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Self-Consistent Inclusion of Space-Charge in the Traveling Wave Tube

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SUMMARY

We show how the complete field of the electron beam may be incorporated into the transmission line model theory of the traveling wave tube (TWT). The fact that the longitudinal component of the field due to the bunched beam is not used when formulating the beam-to-circuit coupling equation is not well-known. The fundamental partial differential equation for the traveling wave field is developed and compared with the older (now standard) ones. The new equation can be solved numerically using the same algorithms, but now the coefficients can be updated continuously as the calculation proceeds down the tube. The coefficients in the older equation are primarily derived from preliminary measurements and some trial and error. The newer coefficients can be found by a recursive method, since each has a well defined physical interpretation and can be calculated once a reasonable first trial solution is postulated. We compare the results of the new expression with those of the older forms, as well as to a field theory model to show the ease in which a reasonable fit to the field prediction is obtained. A complete summary of the existing transmission line modeling of the TWT is given to explain the somewhat vague ideas and techniques in the general area of drifting carrier-traveling circuit wave interactions. The basic assumptions and inconsistencies of the existing theory and areas of confusion in the general literature are examined and hopefully cleared up.

INTRODUCTION

The main thrust of this report is to indicate how the forces of the space-charge in the electron beam may be incorporated into the general theory of the traveling wave tube (TWT) in a self-consistent manner. In earlier analyses, the inclusion of the space-charge forces has been done in a hueristic manner. This fact, however, is seldom stated in most of the literature. It should be mentioned that nearly all the theories and analyses available in the open literature to date, rely on this somewhat subtle, but ad hoc method of incorporating the "space-charge effect." Perhaps the earliest clear explanation of the neglect of space-charge forces was given by Nordsieck (ref. 29). The paper was concerned with the large signal analysis of the TWT, we quote, "Probably the most serious approximation made in the work is the assumption that the electrons are accelerated by the electric fields of the circuit only, not by the fields of neighboring electrons (neglect of space-charge forces). No satisfactory way has been found to include these space-charge forces in the calculation without an unwarranted increase in labor. Thus the results are strictly applicable only to very small experimental beam densities; however, it is felt that at all beam current densities used in practice, the only serious effect of the space-charge forces is to reduce the high-harmonic content

of the beam below that calculated." The paper then gives the partial differential equation for the voltage wave induced on the circuit by the bunched electron beam

$$\frac{\partial^2 V}{\partial t^2} - u_o^2 \frac{\partial^2 V}{\partial z^2} = u_o Z_o \frac{\partial^2 \rho_\ell}{\partial t^2} \tag{1}$$

where $u_o^2 = 1/(L_o C_o)$, $Z_o = \sqrt{L_o/C_o}$, have been used. Here V is the voltage induced on the equivalent transmission line which models the actual slow-wave circuit, L_o and C_o are the effective inductance and capacitance per unit length of this transmission line. The forcing function on the right side ρ_ℓ is the linear charge density (charge per unit length) of the bunched electron beam.

A very detailed large-signal study of the TWT was performed by Rowe (ref. 43) and Detweiler (ref. 30), and the resulting computer program has become the basis of some analytical procedures used in tube design at both NASA and Hughes Aircraft (private communications). The method of handling space-charge is again very approximate for the following reasons. First of all the basic equation for the induced voltage can take on two different forms. The most basic one being

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{u_o^2} \frac{\partial^2 V}{\partial t^2} - \frac{2\omega Cd}{u_o^2} \frac{\partial V}{\partial t} = - \frac{Z_o}{u_o} \left[\frac{\partial^2 \rho_\ell}{\partial t^2} + 2\omega Cd \frac{\partial \rho_\ell}{\partial t} \right]$$

where C and d are the gain and loss parameters respectively. Inspection shows this form reduces to the earlier equation when the loss parameter goes to zero. A second form for this equation is

$$\frac{\partial^2 V}{\partial t^2} - u_o^2 \frac{\partial^2 V}{\partial z^2} + 2\omega Cd \frac{\partial V}{\partial t} = u_o Z_o \left[\frac{\partial^2 \sigma_\ell}{\partial t^2} + 2\omega Cd \frac{\partial \sigma_\ell}{\partial t} \right]$$

where now σ_ℓ is an equivalent induced charge density on the circuit due to the bunched beam. The term σ_ℓ is calculated as follows; the defining expression is

$$\sigma_\ell = 2\pi \int_0^{b'} \psi(r) \rho_o r_o \left| \frac{\partial z_o}{\partial z} \right| dr_o \quad .$$

Here $\psi(r)$ is a coupling function that must be approximated, and it is found in the limit where ρ_ℓ is zero. The angular frequency is ω and ρ_o, r_o are the dc charge density and beam radius at the entrance to the tube. The term b' is the beam diameter, z_o is a reference plane, while z is the distance along the tube. We quote from the thesis (ref. 30) on p. 66, "this coupling function is seen to have the same form as that for a thin hollow stream (cf. Rowe, ref. 10) which is a consequence of neglecting the r-f charge in the stream. Admittedly, neglect of the r-f stream charge in determining the

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coupling functions is an approximation. However, the overall effect of this approximation should be small and seems well justified, especially in view of the considerable complexity of solving the problem exactly." A similar statement appears on p. 51 of the review paper (ref. 31) by Rowe and Detweiler.

