Delta} & | & K_{NN} \end{bmatrix} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \underset{\sim}{x}_{I1} \\ \underset{\sim}{x}_{I2} \\ \vdots \\ \underset{\sim}{x}_{I\Delta} \\ \hline \underset{\sim}{x}_N \end{array} \right\} \tag{10}$$

The mass matrix has a similar form. Another form corresponding to vector $\underset{\sim}{x}$ in Eq. (9) is:

$$M = \begin{bmatrix} M_{II} & M_{IN} \\ M_{NI} & M_{NN} \end{bmatrix} \quad , \quad K = \begin{bmatrix} K_{II} & K_{IN} \\ K_{NI} & K_{NN} \end{bmatrix} \tag{11}$$

with partitioning corresponding to Eq. (9).

The form of Eq. (10) suggests that

$$S_{NIj} = - K_{NN}^{-1} K_{NIj} \quad , \quad j = 1, 2, \cdots, d \tag{12}$$

and

$$\underset{\sim}{x} = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ \hline S_{NI1} & S_{NI2} & \cdots & S_{NId} & \bar{\phi}_N \end{bmatrix} \begin{Bmatrix} \underset{\sim}{x}_{I1} \\ \underset{\sim}{x}_{I2} \\ \vdots \\ \underset{\sim}{x}_{Id} \\ \bar{q}_N \end{Bmatrix} \tag{13}$$

or,

$$\underset{\sim}{x} = \begin{bmatrix} I & 0 \\ \hline S & \bar{\phi}_N \end{bmatrix} \begin{Bmatrix} \underset{\sim}{x}_I \\ \bar{q}_N \end{Bmatrix} \tag{14}$$

with

$$\bar{x}_N = \bar{\phi}_N \bar{q}_N \tag{15}$$

where $\bar{\phi}_N$ are IRNM and the partitions in Eq. (14) correspond to Eq. (13).

Using Eqs. (11), (12) and (14) the mass and stiffness matrices for an element corresponding to coordinates $\underset{\sim}{x}_I, \bar{q}_N$ are:

$$\overline{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & I \end{bmatrix} \leftrightarrow \begin{Bmatrix} \ddot{\underset{\sim}{x}}_I \\ \ddot{\bar{q}}_N \end{Bmatrix} \tag{16}$$

$$\overline{K} = \begin{bmatrix} K_{11} & 0 \\ 0 & \bar{\omega}^2 \end{bmatrix} \leftrightarrow \begin{Bmatrix} \underset{\sim}{x}_I \\ \bar{q}_N \end{Bmatrix} \tag{17}$$

where

$$M_{11} = M_{II} + M_{IN} S + S^T (M_{IN}^T + M_{NN} S) \tag{18}$$

$$M_{12} = (M_{IN} + S^T M_{NN}) \bar{\phi}_N \tag{19}$$

$$K_{11} = K_{II} + K_{IN} S \tag{20}$$

and where we made use of the fact that

$$K_{IN} + S^T K_{NN} = 0 \tag{21}$$

$$\left. \begin{array}{l} \bar{\phi}_N^T M_{NN} \bar{\phi}_N = I \\ \bar{\phi}_N^T K_{NN} \bar{\phi}_N = \bar{\omega}^2 \end{array} \right\} \quad \text{IRNM orthonormality conditions} \tag{22}$$

The approach followed to model the element reveals a unique perspective on a serious shortcoming of the standard finite element techniques when used to construct element mass matrices. Indeed, it is clear that a standard finite element mass matrix (consistent) represents a Guyan reduction in which the internal degrees of freedom (non-interface) are eliminated, i.e. all $\bar{q}_N$ coordinates are neglected. As will be seen later, such an approximation is valid only when the elements are "small" enough so that indeed the coordinates $\bar{q}_N$ have no effect on the response of the structure.

In the next section we will discuss the formation of the system equations.

