

**NASA Contractor Report 178267**

**ICASE REPORT NO. 87-15**

# ICASE

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**Contract No. NAS1-18107**

**March 1987**

**(NASA-CR-178267) A SURVEY OF MIXED FINITE  
ELEMENT METHODS Final Report (NASA) 33 p  
Avail: NTIS HC AC3/MF A01 CSCL 12A**

**N87-23229**

**Unclas  
G3/64 0077635**

**INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING  
NASA Langley Research Center, Hampton, Virginia 23665**

**Operated by the Universities Space Research Association**



**National Aeronautics and  
Space Administration**

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# **A SURVEY OF MIXED FINITE ELEMENT METHODS**

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## **ABSTRACT**

This paper is an introduction to and an overview of mixed finite element methods. It discusses the mixed formulation of certain basic problems in elasticity and hydrodynamics. It also discusses special techniques for solving the discrete problem.

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This research was supported under NASA Contract No. NAS1-18107 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

## 1. INTRODUCTION

The aim of this paper is to present an introductory survey of mixed finite element methods. We shall deal first with the so-called mixed formulation of some problems arising in elasticity and hydrodynamics. Then we shall analyze the difficulties connected with the choice of appropriate finite element discretizations for a mixed formulation. Finally, we shall discuss some special techniques that are often helpful for solving the discretized problem.

The notation of Ciarlet [17] is followed throughout.

## 2. MIXED FORMULATIONS

A precise and satisfactory definition of "mixed method" (or of "mixed formulation") does not exist. The term started in the engineering literature (Herrmann [33], [34]; Hellan [32]) in connection with the elasticity theory to denote methods based on the Hellinger-Reissner principle, in which both displacements and stresses were approximated simultaneously. Even among the mathematicians, such as Glowinski [29], Babuska [6], Crouzeix-Raviart [18], Johnson [36], whose work can now be considered as pioneering, the term "mixed" was used only by Johnson in the context of plate bending problems. The term is now used in a much wider sense and has become rather vague. Here we will live with such vagueness, and we shall not try a new unsatisfactory definition. Instead, we present a few cases: linear elliptic problems, Stokes equations for incompressible fluids, and linear elasticity problems. The first case will be dealt with in more detail because it is formally much simpler, while only a few essential points will be stressed for the other two cases.

Example 1. Linear elliptic operators

The use of mixed formulations for linear elliptic operators is rather recent and, as we shall see, is recommended only in some special cases. However, its presentation is very simple and this makes it an ideal example. Consider the model problem:

$$(2.1) \quad \operatorname{div}(A(\underline{x}) \underline{\operatorname{grad}} u) = f \quad \text{in } D \subset \mathbb{R}^d,$$

$$(2.2) \quad (A(\underline{x}) \underline{\operatorname{grad}} u) \cdot \underline{n} = g_1 \quad \text{on } \Gamma_{\text{Neu}},$$

$$(2.3) \quad u = g_0 \quad \text{on } \Gamma_{\text{Dir}},$$

where: i)  $\Gamma_{\text{Dir}} \cup \Gamma_{\text{Neu}} = \Gamma = \partial D$  is a splitting of  $\partial D$ , the boundary of the domain  $D$ , ii)  $A(\underline{x})$  is a smooth function on  $\bar{D}$ , with  $A(\underline{x}) \geq \alpha > 0$  for every  $\underline{x}$  in  $D$ , iii)  $\underline{n}$  is the unit outward normal to  $\partial D$ , and iv)  $f, g_1, g_0$  are given smooth functions in  $D$  and on  $\Gamma_{\text{Neu}}, \Gamma_{\text{Dir}}$  respectively. Introducing for  $g = g_0$  or  $g = 0$  the manifold

$$H_g^1(D) = \{v \mid v \in H^1(D); v = g \text{ on } \Gamma_{\text{Dir}}\},$$

we can write the variational formulation of (2.1) - (2.3) as follows:

$$\text{find } u \in H_g^1 \quad \text{such that}$$

$$\int_D A(\underline{x}) \underline{\operatorname{grad}} u \cdot \underline{\operatorname{grad}} v \, d\underline{x} = - \int_D f v \, d\underline{x} + \int_{\Gamma_{\text{Neu}}} g_1 v d\Gamma \quad \forall v \in H_0^1(D).$$

In order to derive the mixed formulation of (2.1) - (2.3) we introduce the variable:

$$(2.4) \quad \underline{p} = A(\underline{x}) \underline{\text{grad}} u \quad \text{in } D,$$

so that (2.1) and (2.2) become, respectively:

$$(2.5) \quad \text{div } \underline{p} = f \quad \text{in } D,$$

$$(2.6) \quad \underline{p} \cdot \underline{n} = g_1 \quad \text{on } \Gamma_{\text{Neu}}.$$

The formulation (2.3) - (2.6) is often called the mixed formulation of (2.1) - (2.3). Two reasonable variational formulations for (2.3) - (2.6) are now possible. The first one is:

find  $u \in H_{g_0}^1(D)$  and  $\underline{p} \in (L^2(D))^d$  such that:

$$(2.7) \quad \int_D (A(\underline{x}))^{-1} \underline{p} \cdot \underline{q} \, d\underline{x} - \int_D \underline{q} \cdot \underline{\text{grad}} u \, d\underline{x} = 0 \quad \forall \underline{q} \in (L^2(D))^d,$$

$$(2.8) \quad - \int_D \underline{p} \cdot \underline{\text{grad}} v \, d\underline{x} = \int_D f v \, d\underline{x} - \int_{\Gamma_{\text{Neu}}} g_1 v \, d\Gamma \quad \forall v \in H_0^1(D).$$