It is not clear to the present author that the assumptions of either completely ignoring the space-charge or finding the coupling function in the limit when the space-charge density goes to zero, are easily justified. It appears that since no way could be found to include it, it was just convenient to make whatever assumptions necessary to obtain a solution. The main objective of this report is to include the space-charge into the theory in a self-consistent manner, and to test the new results against the older ones to find out if indeed the older assumptions are acceptable. It is surprising that, to the author's knowledge, only one published paper in the literature (ref. 35) has attempted to compare the computed tube parameters with those obtained from measurements. This paper will be discussed in a later section. It is perhaps surprising that the TWT theory is not on a firmer foundation than that indicated above. In view of this, the considerable length of the report is thought to be necessary to state our position, as well as explain our contribution to the theory.

In most textbooks the theory and analysis of the TWT is presented along the lines of Pierce's transmission line model (refs. 1 to 19). This model has also become the standard for many discussions of the TWT and related beam devices in the general literature. The central expression for the theory is the beam-to-circuit (BTC) coupling equation stated earlier.

$$\frac{\partial^2 V}{\partial z^2} - L_0 C_0 \frac{\partial^2 V}{\partial t^2} = - L_0 \frac{\partial^2 \rho_\ell}{\partial t^2}$$

When variation of the form $e^{j\omega t - \Gamma z}$ is used, we find

$$E_z^{CKT} = - \frac{\partial V}{\partial z} = \frac{-K \Gamma_1 \Gamma^2}{\Gamma^2 - \Gamma_1^2} i$$

where i is the ac beam current, and K is the characteristic impedance of the line. The theory is actually internally inconsistent in that the approximations needed to develop all of the above forms for the BTC (including the second form by Detweiler using the coupling function $\psi(r)$, essentially neglect the longitudinal flux of the space-charge. However, this flux is included in the equation of motion for the electron beam. To be consistent, this flux and its forces should be included in the BTC as well as the equation of motion in a self-consistent manner. One reason this compatibility problem has gone either unnoticed or neglected, is due to the extreme flexibility of the model. This model, by necessity, must have at least two fit parameters, which may be continuously adjusted to match theory to experiment. Thus any consistency problems are effectively screened by appropriate adjustment in the parameters. Thus the basic parameters of the theory, C the gain parameter, and QC the space-charge parameter, really are curve-fit derived quantities, and are not really based on fundamental concepts as is often assumed or implied in the literature. It is also interesting that the three waves predicted by the

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model have never been directly verified by any published experimental results (private communication with A.S. Gilmour).

When the longitudinal flux of the space-charge is included in the analysis in a self-consistent manner, the BTC can be written in any of the following forms;

$$\frac{\partial^2 V}{\partial z^2} - L_o C_o \frac{\partial^2 V}{\partial t^2} = - L_o(1 - R^2) \frac{\partial^2 \rho_\ell}{\partial t^2} \tag{2a}$$

$$\frac{\partial^2 V}{\partial z^2} - L_o C_o \frac{\partial^2 V}{\partial t^2} = - L_o \frac{\partial^2 \rho_\ell}{\partial t^2} - L_o \epsilon_o A_1 g \frac{\partial^4 V}{\partial t^2 \partial z^2} \tag{2b}$$

$$\frac{\partial^2 V}{\partial z^2} - L_o C_o \frac{\partial^2 V}{\partial t^2} = - L_o \frac{\partial^2 \rho_\ell}{\partial t^2} - L_o \epsilon_o A_1 f \frac{\partial^3 \rho_\ell}{\partial t^3} \tag{2c}$$

where ϵ_o and A_1 are the permittivity and cross-sectional area of the beam. The quantity R is the space-charge reduction factor, and the factors g and f will be defined later. In the next sections we will develop equation (2) and compare the propagation constants obtained from it with those developed from the original "Pierce" technique, where C and QC are the fit parameters, along with the "conventional" approach, where the fit parameters are C and R. Then these results will be compared to a "field theory" analysis developed by Chu and Jackson (ref. 32). The comparisons will be carried out using data for the Western Electric 444A, 5W, 6 GHz tube.

The basic "Pierce" theory, being rather old and reasonably established, due primarily to its longevity, has been used to study the interaction of drifting carriers in semiconductors and external traveling waves; both acoustic (ref. 20) and electromagnetic (refs. 21 to 28). The neglect of inter-bunch forces (the "space-charge effect," in tube discussions) is highly questionable in solid-state plasmas, where the charge density is very high. It appears the authors of these papers were not aware of the seldom (if ever) mentioned restriction of neglect of the longitudinal flux (inter-bunch forces) in TWT theory. This fact punctuates the notion that the internal inconsistency of the TWT analysis is generally not considered important, or more probably not known to the general electrical engineering community. As a matter of fact, the two rather clear quotes cited above are the only ones known to the author that clearly spell out the true nature of the approximations in the theory.

The report traces the rather sketchy and uneven development of the theory to date. The reasons why the inter-bunch flux was originally omitted (initially unknown to Pierce) is explained in detail. The subsequent attempts to include it, by incorporating it into just the equation of motion is presented. The inconsistency of this procedure is thoroughly explained. These points, though important, are treated in separate appendices, in an attempt to sort out some very confusing and incorrect statements in the literature. We show that the transmission line model (TLM) for the helix follows directly from Maxwell's equations, and in that respect, rests on firm footing. Surprisingly Pierce chose to abandon any real faith in the equivalent transmission line (see p. 112 of his text, ref. 3), as he and others were attempting to put the

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initial ideas onto a firmer foundation. This vacillation is in part the cause
of some of the confusion in the theory. The structure of the report (many
appendices), is due in part to the rather disjointed manner in which the
theory has evolved. We attempt to clarify most, if not all, of the confusion
in this theory.