IV. The System Equations

1. Coupling of Two Elements

In this section we will demonstrate how elements are assembled into global components. As an example, consider the three elements in Figure 2. Equations (16-20) represent the general form of the element equations. The displacement vector for element i can be written as

$$\left\{ \begin{array}{c} \underset{\sim}{x}^i_I \\ \bar{q}^i_N \end{array} \right\}$$ (23)

In general, the vector $\underset{\sim}{x}^i_I$ represents a set of <u>local</u> displacement coordinates. In order to enforce displacement compatibility between the element interfaces it is necessary to transform $\underset{\sim}{x}^i_I$ into a set of <u>global</u> coordinates $\underset{\sim}{X}^i_I$,

$$\left\{ \begin{array}{c} \underset{\sim}{x}^i_I \\ \bar{q}^i_N \end{array} \right\} = \left[\begin{array}{cc} Q^i_I & 0 \\ 0 & I \end{array} \right] \left\{ \begin{array}{c} \underset{\sim}{X}^i_I \\ \bar{q}^i_N \end{array} \right\}$$ (24)

where Q^i_I is a geometric transformation matrix. Note that no transformation is necessary for $\bar{q}^i_N$ because elements only connect through interfaces. Using equations of the type Eq. (24) we can rewrite Eqs. (16-20) in the following form

$$\bar{M}^i = \left[\begin{array}{cc} M^i_{11} & M^i_{12} \\ M^i_{21} & I \end{array} \right] \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \ddot{\underset{\sim}{X}}^i_I \\ \ddot{\bar{q}}^i_N \end{array} \right\}$$ (25)

$$\bar{K}^i = \left[\begin{array}{cc} K^i_{11} & 0 \\ 0 & \bar{\omega}^2_i \end{array} \right] \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \underset{\sim}{X}^i_I \\ \bar{q}^i_N \end{array} \right\}$$ (26)

where M^i_{11}, M^i_{12} and K^i_{11} are easily found.

The first step in the assembly process is to connect element 1 to element 2 by requiring that

$$\underset{\sim}{X}^1_I = \underset{\sim}{X}^2_{I1} \quad , \qquad \text{for all times}$$ (27)

The <u>uncoupled</u> equations of motion for the 1-2 component can be written as follows

$$\left[\begin{array}{c|ccc} \begin{array}{cc} M^1_4 & M^1_{12} \\ M^1_{21} & I \end{array} & & 0 & \\ \hline & m^2_{11} & m^2_{12} & m^2_{13} \\ 0 & m^2_{21} & m^2_{22} & m^2_{23} \\ & m^2_{31} & m^2_{32} & I \end{array} \right] \left\{ \begin{array}{c} \ddot{\underset{\sim}{X}}^1_I \\ \ddot{\bar{q}}^1_N \\ \ddot{\underset{\sim}{X}}^2_{I1} \\ \ddot{\underset{\sim}{X}}^2_{I2} \\ \ddot{\bar{q}}^2_N \end{array} \right\} + \left[\begin{array}{c|ccc} \begin{array}{cc} K^1_{11} & 0 \\ 0 & \bar{\omega}^2_1 \end{array} & & 0 & \\ \hline & k^2_{11} & k^2_{12} & 0 \\ 0 & k^2_{21} & k^2_{22} & 0 \\ & 0 & 0 & \bar{\omega}^2_2 \end{array} \right] \left\{ \begin{array}{c} \underset{\sim}{X}^1_I \\ \bar{q}^1_N \\ \underset{\sim}{X}^2_{I1} \\ \underset{\sim}{X}^2_{I2} \\ \bar{q}^2_N \end{array} \right\} = 0$$ (28)

where the damping and forcing terms are omitted since we are only interested in the eigenvalue problem. Note that for element 2 we have two interfaces and we denoted

$$M^2_{11} = \left[\begin{array}{cc} m^2_{11} & m^2_{12} \\ m^2_{21} & m^2_{22} \end{array} \right], \quad M^2_{12} = \left[\begin{array}{c} m^2_{13} \\ m^2_{23} \end{array} \right], \quad K^2_{11} = \left[\begin{array}{cc} k^2_{11} & k^2_{12} \\ k^2_{21} & k^2_{22} \end{array} \right]$$ (29)

Taking into account Eq. (27) we can write the coupled equations of motion for component 1-2 as

