EXTENSION OF KIRCHHOFF'S FORMULA
TO RADIATION FROM MOVING SURFACES

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ABSTRACT

Kirchhoff's formula for radiation from a closed surface has been used recently for prediction of the noise of high speed rotors and propellers. Because the closed surface on which the boundary data are prescribed in these cases is in motion, an extension of Kirchhoff's formula to this condition is required. In this paper such a formula, obtained originally by Morgans for the interior problem, is derived for regions exterior to surfaces moving at speeds below the wave propagation speed by making use of some results of generalized function theory. It is shown that the usual Kirchhoff formula is a special case of the main result of the paper. The general result applies to a deformable surface. However, the special form it assumes for a rigid surface in motion is also noted. Some possible areas of application of the formula to problems of current interest in aeroacoustics are discussed.

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INTRODUCTION

Kirchhoff's formula, published in 1882, is used in the theory of diffraction of light and in other electromagnetic problems [1-2]. It also has many applications to problems of wave propagation in acoustics [3]. One of the novel uses of this formula was proposed by Hawkings for predicting the noise of high speed propellers and helicopter rotors [4]. His idea involves surrounding the rotating blades by a closed surface S which moves with the forward speed of the machine. Inside this surface, nonlinear aerodynamic calculations are carried out which give the blade loads and the pressure history and its spatial and temporal derivatives on the surrounding surface S. In the exterior of this surface, it is proposed that a formula similar to Kirchhoff's be used to calculate propagation of sound in terms of these surface values. This necessitates extension of the Kirchhoff formula to apply to moving surfaces. In 1930, such an extension was derived for the interior problem by Morgans [5]. His analysis was lengthy and somewhat complicated, and, in addition, it does not seem to be well known among acousticians. For these reasons, a modern derivation of a generalized Kirchhoff formula applicable to moving surfaces is presented here. It is very likely that specialized versions of this formula have been rederived by many researchers in the context of specific applications. One such case known to the authors is discussed in the last section of the current paper.

In the following section the method used in the derivation of the main result is illustrated by deriving the Kirchhoff formula for radiation into the region exterior to a stationary surface S. The method involves application of the wave operator to a function ϕ which is set equal to zero outside the domain of interest. The function ϕ is thus discontinuous across the surface S. The derivatives in the wave operator are considered as generalized derivatives

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[6-9] which are defined everywhere. The resulting expression then contains terms which involve the jumps in φ and its derivatives multiplied by Dirac delta functions whose supports are on S. These terms can be viewed as source terms for a wave equation which is valid in the entire unbounded space. Kirchhoff's formula is then derived using the Green's function of the wave equation for unbounded space.

In the main section of this paper, the above method is extended to a smooth moving surface. It will be observed that essentially no major difficulties arise in this case. One thus avoids the conventional approach which can become complicated in the case of moving surfaces. The resulting general Kirchhoff formula is then written for piecewise smooth rigid surfaces. It is also shown that the usual formula for a stationary surface is a special case of the general result presented here.

In this paper the wave propagation problem is treated as an acoustic problem and the terminology of acoustics is used. The main result is of course applicable to many other wave propagation problems. Immediate applications to the problem of the prediction of the noise of high speed propellers and rotors are expected.

EXAMPLE OF THE MATHEMATICAL APPROACH

In this section the mathematical method of deriving the main result is illustrated by utilizing it to obtain the Kirchhoff formula for a stationary surface [1-3]. Consider the closed and bounded smooth surface S described by $f(x)=0$ such that $f > 0$ in the exterior of this surface. Assume that φ and its first derivatives are continuous in $f > 0$ and that φ satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi \equiv \square^2 \phi = 0 \tag{1}$$

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in the exterior of surface S, and that Φ, Φ_t and Φ_n are given on S for

$t \epsilon(-\infty, \infty)$. Extend Φ to the interior of $S(f < 0)$ by assuming that $\Phi = 0$ inside

S. To reduce confusion between Φ and this extended function, the latter

function $\tilde{\Phi}(\underset{\sim}{x}, t)$ is defined as follows:

$$\tilde{\Phi}(\underset{\sim}{x}, t) = \begin{cases} \Phi(\underset{\sim}{x}, t) & f > 0 \quad \text{(exterior of S)} \\ 0 & f < 0 \quad \text{(interior of S)} \end{cases} \tag{2}$$

The function $\tilde{\Phi}$ is a discontinuous function whose derivatives do not exist

on $f=0$. One can, however, define generalized derivatives for such a function

[6-9]. In this paper operators involving generalized derivatives will be de-

noted by a bar over the operator symbol. It follows from the definition of

$\tilde{\Phi}$ that wherever ordinary derivatives of $\tilde{\Phi}$ exist (i.e., interior or exterior to

S) one has

$$\square^2 \tilde{\Phi} = 0. \tag{3}$$

However, one can show, as is done below, that

$$\bar{\square}^2 \tilde{\Phi} = \frac{1}{c^2} \frac{\bar{\partial}^2 \tilde{\Phi}}{\partial t^2} - \bar{\nabla}^2 \tilde{\Phi} \neq 0 \tag{4}$$

It will be seen that what appears on the right side of Eq. (4) leads to the

Kirchhoff formula. For the reader who is not familiar with generalized func-

tion theory, it is mentioned that one requires considerable conceptual develop-

ment in going from Eq. (3) to Eq. (4). Examples of applications given in

references [7] and [8] may help in understanding the following steps.

