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A New Approach to State Estimation in Deterministic Digital Control Systems

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I. INTRODUCTION

Books on modern digital control systems usually address the problem of controlling a continuous-time plant driven by a zero-order-hold with a sampled output, as shown in Figure 1. For example, see Reference 1, page 126. A common solution to this problem is to estimate the state of the system at the sampling instant using a state observer and then feedback the estimated state [1, p. 195]. However, the state observer has two undesirable characteristics. First of all, it is a dynamical system in itself and, hence, adds additional states, eigenvalues, and dynamics to the system. This can affect system stability. Secondly, as a consequence, the estimated state is normally an approximation to the true state and is usually not a good one early in the estimation process. This paper presents a new approach to state estimation which has neither of these problems. The new estimator adds no new states, eigenvalues, or dynamics to the system and if the plant parameters are known exactly, the estimated state is actually equal to the true state. Useful in the development of the new state estimator are some results to date for continuous-time plants driven by a zero-order-hold with sampled outputs. These are reviewed in Section II, prior to the development of the new state estimator presented in Section III. Section IV presents an example which illustrates the procedure to follow to completely design the new estimator. Section V contains the conclusions and final comments.

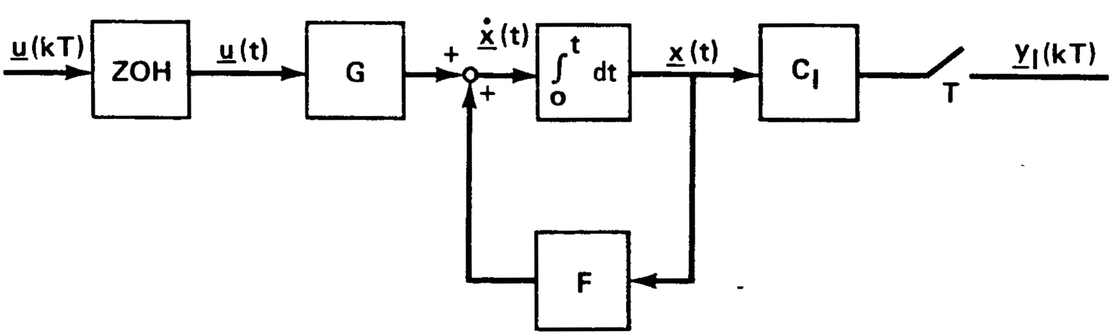

Figure 1. Continuous-time plant driven by a zero-order-hold with instantaneous measurements.

II. PRELIMINARY

For the plant in Figure 1, $\underline{x}(t)$ is an nxl state vector, $\underline{u}(k)$ is an rxl control input vector, $\underline{y}_I(k)$ is an mxl output or measurement vector, F is an nxn system matrix, G is an nxr control matrix, and C_I is an mxn output matrix. Since $\underline{y}_I(k) = C_I\,\underline{x}(k)$, where k is the usual shorthand notation for time kT, $\underline{y}_I(k)$ represents an instantaneous measure of the system at the sampling instant kT. Hence, the plant in Figure 1 will be regarded as having instantaneous measurements for outputs. It is well known that this system can be modeled at the sampling instants by the discrete state equations [1, p. 126]

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{u}(k) \tag{1}$$

$$\underline{y}_I(k) = C_I\,\underline{x}(k) \tag{2}$$

where

$$\phi(t) = \mathcal{L}^{-1}[(sI-F)^{-1}] \quad , \tag{3}$$

$$A = \phi(T) \quad , \tag{4}$$

and

$$B = \left[\int_0^T \phi(\lambda)\,d\lambda\right] G \quad . \tag{5}$$

$\phi(t)$ is the nxn state transition matrix. A and B are the nxn system matrix and the nxr control matrix, respectively, for the discrete state equations (1) and (2).

A and B can be determined analytically using equations (3) to (5). An alternative approach, which is also quite suitable for numerical computation, is as follows [2]. $\phi(t)$ and $\int_0^t \phi(\lambda)d\lambda$ can be expressed in the form of matrix exponential series as

$$\phi(t) = \sum_{i=0}^{\infty} \frac{F^i t^i}{i!} \tag{6}$$

and

$$\int_0^t \phi(\lambda)d\lambda = \sum_{i=0}^{\infty} \frac{F^i\,t^{i+1}}{(i+1)!} \quad , \tag{7}$$

respectively. From equations (6) and (7),

$$\phi(t) = I + F\left[\int_0^t \phi(\lambda)\,d\lambda\right] \tag{8}$$

where I is an nxn identity matrix. Hence, $\int_0^T \phi(\lambda)\, d\lambda$ can be determined using equation (7) with t = T and this result substituted into equation (8) to get $\phi(T)$. With these results, A and B can be found using equations (4) and (5).

