Thermocapillary Bubble Migration for Large Marangoni Numbers

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SUMMARY

The thermocapillary motion of spherical bubbles present in an unbounded liquid with a linear temperature distribution is analyzed, when the Reynolds number and the Marangoni number are large. Previous calculations of the terminal velocity performed for this parametric range did not take into complete consideration the thermal boundary layer present near the surface of the bubble. In the present study, a scaling analysis is presented for this problem. The thermal boundary layer is analyzed by an integral method. The resulting terminal velocity is lower than the one previously calculated, though it is of the same order of magnitude.

INTRODUCTION

In the absence of gravity, bubbles present in a host fluid with a temperature gradient will move towards the hotter portion of the fluid. This is due to the shear stress at the interface that is generated by the temperature induced surface tension gradient. Many studies, both theoretical and experimental, have been performed on this phenomenon (refs. 1 to 7) where the objective is to determine the terminal velocity of migration and the shape of the bubble.

Two of the most important parameters that influence the flow and the heat transfer in this problem are the Reynolds number R_σ and the Marangoni number Ma. The Marangoni number is really a Peclet number. Other parameters (Weber number, Capillary number, etc.) control the shape of the bubble (ref. 6). Most analyses are restricted to small and unit order R_σ and Ma as they use the creeping flow equations or perturbations to it. In reference 6, a solution was found for any R_σ, so long as Ma is small. Crespo and Manuel (ref. 7) have calculated the terminal velocity for large Ma, where no restriction is stated on R_σ explicitly. However, a large R_σ is implied as a thin flow boundary layer is assumed. The authors in reference 7 use a mechanical energy argument, first used by Levich (ref. 8) in his analysis of the rise of bubbles in a liquid in a gravitational field, for large Reynold's numbers. In reference 7, the temperature field is not analyzed at all and the work of the surface tension forces at the interface that was calculated is suspect, especially at the front and rear stagnation points. In the present study, the analysis in reference 7 is extended and the temperature field is analyzed by an integral method. This results in a decrease in the calculated terminal velocity compared to the value reported in reference 7.

NOMENCLATURE

A   temperature gradient far away from the bubble

Ca   Capillary number, $\mu V_R / \sigma$

f reference quantity for the ratio V'/V_1

Ma Marangoni number, $(-\sigma_T)AR_1^2/(\mu\alpha)$

Pr Prandtl number, v/α

p,P dimensionless and dimensional pressure

R_1 bubble radius

Re Reynolds number, $V_R R_1/v$

R_σ surface tension Reynolds number, $(-\sigma_T)AR_1^2/(\mu v)$

r,R dimensionless and dimensional radial coordinate

T temperature

$\bar{T}'$ dimensionless transformed temperature

T_S bubble surface temperature

t time

u,U dimensionless and dimensional radial velocity

v,V dimensionless and dimensional tangential velocity

$\underset{\sim}{V}$ velocity vector

Wb Weber number, $\rho V_R^2 R_1/\sigma$

α thermal diffusivity

δ reference quantity for the boundary layer thickness

θ tangential coordinate

λ inverse of the boundary layer thickness at any θ

μ viscosity

v kinematic viscosity

ξ stretched radial coordinate

ρ density

σ surface tension

σ_T temperature coefficient of surface tension

τ viscous stress

ϕ azimuthal coordinate

<u>Superscripts</u>

'   correction fields in the boundary layer

<u>Subscripts</u>

1   inviscid fields

R   reference values

∞   values far from the bubble

<div align="center">Formulation</div>

As the bubble moves through the fluid at its terminal velocity, it is convenient to choose a coordinate system on the bubble with the origin at its center of mass. In this system, the bubble is stationary and the fluid outside approaches the bubble at the terminal velocity. The velocity field is steady. The temperature field is not steady, as the bubble constantly moves to a warmer region. However, the gradient of the temperature field is steady and it is easy to transform to another temperature that is steady (ref. 5).