One can always define the surface f=0 in such a way that $\left|\nabla f\right|$=1 on this surface. This is done by taking df=dn where dn is distance from the surface f=0 along its local normal. With this definition one has $\nabla f = \underset{\sim}{n}$ where $\underset{\sim}{n}$ is the local unit normal pointing into the exterior region. Then, using generalized derivatives [8], one can write at once

$$\frac{\overline{\partial}\tilde{\phi}}{\partial t} = \frac{\partial\tilde{\phi}}{\partial t} + \tilde{\phi}\,\frac{\partial f}{\partial t}\,\delta(f) = \frac{\partial\tilde{\phi}}{\partial t}\ , \tag{5-a}$$

$$\frac{\overline{\partial}^2\tilde{\phi}}{\partial t^2} = \frac{\partial^2\tilde{\phi}}{\partial t^2}\ , \tag{5-b}$$

$$\overline{\nabla}\tilde{\phi} = \nabla\tilde{\phi} + \phi\,\underset{\sim}{n}\,\delta(f)\ , \tag{5-c}$$

$$\overline{\nabla}^2\tilde{\phi} = \nabla^2\tilde{\phi} + \phi_n\,\delta(f) + \overline{\nabla}\cdot[\phi\underset{\sim}{n}\delta(f)]\ , \tag{5-d}$$

where in Eq. (5-c) the function ϕ on the right side is defined as

$$\phi \equiv \lim_{f\to0+}\ \phi(\underset{\sim}{x},t) \tag{6}$$

The function ϕ_n in Eq. (5-d) is the normal derivative of ϕ, and is defined similarly. The Dirac delta function is denoted $\delta(f)$. From Eq. (5) one obtains

$$\overline{\Box}^2\tilde{\phi} = \Box^2\tilde{\phi} - \phi_n\,\delta(f) - \overline{\nabla}\cdot[\phi\underset{\sim}{n}\delta(f)] = -\phi_n\,\delta(f) - \overline{\nabla}\cdot[\phi\underset{\sim}{n}\delta(f)]\ . \tag{7}$$

Here, Eq. (3) is used in the last step. This completes the determination of the right side of Eq. (4).

Equation (7) is valid in the entire unbounded space. One can now use the Green's Function for the wave equation in unbounded space, $\delta(g)/4\pi r$, where

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$q = \tau - t + r/c$ and $r = |x - y|$. The space-time variables of the observer and source are denoted (x, t) and (y, τ), respectively. The speed of sound is c. Equation (7) then yields

$$4\pi\widetilde{\Phi}(x, t) = -\int \frac{\Phi_n}{r} \delta(f) \delta(g) dy d\tau - \bar{\nabla} \cdot \int \frac{\Phi_n}{r} \delta(f) \delta(g) dy d\tau. \tag{8}$$

The space integrals in Eq. (8) are over the entire unbounded space and the time integral is over $(-\infty, t]$. The divergence operator on the right acts on the variable x. Use the fact that the volume element $dy = df dS$, where dS is the element of area of the surface S: $f = 0$. Then let $\tau \to g$ noting that the Jacobian of the transformation $|\partial\tau/\partial g| = |\partial g/\partial\tau|^{-1}$ is unity. Integration over f, by virtue of the delta function $\delta(f)$, restricts the expressions to $f = 0$. Subsequent integration over g restricts the expressions to $g = 0$, which introduces the retarded time. Equation (8) thus becomes

$$4\pi\widetilde{\Phi}(x, t) = -\int_{f=0} \frac{[\Phi_n]}{r} dS - \bar{\nabla} \cdot \int_{f=0} \frac{[\Phi]}{r} n \, dS \tag{9}$$

Here the square bracket stands for evaluation of the functions at the retarded time; i.e., $[\Phi] \equiv \Phi(y, t - r/c)$.