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$$
\begin{bmatrix}
m_{22}^2 & m_{21}^2 & 0 & m_{23}^2 \\
m_{12}^2 & M_{11}'+m_{11}^2 & M_{12}' & m_{13}^2 \\
0 & M_{21}' & I & 0 \\
m_{32}^2 & m_{31}^2 & 0 & I
\end{bmatrix}
\begin{Bmatrix}
\ddot{\underset{\sim}{X}}_{I2}^2 \\
\ddot{\underset{\sim}{X}}_{I}' \\
\ddot{\underset{\sim}{q}}_N' \\
\ddot{\underset{\sim}{q}}_N^2
\end{Bmatrix}
+
\begin{bmatrix}
k_{22}^2 & k_{21}^2 & 0 & 0 \\
k_{12}^2 & K_{11}'+k_{11}^2 & 0 & 0 \\
0 & 0 & \bar{\omega}_I^2 & 0 \\
0 & 0 & 0 & \bar{\omega}_2^2
\end{bmatrix}
\begin{Bmatrix}
\underset{\sim}{X}_{I2}^2 \\
\underset{\sim}{X}_I' \\
\underset{\sim}{q}_N' \\
\underset{\sim}{q}_N^2
\end{Bmatrix}
\qquad (30)
$$

At this point, element 3 should be connected to the 1-2 component. Before doing this, Equation (30) will be manipulated into a form identical to Equations (16-17). This is an important feature of the presented technique. Indeed, it will allow us to consider the 1-2 component as a standard element as defined in Section I. In other words, the resulting equations will represent the dynamics of element 1-2. In addition, the equations will be in a form suitable to apply an escalator type eigensolver to be discussed later.

2. Transformation of the Interface Displacement Vector

The first step in reducing the 1-2 component into a single element is to introduce the following transformation

$$
\underset{\sim}{X}_I' = S_{II2}\,\underset{\sim}{X}_{I2}^2 + \bar{\underset{\sim}{X}}_I' \qquad (31)
$$

with

$$
S_{II2} = -\left[K_{11}' + k_{11}^2\right]^{-1}k_{12}^2 \qquad (32)
$$

This transformation is entirely equivalent to Eq. (2). The term $S_{II2}\,\underset{\sim}{X}_{I2}^2$ represents that part of $\underset{\sim}{X}_I'$ which is due to the displacement of the interface I2. The term $\bar{\underset{\sim}{X}}_I'$ is the displacement of I with respect to the interface I2 as seen by an absolute observer. Note that the matrix $\left[K_{11}'+k_{11}^2\right]$ is never singular because it represents the stiffness matrix of a rigid body restrained system.

The displacement vector $\bar{\underset{\sim}{X}}_I'$ can be written as a linear combination of IRNM as follows

$$
\bar{\underset{\sim}{X}}_I' = \bar{\Phi}_I'\,\bar{\underset{\sim}{q}}_I' \qquad (33)
$$

where the eigenvalue problem solved is

$$
\bar{\omega}_{I1}^2\left[M_{11}' + m_{11}^2\right]\bar{\underset{\sim}{\Phi}}_I' = \left[K_{11}' + k_{11}^2\right]\bar{\underset{\sim}{\Phi}}_I' \qquad (34)
$$

The following transformation

$$
\begin{Bmatrix}
\underset{\sim}{X}_{I2}^2 \\
\underset{\sim}{X}_I' \\
\underset{\sim}{q}_N' \\
\underset{\sim}{q}_N^2
\end{Bmatrix}
=
\begin{bmatrix}
I & 0 & 0 & 0 \\
S_{II2} & \bar{\Phi}_I' & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\begin{Bmatrix}
\underset{\sim}{X}_{I2}^2 \\
\bar{\underset{\sim}{q}}_I' \\
\underset{\sim}{q}_N' \\
\underset{\sim}{q}_N^2
\end{Bmatrix}
\qquad (35)
$$