The next section summarizes the TLM originally developed by Pierce. Then
we develop the new form for the BTC coupling equation, which is equation (2),
which is the main objective of the paper. Next we compare the above models
with a field theory derived result due to Chu and Jackson (ref. 32). Then the
summary and final comments are presented. Finally the ten appendices address
specific points not covered in detail elsewhere. One reason for these is to
show without question, that the existing theory is not a closed and rigorous
one, as is apparently assumed in the general electrical engineering community.

TLM CONCEPTS

The Transmission Line Model (TLM) is that of an almost ordinary transmis-
sion line coupled to a nearby electron stream as shown in figure 1. It may be
beneficial to read the appendices before reading further, if one is not
extremely familiar with the circuit approach to TWT theory. The field of the
wave guided by the circuit, bunches the electrons in the beam, producing an ac
convection current i. The bunched beam reinduces the signal back onto the
circuit via the impressed current J^i. The normal modes of the beam (classic
space-charge waves), couple to the spatial harmonic with phase velocity near
the beam drift velocity u_0. The system is treated as a coupled mode problem
with weak coupling. The model has been popular due to the ease with which the
propagation constants may be found. For variation of the form $e^{j\omega t - \Gamma z}$, the
propagation constants Γ_i must satisfy

$$- \frac{\partial}{\partial z} \rightarrow \Gamma \tag{3}$$

$$a_4 \Gamma^4 + a_3 \Gamma^3 + a_2 \Gamma^2 + a_1 \Gamma + a_0 = 0 \ . \tag{4}$$

The system predicts three forward waves (thus the name "three-wave theory"),
and one backward traveling wave. The overall coupling strength is reflected
in the gain parameter C, which is on the order of 10^{-2}. The space-charge fac-
tor Q is on the order of five. Ultimately these two fit parameters must be
found by comparing theory to an actual tube, although some efforts have been
made to calculate them. The need for two essentially free fit parameters is
the price one pays for the simplicity of the dispersion relation (eq. (4)).

The development of the TLM may be found in the papers and text by Pierce
(refs. 1 to 3). For clarity, however, we will develop the major results in an
order and manner different from his presentation. A review of the evolution
of the theory may be found in Appendix A. There, the reasons for the incon-
sistent inclusion of E_z^{sc} (the longitudinal field of the space-charge) may be
found. The three-wave theory has received very detailed treatment in some of
the texts cited earlier, and the reader should consult them for more expanded
explanations than those given here. The basic idea is illustrated in figure 1.
The bunched beam forms the ac convection current i. Some of the flux lines
from the bunches terminate on the circuit, while others terminate within the

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beam itself. Those from the beam to the circuit represent the displacement current that couples beam energy into the circuit proper. Those that are confined solely to the beam represent inter-bunch forces, and collectively they constitute the so-called "space-charge" effect. The composite picture is supposed to represent the coupling of the normal modes of both the beam and circuit. The coupling is assumed weak, so a linear theory is applicable.

The main equation to be solved is that for the motion of the drifting electrons

$$\frac{Dv}{Dt} = - |\bar{\eta}| \left(E_z{}^{ckt} + E_z{}^{sc} \right)$$ (5)

where $E_z{}^{ckt}$ and $E_z{}^{sc}$ represent the electric fields of the circuit and beam respectively. The superscript sc means "space-charge." The capital D's on the left side denote the fluid or "substantial" derivative. The quantity $|\bar{\eta}|$ is the charge to mass ratio of an electron. The circuit field is the field on the slow wave structure when the beam is absent. The space-charge field is that due to the bunched beam when the effects of the circuit have somehow been removed (no signal on the line). Using the definition of ac convection current and the continuity equation, the above becomes

$$\frac{u_0{}^2 (j\beta_e - \Gamma)^2}{j\omega\rho_0 A_1} i = - |\bar{\eta}| \left(E_z{}^{ckt} + E_z{}^{sc} \right)$$ (6)

To obtain the dispersion equation, one must relate $E_z{}^{ckt}$ and $E_z{}^{sc}$ to the convection current i. The circuit field is first related to the impressed current density J^i by:

$$E_z{}^{ckt} = \frac{-K\Gamma\Gamma_1}{\Gamma^2 - \Gamma_1{}^2} J^i$$ (7)

This equation is the key one for the TLM, and its development may be found in any of the texts cited earlier. However, in all of them, the impressed current is expressed as $J^i = - \partial i/\partial z$ which we will show implies the neglect of the longitudinal field of the space-charge bunches. Substituting this into equation (6) yields

$$\frac{u_0{}^2 (j\beta_e - \Gamma)^2}{j\omega\rho_0 A_1} i = - |\bar{\eta}| \left[\frac{-K\Gamma\Gamma_1}{\Gamma^2 - \Gamma_1{}^2} J^i + E_z{}^{sc} \right]$$ (8)

Now we must find equations relating J^i and $E_z{}^{sc}$ to the convection current i. When this is accomplished, i cancels on both sides, leaving the dispersion equation, (the fourth order polynomial given earlier, see eq. (4)). The next section obtains self-consistent relationships between J^i, $E_z{}^{sc}$ and i.