In order to introduce the second variational formulation of (2.3) - (2.6) we define, for  $g = g_1$  or  $g = 0$ , the manifold:

$$H_g(\text{div}; D) = \{ \underline{q} \mid \underline{q} \in (L^2(D))^d; \text{div } \underline{q} \in L^2(D); \underline{q} \cdot \underline{n} = g \text{ on } \Gamma_{\text{Neu}} \}.$$

The second variational formulation of (2.3) - (2.6) is now:

find  $u \in L^2(D)$  and  $\underline{p} \in H_{g_1}(\text{div}; D)$  such that:

$$(2.9) \int_D (A(\underline{x}))^{-1} \underline{p} \cdot \underline{q} \, d\underline{x} + \int_D u \, \text{div} \, \underline{q} \, d\underline{x} = \int_{\Gamma_{\text{Dir}}} g_0 \underline{q} \cdot \underline{n} \, d\Gamma \quad \forall \underline{q} \in H_0(\text{div}; D),$$

$$(2.10) \quad \int_D v \, \text{div} \, \underline{p} \, d\underline{x} = \int_D f v \, d\underline{x} \quad \forall v \in L^2(D).$$

The difference between (2.7) - (2.8) and (2.9) - (2.10) is clearly a simple integration by parts (or, if you prefer, a Green's formula). However, it must be pointed out that the regularity required a priori for  $u$  and  $\underline{p}$  is somehow interchanged. This implies that, in discretizing (2.7) - (2.8), one has to use a continuous " $u$ " and can use a discontinuous " $\underline{p}$ ", while in discretizing (2.9) - (2.10) one can use a discontinuous " $u$ " but must use a " $\underline{p}$ " with divergence in  $L^2(D)$  (and hence  $\underline{p} \cdot \underline{n}$  continuous at the interfaces). Note also the inversion in the treatment of the boundary conditions.

It is questionable whether (2.7) - (2.8) should be called a mixed formulation for problem (2.1) - (2.3). On the other hand, it seems acceptable to call the formulation (2.9) - (2.10) "mixed." In general, the original formulation (2.1) - (2.3) is to be preferred. It is simpler, it uses just one variable, and many robust methods are based on this approximation. However, in some applications, the "auxiliary" unknown  $\underline{p}$  defined in (2.4) is actually the more relevant physical variable and/or is the only information that has to be transferred into other equations that are coupled with (2.1) - (2.3). In such cases, the use of a mixed formulation might be preferred, as far as it

provides (as it often does) a better accuracy for  $\underline{p}$ . In general, the formulation (2.9) - (2.10) is then used, since it deals with a smoother vector field  $\underline{p}$ . It is often said that the crucial feature in the mixed approach is that it averages  $(A(\underline{x}))^{-1}$  instead of  $A(\underline{x})$ . This is surely a better thing to do, at least in one dimension, being connected with the homogenization theory; see for instance Babuska-Osborn [7]. However, dramatic improvements have been obtained by using (2.9) - (2.10) with a constant  $A(\underline{x})$ ; see for instance Marini-Savini [40]. The true reason (if any) for the better behavior of the mixed formulations over the classical ones is still not completely understood. Practical experiences suggest the use of a mixed formulation for "bad behaved" problems, in which the variable  $\underline{p}(\underline{x})$  is expected to be "smoother" than the variable  $u(\underline{x})$ , but clearly this is not the whole story.

#### Example 2. Incompressible fluids

The Stokes equations for incompressible fluids are of the type:

$$(2.11) \quad -\Delta \underline{u} + \underline{\text{grad}} \, p = \underline{f} \quad \text{in } D \subset \mathbb{R}^d$$

$$(2.12) \quad \text{div } \underline{u} = 0 \quad \text{in } D.$$

Various kinds of boundary conditions can be used in connection with (2.11), (2.12). For the sake of simplicity, we shall consider only the (physically uninteresting) Dirichlet boundary conditions

$$(2.13) \quad \underline{u} = 0 \quad \text{on } \Gamma = \partial D.$$

The natural variational formulation of (2.11) - (2.13) is:

find  $\underline{u} \in (H_0^1(D))^d$  and  $p \in L^2(D)$  such that:

$$(2.14) \quad \int_D \underline{\text{grad}} \underline{u} : \underline{\text{grad}} \underline{v} \, d\underline{x} - \int_D p \, \text{div} \, \underline{v} \, d\underline{x} = \int_D \underline{f} \cdot \underline{v} \, d\underline{x} \quad \forall \underline{v} \in (H_0^1(D))^d,$$

$$(2.15) \quad \int_D q \, \text{div} \, \underline{u} \, d\underline{x} = 0 \quad \forall q \in L^2(D).$$

The formulation (2.14) - (2.15) had been used for years before the term "mixed method" came into use; however, it is recognized now that (2.14) - (2.15) behaves like a mixed formulation as far as the difficulties in finding good approximations are concerned. We shall also see that (2.14) - (2.15) easily falls into the same abstract framework that is commonly used for mixed methods. Hence, we are somehow allowed to consider (2.14) - (2.15) as a mixed formulation.