Taking the gradient operator inside the second integral of Eq. (9) and collecting terms of $O(1/r)$ and $O(1/r^2)$, one gets

$$4\pi\widetilde{\Phi}(x, t) = \begin{cases} 4\pi\Phi(x, t) & f > 0 \\ 0 & f < 0 \end{cases}$$

$$= \int_{f=0} \frac{[c^{-1}\Phi_\tau \cos\theta - \Phi_n]}{r} dS + \int_{f=0} \frac{[\Phi]\cos\theta}{r^2} dS \tag{10}$$

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Here $\phi_\tau = \partial\phi(\chi,\tau)/\partial\tau$ and θ is the angle between $\underset{\sim}{n}$ (pointing into the exterior region) and the radiation direction $\underset{\sim}{r}=\underset{\sim}{x}-\underset{\sim}{y}$. For the exterior region $f > 0$, this is Kirchhoff's formula for a stationary surface S. The present method automatically shows that for the observer inside the surface S the integrals in Kirchhoff's formula yield a zero (null) field. This means that ϕ, ϕ_n and ϕ_τ are not independent on the surface S. Equation (10) is an identity which may be utilized in various ways. As it stands it may be considered as an integral representation of ϕ at points exterior to S in terms of surface quantities. Another interpretation follows if the observer point $\underset{\sim}{x}$ is taken on S itself; in that case, Eq. (10) becomes an integral equation governing ϕ on S. This latter interpretation is the basis of modern boundary element methods. It is seen that the integrals of Eq. (10) remain valid for a piecewise smooth surface S.

It is noted that in the usual applications of Eq. (10) ϕ is taken as a complex harmonic function of time so that the temporal part of ϕ disappears in Eq. (10); the resulting Kirchhoff formula pertains to the complex amplitude of ϕ [1-3]. The amplitude satisfies Helmholtz' equation. In this case one could begin the above procedure from a Helmholtz equation rather than from the wave equation as is done above.

THE GENERAL KIRCHHOFF FORMULA

In this section, the Kirchhoff formula for moving surfaces will be derived. Let $f(\underset{\sim}{x},t) = 0$ describe the moving surface S which is assumed to be piecewise smooth. The function f is defined as before such that $f > 0$ outside S and $\nabla f = \underset{\sim}{n}$ on $f = 0$. The function ϕ and its first derivatives are assumed to be continuous and ϕ satisfies Eq. (1). This function is again extended to $\tilde{\phi}$ by

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Eq. (2) so that Eq. (3) is satisfied both inside and outside S. The terms on the right side of Eq. (4) are found by the same technique as presented above.

Taking the generalized derivative of $\tilde{\phi}$ with respect to time gives

$$\frac{\bar{\partial}\tilde{\phi}}{\partial t} = \frac{\partial\tilde{\phi}}{\partial t} + \phi\,\frac{\partial f}{\partial t}\,\delta(f) = \frac{\partial\tilde{\phi}}{\partial t} - \phi\,v_n\,\delta(f) \tag{11}$$

where $v_n = -\partial f/\partial t$ is the local normal velocity of S with respect to the undisturbed medium. Differentiating both sides of Eq. (11) once more yields

$$\frac{\bar{\partial}^2\tilde{\phi}}{\partial t^2} = \frac{\partial^2\tilde{\phi}}{\partial t^2} - \phi_{t(x)}v_n\delta(f) - \frac{\bar{\partial}}{\partial t}[\phi v_n\delta(f)] \tag{12}$$

where $\phi_{t(x)} = \partial\phi/\partial t\big|_{\underset{\sim}{x}}$. The generalized Laplacian of ϕ is given by Eq. (5-d) as before. From this and Eq. (12), one obtains

$$\bar{\square}^2\tilde{\phi} = -(\phi_n + \frac{1}{c}M_n\phi_{t(x)})\delta(f) - \frac{1}{c}\frac{\bar{\partial}}{\partial t}[M_n\phi\delta(f)] - \bar{\nabla}\cdot[\phi\,\underset{\sim}{n}\,\delta(f)] \tag{13}$$

where $M_n = v_n/c$ is the local normal Mach number on S. In the following, partial derivatives with respect to time holding different sets of space variables fixed will appear. The notation introduced above (e.g. $\phi_{t(x)}$) will be maintained wherever necessary in order to prevent confusion. The source terms in Eq. (13) can be written in other forms by performing the temporal and spatial derivatives in the second and the third terms, respectively. The above form, however, seems to require the fewest algebraic manipulations in the following analysis.

Upon use of the Green's Function of the wave operator in unbounded space, Eq. (13) gives

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$$4\pi\widetilde{\Phi}(\underset{\sim}{x},t) = -\int \frac{1}{r}(\Phi_n + \frac{1}{c} M_n \Phi_{\tau(y)})\delta(f)\delta(q)d\underset{\sim}{y}d\tau - \frac{1}{c}\frac{\overline{\partial}}{\partial t}\int \frac{1}{r} M_n \Phi \delta(f)\delta(g) d\underset{\sim}{y} d\tau$$

$$- \overline{\nabla}\cdot\int \frac{1}{r} \Phi \underset{\sim}{n} \delta(f)\delta(q) d\underset{\sim}{y} d\tau \tag{14}$$

where $\Phi_{\tau(y)} = \partial\Phi(\underset{\sim}{y},\tau)/\partial\tau$.