Now consider the plant in Figure 2, which is a generalization of the one in Figure 1. In addition to the instantaneous measurement vector $\underline{y}_I(kT)$, the plant in Figure 2 has the measurement vector $\underline{y}'_F(kT)$ generated as follows. First, the continuous-time output $\underline{z}(t)$ is sampled every T/N seconds. Every N samples are multiplied by the weighting matrices H_j, j = 0,1,...,N-1, and then summed to generate the output $\underline{y}_F(kT)$, every T seconds. Functionally, this is equivalent to passing the discrete measurements generated every T/N seconds through a multi-input/multi-output moving average (MA) process with coefficient matrices H_j, j = 0,1,...,N-1 [3]. The output of the MA prefilter is sampled every T seconds to generate $\underline{y}_F(kT)$. Then, $\underline{y}_F(kT)$ has subtracted from it $E_- \underline{u}[(k-1)T]$, where E_- is a constant matrix, to produce the modified MA-prefiltered measurement vector $\underline{y}'_F(kT)$. In Figure 2, C_F is a pxn output matrix and $\underline{z}(t)$ is a pxl vector. The weighting matrices H_j, j = 0,1,...,N-1 are each qxp. Hence, $\underline{y}_F(kT)$ is a qxl intermediate output vector and $\underline{y}'_F(kT)$ a qxl output vector. Since $\underline{u}[(k-1)T]$ is an rxl delayed input vector, E_- is a qxr matrix.

Previously, Polites [4] showed that when

$$E_- = \left[\sum_{j=0}^{N-1} H_j\, C_F \int_0^{-j(T/N)} \phi(\lambda)\, d\lambda \right] G \quad , \tag{9}$$

the discrete state equations for the plant in Figure 2 become

$$\underline{x}(k+1) = A\, \underline{x}(k) + B\, \underline{u}(k) \tag{10}$$

$$\underline{y}_T(k) = \begin{bmatrix} \underline{y}_I(k) \\ \underline{y}'_F(k) \end{bmatrix} = \begin{bmatrix} C_I \\ D_- \end{bmatrix} \underline{x}(k) \tag{11}$$

where D_- is a qxn matrix given by

$$D_- = \sum_{j=0}^{N-1} H_j\, C_F\, \phi(-j\, \tfrac{T}{N}) \quad . \tag{12}$$

An alternative expression for equation (9) is

$$E_- = H\, \beta \tag{13}$$

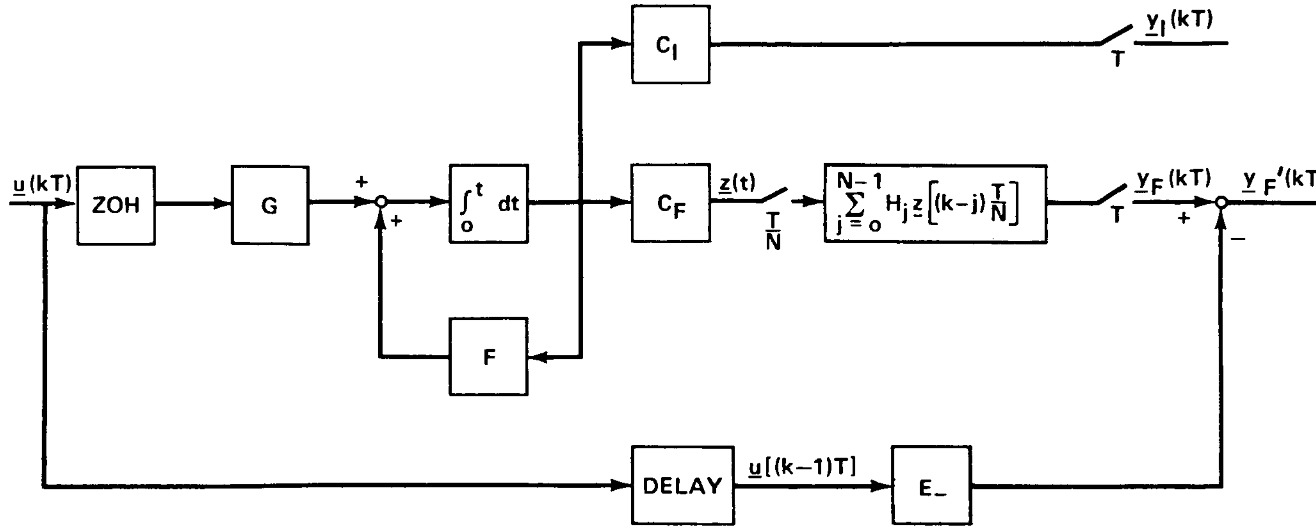

Figure 2. Continuous-time plant driven by a zero-order-hold with instantaneous and modified MA-prefiltered measurements.