The flow is considered to be incompressible and laminar. All physical properties other than the surface tension are taken to be independent of temperature and are hence spatially constant. The bubble is assumed to retain its spherical shape. From the geometry and the boundary conditions, the problem is symmetric about the flow direction. It is assumed that the viscosity and thermal conductivity of the gas inside the bubble are negligible compared to those in the liquid outside it. Hence, the flow and the heat transfer within the bubble is not analyzed. The coordinate system is R, θ, ϕ with the origin at the center of the bubble (fig. 1). θ is measured counterclockwise from the point of incidence of the flow. The basic equations and boundary conditions describing the flow are the same as in reference 6 and are reproduced below in equations (1) to (10).

$$r = \frac{R}{R_1}, \qquad u = \frac{U}{V_R}, \qquad v = \frac{V}{V_R},$$

$$p = \frac{P}{\rho V_R^2}, \qquad T' = \frac{(T - T_0 - AV_\infty t)}{AR_1}$$

(1)

where R_1 is the radius of the bubble, A is the temperature gradient in the liquid far away from the bubble, $V_R = (-\sigma_T)AR_1/\mu$, is a reference velocity determined from the shear stress condition at the interface, T' is the dimensionless transformed temperature which is steady. In what follows, unless otherwise mentioned, T will be used to denote T', for convenience. $\sigma_T = d\sigma/dT$ is the rate of change in surface tension with temperature and is taken to be a constant. It is usually negative. V_∞ is the terminal velocity. The basic equations are

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v \sin \theta) = 0 \tag{2}$$

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$$u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{R_\sigma} \left[\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \theta} - \frac{2v}{r^2} \cot \theta \right] \tag{3}$$

$$u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{R_\sigma} \left[\nabla^2 v + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r^2 \sin^2 \theta} \right] \tag{4}$$

$$v_\infty + u \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \theta} = \frac{1}{Ma} \nabla^2 T \tag{5}$$

$$\nabla^2 \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right)$$

where $R_\sigma = V_R R_1 / v = (-\sigma_T) A R_1^2 / (\mu v)$ is the surface-tension Reynolds number, $Ma = Pr \, R_\sigma$ is the Marangoni number, and $v_\infty = V_\infty / V_R$.

The boundary conditions are

At $r = 1$,

$$u = 0 \tag{6}$$

$$\frac{\partial}{\partial r} \left(\frac{v}{r} \right) = \frac{\partial T}{\partial \theta} \tag{7}$$

$$\frac{\partial T}{\partial r} = 0 \tag{8}$$

As $r \to \infty$,

$$u \to -v_\infty \cos \theta, \qquad v \to v_\infty \sin \theta, \qquad p \to \frac{P_\infty}{\rho V_R^2}, \qquad T \to r \cos \theta \tag{9}$$

In addition to the above conditions, we have another condition resulting from the fact that the total force acting on the bubble is zero, as the bubble is moving at a constant velocity. This condition is used to determine v_∞, the dimensionless terminal velocity.

$$\int_0^\pi \left[\tau_{R\theta} \sin^2 \theta - \tau_{RR} \cos \theta \sin \theta \right]_{R=R_1} d\theta = 0 \tag{10}$$

where $(\tau_{R\theta})_{R=R_1} = \mu \left[R \frac{\partial}{\partial R} \left(\frac{v}{R} \right) \right]_{R=R_1}$ and $(\tau_{RR})_{R=R_1} = 2\mu \left(\frac{\partial u}{\partial R} \right)_{R=R_1} - P(R_1, \theta)$.

As an alternative to the net force condition, Crespo and Manuel (ref. 7), following Levich (ref. 8), have used a condition arising from the conservation of mechanical energy. In a reference frame in which the bubble is stationary, conservation of mechanical energy implies that the rate of energy dissipated in the liquid by viscous forces must equal the rate of work done by the surface tension forces at the interface. Hence, this condition may be written as (refs. 7 to 9).

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$$\sigma_T \int_S \left(V\, \frac{\partial T}{\partial \theta}\right)_{R=R_1} ds = \mu \int_{\mathscr{V}} \Phi \, d\mathscr{V} = \mu \int_{\mathscr{V}} \left[\text{curl } \underset{\sim}{V} \cdot \text{curl } \underset{\sim}{V}\right] d\mathscr{V}$$

$$+ 2\mu \int_S \left[\underset{\sim}{n} \cdot (\underset{\sim}{V} \times \text{curl } \underset{\sim}{V}) - \frac{1}{2}\frac{\partial}{\partial n}(\underset{\sim}{V} \cdot \underset{\sim}{V})\right] ds \qquad (11)$$