If the spatial variables $\underset{\sim}{y}$ are transformed to Lagrangian variables $\underset{\sim}{\eta}$, and if the transformation $\tau \to q$ is employed and the delta functions integrated out, then Eq. (14) can immediately be written in the form of Eq. (5.3) of Reference [10]. Such an approach leaves the divergence and the time derivatinve operators outside the integrals, however, which renders the resulting formula unsuitable for numerical application. In the current paper Eq. (14) will be manipulated differently to lead, as will be seen, to a more practically useful result.

Before interpreting the integrals in Eq. (14), the following relation is used to convert the space derivatives in the last integral to a time derivative:

$$\frac{\overline{\partial}}{\partial x_i}[\frac{\delta(q)}{r}] = -\frac{1}{c}\frac{\overline{\partial}}{\partial t}[\frac{\hat{r}_i \delta(g)}{r}] - \frac{\hat{r}_i \delta(g)}{r^2} \tag{15}$$

where $\underset{\sim}{\hat{r}} = (\underset{\sim}{x}-\underset{\sim}{y})/r$ is the unit radiation vector. Note that in this relation the source variables $(\underset{\sim}{y},\tau)$ are held fixed. Equation (14) can now be written as

$$4\pi\widetilde{\Phi}(\underset{\sim}{x},t) = -\int \frac{1}{r}(\Phi_n + \frac{1}{c} M_n \Phi_{\tau(y)})\delta(f)\delta(q)d\underset{\sim}{y} d\tau + \int \frac{1}{r^2} \Phi \cos\theta \delta(f)\delta(g)d\underset{\sim}{y} d\tau$$

$$+ \frac{1}{c}\frac{\overline{\partial}}{\partial t} \int \frac{1}{r}(\cos\theta - M_n)\Phi \delta(f)\delta(q)d\underset{\sim}{y} d\tau \tag{16}$$

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There are two types of integrals in Eq. (16), which will be denoted

$$I_1 = \int Q_1 \delta(f)\delta(q)d\underset{\sim}{y} \, d\tau \tag{17}$$

$$I_2 = \frac{1}{c}\frac{\bar{\partial}}{\partial t}\int Q_2 \, \delta(f)\delta(q)d\underset{\sim}{y} \, d\tau \tag{18}$$

The technique of evaluation of these integrals will be given below.

Although in many applications of the Kirchhoff formula in acoustics the surface S is taken as rigid, for completeness S is here assumed to be deformable. Let S be described by the surface coordinates (u^1, u^2) and assume $u^3=f$. This mapping is only required in the vicinity of $f=0$. Let the mapping $(u^1, u^2, u^3) \rightarrow \underset{\sim}{y}$ be a differentiable function of time τ. Denote the determinant of the coefficients of the first fundamental form on S by $g_{(2)} = g_{11}g_{22} - g_{12}{}^2$, where the $g_{ij}(i,j = 1,2)$ are the metric tensor components. Then $g_{(2)}$ is a function of time τ in addition to (u^1,u^2). For example, consider a sphere of radius $R(\tau)$ whose center is moving at speed $\underset{\sim}{U}(\tau)$. Let (u^1,u^2) be the spherical surface coordinates on this sphere as shown in Fig. 1. Then if at $\tau=0$ the center of the sphere is at the origin of the $\underset{\sim}{y}$-frame, it follows that

$$y_1 = y_1^0(\tau) + [u^3+R(\tau)]\cos u^1 \sin u^2$$

$$y_2 = y_2^0(\tau) + [u^3+R(\tau)]\sin u^1 \sin u^2 \tag{19}$$

$$y_3 = y_3^0(\tau) + [u^3+R(\tau)]\cos u^2$$

where

$$\underset{\sim}{y}^0(\tau) = \int_0^\tau \underset{\sim}{U}(\tau') \, d\tau' \tag{20}$$

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Also, it is seen that on S

$$\sqrt{q_{(2)}} = R^2(\tau) \sin u^2 \qquad (21)$$

To evaluate I_1, the following successive transformations are carried out: $\chi \to (u^1, u^2, u^3)$ and then $\tau \to q$. The Jacobians of transformations are 1 and $1/(1-M_r)$, respectively, where $M_r = \underset{\sim}{v} \cdot \hat{\underset{\sim}{r}}/c$. Here $\underset{\sim}{v} = \partial\underset{\sim}{\chi}/\partial\tau$ holding (u^1, u^2, u^3) fixed. After the first transformation and integration with respect to f(i.e. u^3), I_1 is given by

$$I_1 = \int_{-\infty}^{t} \int_{D(S)} \underset{1}{\rho}(u^1, u^2, 0, \tau) \; \delta(q) \; \sqrt{q_{(2)}} \; du^1 du^2 d\tau \qquad (22)$$

where $D(S)$ is the domain of S in (u^1, u^2)-space. Note that even if S is deformable, the limits of the inner integrals can be made time independent and are assumed so here (see Fig. 1). The deformation of S appears in the variation of $q_{(2)}$ with τ. The second transformation $\tau \to q$ and the subsequent integration with respect to q restrict the integrand to q = 0 and yield

$$I_1 = \int_{D(S)} \left[\frac{\underset{1}{\rho} \sqrt{q_{(2)}}}{1-M_r} \right]_{q=0} du^1 du^2 \qquad (23)$$