DEVELOPMENT OF J^i

Physically, J^i is to represent the displacement current from the beam to the circuit, note it has units of amps/meter. With reference to figure 2, the

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total displacement current into the circuit from the beam segment of length Δz is

$$J^i \, \Delta z = j\omega\epsilon_0 E_r{}^{SC} A_2 \tag{9}$$

where $E_r{}^{SC}$ is the radial electric field component, and A_2 is the area of the sidewall of the cylinder surrounding the beam and nearly touching the circuit. From Maxwell's equations in cylindrical coordinates, we may express $j\omega\epsilon_0 E_r{}^{SC}$ (at a point where particle current is absent) as

$$j\omega\epsilon_0 E_r{}^{SC} = -\frac{\partial H\varphi}{\partial z} \tag{10}$$

Using Ampere's law to find $H\varphi$ when the total current consists of both convection and displacement parts

$$H_\varphi = \frac{i}{2\pi r_a} + j\omega\epsilon_0 \frac{r_b{}^2}{2r_a} E_z{}^{SC} \tag{11}$$

and further manipulation yields

$$J^i = -\frac{\partial i}{\partial z} - j\omega\epsilon_0 A_1 \frac{\partial E_z{}^{SC}}{\partial z} \tag{12}$$

where A_1 is the cross-sectional area of the beam. A second method to develop J^i is given in Appendix B, where it is shown that J^i may also be written as

$$J^i = j\omega\epsilon_0 A_1 \frac{\partial E_y{}^{SC}}{\partial y} \tag{13}$$

The original form for J^i, developed by Pierce (ref. 3), drops the second term in equation (12) that is,

$$J^i = -\partial i/\partial z \tag{12a}$$

which neglects the inter-bunch forces ($E_z{}^{SC} \rightarrow 0$). The equation of motion now becomes

$$\frac{u_0{}^2(j\beta_e - \Gamma)^2}{j\omega\rho_0 A_1} i = -|\bar{n}| \left[\frac{-K\Gamma\Gamma_1}{\Gamma^2 - \Gamma_1{}^2} \left(\Gamma i + j\omega\epsilon_0 A_1 \Gamma E_z{}^{SC} \right) + E_z{}^{SC} \right] \tag{14}$$

where it is apparent that we need only relate $E_z{}^{SC}$ to i to complete the calculation. We show in the next section that one may use three equivalent methods to close the calculation.

SELF-CONSISTENT METHODS

We present three alternative self-consistent methods to eliminate $E_z{}^{SC}$ from equation (14). The first involves the space charge reduction factor R.

For a review of the concepts associated with R, see appendix C. From Poisson's equation

$$\epsilon_o \frac{\partial E_y^{sc}}{\partial y} + \epsilon_o \frac{\partial E_z^{sc}}{\partial z} = \rho \qquad (15)$$

define

$$\epsilon_o \frac{\partial E_z^{sc}}{\partial z} = R^2 \rho \qquad (16)$$

then

$$\epsilon_o \frac{\partial E_y^{sc}}{\partial y} = \rho(1 - R^2) \qquad (17)$$

Use the continuity equation to eliminate ρ in equation (16),

$$\epsilon_o \frac{\partial E_z^{sc}}{\partial z} = \frac{jR^2}{\omega A_1} \left(\frac{\partial i}{\partial z}\right)$$

assume the integration constant is zero, then

$$E_z^{sc} = \frac{jR^2}{\omega \epsilon_o A_1} i \qquad (18)$$

which relates E_z^{sc} to the convection current via R. Substituting into equation (14) yields

$$\frac{u_o^2(j\beta_e - \Gamma)^2}{j\omega \rho_o A_1} i = |\bar{n}| \left[\frac{-K\Gamma \Gamma_1}{\Gamma^2 - \Gamma_1^2} \left(\Gamma i - R^2 \Gamma i\right) + \frac{jR^2 i}{\omega \epsilon_o A_1} \right] \qquad (19)$$

Rearrange equation (19) and list it with the "conventional" (see appendix D) space-charge reduction factor approach, along with Pierce's original QC approach (see appendix A).

$$\left[\Gamma^2 - \Gamma_1^2\right] \left[(j\beta_e - \Gamma)^2 + \frac{(R\omega_p)^2}{u_o^2}\right] = j2\beta_e \Gamma_1 \Gamma^2 C^3 (1 - R^2) \qquad (20a)$$

$$\left[\Gamma^2 - \Gamma_1^2\right] \left[(j\beta_e - \Gamma)^2 + \frac{(R\omega_p)^2}{u_o^2}\right] = j2\beta_e \Gamma_1 \Gamma^2 C^3 \qquad (20b)$$

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$$\left[\Gamma^2 - \Gamma_1^2 \right]\left[(j\beta_e - \Gamma)^2 - c^2\Gamma^2(4QC) \right] = -j2\beta_e\Gamma_1\Gamma^2c^3 \qquad (20c)$$

Notice that all three may be made to yield nearly the same numerical roots by appropriate choice of the fit parameters. The fit pairs are (R,C) in the first two, and (Q,C) in the latter. Notice that the "conventional" reduction factor method equation (20b) permits finite R on the left side of the equation, while assuming it to be zero on the right. Alternatively one may say the factor $(1 - R^2)$ is just absorbed into the effective gain parameter c^3. Pierce's QC method also absorbs the term $(1 - R^2)$ into the effective gain parameter c^3, and the Q factor may be adjusted on the left side such that the last terms in the brackets of equations (20a) and (20c) are equal.

The reduction factor method gives J^i the form

$$J^i = - \frac{\partial i}{\partial z} (1 - R^2) \qquad (21)$$

which when used in equation (7), gives the differential equation relating the circuit voltage and charge density as

$$\frac{\partial^2 V}{\partial z^2} - L_0 C_0 \frac{\partial^2 V}{\partial t^2} = - L_0 (1 - R^2) \frac{\partial^2 \rho_\ell}{\partial t^2} \qquad (22)$$

comparing this with the older form wherein J^i is given by equation (12a),

$$\frac{\partial^2 V}{\partial z^2} - L_0 C_0 \frac{\partial^2 V}{\partial t^2} = - L_0 \frac{\partial^2 \rho_\ell}{\partial t^2} \qquad (23)$$

Thus only the inductance on the right hand side is modified by the factor $(1 - R^2)$ when space-charge is included in a self-consistent manner. This shows the transmission line model has great flexibility; the older approach of neglecting $E_z{}^{sc}$ in the impressed current J^i yet keeping it in the equation of motion, is completely screened by appropriate choice of the fit parameter c^3, which, if consistency is strictly adhered to, should be c^3 multiplied by the factor $(1 - R^2)$.