can now be substituted into Eq. (30), yielding

$$
\begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
m_{21} & I & m_{23} & m_{24} \\
m_{31} & m_{32} & I & 0 \\
m_{41} & m_{42} & 0 & I
\end{bmatrix}
\begin{Bmatrix}
\ddot{\underset{\sim}{X}}_{I2}^2 \\
\ddot{\bar{\underset{\sim}{q}}}_I' \\
\ddot{\underset{\sim}{q}}_N' \\
\ddot{\bar{\underset{\sim}{q}}}_N^2
\end{Bmatrix}
+
\begin{bmatrix}
k_{11} & 0 & 0 & 0 \\
0 & \bar{\omega}_{I1}^2 & 0 & 0 \\
0 & 0 & \bar{\omega}_I^2 & 0 \\
0 & 0 & 0 & \bar{\omega}_2^2
\end{bmatrix}
\begin{Bmatrix}
\underset{\sim}{X}_{I2}^2 \\
\bar{\underset{\sim}{q}}_I' \\
\underset{\sim}{q}_N' \\
\bar{\underset{\sim}{q}}_N^2
\end{Bmatrix}
\qquad (36)
$$

where $m_{11}, m_{12}, \ldots, m_{23}, m_{24}$ and k_{11} can be easily found.

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3. The IRNM Eigenvalue Problem

The second step in reducing Eq. (30) to a form identical to Eqs. (16-17) is the solution of the eigenvalue problem

$$\bar{\omega}^2 \begin{bmatrix} I & m_{23} & m_{24} \\ m_{32} & I & 0 \\ m_{42} & 0 & I \end{bmatrix} \underset{\sim}{\Phi} = \begin{bmatrix} \bar{\omega}_I^2 & 0 & 0 \\ 0 & \bar{\omega}_I^2 & 0 \\ 0 & 0 & \bar{\omega}_2^2 \end{bmatrix} \underset{\sim}{\Phi} \tag{37}$$

Using the transformation

$$\begin{Bmatrix} \bar{q}'_I \\ \bar{q}'_N \\ \bar{q}_N^2 \end{Bmatrix} = \bar{\Phi} \, \bar{q} \tag{38}$$

in Eq. (36) yields

$$\begin{bmatrix} \bar{M}_{11} & \widetilde{M}_{12} \\ \bar{M}_{21} & I \end{bmatrix} \begin{Bmatrix} \ddot{X}_{I2}^2 \\ \ddot{\bar{q}} \end{Bmatrix} + \begin{bmatrix} \widetilde{K}_{11} & 0 \\ 0 & \bar{\omega}^2 \end{bmatrix} \begin{Bmatrix} X_{I2}^2 \\ \bar{q} \end{Bmatrix} = \underset{\sim}{0} \tag{39}$$

with

$$\bar{M}_{11} = m_{11} \quad , \quad \bar{M}_{12} = [\, m_{12} \; m_{13} \; m_{14} \,] \, \bar{\Phi}$$

$$\bar{K}_{11} = k_{11} \tag{40}$$

Equation (39) is now equivalent in form to Eqs. (16-17). It essentially represents the equations of motion for the 1-2 component viewed as one single element. If along the way no modes are truncated then Eq. (39) involves no unusual approximations. In particular, for the continuous equivalent, the equations corresponding to Eq. (39) are still "exact".

The eigenvalue problem (37) has a very special form and an optimum solution will be discussed in Section V.

An important aspect of the present technique is the truncation of the mode set $\bar{\Phi}$ according to a preset frequency. This preset frequency must reflect the frequency content in an element necessary to obtain the desired fidelity in the overall system model. How the "element" cut-off frequency compares to the "system" cut-off frequency is still a matter of research or "experience".

Once the appropriate truncation is performed, a third element can be added through interface 12. Note that the number of degrees of freedom is already reduced. The eigenvalue problem (37) is small and can be solved very efficiently as will be shown.

It is also important to note that because of the transformation (31) we do not loose any accuracy in the rigid body and static properties of the system when modes are truncated.