The interpretation of the integrand is as follows. If the relation q=0 is explicitly written (remembering that $u^3=f=0$ also in the integrand)

$$q = \tau - t + \left| \underset{\sim}{x} - \underset{\sim}{y}(u^1, u^2, 0, \tau) \right|/c = 0 , \qquad (24)$$

then the solution in source time, τ^*, of this equation is the emission time which must be used in the integrand of Eq. (23). Note that since $\left| \underset{\sim}{v} \right| < c$ by

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assumption, Eq. (24) has a unique solution. Equation (23) will thus be written as

$$I_1 = \int_{D(S)} \left[\frac{Q_1 \sqrt{g_{(2)}}}{1-M_r}\right]_{\tau*} du^1 du^2. \tag{25}$$

Since D(S) is independent of t, I_2 follows immediately from Eq. (18) as

$$I_2 = \frac{1}{c} \int_{D(S)} \frac{\partial}{\partial t} \left[\frac{Q_2 \sqrt{g_{(2)}}}{1-M_r}\right]_{\tau*} du^1 du^2 = \frac{1}{c} \int_{D(S)} \left[\frac{1}{1-M_r} \frac{\partial}{\partial \tau} \left\{\frac{Q_2 \sqrt{g_{(2)}}}{1-M_r}\right\}\right]_{\tau*} du^1 du^2 \tag{26}$$

where the relation $\partial/\partial t = (1-M_r)^{-1} \partial/\partial \tau|_{\underset{\sim}{u}}$, obtained from Eq. (24), is used in the last expression. Note carefully that in taking $\partial/\partial \tau$ the variables u^1 and u^2 are kept fixed.

After using the results of Eqs. (25) and (26), Eq. (16) can be written as

$$4\pi \widetilde{\Phi}(\underset{\sim}{x},t) = -\int_{D(S)} \left[\frac{(\Phi_n + c^{-1}M_n \Phi_{\tau(y)})\sqrt{g_{(2)}}}{r(1-M_r)}\right]_{\tau*} du^1 du^2 + \int_{D(S)} \left[\frac{\Phi\sqrt{g_{(2)}}\cos\theta}{r^2(1-M_r)}\right]_{\tau*} du^1 du^2$$

$$+ \frac{1}{c} \int_{D(S)} \left[\frac{1}{1-M_r} \frac{\partial}{\partial \tau} \left\{\frac{(\cos\theta - M_n)\Phi\sqrt{g_{(2)}}}{r(1-M_r)}\right\}\right]_{\tau*} du^1 du^2 \tag{27}$$

This result is equivalent to that of Morgans [5] and is referred to here as the general Kirchhoff formula. The notation of the present paper, however, differs from Morgans', who addressed the interior problem.

There is a somewhat subtle point which should be made, however, about the result given in Reference [5]. The quantity $[\sqrt{g_{(2)}}]_{\tau*} du^1 du^2$ is there written dS, but this is not the element of area of the physical surface S which is defined by fixing τ in $f(\underset{\sim}{y}, \tau) = 0$. That element is $\sqrt{g_{(2)}} \, du^1 du^2$. The difference arises because as S deforms in space any element specified by (u^1, u^2) has dif-

ferent emission times τ^* for a fixed $\underset{\sim}{x}$ and t. Corresponding to each τ^* the

surface S has a different shape. Although the main result of Morgans is

correct if interpreted as in the present paper, in some of his elementary

examples there is an ambiguity in the meaning of dS. It should be noted that

Morgans' result (27) is claimed to be erroneous by the authors of Reference

[10]. However, they do not specifically point out the nature of the error.

The modern derivation given above indicates that Morgans' formula is, in fact,

correct, a conclusion which was made also by Munro [11].

Finally, it must be mentioned that the moving surface S in the above

analysis can be unbounded provided that it divides the space into interior and

exterior parts. An open bounded or unbounded surface can be considered as the

limiting case of a closed surface of infinitely small thickness to which the

Kirchhoff formula can be applied.

If S is stationary, then M_n and M_r are zero and $q_{(2)}$ is time independent.

Also τ^* is simply t-r/c. Under these conditions, Eq. (27), reduces to the

Kirchhoff formula for stationary surfaces, Eq. (10).