### Example 3. Linear elasticity problems

For a vector valued function  $\underline{v}(\underline{x})$  we define  $\underline{\epsilon}(\underline{v})$  by:

$$(2.16) \quad \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (i, j = 1, \dots, d).$$

The linear elasticity equations are now:

$$(2.17) \quad \underline{\sigma} = E : \underline{\epsilon}(\underline{u}) \quad (\text{i.e., } \sigma_{ij} = \sum_{r=1}^d \sum_{s=1}^d E_{ijrs} \, \epsilon_{rs}(\underline{u})),$$

$$(2.18) \quad \text{div} \, \underline{\sigma} = \underline{f} \quad \text{in } D.$$



Substituting (2.16), (2.17) into (2.18) gives a second order linear elliptic system in the unknowns  $\underline{u}$ . Clearly,  $E$  is the elasticity tensor and is assumed here to have constant coefficients (and nice "ellipticity" properties). Its inverse (compliance tensor) will be denoted by  $C$ . Hence:

$$(2.19) \quad \underline{\tau} = E:\underline{\varepsilon}(v) \iff \underline{\varepsilon}(v) = C:\underline{\tau}.$$

We assume again the simple boundary conditions:

$$(2.20) \quad u = 0 \quad \text{on } \partial D.$$

This, of course, is strongly unrealistic: usually one has  $\underline{u} = \tilde{\underline{u}}$  given on  $\Gamma_{Dir}$  and  $\underline{\sigma} \cdot \underline{n} = \underline{t}$  given on  $\Gamma_{Neu}$ . However, the proper way of dealing with realistic boundary conditions coincides with the one used in Example 1: we chose then to give more details (with simpler notations) while discussing the linear elliptic problem.

One can notice here that the splitting of the problem in more than one unknown is natural with solid physical reasons. This is probably why the first mixed formulations were used in elasticity theory. We shall present here only one mixed formulation, which is similar to the formulation (2.9) - (2.10) for a single elliptic equation. We set:

$$H(\underline{\text{div}}; D) = \{ \underline{\tau} \mid \underline{\tau} \in (L^2(D))^{d^2}; \tau_{ij} = \tau_{ji} \quad \forall i, j; \underline{\text{div}} \underline{\tau} \in (L^2(D))^d \}$$

and we consider the problem:

Find  $\underline{u} \in (L^2(D))^d$  and  $\underline{\sigma} \in H(\text{div}; D)$  such that

$$(2.21) \quad \int_D (C:\underline{\sigma}):\underline{\tau} \, d\mathbf{x} + \int_D \underline{u} \cdot \text{div} \, \underline{\tau} \, d\mathbf{x} = 0 \quad \forall \, \underline{\tau} \in H(\text{div}; D),$$

$$(2.22) \quad \int_D \underline{v} \cdot \text{div} \, \underline{\sigma} \, d\mathbf{x} = \int_D \underline{f} \cdot \underline{v} \, d\mathbf{x} \quad \forall \, \underline{v} \in (L^2(D))^d.$$

One can see that (2.21) - (2.22) practically coincide with the variational formulation of the Hellinger-Reissner principle. The use of this principle in the framework of finite elements can be traced back to the pioneering work of Herrmann [33], [34], and Hellan [32]. The interest in using the stress field  $\underline{\sigma}$  as an independent variable is questionable in as simple a case as the present one, but its use is clear in more general and more complicated problems involving nonlinearities, plasticity and so on.

We shall now state an abstract existence theorem that is a simplified version of a more general result proved in [11].

**Theorem 1:** Let  $E$  and  $\Psi$  be real Hilbert spaces,  $a(\xi_1, \xi_2)$  a bilinear form on  $E \times E$  and  $b(\xi, \psi)$  a bilinear form on  $E \times \Psi$ . Set:

$$K = \{\xi \mid \xi \in E, b(\xi, \psi) = 0 \quad \forall \, \psi \in \Psi\},$$

and assume that

$$(2.23) \quad \exists \, \alpha > 0 \quad \text{s.t.} \quad a(\xi, \xi) \geq \alpha \|\xi\|_E^2 \quad \forall \, \xi \in K,$$

$$(2.24) \quad \exists \, \beta > 0 \quad \text{s.t.} \quad \sup_{\xi \in E - \{0\}} \frac{b(\xi, \psi)}{\|\xi\|_E} \geq \beta \|\psi\|_\Psi \quad \forall \, \psi \in \Psi.$$

Then for every  $\ell_1 \in \Xi'$  and  $\ell_2 \in \Psi'$  there exists a unique solution  $(\bar{\xi}, \bar{\psi})$  of the problem:

$$(2.25) \quad a(\bar{\xi}, \xi) + b(\xi, \bar{\psi}) = \langle \ell_1, \xi \rangle \quad \forall \xi \in \Xi,$$

$$(2.26) \quad b(\bar{\xi}, \psi) = \langle \ell_2, \psi \rangle \quad \forall \psi \in \Psi. \quad \blacksquare$$

Remark: Actually a stronger result is proved in [11]. Namely: {problem (2.25), (2.26) has a unique solution for every  $\ell_1 \in \Xi'$  and  $\ell_2 \in \Psi'$ } if and only if {(2.24) holds and the bilinear form  $a(\xi_1, \xi_2)$ , restricted to  $K$ , is nonsingular (in the sense that it induces an isomorphism from  $K$  on  $K'$ )}. Clearly, if one assumes that  $a(\xi_1, \xi_2)$  is symmetric and positive semidefinite, then (2.23) and (2.24) are necessary and sufficient for the existence and uniqueness of the solution of (2.25) - (2.26).

Remark: It is clear that, if  $a(\xi_1, \xi_2)$  is symmetric, the solution  $(\bar{\xi}, \bar{\psi})$  of (2.25) - (2.26) minimizes the functional

$$(2.27) \quad J(\xi) = \frac{1}{2} a(\xi, \xi) - \langle \ell_1, \xi \rangle,$$

on the subspace of  $\Xi$ :

$$(2.28) \quad K(\ell_2) = \{\xi \mid b(\xi, \psi) = \langle \ell_2, \psi \rangle \quad \forall \psi \in \Psi\},$$

and the formulation (2.25) - (2.26) corresponds to the introduction in (2.27) - (2.28) of the Lagrange multiplier  $\bar{\psi}$ .