Simplifying and using $U = 0$ at $R=R_1$ this becomes

$$R_1 \sigma_T \int_0^\pi \left(V\, \frac{\partial T}{\partial \theta}\right)_{R=R_1} \sin\theta \, d\theta = \mu \int_0^\pi\!\!\int_{R_1}^\infty \left[\frac{\partial}{\partial R}(RV) - \frac{\partial U}{\partial \theta}\right]^2 \sin\theta \, dR \, d\theta$$

$$+ 2\mu R_1 \int_0^\pi (V^2)_{R=R_1} \sin\theta \, d\theta \qquad (12)$$

where $[\partial/\partial R(RV) - \partial U/\partial \theta]^2 = |\text{curl } \underset{\sim}{V}|^2 =$ the square of the magnitude of the vorticity and T is the dimensional transformed temperature.

While in typical boundary layer problems (like flow over a flat plate), using an energy method such as the one above to determine the drag is more difficult than directly integrating the surface stresses, it appears that in problems with spherical bubbles, the energy method is easier to apply. Levich (ref. 8) and Moore (ref. 10) have used such a method to predict the terminal velocity of rise of bubbles in a gravitational field, when the Reynolds number is large.

Analysis

The boundary value problem that has been formulated is very difficult to solve. In the introduction, the various conditions have been mentioned under which solutions have been obtained. In the present study, our interest is to predict the terminal velocity for large R_σ and Ma. As a first step, an estimate for the terminal velocity, i.e., V_R is determined for large R_σ and Ma. The previous estimate $V_R = (-\sigma_T)AR_1/\mu$ (ref. 6) is valid only for small Ma, as the shear stress was assumed to be of order $\mu V_R/R_1$, which excludes the presence of sharp gradients, i.e., the presence of boundary layers. We will determine V_R by scaling analysis. For large R_σ and Ma, we expect a boundary layer to be present near the bubble surface. We will represent the velocity field V as the sum of the inviscid velocity field V_1 and a boundary layer correction velocity field V', i.e.,

$$\underset{\sim}{V} = \underset{\sim}{V_1} + \underset{\sim}{V}' \qquad (13)$$

The order of magnitude of V_1 is denoted to be V_R and that of V' to be fV_R . The correction field V' is nonzero within the boundary layer and is zero outside it. δ denotes thickness of the boundary layer. V_R , f, and δ are all unknown and must be determined. These are determined below by appropriate balance of terms in the equations and boundary conditions.

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Nondimensional quantities are defined as follows

$$\xi = \frac{R - R_1}{\delta R_1}, \qquad r = \frac{R}{R_1}, \qquad u_1 = \frac{U_1}{V_R}, \qquad v_1 = \frac{V_1}{V_R},$$

$$u' = \frac{U'}{\delta f V_R}, \qquad v' = \frac{V'}{f V_R}, \qquad p' = \frac{P - P_1}{f \rho V_R^2} \qquad (14)$$

where U_1, V_1, and P_1 are the dimensional inviscid velocity components and the pressure and U' and V' are the correction velocities in the boundary layer. Using equation (14) in equations (2) to (5), the following equations are obtained

$$\frac{1}{(1 + \delta\xi)^2} \frac{\partial}{\partial\xi} [(1 + \delta\xi)^2 u'] + \frac{1}{(1 + \delta\xi)\sin\theta} \frac{\partial}{\partial\theta} (v' \sin\theta) = 0 \qquad (15)$$

$$u_1 \frac{\partial u'}{\partial\xi} + u' \frac{\partial u_1}{\partial\xi} + \delta f u' \frac{\partial u'}{\partial\xi} + \frac{\delta}{(1 + \delta\xi)} \left[v_1 \frac{\partial u'}{\partial\theta} + v' \frac{\partial u_1}{\partial\theta} + f v' \frac{\partial u'}{\partial\theta} \right] - f^2 v'^2 - 2 v_1 v'$$

$$= -\frac{1}{\delta} \frac{\partial p'}{\partial\xi} + \frac{1}{\delta Re} \frac{1}{(1 + \delta\xi)^2} \left\{ \frac{\partial}{\partial\xi} \left[(1 + \delta\xi)^2 \frac{\partial u'}{\partial\xi} \right] + \frac{\delta^2}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u'}{\partial\theta} \right) \right.$$

$$\left. - 2\delta^2 u' - 2\delta \frac{\partial v'}{\partial\theta} - 2\delta v' \cot\theta \right\} \qquad (16)$$