where

$$H = [H_0 \mid H_1 \mid \cdots \mid H_{N-1}] \tag{14}$$

and

$$\beta = \begin{bmatrix} C_F \left[\int_0^{0} \phi(\lambda)\, d\lambda\right] G \\ C_F \left[\int_0^{-T/N} \phi(\lambda)\, d\lambda\right] G \\ \vdots \\ C_F \left[\int_0^{-(N-1)(T/N)} \phi(\lambda)\, d\lambda\right] G \end{bmatrix} . \tag{15}$$

In equation (14), H is a qx(Np) matrix and in equation (15), β is an (Np)xr matrix. An alternative expression for equation (12) is

$$D_{-} = H\alpha \tag{16}$$

where

$$\alpha = \begin{bmatrix} C_F\ \phi(0) \\ C_F\ \phi(-\frac{T}{N}) \\ \vdots \\ C_F\ \phi[-(N-1)\frac{T}{N}] \end{bmatrix} \quad . \tag{17}$$

In equation (17), α is an (Np)xn matrix.

E_- and D_- can be evaluated analytically using equations (3) and (13) to (17). An alternative approach, which can be either analytical or numerical, is as follows. Let $t = -j(T/N)$ where $j = 0,1,\ldots,N-1$ and use equation (7) to determine $\int_0^{-j(T/N)} \phi(\lambda)d\lambda, j = 0,1,\ldots,N-1$. Substitute these results into equation (8) to get $\phi[-j(T/N)], j = 0,1,\ldots,N-1$. At this point, E_- and D_- can be found using equations (13) to (17).

III. THE NEW STATE ESTIMATOR

A general block diagram of the plant and the new state estimator is shown in Figure 3. Observe that this is Figure 2 with

$$C_I = 0 \quad . \tag{18}$$

Thus, if E_- is given by equation (13), the discrete state equations for the complete system in Figure 3 are

$$\underline{x}(k+1) = A\ \underline{x}(k) + B\ \underline{u}(k) \tag{19}$$

$$\underline{y}_F'(k) = [H\alpha]\ \underline{x}(k) \quad . \tag{20}$$

This follows from equations (10), (11), (16), and (18). Recall that α is an (Np)xn matrix. If $Np \geq n$, or equivalently $N \geq (n/p)$, and α has maximal rank (i.e., rank n), then $(\alpha^T\alpha)$ is positive definite and nonsingular. In this case, H can be given by the pseudo inverse of α or [5]

$$H = (\alpha^T\alpha)^{-1}\ \alpha^T \quad . \tag{21}$$

In general, H is a qx(Np) matrix. However, when it is given by equation (21), it is an nx(Np) matrix. This implies that q = n in this case. Consequently, $\underline{y}_F'(kT)$, which is a qxl vector in general, is an nxl vector in this case. From equation (19) to (21), the discrete state equations for the system in Figure 3 become

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{u}(k) \tag{22}$$

$$\underline{y}_F'(k) = \underline{x}(k) \quad . \tag{23}$$

Hence, the measurement vector $\underline{y}_F'(k)$ equals the true state vector $\underline{x}(k)$. For H to be given by equation (21) and E_- by equation (13), the matrices F, G, and C_F must be known. Given H, the coefficient matrices H_j, j = 0,1,...,N-1 in the MA prefilter are determined by partitioning H into N nxp submatrices. This follows from equation (14).

At last, the new state estimation scheme, and the procedure for selecting the parameters in it, can be summarized as follows. A general block diagram of the new state estimator and the plant whose states are to be estimated is shown in Figure 3. The MA-prefilter coefficient matrices H_j, j = 0,1,....,N-1 and the matrix E_- are to be calculated so the discrete state equations for the entire system in Figure 3 are given by equations (22) and (23). Consequently, the output vector $\underline{y}_F'(k)$ does more than estimate the state vector $\underline{x}(k)$; it equals it! Realize that for H_j, j = 0,1,...,N-1 and E_- to be calculated so this is true requires that F, G, and C_F in the plant be known precisely. Recall that C_F is a pxn matrix. Choose the length of the MA prefilter, N, so that N $\geq$ (n/p). Knowing N and F, $\int_0^{-j(T/N)} \phi(\lambda)d\lambda$ and $\phi[-j(T/N)]$, j = 0,1,...,N-1, can be calculated using either of the methods described in Section II. Having these and knowing G and C_F, α and β can be determined using equations (17) and (15), respectively. Test α to be sure that it has maximal rank (i.e., rank n). If so, let H be given by equation (21). The matrix E_- can now be determined using equation (13). Partitioning H into N nxp submatrices as in equation (14) reveals the MA prefilter coefficient matrices H_j, j = 0,1,...,N-1. The estimator is now completely defined.