ALTERNATE FORM FOR APPLICATIONS

Because Eq. (27) contains $\phi_{\tau(y)}$ and also a time derivative of a compli-

cated function in the last integrand, it is still not in a form which is

readily applied for numerical analysis. In order to obtain a result which is

practically useful it is essential that the time derivative in the integrand of

the last term be evaluated analytically and the entire result written in terms

of geometric and kinematic quantities which are available or can be readily

measured. To this end the following relations will be used:

$$\frac{1}{c}\frac{\partial r}{\partial \tau} = \frac{1}{c}\frac{\partial}{\partial \tau}\left|\underset{\sim}{x} - \underset{\sim}{y}(u^1,u^2,0,\tau)\right| = -M_r \qquad (28\text{-}a)$$

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$$\frac{1}{c} \frac{\partial M_r}{\partial \tau} = \frac{1}{c} \overset{\bullet}{\underset{\sim}{M}} \cdot \overset{\wedge}{\underset{\sim}{r}} + \frac{1}{r}(M^2 - M_r^2) \qquad (28\text{-b})$$

$$\frac{1}{c} \frac{\partial \cos\theta}{\partial \tau} = \frac{1}{c} \overset{\bullet}{\underset{\sim}{n}} \cdot \overset{\wedge}{\underset{\sim}{r}} + \frac{1}{r}(M_r \cos\theta - M_n) \qquad (28\text{-c})$$

$$\frac{1}{c} \frac{\partial M_n}{\partial \tau} = \frac{1}{c}(\overset{\bullet}{\underset{\sim}{M}} \cdot \underset{\sim}{n} + \underset{\sim}{M} \cdot \overset{\bullet}{\underset{\sim}{n}}) \qquad (28\text{-d})$$

where the dot over $\underset{\sim}{M}$ and $\underset{\sim}{n}$ denote source time derivative keeping (u^1, u^2) fixed. In addition, the following definitions are introduced:

$$\overset{\bullet}{M}_r = \overset{\bullet}{\underset{\sim}{M}} \cdot \overset{\wedge}{\underset{\sim}{r}} \qquad (29\text{-a})$$

$$\overset{\bullet}{n}_r = \overset{\bullet}{\underset{\sim}{n}} \cdot \overset{\wedge}{\underset{\sim}{r}} \qquad (29\text{-b})$$

$$\overset{\bullet}{M}_n = \overset{\bullet}{\underset{\sim}{M}} \cdot \underset{\sim}{n} \qquad (29\text{-c})$$

$$\overset{\bullet}{n}_M = \overset{\bullet}{\underset{\sim}{n}} \cdot \underset{\sim}{M} \qquad (29\text{-d})$$

$$\frac{1}{\sqrt{g_{(2)}}} \frac{\partial \sqrt{g_{(2)}}}{\partial \tau} = \overset{\bullet}{\sigma} \qquad (29\text{-e})$$

Note that in the last integral of Eq. (27), a new time rate of change of Φ will appear when (u^1, u^2) is kept fixed. This is denoted $\overset{\bullet}{\Phi}$. The relation between $\Phi_{\tau(y)}$ and $\overset{\bullet}{\Phi}$ is obtained as follows:

$$\overset{\bullet}{\Phi} = \{\frac{\partial}{\partial \tau} \Phi[\underset{\sim}{y}(\underset{\sim}{u}, \tau), \tau]\}_{u_3 = 0} = \{\frac{\partial \Phi}{\partial y_i} \frac{\partial y_i}{\partial \tau} + \frac{\partial \Phi}{\partial \tau}\} u_3 = 0 \qquad (30)$$

$$= \{\underset{\sim}{v} \cdot \nabla\Phi + \Phi_{\tau(y)}\}_{u_3 = 0} = cM_n \Phi_n + c\underset{\sim}{M}_t \cdot \nabla_2 \Phi + \Phi_{\tau(y)}$$

where $\underset{\sim}{u} = (u^1, u^2, u^3)$, $\underset{\sim}{v} = \partial\chi/\partial\tau\big|_{\underset{\sim}{u}}$ and $\underset{\sim}{M}_t$ is the tangential component of the Mach number vector $\underset{\sim}{M} = \underset{\sim}{v}/c$ on S. Also ∇_2 stands for the surface gradient operator.

Equation (27) can now be written as

$$4\pi\widetilde{\Phi}(\underset{\sim}{x},t) = \int_{D(S)} \left[\frac{E_1\sqrt{q_{(2)}}}{r(1-M_r)}\right]_{\tau^*} du^1 du^2 + \int_{D(S)} \left[\frac{\Phi E_2\sqrt{q_{(2)}}}{r^2(1-M_r)}\right]_{\tau^*} du^1 du^2 \qquad (31)$$

where the expressions E_1 and E_2 are

$$E_1 = (M_n^2 - 1)\,\Phi_n + M_n\underset{\sim}{M}_t\cdot\nabla_2\Phi - c^{-1}M_n\dot{\Phi} + \frac{c^{-1}}{1-M_r}[(n_r - M_n - n_M)\Phi$$

$$+ (\cos\theta - M_n)\dot{\Phi} + (\cos\theta - M_n)\dot{\Phi}\sigma] + \frac{c^{-1}}{(1-M_r)^2}[\dot{M}_r(\cos\theta - M_n)\Phi] ; \qquad (32\text{-}a)$$

$$E_2 = \cos\theta + \frac{1}{1-M_r}[2M_r\cos\theta - M_n M_r - M_n] + \frac{1}{(1-M_r)^2}[(\cos\theta - M_n)(M_r^2 - M^2)] \qquad (32\text{-}b)$$