$$u_1 \frac{\partial v'}{\partial\xi} + \delta u' \frac{\partial v_1}{\partial\xi} + \delta f u' \frac{\partial v'}{\partial\xi} + \frac{\delta}{(1 + \delta\xi)} \left[v_1 \frac{\partial v'}{\partial\theta} + v' \frac{\partial v_1}{\partial\theta} + f v' \frac{\partial v'}{\partial\theta} \right.$$

$$\left. + u_1 v' + \delta v_1 u' + \delta f u' v' \right]$$

$$= -\frac{\delta}{(1 + \delta\xi)} \frac{\partial p'}{\partial\theta} + \frac{1}{\delta Re} \frac{1}{(1 + \delta\xi)^2} \left\{ \frac{\partial}{\partial\xi} \left[(1 + \delta\xi)^2 \frac{\partial v'}{\partial\xi} \right] \right.$$

$$\left. + \frac{\delta^2}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial v'}{\partial\theta} \right) + 2\delta^3 \frac{\partial u'}{\partial\theta} - \delta^2 \frac{v'}{\sin^2\theta} \right\} \qquad (17)$$

$$v_\infty + (u_1 + \delta f u') \frac{\partial T}{\partial r} + \frac{1}{r} (v_1 + f v') \frac{\partial T}{\partial\theta} = \frac{1}{Pr\ Re} \nabla^2 T \qquad (18)$$

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where

$$Re = \frac{V_R R_1}{\nu}, \qquad v_\infty = \frac{V_\infty}{V_R}$$

and

$$u_1(\xi,\theta) = - \frac{\delta v_\infty \cos\theta}{(1 + \delta\xi)^3} [3\xi + 3\delta\xi^2 + \delta^2\xi^3]$$

$$v_1(\xi,\theta) = v_\infty \sin\theta \left[1 + \frac{1}{2(1 + \delta\xi)^3}\right] \tag{19}$$

The shear stress condition (eq. (7)) becomes

$$\frac{f}{\delta} \frac{\partial}{\partial\xi}\left(\frac{v'}{1 + \delta\xi}\right)_{\xi=0} = \frac{(-\sigma_T)AR_1}{\mu V_R} \frac{\partial T}{\partial\theta} + 3v_\infty \sin\theta \tag{20}$$

The mechanical energy condition (eq. (12)) becomes

$$\frac{(-\sigma_T)AR_1}{\mu V_R} \int_0^\pi - \left[(v_1 + fv')\frac{\partial T}{\partial\theta}\right]_{\xi=0} \sin\theta \, d\theta$$

$$= \frac{f^2}{\delta} \int_0^\pi \int_0^\infty \left\{\frac{\partial}{\partial\xi}[(1 + \delta\xi)v'] - \delta\frac{\partial u'}{\partial\theta}\right\}^2 \sin\theta \, d\theta \, d\xi$$

$$+ 2\int_0^\pi \left[v_1^2 + 2fv_1v' + f^2v'^2\right]_{\xi=0} \sin\theta \, d\theta \tag{21}$$

Three conditions are needed to determine the unknowns δ, f, and V_R. These are described below

(1) Inertia and viscous forces are of the same order of magnitude in the boundary layer. Hence $u_1 \frac{\partial v'}{\partial\xi} \sim \frac{1}{\delta Re} \frac{1}{(1 + \delta\xi)^2} \frac{\partial}{\partial\xi}\left[(1 + \delta\xi)^2 \frac{\partial v'}{\partial\xi}\right]$.

Since $u_1 \sim \delta$ for $\xi \sim 1$, $\delta \sim \frac{1}{\delta Re} \Rightarrow \delta \sim \frac{1}{\sqrt{Re}}$

(2) The left-hand side of the mechanical energy condition (eq. (21)) represents the rate of work due to surface tension forces. The first term on the right-hand side represents the energy dissipated in the boundary layer (since curl $\underset{\sim}{V} = 0$ outside the boundary layer) and the second term represents the energy dissipated by the potential flow and an additional boundary layer

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contribution (note that the flow considered here is potential because it is irrotational and not because it is inviscid; hence the potential flow does dissipate energy). We expect $f \sim 0(1)$ or less and $\delta \ll 1$. Hence f^2/δ represents the largest portions of the energy dissipated in the boundary layer. Hence

$$\frac{(-\sigma_T)AR_1}{\mu V_R} \sim \text{ larger of } \left[\frac{f^2}{\delta} \text{ and } 1\right].$$