The second method to eliminate $E_z{}^{sc}$ is to just assume

$$E_z{}^{sc} = fi \qquad (24)$$

where f is a complex fit parameter. This allows a little more flexibility in matching up with the field analysis results. We are nearly forced into this assumption, since we have "used up" all of the standard equations relating field, charge, current, etc., and it is not apparent what additional constraints may be found to tie down the relationship between $E_z{}^{sc}$ and i. The units of f are ohm/meter, so it is a distributed impedance. Substituting equation (24) into equation (14) and rearranging yields

$$\left[\Gamma^2 - \Gamma_1^2 \right]\left[(j\beta_e - \Gamma)^2 - (\omega_p/u_0)^2 j\omega\epsilon_0 A_1 f \right] = -j2\beta_e\Gamma_1\Gamma^2c^3(1 + j\omega\epsilon_0 A_1 f) \qquad (25)$$

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Comparing with equation (20a) we notice this choice is equivalent to the reduction factor method where

$$R^2 = -j\omega\epsilon_o A_1 f \tag{26}$$

which means the reduction factor is now complex. In the past R has always been defined to be pure real.

The third scheme is to assume the circuit $E_z{}^{ckt}$ and space charge field $E_z{}^{sc}$ are related by a complex factor g.

$$E_z{}^{sc} = g\ E_z{}^{ckt} \tag{27}$$

The phase of g may be estimated from physical reasoning concerning the energy exchange from the beam to the circuit. Since the bunches should be in the decelerating phase of the circuit wave during the exchange, the phase of g should be somewhere between 90° and 180° (fig. 3). After substituting into equation (6) and using equations (7) and (9), we get

$$\left[\Gamma^2 - \Gamma_1{}^2\right]\left[(j\beta_e - \Gamma)^2\right] = -j\tau\Gamma_1\Gamma^2\left[(\omega_p/u_o)^2(1 + g) + g(j\beta_e - \Gamma)^2\right] \tag{28}$$

$$\tau = \omega\epsilon_o A_1 K \qquad K = (L_o/C_o)^{1/2}$$

The differential equation for this case is

$$\frac{\partial^2 V}{\partial z^2} - L_o C_o \frac{\partial^2 V}{\partial t^2} = -L_o \frac{\partial^2 \rho_\ell}{\partial t^2} - L_o \epsilon_o A_1 g \frac{\partial^4 V}{\partial t^2 \partial z^2} \tag{29}$$

Thus while the original space charge reduction factor approach seemed to close the calculation (since R could be found from purely geometrical considerations), it was not incorporated correctly into the theory. We have shown that not only must it appear in two places in the dispersion polynomial (eq. (20a)) it is also generally complex (eq. (26)). The complex factors f or g may be used interchangeably, and they are related by the following equation.

$$g = \frac{(\Gamma_1{}^2 - \Gamma^2)}{K\Gamma^2\Gamma_1(1 + j\omega\epsilon_o A_1)} f \tag{30}$$

Thus either a complex "reduction factor" R or f, or g may be used to model the interaction; the choice depends on personal preference or ease in which their respective partial differential equations may be solved numerically. The next sections compare the root behavior of the above quartic polynomials with that of the field analysis by Chu and Jackson (ref. 32).

COMPARISON OF MODELS

A more fundamental approach to analyze the TWT is to solve Maxwell's equations subject to appropriate boundary conditions. One of the earliest field-type of analyses was performed by Chu and Jackson (ref. 32), where the helix

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was modeled as a helically conducting sheath. After using the normal small-signal approximations for the electron beam, the propagation constants were found to satisfy the following set of equations:

$$\eta^2 = p^2 \left[1 + (\omega_p/u_0)^2 \Big/ (j\beta_e - \Gamma)^2 \right] \tag{31}$$

$$p^2 = -(\Gamma^2 + k^2) \tag{32}$$

$$\eta = \frac{I_0(\eta b)}{I_1(\eta b)} \, p \left[\frac{I_1(pb) - K_1(pb)\{C_4/C_3\}}{I_0(pb) + K_0(pb)\{C_4/C_3\}} \right] \tag{33}$$

$$\frac{C_4}{C_3} = \frac{\left[(k/p)^2 \cot^2\psi\right] I_1(pa)K_1(pa) - I_0(pa)K_0(pa)}{K_0^2(pa)} \tag{34}$$

where (see fig. 1)

Γ	a complex propagation constant $= \Gamma_r + j\Gamma_i$
k	ω/c, the free space wavenumber
ω_p	the plasma frequency
u_0	the electron drift velocity
β_e	ω/u_0, the electronic wavenumber
η	a complex separation constant
b	radius of beam
a	radius of helically conducting sheath
ψ	angle of perfect conduction on the sheath
$I_0, I_1,$ K_0, K_1	are the modified Bessel functions

Unfortunately, the propagation constants are tied up in the arguments of the modified Bessel functions. For such a complicated set of transcendental equations, it is not immediately obvious how many propagation constants Γ_i, i = 1, 2, ..., may be found for a given set of tube parameters. Not much work has been done with the above set, and even with computer aids, the work is tedious. Collin (ref. 33) presents results for a few limiting cases; that of the beam completely filling the sheath, and that of finite separation between the beam and helix, but with the helical sheath replaced by a perfectly conducting cylinder. The complexity of the above set is in part why the transmission line approach has enjoyed such popularity. The authors of most texts, as well as those of scores of journal articles have chosen the TLM as the preferred analysis tool for the TWT. Snyder (ref. 34) has addressed the rather

large disparity between the field and transmission line models, and has developed formulas that help bridge the wide gap between the two approaches.