4. Coupling of Element 1-2 and Element 3.

The next step is to couple element 3 to element 1-2. This process is very similar to the one already described. From Eqs. (25-26) we can obtain for $i = 3$

$$\bar{M}^3 = \begin{bmatrix} M_{11}^3 & M_{12}^3 \\ M_{21}^3 & I \end{bmatrix} \quad , \quad \bar{K}^3 = \begin{bmatrix} K_{11}^3 & 0 \\ 0 & \bar{\omega}_3^2 \end{bmatrix} \longleftrightarrow \begin{Bmatrix} X_I^3 \\ \bar{q}_N^3 \end{Bmatrix} \tag{41}$$

The uncoupled equations of motion for the 12-3 component can be written as

$$\begin{bmatrix} \begin{array}{cc|c} \widetilde{M}_{11} & \widetilde{M}_{12} & 0 \\ \widetilde{M}_{21} & I & \\ \hline & & M_{11}^3 \;\; M_{12}^3 \\ 0 & & M_{21}^3 \;\; I \end{array} \end{bmatrix} \left\{ \begin{array}{c} \ddot{\underset{\sim}{X}}{}^2_{I2} \\ \ddot{\underset{\sim}{q}} \\ \hline \ddot{\underset{\sim}{X}}{}^3_{I1} \\ \ddot{\underset{\sim}{q}}{}^3_N \end{array} \right\} + \begin{bmatrix} \begin{array}{cc|c} \widetilde{K}_{11} & 0 & 0 \\ 0 & \bar{\omega}^2 & \\ \hline & & K_{11}^3 \;\; 0 \\ 0 & & 0 \;\; \bar{\omega}_3^2 \end{array} \end{bmatrix} \left\{ \begin{array}{c} \underset{\sim}{X}{}^2_{I2} \\ \underset{\sim}{q} \\ \hline \underset{\sim}{X}{}^3_{I1} \\ \underset{\sim}{q}{}^3_N \end{array} \right\} = 0 \tag{12}$$

where we used Eqs. (39) and (41). Note that the interface I2 of element 3 is part of the model if no other elements are to be connected. However, if other elements are to be connected, this interface must be made explicit.

We now must impose the compatibility condition

$$\underset{\sim}{X}{}^2_{I2} = \underset{\sim}{X}{}^3_{I1} \tag{43}$$

Because this is the last element, no further transformations of the interface displacement vector is necessary. Using Eq. (34) we can write

$$\begin{bmatrix} \widetilde{M}_{11} + M_{11}^3 & \widetilde{M}_{12} & M_{12}^3 \\ \widetilde{M}_{21} & I & 0 \\ M_{21}^3 & 0 & I \end{bmatrix} \left\{ \begin{array}{c} \ddot{\underset{\sim}{X}}{}^3_{I1} \\ \ddot{\underset{\sim}{q}} \\ \ddot{\underset{\sim}{q}}{}^3_N \end{array} \right\} + \begin{bmatrix} \widetilde{K}_{11} + K_{11}^3 & 0 & 0 \\ 0 & \bar{\omega}^2 & 0 \\ 0 & 0 & \bar{\omega}_3^2 \end{bmatrix} \left\{ \begin{array}{c} \underset{\sim}{X}{}^3_{I1} \\ \underset{\sim}{q} \\ \underset{\sim}{q}{}^3_N \end{array} \right\} = \underset{\sim}{0} \tag{44}$$

which represent the coupled equations for elements 1-2 and 3.

Next, we can solve the small eigenvalue problem.

$$\omega_{I2}^2 \left[\widetilde{M}_{11} + M_{11}^3 \right] \underset{\sim I}{\phi}{}^2 = \left[\widetilde{K}_{11} + K_{11}^3 \right] \underset{\sim I}{\phi}{}^2 \tag{45}$$

so that

$$\underset{\sim}{X}{}^3_{I1} = \underset{I}{\phi}{}^2 \underset{I}{q}{}^2 \tag{46}$$

Note that no "bar" is necessary over the above quantities, because if the system is free, the rigid body properties will be incorporated in Eqs. (45) and (46).