(3) Similarly, from the shear stress condition,

$$\frac{f}{\delta} \sim \text{ larger of } \left[\frac{(-\sigma_T)AR_1}{\mu V_R} \text{ and } 1\right]$$

It is assumed that $(\partial T/\partial \theta)_{r=1} \sim 1$. Physically, this means that the temperature between the two stagnation points on the bubble surface is of order AR_1. Solving these three conditions for δ, f, and V_R, we find that there are two possibilities

$$(I) \qquad\qquad \delta = f \sim \frac{1}{\sqrt{R_\sigma}}, \qquad V_R \sim \frac{(-\sigma_T)AR_1}{\mu} \qquad\qquad (22)$$

Since δ must be small compared to one, $R_\sigma \gg 1$.

$$(II) \qquad\qquad \delta \sim \frac{1}{R_\sigma^{1/3}}, \qquad f \sim 1, \qquad V_R \sim \left[\frac{(-\sigma_T)^2 A^2 R_1 v}{\mu^2}\right]^{1/3} \qquad\qquad (23)$$

Again, $\delta \ll 1 \Rightarrow R_\sigma \gg 1$.

For the first possibility, $f^2/\delta = 1/\sqrt{R_\sigma}$ which is small compared to one. Hence all the work that is done by the surface tension forces is dissipated by the potential flow, with negligible amounts dissipated in the boundary layer. For the second possibility, the converse is true and hence the predominant dissipation of energy occurs within the boundary layer, with negligible amounts dissipated outside it by the potential flow. Also, according to the first possibility, the correction velocity within the boundary layer is small compared to the potential flow velocity, whereas, according to the second possibility, the two are comparable. Crespo and Manuel assumed that possibility I is correct, as they neglected energy dissipation in the boundary layer, referring to arguments by Levich (ref. 8, Ch 7, § 82) for justification. The velocity scale in the second possibility is the one frequently used in literature for thermocapillary flows with large R_σ and was first obtained by Ostrach (ref. 11).

So far, the author has not been able to produce an argument which tells a priori which of the two possibilities is the one that is physically correct and is chosen by nature. Therefore, both possibilities were pursued. It was found that the second possibility, analyzed by a boundary layer integral method similar to the Karman-Pohlhausen method, was never able to predict any terminal

velocity. While the scaling analysis for possibility II revealed that all the work done by the surface tension forces could be dissipated within the boundary layer, calculations by the integral method revealed that for any terminal velocity, the boundary layer dissipation was always less than (and hence could never equal) the rate of work done by the surface tension forces. Thus possibility II was rejected. Therefore, in what follows, possibility I is chosen to be physically correct and will be pursued in the remainder of this work.

Using equation (22) in the energy equation (eq. (18)) and the mechanical energy condition (eq. (21)), we may neglect terms containing u' and v', for large R_σ. The following simplified problem for T and v_∞ is then obtained

$$v_\infty + u_1 \frac{\partial T}{\partial r} + \frac{v_1}{r} \frac{\partial T}{\partial \theta} = \frac{1}{Ma} \nabla^2 T \tag{24}$$

$$\text{At } r = 1, \qquad \frac{\partial T}{\partial r} = 0 \tag{25}$$

$$\text{As } r \to \infty, \qquad T \to r \cos \theta \tag{26}$$

$$-\int_0^\pi \left(v_1 \frac{\partial T}{\partial \theta}\right)_{r=1} \sin \theta \, d\theta = 2 \int_0^\pi (v_1^2)_{r=1} \sin \theta \, d\theta \tag{27}$$

where

$$u_1 = -v_\infty \cos \theta \left(1 - \frac{1}{r^3}\right) \text{ and } v_1 = v_\infty \sin \theta \left(1 + \frac{1}{2r^3}\right) \tag{28}$$

Solution

The above equation for T, equation (24), though linear, is not easy to solve. Crespo and Manuel (ref. 7) neglected the right-hand side of equation (24) for large Ma, yielding

$$v_\infty + u_1 \frac{\partial T}{\partial r} + \frac{v_1}{r} \frac{\partial T}{\partial \theta} = 0 \tag{29}$$