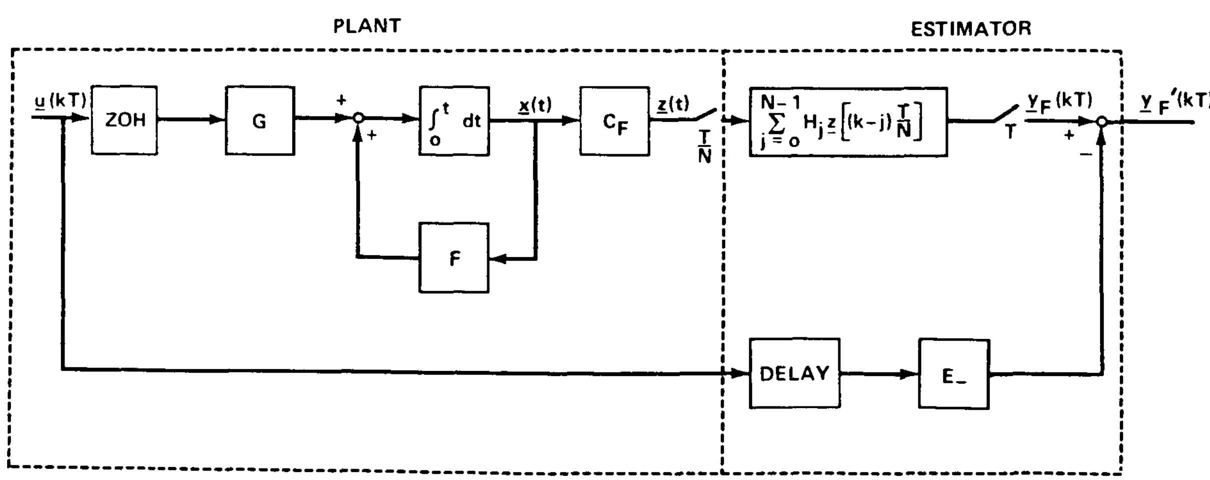

Figure 3. General block diagram of the plant and the new state estimator.

IV. AN EXAMPLE

Consider the double integrator plant driven by a zero-order-hold as shown in Figure 4. The continuous-time output is sampled every T/N seconds and input into the new state estimator along with the control input u(kT). The parameters in the estimator are to be chosen so the discrete state equations for the complete system are given by equations (22) and (23).

Manipulating the plant in Figure 4 into the format of Figure 3 yields

$$[F] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \tag{24}$$

$$[G] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \tag{25}$$

and

$$[C_F] = [1 \quad 0] \quad . \tag{26}$$

Using equations (24) and (25) and the formulas presented in Section II,

$$\phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad , \tag{27}$$

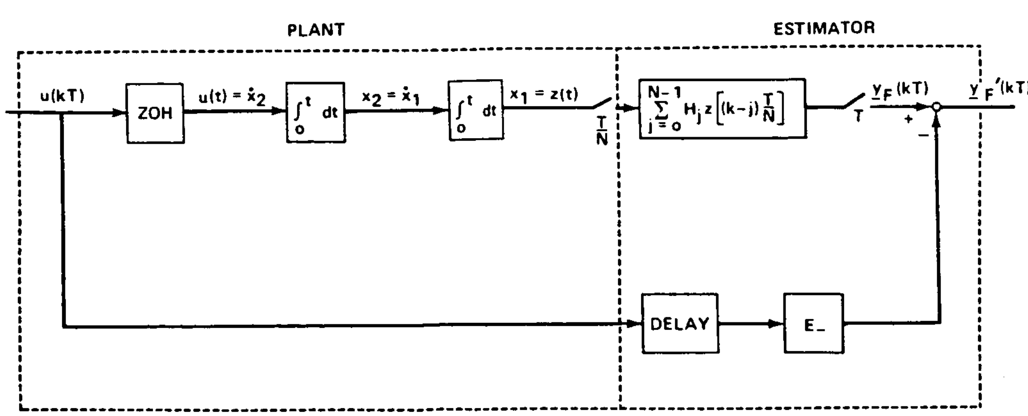

Figure 4. Plant and new state estimator for the example.