Incorporating Eq. (46) into Eq. (44) yields

$$\begin{bmatrix} I & \widetilde{m}_{12} & \widetilde{m}_{13} \\ \widetilde{m}_{12}^T & I & 0 \\ \widetilde{m}_{13}^T & 0 & I \end{bmatrix} \left\{ \begin{array}{c} \ddot{\underset{\sim I}{q}}{}^2 \\ \ddot{\underset{\sim}{q}} \\ \ddot{\underset{\sim}{q}}{}^3_N \end{array} \right\} + \begin{bmatrix} \omega_{I2}^2 & 0 & 0 \\ 0 & \bar{\omega}^2 & 0 \\ 0 & 0 & \bar{\omega}_3^2 \end{bmatrix} \left\{ \begin{array}{c} \underset{\sim I}{q}{}^2 \\ \underset{\sim}{\bar{q}} \\ \underset{\sim}{\bar{q}}{}^3_N \end{array} \right\} = \underset{\sim}{0} \tag{47}$$

with

$$\widetilde{m}_{12} = \underset{I}{\phi}{}^{2^T} \widetilde{M}_{12} \, , \quad \widetilde{m}_{13} = \underset{I}{\phi}{}^{2^T} M_{12}^3 \tag{48}$$

Finally, note the simple form of this system of equation (47). First, generally truncation will reduce the size significantly. In addition, the specific form will allow for application of the escalator algorithm as discussed in Section V. This eigenvalue solver will yield a set of system frequencies $\bar{\omega}^2$ and system modes $\bar{\phi}$ with a minimum of effort and cost.

Before we discuss the special eigenvalue solver, remarks are in order:

(1) So far, we described how the elements are coupled together. It is a matter of repeating the same procedure for each added element. Each time truncation is used on the element level as well as on the level of the current system. A series of relatively small eigenvalue problems is

solved using a very efficient eigenvalue problem solver. The entire system is gradually built up, keeping only the necessary frequency fidelity at each stage. Several interesting questions arise, for example: (a) what is the ideal trade-off between element frequency truncation and current system frequency truncation, and how do both relate to the overall desired fidelity? (b) Which is the optimum way of solving the successive eigenvalue problems? Should we wait until several elements are collected before a current system eigenvalue problem is solved? This is an important question since it affects storage, cost and accuracy.

(2) The manner in which the elements are coupled together makes this technique ideally suited to handle spatial periodicity. Indeed, Eq. (39) shows that each current system can be considered as an "element". If care is taken, it is possible to use the same element over again, without adding significantly to the cost. For example, a periodic truss can be started with one element which is truncated according to a desired frequency. This model can now be doubled and truncated again. This truncated 2-element model can now be doubled again, to yield a 4-element truss, etc.

V. The Eigenvalue Problem Solver

In this section we will describe an eigenvalue problem solution technique which is particularly suited for our purposes.

First, let us consider the following special eigenvalue problem:

$$\lambda \begin{bmatrix} 1 & m_1 & m_2 & \cdots & m_n \\ m_1 & 1 & 0 & \cdots & 0 \\ m_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ m_n & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{Bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_n \end{Bmatrix} = \begin{bmatrix} k_0 & 0 & \cdots & 0 \\ 0 & k_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix} \begin{Bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_n \end{Bmatrix} \tag{19}$$

This problem has a diagonal stiffness matrix and a unit mass matrix except for the first row and column. The m_i values are such that the mass matrix is positive definite.

The characteristic equation of this problem can be written as follows

$$(\lambda - k_0) - \lambda^2 \sum_{i=1}^{n} \frac{m_i^2}{\lambda - k_i} = 0 \tag{50}$$

This assumes implicitly that $k_i \neq k_j$ (for $i \neq j$ and $i, j = 1, \cdots, n$) and also that $k_0 \neq 0$. If $k_i = k_j$ then it can be shown that there is a root $\lambda = k_i$. It can also be shown that all roots satisfy the inequality

$$\lambda_0 < k_0 < k_1 < \lambda_1 < k_2 < \lambda_2 < \cdots < \lambda_{n-2} < k_{n-1} < \lambda_{n-1} < k_n < \lambda_n \tag{51}$$

In other words, we have isolated the eigenvalues of the system represented by Eq. (19). Note that property (51) again shows that for $k_i = k_j$, k_i becomes a root of the system. Property (51) allows us to use for example the Newton-Raphson technique to find the actual eigenvalues λ_i. This iteration scheme converges quadratically provided a good initial value is found. Without going into detail, at this point we can say that property (51) allows for a very accurate initial value for each of the eigenvalues . Therefore, convergence is extremely fast, often after two or three iterations.