### 3. DISCRETIZING A MIXED FORMULATION

Let us deal first with the abstract framework (2.25) - (2.26). Assume that we are given two sequences  $\{\Xi_h\}_{h>0}$  and  $\{\Psi_h\}_{h>0}$  of subspaces of  $\Xi$  and  $\Psi$  respectively. We set

$$(3.1) \quad K_h = \{\xi_h \mid \xi_h \in \Xi_h, b(\xi_h, \psi_h) = 0 \quad \forall \psi_h \in \Psi_h\}.$$

We have the following approximation theorem ([11]).

**Theorem 2:** Assume that

$$(3.2) \quad \alpha_h > 0 \quad \text{s.t.} \quad a(\xi, \xi) \geq \alpha_h \|\xi\|_{\Xi}^2 \quad \forall \xi \in K_h,$$

$$(3.3) \quad \beta_h > 0 \quad \text{s.t.} \quad \sup_{\xi \in \Xi_h - \{0\}} \frac{b(\xi, \psi)}{\|\xi\|_{\Xi}} \geq \beta_h \|\psi\|_{\Psi} \quad \forall \psi \in \Psi_h.$$

Then for every  $\ell_1 \in \Xi'$  and  $\ell_2 \in \Psi'$  and for every  $h > 0$  the discrete problem:

$$(3.4) \quad a(\bar{\xi}_h, \xi) + b(\bar{\xi}_h, \bar{\psi}_h) = \langle \ell_1, \xi \rangle \quad \forall \xi \in \Xi,$$

$$(3.5) \quad b(\bar{\xi}_h, \psi) = \langle \ell_2, \psi \rangle \quad \forall \psi \in \Psi_h,$$

has a unique solution. Moreover, there exists a constant  $\gamma_h(\alpha_h, \beta_h) > 0$  such that

$$(3.6) \quad \|\bar{\xi} - \bar{\xi}_h\|_{\Xi} + \|\bar{\psi} - \bar{\psi}_h\|_{\Psi} \leq \gamma_h \left( \inf_{\xi_h \in \Xi_h} \|\bar{\xi} - \xi_h\|_{\Xi} + \inf_{\psi_h \in \Psi_h} \|\bar{\psi} - \psi_h\|_{\Psi} \right).$$

The dependence of  $\gamma_h$  on  $\alpha_h$  and  $\beta_h$  can be easily traced (see [11]). Clearly if (3.2) and (3.3) hold with constants  $\bar{\alpha}, \bar{\beta}$  independent of  $h$ , then (3.6) holds with a constant  $\bar{\gamma}$  independent of  $h$ . More general versions of Theorem 2 (and also of Theorem 1) can be found for instance in Falk-Osborn [21] or in Bernardi-Canuto-Maday [9].

We are now going to see the implications of Theorem 2 in the examples of the previous section.

Example 1h. Discretizations of the mixed formulations for linear elliptic operators.

Many examples of successful discretizations of (2.9) - (2.10) are known. The first ones were introduced by Raviart and Thomas [44] and then elaborated and extended to more general cases by Nedelec [41]. Other families of possible discretizations were introduced years later by Brezzi, Douglas, and Marini [15] and then elaborated and extended in several more recent papers (see, e.g., [13], [42], [14]). All of them share a very helpful property, the so-called "commuting diagram property", whose importance was first fully recognized in Douglas-Roberts [19]. Let us discuss it in a particular case: the BDM (Brezzi, Douglas, Marini) element of degree 2 for two-dimensional problems ( $D \subset \mathbb{R}^2$ ). Let  $\mathcal{T}_h$  be a regular sequence of decompositions of  $D$  into triangles. We assume for the sake of simplicity that  $\Gamma_{\text{Neu}} = \emptyset$  in (2.2) and  $A(\underline{x}) = 1$ . As a discretization of  $H(\text{div}; D)$  and  $L^2(D)$  respectively we take

$$(3.7) \quad \Xi_h = \{ \underline{q} \mid \underline{q} \in H(\text{div}; D); \underline{q}|_T \in (P_2)^2 \quad \forall T \in \mathcal{T}_h \},$$

$$(3.8) \quad \Psi_h = \{v \mid v \in L^2(D); v|_T \in P_1 \quad \forall T \in \mathcal{T}_h\}.$$

Here and in the following  $P_k(S)$  (or simply  $P_k$ ) will denote the set of polynomials of degree  $\leq k$  on the set  $S$ . We consider now the discretized problem:

find  $\underline{p}_h \in \Xi_h$  and  $\underline{u}_h \in \Psi_h$  such that

$$(3.9) \quad \int_D \underline{p}_h \cdot \underline{q} \, d\underline{x} + \int_D \underline{u}_h \operatorname{div} \underline{q} \, d\underline{x} = \int_{\partial D} g_0 \underline{q} \cdot \underline{n} \, d\Gamma \quad \forall \underline{q} \in \Xi_h,$$

$$(3.10) \quad \int_D v \operatorname{div} \underline{p}_h \, d\underline{x} = \int_D f v \, d\underline{x} \quad \forall v \in \Psi_h.$$

We define now an operator  $M_h$  from  $(H^1(D))^2$  into  $\Xi_h$  by:

$$(3.11) \quad \left\{ \begin{array}{ll} \text{i)} & \int_e (\underline{q} - M_h \underline{q}) \cdot \underline{n} \, p_2 \, ds = 0 \quad \forall e, \text{ edge in } \mathcal{T}_h, \forall p_2 \in P_2(e) \\ \text{ii)} & \int_T (\underline{q} - M_h \underline{q}) \, d\underline{x} = 0 \quad \forall T, \text{ triangle in } \mathcal{T}_h \\ \text{iii)} & \int_T (\underline{q} - M_h \underline{q}) \cdot \operatorname{rot} b_T \, d\underline{x} = 0 \quad \forall T, \text{ triangle in } \mathcal{T}_h \end{array} \right.$$

where  $b_T := \lambda_1 \lambda_2 \lambda_3$  is the cubic vanishing on  $\partial T$ , and