The above set may be looked upon as two constraints on the complex separation constant η. The first equation is a restriction due to the beam, whereas equation (33) is a circuit restriction. For clarity we label η in equation (31) as η_B and that in equation (33) as η_C. The values of complex Γ that satisfy the above with given tube parameters, such as drift velocity, frequency, etc. were obtained numerically. The Western electric tube, number 444A, was chosen since most of its parameters are given in either of two textbooks (refs. 4 and 12). All results are for the operating frequency of 6.175 GHz. For reference, the other parameters are listed below.

$\omega_p = 4.5373$ GHz

$\varepsilon = 8.854 \times 10^{-14}$ f/cm

A_1 = cross-sectional area of beam, 1.32732×10^{-2} cm^2

$C = 0.058$

$Q = 5.0$

R = the "standard" reduction factor, 0.46

$k = \omega/c = 1.293$ cm^{-1}

$\cot(\psi) = 11.2274$, $\psi = 5.1°$

$a = 0.100$ cm

$b = 0.065$ cm

For simplicity, the circuit was assumed lossless. Table I gives the solutions for specific values of drift velocity u_0. Figure 4 is a plot of the variations of η_B and η_C as the imaginary part of the wavenumber $j\Gamma_i$ is swept over the range from $j15$ to $j16$. The real part is fixed at $\Gamma_r = -0.25$ and u_0 is fixed at 2.6×10^9 cm/sec. In this figure, no solution is found since the curves cross at different values of Γ_i. Figure 5 shows a wide sweep for Γ_i with Γ_r fixed at -0.250, and u_0 as before. This shows the behavior of the system, and in particular, that generally only one solution is possible for a specific drift velocity. The "cold circuit" propagation constant Γ_1 is found from references 32 or 34

$$\left(\frac{ka \cot\psi}{pa}\right)^2 \left[I_1(pa)K_1(pa)\big/(I_0(pa)K_0(pa))\right] = 1 \qquad (35)$$

which yields $pa = 1.219$, $p = 12.19$

Also using $p \cong -j\Gamma$ for forward waves, we have $\Gamma_1 = j12.19$ cm^{-1}.

Figure 6 is a plot of the solutions for η, i.e., when $\eta_B = \eta_C = \eta$ as a function of the drift velocity u_0. Figure 7 shows the root constellation for equation (31) for the values of η obtained in figure 6. Only the root with the small negative real part satisfies the other constraint (eq. (33)). Notice

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equation (31) is a fourth order polynomial with complex coefficients, and is similar to the dispersion polynomials developed with the TLM method (eq. (4)). However, unlike the TLM roots, only one satisfies the complete set of boundary conditions. In other words it says, that one, not three forward waves exists. This fact, however, is a peculiarity due to the sheath helix model. A real helix needs many modes to correctly satisfy Maxwell's equations. So we cannot say exactly how many modes actually exist in the tube. This, however, is not of importance for our study, as we are using the field solution as a standard by which to compare the root behavior of the polynomials developed in other sections. With that proviso explicitly stated, we will assume the smaller growing root to be the "exact" solution for the tube at hand. We can easily see that the TLM model and the "field" model are similar in that they both involve the solution of a fourth order polynomial with complex coefficients. This similarity was briefly discussed in reference 32. Experiments, mainly the Kompfner dip condition, tend to imply several modes exist near the input end of the tube.

Since all of the cases studied reduce to fourth order dispersion equations, we may compare them as shown in Table II. The coefficients a_4, a_3, a_2, a_1, a_0 as given in equation (4) are listed for the cases of Chu, Pierce, Conventional, and f-factor methods. If we try to "match-up" the coefficients to force the TLM results to agree with those of Chu we find the following trends:

Chu-Pierce: $4QC^3 \rightarrow 0$

$$\Gamma_1 \simeq jn \simeq \beta_p^2/(C^3(4Q + j2\beta_e)) \qquad \beta_p = \omega_p/u_0$$

Chu-Conventional:

$$\Gamma_1 \simeq jn = jk/R = -j\beta_p^2(1 - R^2)/2\beta_e C^3$$

Chu-f-factor:

$$\Gamma_1 \simeq jn = jk/R = -j\beta_p^2/2\beta_e C^3$$

First of all we notice that the "cold circuit" propagation of the TLMs Γ_1, basically tracks jn, the separation constant. Thus the concept of Γ_1 being relatively fixed (that is independent of u_0) does not hold very well. Figure 8 depicts the variation of the growing wave using the "Pierce" form in the complex Γ-plane. This form uses the coupled mode matching, or fit parameters C and QC, the gain and space-charge parameters, respectively. Also shown is the root behavior for the "field" or Chu and Jackson solution. The values of drift velocity are shown to aid in the comparison. The upper curve is for $\Gamma_1 = j14.52$ cm^{-1} which is the value of $\omega/v\varphi$ where $v\varphi \doteq c \tan\psi = 2.672 \times 10^9$ cm/sec is the phase velocity of the wave. The lower one is for $\Gamma_1 = j12.19$ cm^{-1} which is the "cold circuit" value obtained in equation (36). In figure 9 we show the result of the "conventional" approach; the matching parameters are C and R. The root trajectory in figure 10 is that of a more involved space charge parameter scheme. This scheme is covered in detail in reference (11), and summarized in appendix D. In essence the space-charge parameter is found using equation (D - 1). The two possible values for R are denoted $R_+ = 0.503$, and $R_- = 0.5698$. In all of the above figures, it is worth noting that only over certain intervals for u_0 did the "growing root" for the TLMs leave the imaginary axis. In other words for u_0 out of the ranges shown, the "growing root" did not have a significant negative real part.