If the surface S is assumed rigid, then $q_{(2)}$ is independent of τ and

$$\dot{\underset{\sim}{n}} = \underset{\sim}{\omega} \times \underset{\sim}{n} \qquad (33)$$

where $\underset{\sim}{\omega}$ is the angular velocity of S. One can now write $dS = \sqrt{q_{(2)}}\,du^1 du^2$ for the element of area of S. Equation (31) can thus be written for a rigid surface as

$$4\pi\widetilde{\Phi}(\underset{\sim}{x},t) = \int_S \left[\frac{E_1}{r(1-M_r)}\right]_{\tau^*} dS + \int_S \left[\frac{\Phi E_2}{r^2(1-M_r)}\right]_{\tau^*} dS \qquad (34)$$

where E_1 and E_2 are again given by Eq. (32) using Eq. (33) where needed. The

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general Kirchhoff formula of Eqs. (31) and (34) is now in a form which is suit-
able for practical applications. These results appear not to have been pub-
lished elsewhere. Once more it can be checked that the Kirchhoff formula for
stationary surfaces, Eq. (10), can be obtained from these equations. Again it
can be seen that both Eqs. (31) and (34) remain valid for a piecewise smooth
surface S.

REMARKS ON EXTENSION TO SUPERSONIC MOTION

A more general Kirchhoff formula can be derived which is valid for deform-
able, piecewise smooth surfaces in arbitrary motion (subsonic or supersonic).
The derivation of this result is very lengthy and complicated, and it will not
be written out here. It relies on the solution of the following wave equation
given by the authors [12]:

$$\bar{\Box}^2 \Phi = \bar{\nabla}_4 \cdot [\underset{\sim}{Q} |\nabla f| \delta(f)]. \tag{35}$$

In this equation $\nabla_4 = (\nabla, c^{-1} \partial/\partial t)$ and $\underset{\sim}{Q}$ is a 4-vector. Note that $|\nabla f| = 1$ in the
present paper. Equation (13) can be written as

$$\bar{\Box}^2 \tilde{\Phi} = -(\Phi_n + \frac{1}{c} M_n \Phi_{t(x)}) \delta(f) - \bar{\nabla}_4 \cdot [\underset{\sim}{Q} \delta(f)] \tag{36}$$

where $\underset{\sim}{Q} = \Phi(\underset{\sim}{n}, M_n)$. Now the Green's function of the wave equation can be applied
to find $\tilde{\Phi}$ utilizing the result of reference [12] for the second term of Eq. (36).
The resulting Kirchhoff formula is more general than the formula presented here.
However, the various forms of the general Kirchhoff formula of the current paper
appear to be adequate for present applications in acoustics.

APPLICATIONS IN ACOUSTICS

As mentioned in the introduction, application of the Kirchhoff formula has been proposed for rotor noise prediction. Attempts have also been made by Forsyth and Korkan [13] to use this formula for high speed propeller noise calculation. These authors have proposed using the Kirchhoff formula for a stationary surface which, as shown above, is very different from the general Kirchhoff formula. The surface S in the work of Forsyth and Korkan is taken as a finite cylinder, fixed with respect to the propeller, with its axis along that of the propeller. This surface is therefore in motion with respect to the unbounded acoustic medium, and the general Kirchhoff formula must be applied to S.

One of the interesting applications of the general Kirchhoff formula may be in the prediction of the noise of high bypass ratio turbofans. These propulsors are currently favored in aircraft designs because of their high efficiency. By using a cylindrical surface S with its axis along the fan axis and its ends at the fan inlet and exhaust, Fig. 2, the main result of the present paper can be applied to predict the radiation pattern. There are several sophisticated turbomachinery aerodynamics codes which can supply the needed input data to the Kirchhoff formula.

Lyrintzis and George have developed a specialized Kirchhoff formula to calculate the far field noise from a single helicopter blade interacting with a vortex [14]. Their formula applies to uniform motion of a rigid surface and is derived by using Green's theorem. It is equivalent, in their case, to the general Kirchhoff formula derived here, Eq. (34).