$\operatorname{rot} \phi := (-\partial \phi / \partial x_1, \partial \phi / \partial x_2)$ . We also define an operator  $P_h$  from  $L^2(D)$  into  $\Psi_h$  by:

$$(3.12) \quad \int_T (v - P_h v) p_1 \, d\underline{x} = 0 \quad \forall T, \text{ triangle in } \mathcal{T}_h, \forall p_1 \in P_1(T).$$

Let us check now that  $\operatorname{div} M_h \underline{q} = P_h \operatorname{div} \underline{q}$  for all  $\underline{q} \in (H^1(D))^2$ .

Actually, for all  $\underline{v}_h \in \psi_h$  we have:

$$\begin{aligned}
 (3.13) \quad \int_T \underline{v}_h \operatorname{div} M_h \underline{q} \, d\underline{x} &= \int_{\partial T} \underline{v}_h (M_h \underline{q} \cdot \underline{n}) \, ds - \int_T \operatorname{grad} \underline{v}_h \cdot M_h \underline{q} \, d\underline{x} \\
 &= \int_{\partial T} \underline{v}_h \underline{q} \cdot \underline{n} \, ds - \int_T \operatorname{grad} \underline{v}_h \cdot \underline{q} \, d\underline{x} = \int_T \underline{v}_h \operatorname{div} \underline{q} \, d\underline{x} = \int_T \underline{v}_h P_h \operatorname{div} \underline{q} \, d\underline{x}.
 \end{aligned}$$

It is also easy to check that the divergence operator is linear, continuous and surjective from  $(H^1(D))^2$  onto  $L^2(D)$ . This can be summarized in the following diagram:

$$(3.14) \quad \begin{array}{ccccc}
 (H^1(D))^2 & \xrightarrow{\operatorname{div}} & L^2(D) & \longrightarrow & 0 \\
 \downarrow M_h & & \downarrow P_h & & \\
 \Xi_h & \xrightarrow{\operatorname{div}} & \Psi_h & \longrightarrow & 0
 \end{array} .$$

It is easy to check that (3.14) implies, in particular, (3.2) and (3.3), but it is much more powerful than that. For instance, it implies:

$$(3.15) \quad \|\underline{p} - \underline{p}_h\|_{(L^2(D))^2}^2 \leq \gamma_1 \|\underline{p} - M_h \underline{p}\|_{(L^2(D))^2}^2,$$

$$(3.16) \quad \|u - u_h\|_{L^2(D)} \leq \gamma_2 (\|\underline{p} - M_h \underline{p}\|_{(L^2(D))^2} + \|u - P_h u\|_{L^2(D)}),$$

with  $\gamma_1, \gamma_2$  are independent of  $h$  (whenever  $\underline{p} \in (H^1(D))^2$ ). In particular, with the choice (3.7) - (3.8) this yields:

$$(3.17) \quad \|\underline{p} - \underline{p}_h\|_{(L^2(D))^2}^2 \leq \gamma_1 h^3 \|\underline{p}\|_{(H^3(D))^2}^2,$$

$$(3.18) \quad \|u - u_h\|_{L^2(D)} \leq \gamma_2 h^2 (\|u\|_{H^2(D)} + \|\underline{p}\|_{(H^2(D))^2}).$$

Note that (3.17) does not follow from the abstract error estimate (3.6).

The commuting diagram has other nice properties. For instance, it allows a simple proof of error estimates in dual norms, as in Douglas-Roberts [20] or in Brezzi-Douglas-Marini [15]. Error estimates in  $L^\infty$  norms are also available: see for instance Scholz [45], [46] and Gastaldi-Nochetto [26], [27].

The most popular scheme for (2.9) - (2.10), that is, the "lowest order Raviart-Thomas", can be obtained by using, instead of (3.7), (3.8):

$$(3.19) \quad \begin{cases} \Xi_h = \{\underline{q} \mid \underline{q} \in H(\text{div}; D); \underline{q}|_T \in (P_1(T))^2 \quad \forall T; \underline{q} \cdot \underline{n}|_e \in P_0(e) \quad \forall e\}. \\ \Psi_h = \{v \mid v \in L^2(D); v|_T \in P_0 \quad \forall T \in \mathcal{T}_h\}. \end{cases}$$

Accordingly, one then uses  $P_0(e)$  instead of  $P_2(e)$  in (3.11)i) and drops (3.11)ii) and iii); similarly one uses  $P_0(T)$  instead of  $P_1(T)$  in (3.12). It follows immediately that (3.13) still holds, and then (3.14) also holds. Clearly, only an  $O(h)$  rate can now be achieved in both (3.17) and (3.18).

#### Example 2h. Discretizations of the Stokes equations.

Life is much harder when we go from (2.9) - (2.10) to (2.14) - (2.15). The only positive aspect is that now the bilinear form  $a(\underline{u}, \underline{v})$  is such that (2.23) actually holds in the whole  $(H_0^1(D))^2$  (our present  $\Xi$ ) so that (3.2) also holds regardless of the choice of the discretization. This might



partially excuse all the "Stokes-thinking" people who consider (3.3) as the condition for mixed methods. If one tries to discretize even the easy (2.9) - (2.10) with a scheme that does not satisfy (3.14), one will see that (3.2) can cause great difficulty.