The match-up using the f-factor approach is shown in figure 11. The value for f that gave this reasonable fit was found by trial and error. This value is $f = 100 -j220$. The value for C is $0.1268 + j.141$. The corresponding "complex" space charge reduction factor R (from eq. (26)) is $0.022 -j0.103$. Since there cannot be a perfect one-to-one match-up between the transmission line models and the field solution, the values of the fit parameters are average ones that give a reasonable curve. Figure 12 shows how the f-factor would change as the drift velocity changes. This forces the root of the f-factor equation to coincide with the field solution root. Figure 13 gives the corresponding change in the complex C parameter. Figure 14 shows the variation of the g-factor with u_0. Notice that for the lower values of u_0 the phase is near 90° which puts the charge bunches in the maximum retarding circuit field. At the other end of the u_0 range, they fall back into a lesser retarding field (fig. 3). However, the phase is always in the physically intuitive correct range; i.e., 90° to 180°. Notice the magnitude of g for this low power tube is on the order of 10^{-3}, which means the horizontal component of the space charge field is much smaller than the circuit field. While this has always been assumed true in TWT analysis, it does not justify neglecting the longitudinal space charge field in equation (12). If anything, it would seem to be more reasonable to drop it in the equation of motion (eq. (5)). It is important to recognize that three field components are dealt with in the TLMs. Namely E_z^{SC}, E_y^{SC}, and E_z^{CKT}. While we assume $|E_z^{CKT}| \gg |E_z^{SC}|$, $|E_y^{SC}|$, we cannot immediately determine the relative magnitudes between E_y^{SC} and E_z^{SC}. The main thrust of this paper is to point out that E_y^{SC} and E_z^{SC} have not been consistently included in the older analyses. When J^i was found, the assumption used was $E_y^{SC} \gg E_z^{SC}$, and actually E_z^{SC} was set to zero. However in the equation of motion E_z^{SC} was added to E_z^{CKT}. At that point E_y^{SC} was not relevant as only z-directed ac motion was assumed due to the focusing of the static magnetic field.

The relative magnitudes of E_y^{SC} and E_z^{SC} can only be found from a field analysis. What we are demonstrating is that neither component needs to be neglected in any step in the TLM, so the analysis can be self-consistent. So while $|E_z^{SC}| \ll |E_z^{CKT}|$ is true, it has no relevance to the relative magnitudes of E_z^{SC} and E_y^{SC}. This ratio is basically related to the complex reduction factor R, and thus indirectly to f via equation (26). One, however, must be careful, since the roots move quite quickly as parameters change by small amounts. Thus dropping E_z^{SC} in equation (5) can cause large variations in root values.

SUMMARY AND FINAL COMMENTS

We have shown how the inter-bunch forces or "space-charge" effects may be incorporated into a transmission line model for the TWT. The departure from previous investigations centers around the specification of the impressed current J^i. Basically

$$J^i = -\frac{\partial i}{\partial z} - j\omega\epsilon_0 A_1 \frac{\partial E_z^{SC}}{\partial z} \tag{37a}$$

$$= -\frac{\partial i}{\partial z}(1 - R^2) \tag{37b}$$

$$= - \frac{\partial i}{\partial z} (1 + j\omega\epsilon_0 A_1 f) \qquad f = \frac{E_z^{sc}}{i} \qquad (37c)$$

$$= - \frac{\partial i}{\partial z} + j\omega\epsilon_0 A_1 g \frac{\partial^2 v}{\partial z^2} \qquad g = \frac{E_z^{sc}}{E_z^{CKT}} \qquad (37d)$$

where the latter three are for the "complex" space-charge reduction factor, f-factor, and g-factor methods. Pierce and the "conventional" method (appendices A and D), used

$$J^i = - \partial i/\partial z$$

but differed in the expression for E_z^{sc} in the equation of motion. Pierce used

$$E_z^{sc} = \frac{i}{\omega\epsilon_0 A_1} i \overset{\Delta}{=} - \frac{j2KQr^2}{\beta_e} i \qquad (38)$$

while the "conventional" method used

$$E_z^{sc} = \frac{jR^2}{\omega\epsilon_0 A_1} i \qquad (39)$$

Thus the basic inconsistency in previous studies revolves around the omission of E_z^{sc} in equation (37a). To show that inconsistent assumptions were indeed made, consider the following argument.

If

$$J^i = - \frac{\partial i}{\partial z} \qquad (40)$$

then equation (37b) says $R \rightarrow 0$. But if in equation (39), $R \rightarrow 0$, then $E_z^{sc} \rightarrow 0$. Thus keeping E_z^{sc} finite, and hence R finite, while assuming equation (40) holds ($R = 0$) is inconsistent. The reason this occurred was due to faulty combining of the "one" and "two" dimensional approximations made in the development of the theory. Equation (40) implies all displacement flux is vertical, whereas, equation (38) ($R = 1$) implies all displacement flux is horizontal. Several authors (refs. 7 and 16) use equations (39) and (40) with the implied, but unstated result that $R = 0$, and $R = 1$ simultaneously. Pierce was able to partially circumvent this extreme by allowing R to vary between 0 and 1. The space-charge parameter Q performed this function; as seen by comparing equations (38) and (39).