$$\int_0^t \phi(\lambda)\, d\lambda = \begin{bmatrix} t & \frac{t^2}{2} \\ 0 & t \end{bmatrix} , \tag{28}$$

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} , \tag{29}$$

and

$$B = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} . \tag{30}$$

Since C_F is a pxn matrix, it follows from equation (26) that $p = 1$ and $n = 2$. Hence, the requirement to select N so that $N \geqslant (n/p)$ can be satisfied by letting

$$N = 4 . \tag{31}$$

Using equations (15), (17), (25) to (28), and (31), α and β are found to be

$$\alpha = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{T}{4} \\ 1 & -\frac{2T}{4} \\ 1 & -\frac{3T}{4} \end{bmatrix} \tag{32}$$

and

$$\beta = \begin{bmatrix} 0 \\ \frac{T^2}{32} \\ \frac{4T^2}{32} \\ \frac{9T^2}{32} \end{bmatrix} , \tag{33}$$

respectively. In equation (32), it is apparent that the rank of α is maximal (i.e., rank = 2). Consequently, H can be given by equation (21). Substituting equation (32) into (21) yields

$$H = [H_0 \mid H_1 \mid H_2 \mid H_3] = \left[\begin{array}{c|c|c|c} \left(\frac{7}{10}\right) & \left(\frac{4}{10}\right) & \left(\frac{1}{10}\right) & \left(-\frac{2}{10}\right) \\ \left(\frac{6}{5T}\right) & \left(\frac{2}{5T}\right) & \left(-\frac{2}{5T}\right) & \left(-\frac{6}{5T}\right) \end{array}\right] \tag{34}$$

where the columns of H are the coefficient matrices for the MA prefilter by virtue of equation (14). Next, $E\_$ can be calculated using equations (13), (33), and (34) and is found to be

$$E\_ = \begin{bmatrix} \left(-\frac{T^2}{32}\right) \\ \left(-\frac{3T}{8}\right) \end{bmatrix} . \tag{35}$$

The estimator is now completely defined. Using equations (32) and (34), one can verify that

$$H\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Hence, the discrete state equations for the entire system in Figure 4, with the estimator parameters defined by equations (31), (34), and (35), are given by equations (22) and (23) where A and B are defined by equations (29) and (30), respectively.

V. CONCLUSIONS

A new state estimator for deterministic digital control systems has been developed which offers two distinct advantages over the widely-used state observer. First of all, it contributes no additional states, eigenvalues, or dynamics to the system and, consequently, will not alter the stability of the system. Secondly, if the plant parameters are known exactly, the estimated state equals the true state and is not just an approximation to it. A disadvantage of this scheme is that the plant output must be sampled at a rate faster than the state observer's. Fortunately, this is not the problem it would have been 20 years ago, considering the speed of today's digital computers. Still, if the sample rate on the output is so fast the computer solving the feedback control algorithms has a problem solving the estimation algorithms also, the following implementation can be used. A microprocessor can be dedicated to sampling the plant output and solving the estimation algorithms to the point where the intermediate output $\underline{y}_F(kT)$ is generated. This can be sent, every T seconds, to the control computer solving the feedback control algorithms. There, the state estimate $\underline{y}_F'(kT)$ can be calculated by subtracting $E_- \underline{u}[(k-1)T]$ from $\underline{y}_F(kT)$. This result can then be input into the feedback control algorithms to generate the control input $\underline{u}(kT)$. This process is repeated every T seconds.

If the work in this paper is extended, the recommendation is to investigate the robustness of the new state estimator and see how it compares with the state observer's. Specifically, the following questions should be addressed. What effect do modeling errors in the plant have on the new estimator and how does this compare with the state observer? What effect do plant process and measurement noise have on the new estimator and how does this compare with the state observer, or even the Kalman filter? How can the robustness of the new estimator be improved? Increasing the length of the MA prefilter, N, may be one possibility. Catenating the new estimator with a state observer or a Kalman filter could be another. This could produce a composite estimator which is better than either individual estimator by itself.

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15. SUPPLEMENTARY NOTES

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16. ABSTRACT

This paper presents a new approach to state estimation in deterministic digital control systems. The scheme is based on sampling the output of the plant at a high rate and prefiltering the discrete measurements in a multi-input/multi-output moving average (MA) process. The coefficient matrices in the MA prefilter are selected so the estimated state equals the true state. An example is presented which illustrates the procedure to follow to completely design the estimator.

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