However, coming back to Stokes, it is true that the only condition to be satisfied by the discretization is (3.3), which now reads:

$$(3.20) \quad \exists \beta_h > 0: \sup_{\underline{v} \in \Xi_h - \{0\}} \frac{\int_D q \operatorname{div} \underline{v} \, d\underline{x}}{\|\underline{v}\|_1} \geq \beta_h \|q\|_{L^2(D)/\mathbb{R}} \quad \forall q \in \Psi_h$$

with, if possible,  $\beta_h$  independent of  $h$ . A sufficient condition is the following so-called Fortin's trick [22]: we have to find a linear operator  $M_h$  from  $(H^1(D))^2$  into  $\Xi_h$  such that:

$$(3.21) \quad \|M_h \underline{v}\|_1 \leq c_1 \|\underline{v}\|_1 \quad \forall \underline{v} \in (H_0^1(D))^2,$$

$$(3.22) \quad \int_D q_h \operatorname{div}(\underline{v} - M_h \underline{v}) \, d\underline{x} = 0 \quad \forall \underline{v} \in (H_0^1(D))^2, \quad \forall q_h \in \Psi_h.$$

We consider one example. Let  $\mathcal{T}_h$  be a decomposition of  $D$  into rectangles,  $R$ , with sides parallel to the axes (the use of isoparametric elements is obviously also allowed, but more complicated to describe), and choose:

$$(3.23) \quad \Xi_h = \{\underline{v} \mid \underline{v} \in (H_0^1(D))^2; \underline{v}|_R \in (Q_2(R))^2 \quad \forall R \in \mathcal{T}_h\},$$

$$(3.24) \quad \Psi_h = \{q \mid q \in L^2(D); \int_D q \, d\underline{x} = 0; q|_R \in P_1(R) \quad \forall R \in \mathcal{T}_h\}.$$

In (3.23)  $Q_2(R)$  means the set of polynomials of degree  $\leq 2$  in each

variable. Let us see how to construct the operator  $M_h$  at least for a smooth  $\underline{v}$ . To deal with a general  $\underline{v}$  in  $(H_0^1(D))^2$  is just technically more complicated, but the philosophy is the same. In each  $R$  we set:

$$(3.25) \quad M_h \underline{v} = \underline{v} \quad \text{at the vertices (8 conditions);}$$

$$(3.26) \quad \int_e (M_h \underline{v} - \underline{v}) ds = 0 \quad \text{on each edge (8 conditions);}$$

$$(3.27) \quad \int_R \operatorname{div}(M_h \underline{v} - \underline{v}) x_i dx = 0 \quad i = 1, 2 \quad (2 \text{ conditions}).$$

We have a total of 18 conditions (note that the dimension of  $Q_2$  is 9). It is easy to check that they are independent. Let us check (3.22); that is, let us check that

$$(3.28) \quad \int_R \operatorname{div}(M_h \underline{v} - \underline{v}) p_1 dx = 0 \quad \forall p_1 \in P_1(R).$$

Clearly, (3.27) implies that (3.28) holds for  $p_1 = x_1$  and  $p_1 = x_2$ . We need only to check  $p_1 \equiv 1$ :

$$(3.29) \quad \int_R \operatorname{div}(M_h \underline{v} - \underline{v}) dx = \int_{\partial R} (M_h \underline{v} - \underline{v}) \cdot \underline{n} ds = 0,$$

due to (3.26). We can now apply (3.6) and obtain:

$$(3.30) \quad \|\underline{u} - \underline{u}_h\|_1 + \|p - p_h\|_{L^2(D)/\mathbb{R}} \leq c h^2 (\|\underline{u}\|_3 + \|p\|_2).$$

There are many other known choices available for obtaining a discretization of

(2.14) - (2.15) that satisfies (3.20). An almost complete list of them can be found in Brezzi-Fortin<sup>1</sup> together with the references. In particular, Scott and Vogelius [47] proved that, under minor restrictions on the decomposition of  $D$  into triangles, one can always use a continuous velocity field of local degree  $k$  and a discontinuous pressure field of local degree  $k-1$ , provided  $k \geq 4$ . For the low degrees, a special headache is provided by the use of bilinear velocities and constant pressures: its convergence has been proved in a variety of cases (see Johnson-Pitkäranta [38], Stenberg [48], Pitkäranta-Stenberg [43]) but not yet in the general case. In any event, a filtering of the pressure field is always required to eliminate the checkerboard modes. General strategies for constructing discretizations that fulfill (3.20) are given in Boland-Nicolaides [10] and Brezzi-Pitkäranta [16]. Modifications of the discrete equations that allow violation of (3.3) have received considerable attention in recent times. Roughly speaking, in the special case of equations (2.14), (2.15), they consist in substituting (2.15) with the "perturbed" equation

$$(3.31) \quad \int_D q \operatorname{div} \underline{u} \, d\underline{x} + h^2 \int_D \underline{\operatorname{grad}} p \cdot \underline{\operatorname{grad}} q \, d\underline{x} = 0 \quad \forall q \in H^1(D)$$

(Brezzi-Pitkäranta [16]) provided that one uses a continuous pressure field in the finite element discretization. A modification of (3.31) of the form:

$$(3.32) \quad \int_D q \operatorname{div} \underline{u} \, d\underline{x} = \sum_{T \in \mathcal{T}_h} \alpha h_T^2 \int_T (\Delta \underline{u} - \underline{\operatorname{grad}} p + \underline{f}) \cdot \underline{\operatorname{grad}} q \, d\underline{x}$$

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<sup>1</sup>F. Brezzi and M. Fortin, Book in preparation.

( $\alpha$  = small parameter to be adjusted;  $h_T$  = diameter of  $T$ ) has been introduced by Hughes-Balestra-Franca [35]. Note that (3.31) is simpler but has a consistency error of order  $h^2$  that is not present in (3.32). For more details and additional results in this direction, see Brezzi-Douglas [12].