Evaluating this equation at $r = 1$, they obtained

$$\left(v_1 \frac{\partial T}{\partial \theta}\right)_{r=1} = -v_\infty \tag{30}$$

Using this in equation (27), the terminal velocity v_∞ is obtained as

$$v_\infty = \frac{1}{3} \tag{31}$$

Thus, they were able to calculate the terminal velocity without solving for the temperature field. However, equation (29) is not strictly valid as it cannot

satisfy equation (25) at the bubble surface. Therefore, a thermal boundary layer must be present near the bubble surface. Also equation (30) is incorrect, especially at the stagnation points, where the left-hand side is zero. We will solve equation (24) by an approximate integral method. Equation (24) may be integrated with respect to r to yield

$$\int_1^\infty [r(v_1 + \delta v')T \sin \theta - v_\infty r^2 \cos \theta \sin^2\theta]\, dr + \frac{v_\infty}{3}(\cos^3\theta - C_1)$$

$$= \frac{1}{Ma}\left[\frac{\sin^2\theta}{2} + \int_1^\infty \left(\sin \theta \frac{\partial T}{\partial \theta} + r \sin^2\theta\right)dr\right] \qquad (32)$$

Evaluating this equation at $\theta = 0$ and assuming that the integrals are zero, $C_1 = 1$. However, at $\theta = \pi$, the second term is nonvanishing. Hence, one of the integrals must be finite at $\theta = \pi$. Physically, there is a thermal wake behind the bubble, for, as the bubble moves into warmer regions, it displaces the warmer fluid. The displaced energy has to be convected downstream in a thermal wake. The integral in the left-hand side of equation (32) must be nonzero at $\theta = \pi$, to account for the energy in the wake.

The following temperature distribution is assumed

$$T(r,\theta) = r \cos \theta - \frac{1}{2}\frac{\cos \theta}{r^2} + \frac{1}{r^3}\frac{1}{(\lambda - 3)}\left[\lambda T_s(\theta) - \frac{(\lambda + 4)}{2} \cos \theta\right]$$

$$+ \frac{1}{(\lambda - 3)}\left[\frac{7}{2} \cos \theta - 3T_s(\theta)\right]e^{-\lambda(\theta)(r-1)} \qquad (33)$$

where $T_s(\theta) = T(1,\theta)$ is the transformed steady temperature at the bubble surface. T_s and λ are unknowns to be determined. The assumed temperature distribution satisfies the boundary conditions (eqs. (25) and (26)) and makes the integrals in equation (32) finite. The exponential function represents a thermal boundary layer (λ is the inverse of the boundary layer thickness) and also accommodates the energy in the wake. The remaining terms in equation (33) represent the temperature distribution outside the boundary layer.

Equations (32) and (24) evaluated at $r = 1$ are used to determine T_s and λ.

$$\frac{9}{8(\lambda - 3)}\left[\lambda T_s - \frac{(\lambda + 4)}{2} \cos \theta\right] - \frac{\cos \theta}{12} + \frac{(\lambda + 1)}{\lambda^2(\lambda - 3)}\left(\frac{7}{2} \cos \theta - 3T_s\right)$$

$$+ \left(\frac{7}{2} \cos \theta - 3T_s\right)\frac{[1 - \lambda e^\lambda Ei(\lambda)]}{2(\lambda - 3)} + \frac{1}{3}\frac{(\cos^3\theta - 1)}{\sin^2\theta} = 0 \qquad (34)$$

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$$V_\infty + \frac{3}{2} V_\infty \sin \theta \, T_s' = \frac{1}{Ma} \left\{ -3 \cos \theta + \frac{12}{(\lambda - 3)} \left[\lambda T_s - \frac{(\lambda + 4)}{2} \cos \theta \right] \right.$$

$$\left. + \frac{\lambda^2}{(\lambda - 3)} \left[\frac{7}{2} \cos \theta - 3 T_s \right] + T_s'' + \cot \theta \, T_s' \right\} \qquad (35)$$

$$T_s'(0) = T_s'(\pi) = 0 \quad \text{from symmetry conditions.} \qquad (36)$$

where $T_s' = dT_s/d\theta$ and $T_s'' = d^2T_s/d\theta^2$. $1/Ma$ terms in equation (34) have been neglected. They are retained in equation (35), as without them, the equation is singular at $\theta = 0$ and π. This system is still not easy to solve. Hence, we will get a solution by satisfying the equations only at $\theta = 0$ and $\pi/2$. It is difficult to satisfy the equations at $\theta = \pi$, as T_s'' is steep there and will not be attempted.