Once the eigenvalues are determined, it is computationally a trivial matter to obtain the corresponding eigenvectors. Also, multiple roots are no problem. This algorithm was programmed but not yet optimized. Nevertheless, on a VAX/780, it requires only 10 seconds of CPU time to solve a problem with n = 450, which is a size far beyond our needs. As part of the presented dynamic analysis technique this eigenvalue problem must be solved for each interface dof.

VI. Summary and Conclusions

The presented research deals with the development of a dynamic analysis method for structural systems. The modeling approach is essentially a finite element method in the sense that the structure

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is divided into n "elements". An "element" is defined as any structural unit whose dofs. can be categorized as either interface or non-interface dofs. The term "element" then, has a rather broad meaning. An element could be a fundamental unit such as a rod, a beam, a plate etc. or it could be an entire structural component. Furthermore, the parameters for the element could be distributed or lumped. The choice of elements is totally arbitrary and is a matter of user convenience. In particular, issues of accuracy and convergence do not enter on the level of element choice as is the case in a standard finite element method. This means for example that bookkeeping is reduced to a minimum.

Each element will be modeled using a set of interface constraint modes (ICM) combined with a set of interface restrained normal modes (IRNM). If the ICM and IRNM can be found with enough accuracy, then a legitimate "element" is defined. For example, entire beams, rods and plates can be considered as one element. Entire components can be made into one element in an off-line manner. Moreover, the entire system can be modeled as if it was represented by partial differential equations. Note that standard finite element techniques in general only use ICM ($=$ shape functions) which leads to very interesting and useful insights into important shortcomings of these techniques. In particular, the problem of an accurate mass distribution is addressed by this new approach.

The element assembly process is essentially the same as in the standard finite element method. However, each combination of elements is automatically converted into a single element. This procedure is based on static condensation without loss of accuracy. This feature is very important and allows for each structural unit to be interpreted as an "element". It also allows for the stiffness matrix to remain diagonal.

The next step is the solution of the system eigenvalue problem. The procedure calls for the sequential solution of a number of small eigenvalue problems based on a truncation principle for IRNM. In addition, the form of these eigenvalue problems is very simple such that an escalator type of eigenvalue problem solver can be used which is extremely cost-effective and fast. The response, loads, etc. calculations are rather standard, but also benefit from the approach in terms of accuracy and cost-effectiveness. The groundwork for this technique is in place and is currently supported by the AFWL/ARBH Kirtland AFB, NM. Some of the advantages of the new technique are: (1) The problem of Order Reduction is believed to be solved. The technique implicitly reduces the system order. Whenever an element is added only information necessary to obtain a prescribed fidelity in the system model is retained. (2) Very accurate. In fact, if desired, "exact" solutions in the distributed parameter sense can be obtained for any structure. (3) Fast and cost-effective. This is due to the small number of elements; the solution of a series of small eigenvalue problems instead of one large problem; the special nature of these small eigenvalue problems combined with the cost-effectiveness of the escalator eigenvalue problem solver. (4) Applicable in general. In particular, extremely large structures do not pose a problem. Once a model is agreed upon, any number of modes and frequencies with any degree of accuracy can be computed. (5) It is anticipated that Micro Computers can be used to solve even the largest of problems. This is due to the small bookkeeping effort and the sequential nature of the solution. (6) Spatial periodicity can be taken into account in a natural manner. (7) It is anticipated that this method will be useful in areas like control optimization, identification and possibly non-linear phenomena. The feasibility of this technique as well as several of the above advantages have been demonstrated with several examples.
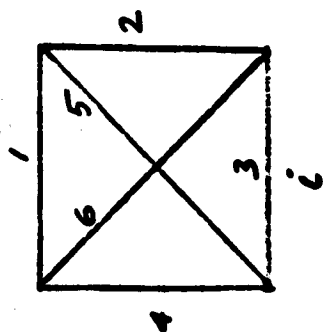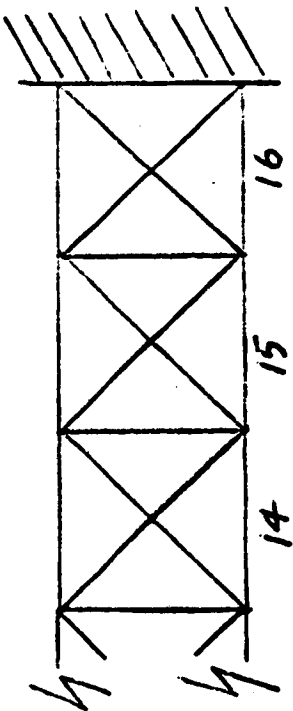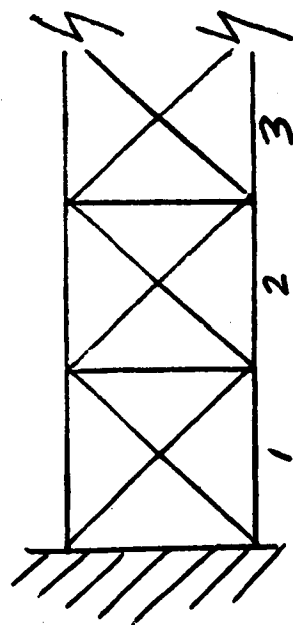
VI. EXAMPLE