For the use of more general boundary conditions, the basic reference is Verfürth [49]; see also the references contained therein.

Additional references for the Stokes and Navier-Stokes equations can be found in Glowinski-Pironneau [30], Glowinski [31], Girault-Raviart [28] and Brezzi-Fortin<sup>2</sup>.

#### Example 3h. Discretizations of linear elasticity problems

It is difficult, in general, to find convenient finite element discretizations for equations (2.21) - (2.22). We shall briefly indicate here four possible ways to proceed. The first possibility is to try to construct spaces that verify the commuting diagram property (as in (3.14)). This has been possible, up to now, only by means of composite elements: that is, each element is split into sub-elements and one uses trial functions that are polynomials in each sub-element (plus suitable continuity requirements from one sub-element to the other). Examples of this approach can be found in Johnson-Mercier [37] or in Arnold-Douglas-Gupta [4]. A second possibility is to give up the symmetry condition that appears in the definition of  $H(\text{div}; D)$  and to enforce it a posteriori by means of a Lagrange multiplier. After discretization we deal then with stress fields having only a weak

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<sup>2</sup>F. Brezzi and M. Fortin, Book in preparation.

symmetry. This idea was first used by Fraeijs de Veubeke [23] and then modified and analyzed by Amara-Thomas [1] and Arnold-Brezzi-Douglas [3]. A third possibility is to change the "auxiliary function" and use a different, nonsymmetric, tensor field instead of  $\underline{\sigma}$ . This will in general produce some trouble at  $\Gamma_{\text{Neu}}$  (if  $\Gamma_{\text{Neu}} \neq \emptyset$ ) that can be treated with the introduction of an additional Lagrange multiplier on  $\Gamma_{\text{Neu}}$ . We refer to Arnold-Falk [5] for more details on this approach. Finally, a fourth possibility is the addition of a stabilizing term in the style of (3.31) or (3.32). For this we refer to Franca-Loula-Hughes-Miranda [25].

It should be pointed out that additional difficulties arise when dealing with nearly incompressible materials. In these cases (2.23) ceases to hold (in the limit) in the whole space but still holds for divergence-free tensor fields. This implies that (3.2) must also be checked if the discretization is such that  $K_h \not\subset K$ . Additional references for the above (and many other) applications can be found in Brezzi-Fortin<sup>3</sup>.

#### 4. NUMERICAL METHODS FOR SOLVING THE DISCRETIZED PROBLEM

The major difficulty that arises in solving a linear problem such as (3.4) (3.5) is that the associated matrix

$$(4.1) \quad \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}$$

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<sup>3</sup>F. Brezzi and M. Fortin, Book in preparation.

is indefinite. There are many ways of overcoming this difficulty, mostly using some particular feature of the problem under consideration in order to rewrite it in a different form. Here we shall briefly sketch two of them: one which is mostly used in Examples 1 and 3 and the other used in Example 2.

The first technique, which is very old (see Fraeijs de Veubeke [24]) starts from the following simple observation. If the space  $\Xi_h$  is made of functions that are completely discontinuous from one element to the other, then the most natural choice of basis functions for  $\Xi_h$  will produce a matrix  $A$  in (4.1) which is block-diagonal. Then the inverse matrix  $A^{-1}$  can be easily computed explicitly. Solving (3.4) element by element for  $\bar{\xi}_h$  and substituting into (3.5) (static condensation) leaves us with the final matrix  $B^T A^{-1} B$  and the only unknown  $\bar{\psi}_h$ . Note that (3.2) and (3.3) will imply that  $B^T A^{-1} B$  is symmetric and positive definite (if  $a(\xi_1, \xi_2)$  is symmetric). Now if instead the functions in  $\Xi_h$  have some continuity properties from one element to the other (as in Example 1h we had  $\underline{p}_h \cdot \underline{n}$  continuous at the interfaces), this cannot be done. However, one can choose to work in a larger space, say  $\tilde{\Xi}_h$ , made of discontinuous functions, and then require the continuity by means of a Lagrange multiplier. Let us see the procedure in the particular case of Example 1h. We set:

$$(4.2) \quad \tilde{\Xi}_h = \{ \underline{q} \mid \underline{q} \in (L^2(D))^2; \underline{q}|_T \in (P_2)^2 \quad \forall T \in \mathcal{T}_h \},$$

$$(4.3) \quad \Lambda_h = \{ \mu \mid \mu|_e \in P_2(e) \quad \forall e, \text{ internal edge in } \mathcal{T}_h, \mu = 0 \text{ on } \partial D \},$$

$$(4.4) \quad c(\underline{q}, \mu) = \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\underline{q} \cdot \underline{n}) \mu \, ds.$$

Clearly, if  $\underline{q} \in \tilde{\Xi}_h$  then

$$(4.5) \quad \underline{q} \in \Xi_h \iff c(\underline{q}, \mu) = 0 \quad \forall \mu \in \Lambda_h.$$