T_s is chosen to be

$$T_s(\theta) = a + b \left(\frac{\theta}{\pi} \right)^2 + c \left(\frac{\theta}{\pi} \right)^3 \qquad (37)$$

This satisfies the condition $T_s'(0) = 0$. The six unknowns a, b, c, $\lambda(0)$, $\lambda(\pi/2)$ and V_∞ are solved from equations (34) and (35) evaluated at $\theta = 0$ and $\pi/2$, the condition $T_s'(\pi) = 0$ and the mechanical energy condition (eq. (27)).

The solution for large Ma is

$$T_s(0) = a = \frac{55}{54}$$

$$b = -\frac{3c}{2} = -\frac{13}{3}$$

$$\lambda(0) = 0.53 \, Ma$$

$$\lambda\left(\frac{\pi}{2}\right) = 0.009 \, Ma \qquad\qquad\qquad (38)$$

$$V_\infty = \frac{13(\pi^2 + 3)}{72\pi^2} = 0.235$$

$$\Delta T_s = T_s(0) - T_s(\pi) = 1.44$$

RESULTS AND DISCUSSION

The results for the terminal velocity, viz, $V_\infty = 0.235 \, (-\sigma_T)AR_1/\mu$ is lower than that obtained by Crespo and Manuel, viz, $V_\infty = 1/3((-\sigma_T)AR_1/\mu)$. However, even though their approach is not fully justified, both results are of the same order of magnitude and only differ in the multiplicative constant in the expression for V_∞. The main difference is that in the present study, the thermal boundary layer has been treated more completely in arriving at the

result for the terminal velocity. For large Ma, v_∞ is independent of the Marangoni number. The result for v_∞ also supports the scaling analysis that has been performed, as the estimate for v_∞ is modified by a multiplicative constant that is only of unit order of magnitude. The temperature on the bubble surface at the front stagnation point is $T_s(0) = 1.01$ and is only slightly different from the temperature of the free stream ($T = 1$) at the same axial location. The thickness of the thermal boundary layer varies as 1/Ma. The boundary layer has a small thickness (1.89/Ma) at $\theta = 0$, is about 58 times thicker at $\theta = \pi/2$ and is infinitely thick (from eq. (34)) at $\theta = \pi$. An interesting conclusion that is supported by the present analysis is the fact that Ma is a singular perturbation parameter for this problem, which has been recognized and considered by Subramanian (ref. 5). For Ma = 0, the solution to T is (ref. 6)

$$T(r,\theta) = r \cos \theta + \frac{1}{2} \frac{\cos \theta}{r^2} \qquad (39)$$

Comparing this to $T(r,\theta)$ in equation (33), we see that the sign of the second term is reversed. The coefficient -1/2 for this term in equation (33) was determined by requiring that the integral in the left-hand side of equation (32) be finite, i.e., that the motion of the bubble does not create an infinite flux of convected energy. Hence, the coefficient of this term must be -1/2 for any nonzero Ma, as otherwise, the flux of energy convected would be infinite. Since the coefficient is +1/2 for Ma = 0, we conclude that this problem is singular with respect to perturbations in Ma (i.e., the inclusion of energy convection terms), as the presence of convection drastically changes the nature of the temperature distribution as $r \to \infty$.

For small Marangoni, Weber, and Capillary numbers and small $\Delta\sigma/\sigma$, the shape of the bubble was obtained in reference 6 to be

$$\eta(\theta) = -\frac{15}{64} \frac{v_\infty^2 R_1 \rho}{\sigma_0} (3 \cos^2\theta - 1) \qquad (40)$$

where the bubble surface is located at $r = 1 + \eta$. The shape of the bubble represented by equation (40) is a spheroid with its minor axis in the flow direction. The same result is also expected to be valid for the large R_σ and Ma that is being considered in the present study for small Wb, Ca, and $\Delta\sigma/\sigma$, because the flow boundary layer is thin and introduces only small changes to the velocity and pressure fields, compared to these fields in potential flow. Since potential flow fields were used in reference 6 to obtain equation (40), the shape of the bubble for the two cases must be the same. This result is not expected to be valid in the vicinity of the rear stagnation point of the bubble, as we expect the boundary layer thickness to be infinite there.