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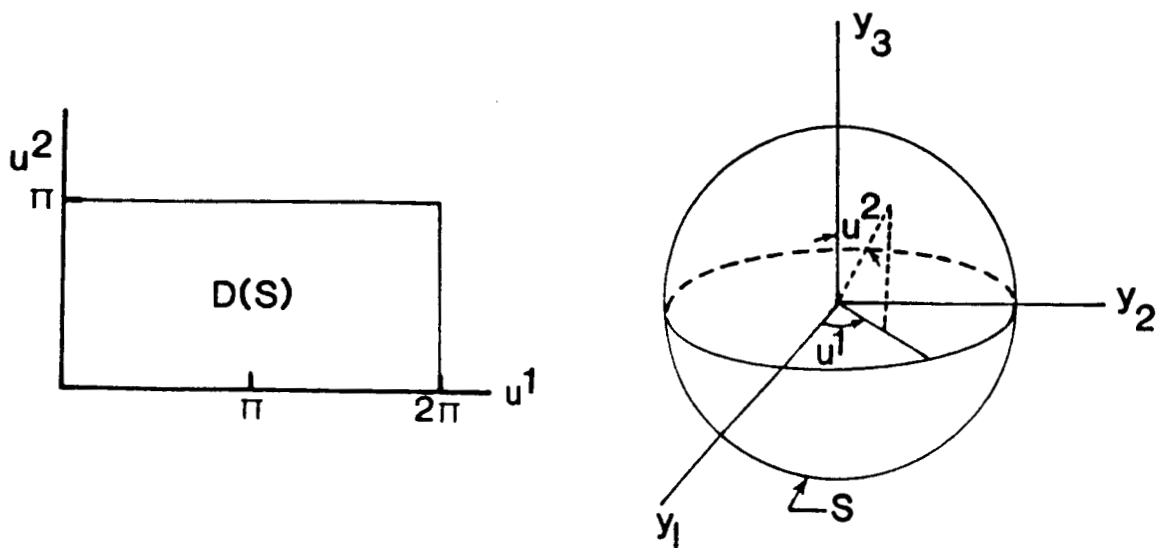
Figure 1: The mapping $y \to (u^1, u^2)$. Domain $D(S)$
is independent of time.

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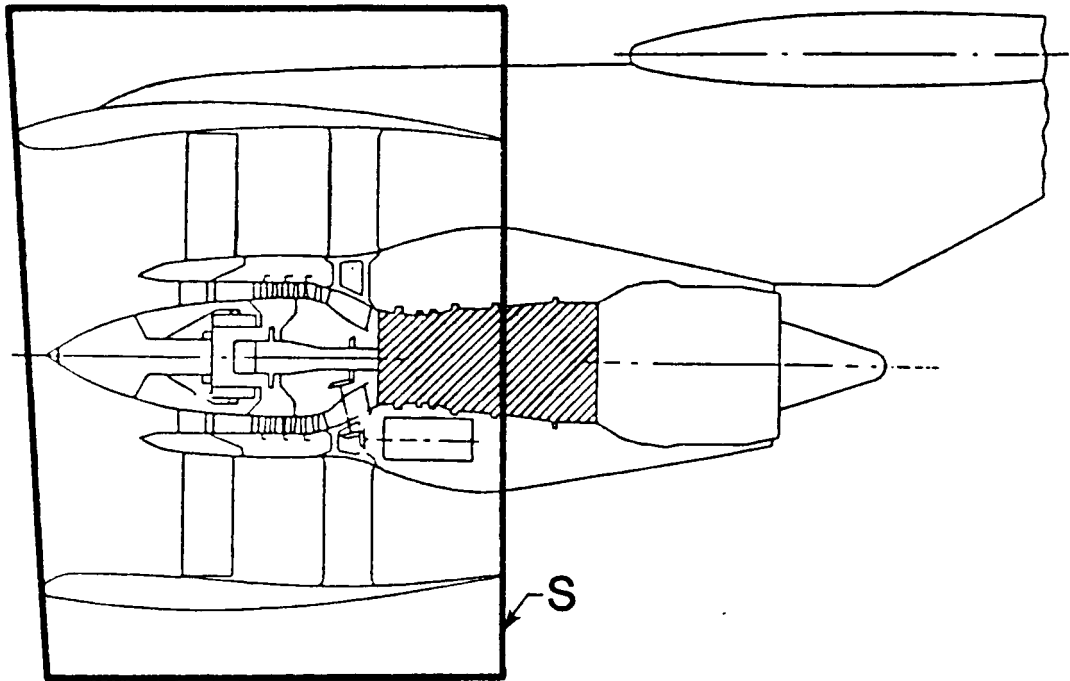
Figure 2: Illustrating a surface S appropriate for turbofan noise analysis.

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16. Abstract

Kirchhoff's formula for radiation from a closed surface has been used recently for
prediction of the noise of high speed rotors and propellers. Because the closed
surface on which the boundary data are prescribed in these cases is in motion, an
extension of Kirchhoff's formula to this condition is required. In this paper such
a formula, obtained originally by Morgans for the interior problem, is derived for
regions exterior to surfaces moving at speeds below the wave propagation speed by
making use of some results of generalized function theory. It is shown that the
usual Kirchhoff formula is a special case of the main result of the paper. The
general result applies to a deformable surface. However, the special form it assumes
for a rigid surface in motion is also noted. Some possible areas of application of
the formula to problems of current interest in aeroacoustics are discussed.

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