FIXED-FIXED TRUSS
16 BAYS

$$L_1 = L_2 = L_3 = L_4 = 50$$
$$L_5 = L_6 = 50\sqrt{2}$$
$$A = 1$$
$$E = 10 \times 10^6$$
$$\rho = 0.1$$
$$\frac{1}{g} = 0.00259$$

VI. EXAMPLE (CONT)

FREQUENCY COMPARISONS

N	EXACT	MSC-NASTRAN	% ERROR	MSC-NASTRAN	% ERROR
1	15.13	15.05	0.5	15.13	-0.002
2	39.28	38.60	1.7	39.28	-0.01
3	72.08	69.72	3.3	72.10	-0.04
4	79.51	79.37	0.2	79.55	-0.06
5	111.2	105.6	5.1	111.3	-0.1
6	155.1	144.2	7	155.4	-0.2
7	159.2	158.1	0.7	159.5	-0.2
8	202.8	184.2	9.1	203.4	-0.3
9	239.1	224.5	6.1	240.3	-0.5
10	253.3	235.4	7.1	254.7	-0.5
40	788.1	691.0	12.3	823.4	-4.5
50	850.9	751.3	11.7	893.4	-5.0

PRESENTED TECHNIQUE →

TAKING PINS AS NODES

$$M_{LONG} = \rho AL \begin{bmatrix} 5/12 & 1/12 \\ 1/12 & 5/12 \end{bmatrix}$$

$$M_{TRANS} = \rho AL \begin{bmatrix} \tfrac{1}{2} & 0 \\ 0 & \tfrac{1}{2} \end{bmatrix}$$

USUAL VERSION

TAKING PINS AS NODES

$$M_{LONG} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M_{TRANS} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

VI. EXAMPLE (CONT)

FREQUENCY COMPARISONS

N	EXACT	MSC-NASTRAN	% ERROR	MSC-NASTRAN	% ERROR
1	15.13	15.05	0.5	15.12	0.07
2	39.28	38.59	1.8	39.24	0.1
3	72.08	69.67	3.3	71.92	0.2
4	79.51	79.31	0.3	79.49	0.03
5	111.2	105.4	5.2	110.8	0.4
6	155.1	143.7	7.4	154.3	0.5
7	159.2	157.6	1	159.1	0.06
8	202.8	183.2	9.7	201.4	0.7
9	239.1	222.6	8.1	238.8	0.1
10	253.3	233.9	7.7	251.0	0.9
40	788.1	649.4	17.6	777.0	1.4
50	850.9	700.1	17.7	837.1	1.6

PRESENTED TECHNIQUE →

→ USE BEAM THEORY TO SIMULATE TRUSS ELEMENTS 277 NODES 534 DOFS (4 SEGMENT–ELEMENT)

→ NOTE, HAS NOT CONVERGED YET IN HIGH FREQS

$$m_{LONG} = \rho AL \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$m_{TRANS} = \rho AL \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$