It is not difficult to check that the new problem

find  $\tilde{\underline{p}}_h \in \Xi_h, \tilde{\underline{u}}_h \in \Psi_h, \lambda_h \in \Lambda_h$  such that

$$(4.6) \quad \int_D \tilde{\underline{p}}_h \cdot \underline{q} \, d\underline{x} + \sum_T \int_T \tilde{\underline{u}}_h \operatorname{div} \underline{q} \, d\underline{x} = \int_{\partial D} g_0 \underline{q} \cdot \underline{n} \, ds + c(\underline{q}, \lambda_h) \quad \forall \underline{q} \in \tilde{\Xi}_h$$

$$(4.7) \quad \sum_T \int_T \underline{v} \operatorname{div} \tilde{\underline{p}}_h \, d\underline{x} = \int_D f \underline{v} \, d\underline{x} \quad \forall \underline{v} \in \Psi_h$$

$$(4.8) \quad c(\tilde{\underline{p}}_h, \mu) = 0 \quad \forall \mu \in \Lambda_h$$

has a unique solution, and that  $\tilde{\underline{p}}_h = \underline{p}_h, \tilde{\underline{u}}_h = \underline{u}_h$ . Now both the unknowns  $\tilde{\underline{p}}_h$  and  $\tilde{\underline{u}}_h$  are a priori discontinuous and they can be eliminated, at the element level, by static condensation. The final matrix, in the unknown  $\lambda_h$ , will be symmetric and positive definite. It is clear that  $\lambda_h$  itself should be an approximation of  $u$  at the interfaces, and it has been used as such by engineers. However, it was only rather recently that it was proved mathematically that  $\lambda_h$  converges to  $u$ , and in general with a better order of convergence than  $\underline{u}_h$  itself (see Arnold-Brezzi [2]). For instance, in the present case, once  $\lambda_h$  is known one can construct, element by element, an  $\underline{u}_h^* \in P_3(T)$  such that

$$(4.9) \quad \left\{ \begin{array}{l} \int_e (u_h^* - \lambda_h) p_2 ds = 0 \quad \forall e \text{ edge of } T, \forall p_2 \in P_2(e) \\ \int_T (u_h^* - u_h) d\underline{x} = 0 \end{array} \right.$$

and show that

$$(4.10) \quad \|u - u_h^*\|_{L^2(D)} \leq O(h^4)$$

instead of (3.18) (for the proof of (4.10) see [15]). A similar result (but only with  $O(h^2)$  in (4.10)) can also be achieved with the lowest order Raviart-Thomas element described at the end of Example 1h. However, the best way to compute the solution for this last element is to solve with the so-called  $P_1$ -nonconforming method and then use the postprocessing of Marini [39]. For additional results on the convergence of  $\lambda_h$  to  $u$  see Brezzi-Douglas-Marini [15], Brezzi-Douglas-Fortin-Marini [14] and Gastaldi-Nochetto [27].

This same idea (disconnect  $\Xi_h$  and use a Lagrange multiplier to force back the continuity) can be used for elasticity problems and in many other cases. However, it has not been possible, so far, to use it, for instance, for the Stokes equations (and more generally when continuity at the vertices is required in  $\Xi_h$ ). Then one can use the following other trick, that was first analyzed by Bercovier [8]. If the space  $\Psi_h$  is made of discontinuous functions (as it was the case in our Example 2h), then one can perturb equation (3.5) into

$$(4.11) \quad b(\bar{\xi}_h, \psi) = \varepsilon(\bar{\psi}_h, \psi)_\Psi + \langle \ell_2, \psi \rangle \quad \forall \psi \in \Psi_h.$$



The corresponding matrix (for (3.4), (4.11)) becomes, roughly,

$$(4.12) \quad \begin{pmatrix} A & B \\ B^T & -\epsilon I \end{pmatrix}.$$

Now the discontinuity in  $\psi_h$  allows us to eliminate  $\bar{\psi}_h$  at the element level. We obtain in that way a matrix  $A + \epsilon^{-1}BB^T$ . If (3.2) and (3.3) are satisfied, this new matrix will be symmetric and positive definite (always if  $a(\xi_1, \xi_2)$  is symmetric). Moreover, calling  $(\xi_h^\epsilon, \psi_h^\epsilon)$  the solution of (3.4) and (4.11), one has

$$(4.13) \quad \|\bar{\xi}_h - \xi_h^\epsilon\|_\Xi + \|\bar{\psi}_h - \psi_h^\epsilon\|_\Psi = O(\epsilon).$$

The method can also be applied when  $\psi_h$  consists of continuous functions, provided that some lumping procedure is used to compute the inner product in (4.11). However, in such cases, one gets for  $\epsilon^{-1}BB^T$  a bandwidth that is generally larger than that of  $A$ , and this is often a considerable drawback.

A different attempt to reduce (3.4) (3.5) to a single equation in the case of the Stokes problem can be found in Bramble<sup>4</sup>.

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<sup>4</sup>J. H. Bramble, to appear.

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# Standard Bibliographic Page

1. Report No. NASA CR-178267 ICASE Report No. 87-15		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle A SURVEY OF MIXED FINITE ELEMENT METHODS				5. Report Date March 1987	
				6. Performing Organization Code	
7. Author(s) F. Brezzi				8. Performing Organization Report No. 87-15	
				10. Work Unit No.	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225				11. Contract or Grant No. NAS1-18107	
				13. Type of Report and Period Covered Contractor Report	
12. Sponsoring Agency Name and Address  National Aeronautics and Space Administration Washington, D.C. 20546				14. Sponsoring Agency Code 505-90-21-01	
15. Supplementary Notes  Langley Technical Monitor: J. C. South  Final Report  Submitted to Proc. Workshop on Finite Element Methods, Theory, and Application, Springer-Verlag					
16. Abstract  This paper is an introduction to and an overview of mixed finite element methods. It discusses the mixed formulation of certain basic problems in elasticity and hydrodynamics. It also discusses special techniques for solving the discrete problem.					
17. Key Words (Suggested by Authors(s))  elasticity equations, Stokes equations, mixed finite element theory				18. Distribution Statement  64 - Numerical Analysis  Unclassified - unlimited	
19. Security Classif.(of this report) Unclassified		20. Security Classif.(of this page) Unclassified		21. No. of Pages 32	
				22. Price A03	