REFERENCES

1. Young, N.O.; Goldstein, J.S.; and Block, M.J.: The Motion of Bubbles in a Vertical Temperature Gradient. J. Fluid Mech., vol. 6, pt. 3, Oct. 1959, pp. 350-356.

2. Bratukhin, Y.K.: Thermocapillary Drift of a Droplet of Viscous Fluid. Fluid Dyn., vol. 10, no. 5, Sept.-Oct. 1975, pp. 833-837.

3. Thompson, R.L.: Marangoni Bubble Motion in Zero Gravity. NASA TM-79264, 1979.

4. Thompson, R.L.; DeWitt, K.J.; and Labus, T.L.: Marangoni Bubble Motion Phenomenon in Zero Gravity. Chem. Eng. Commun., vol. 5, no. 5-6, 1980, pp. 299-314.

5. Subramanian, R.S.: Slow Migration of a Gas Bubble in a Thermal Gradient. AIChE J., vol. 27, no. 4, July 1981, pp. 646-654.

6. Balasubramaniam, R. and Chai, A.T.: Thermocapillary Migration of Droplets: An Exact Solution for Small Marangoni Numbers. J. Colloid Interface Sci., 1987, to be published.

7. Crespo, A. and Manuel, F.: Bubble Motion Under Reduced Gravity. Materials Sciences Under Microgravity, ESA-SP-191, T.D. Guyene and J. Hunt, eds., European Space Agency, 1983, pp. 45-49.

8. Levich, V.G.: Physicochemical Hydrodynamics. Prentice Hall, 1962.

9. Milne-Thomson, L.M.: Theoretical Hydrodynamics, 3rd ed., MacMillan, 1955.

10. Moore, D.W.: The Boundary Layer on a Spherical Gas Bubble. J. Fluid Mech., vol. 16, pt. 2, June 1963, pp. 161-176.

11. Ostrach, S.: Low-Gravity Fluid Flows. Annual Review of Fluid Mechanics, Vol. 14, M. Van Dyke, J.V. Wehausen, and J.L. Lumley, eds., Annual Reviews Inc., Palo Alto, 1982, pp. 313-345.
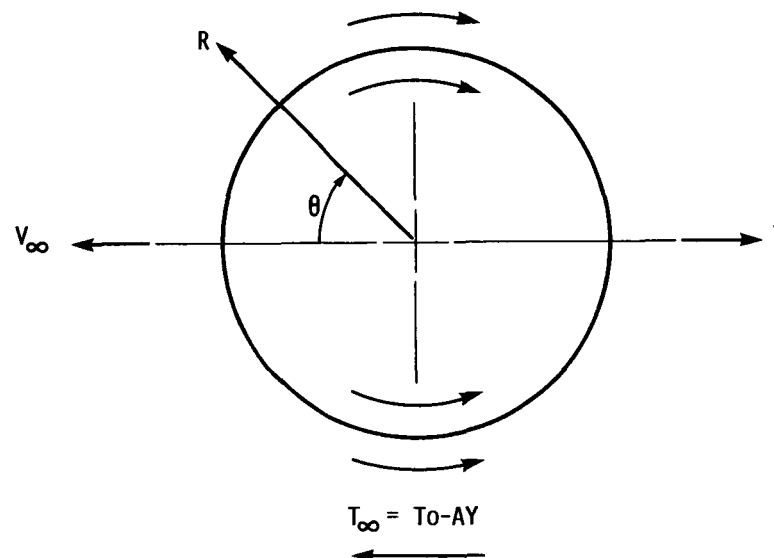
$$T_\infty = T_0 - AY$$

FIGURE 1. - SKETCH OF A MIGRATING DROPLET.

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16. Abstract

The thermocapillary motion of spherical bubbles present in an unbounded liquid with a linear temperature distribution is analyzed, when the Reynolds number and the Marangoni number are large. Previous calculations of the terminal velocity performed for this parametric range did not take into complete consideration the thermal boundary layer present near the surface of the bubble. In the present study, a scaling analysis is presented for this problem. The thermal boundary layer is analyzed by an integral method. The resulting terminal velocity is lower than the one previously calculated, though it is of the same order of magnitude.

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