Surprisingly, only one easily accessible paper concerned with attempts to measure C and QC for a particular TWT exists (ref. 35). By appropriate normalization of the original C and QC, they were able to obtain "excellent" agreement between theory and their measured values. In the light of the previous work, this is not surprising. In essence they were curve fitting the measurements to theory. Figure 4 of the article is revealing; its main structure appears in figure 15. Notice the signal gain measured along the tube is curve-

fitted to a straight line. The slope and intercept are essentially the two fit parameters for all the TLMs. Notice also that the Kompfner dip apparently was not observed; or at least they made no mention of it in their experiments. The ripple pattern is assumed caused by interference of a relatively unattenuated backward traveling wave generated by a slight mismatch at the load end. Further comments and details are left to the appendices; there we expand on topics referred to in the abstract and introduction.

APPENDIX A - PIERCE'S ORIGINAL WORK

The development of the theory may be found in two articles (refs. 1 and 2), and the text by Pierce (ref. 3). In the first paper, he did not specifically separate the electric field into circuit and space charge parts; instead the BTC equation was presented in the form

$$E = q \sum_n \frac{\Gamma_n}{\psi_n^* \left(\Gamma^2 - \Gamma_n^2 \right)}$$

where

q   the impressed alternating convection current (q = i)

E   the electric field acting on the beam in the z-direction

ψ_n^* $2P_n/EE^*$ a constant (units of mho.m^2)

P_n the power flow in one of the nth pair of modes

The footnote accompanying the expression was: "Expression (1) was given to the writer by S.A. Schelkunoff of these laboratories with no guarantees as to its range of applicability or convergence. A rationalization of it in terms of more familiar concepts will be found in appendix A." The expression was intended to represent the excitation of modes on the helix by the beam current q. The fundamental mode with propagation constant Γ_0 was assumed to be the one primarily coupled to the beam, and its growth while traveling in near synchronism with the beam accounted for the experimentally observed signal gain. This term was singled out by writing the above as

$$E = q \left[\frac{\Gamma_0}{\psi_0^* \left(\Gamma^2 - \Gamma_0^2 \right)} + \sum_{n=1} \frac{\Gamma_n}{\psi_n^* \left(\Gamma^2 - \Gamma_n^2 \right)} \right]$$

The sum of the terms, other than the fundamental, was written as the second term in the bracket. These terms were interpreted as both cut-off modes excited by the beam, as well as any propagating modes not in synchronism with the beam. The term was rewritten in the form $j/m_1\beta$, for compactness of notation. The units of m are the same as those for ψ_n^* (the paper mistakenly omits the meter^2 unit). Pierce then develops the propagation constants in the limit where $j/m_1\beta$ is ignored. In later sections he addresses the "space-charge" effects (the $j/m_1\beta$ term) on the tube operation. The "space-charge parameter" Q was defined as

$$Q = \psi_0^*/2m_1$$

and he found its value to be about 13.7 for a specific experimental tube. The term $j/m_1\beta$ was called the "local concentration of charge," which was not the charge distribution developed by the fundamental mode. Rather it was the charge developed by nonsynchronous and cutoff modes collectively.

The second paper (ref. 2) altered the BTC as follows

$$E = q \left[\sum_n \frac{\Gamma_n}{\psi_n{}^*\left(\Gamma^2 - \Gamma_n{}^2\right)} + \frac{i}{\omega\epsilon A} \right]$$

The change from the first form is the addition of the second term, which was to reflect the mutual repulsion between the space-charge bunches. He states in the paper's appendix that the omission of the term was brought to his attention by several readers of the first paper. The physical rationale for the term is as follows. With reference to figure 1, the electric flux lines linking the beam bunches and the circuit are to represent the displacement current between the beam and circuit. The net flow to be directed from the beam to the circuit during the energy exchange and subsequent circuit wave growth. These lines are represented by the impressed current term J^1 in the augmented transmission line equations. The lines between the bunches represent the inter-bunch repulsive forces. These lines are represented by the new term added to the BTC equation. Basically this equation is just the expression for the total field as the sum of circuit and space-charge parts. This is the right hand side of the equation of motion (eq. (5)).

He then rewrites the above as

$$E = q \left[\frac{\Gamma_o}{\psi_o{}^*\left(\Gamma^2 - \Gamma_o{}^2\right)} + \frac{i}{m_1\beta} \right] \tag{A1}$$

which is the same form as used in the first paper. However, the interpretation of the $j/m_1\beta$ term has been expanded to include the inter-bunch repulsion term. The inter-bunch term is developed in a verbal manner in the appendix of the paper. We will develop this term in more detail later. For clarity, we continue with the review of Pierce's work.

With the above BTC coupling equation relating the total field E to the convection current i, $(q = i)$, as well as the equation of motion also relating them (eq. (6)); Pierce was able to eliminate both variables E and i and arrive at the dispersion relation.

$$1 = \left[\frac{\Gamma_o}{\psi_o{}^*\left(\Gamma^2 - \Gamma_o{}^2\right)} + \frac{i}{m_1\beta} \right] \frac{j\beta I_o}{2V_o(-\Gamma + j\beta)^2} \tag{A1}$$

This paper was published in August of 1948. In 1950 Pierce published a text (ref. 3) on the TWT, which has become the standard work for the tube. In it he rearranges the development of the theory in a form unlike that of the two previous papers. In chapter II he develops the BTC equation for the first time using the augmented transmission line equations (see appendix E). In this development only the first term of equation (A1) results, which is due to the form used for J^1. With reference to figure 2, one may write a "node" equation at some small volume in the region where the beam and circuit nearly make contact. Using continuity and Gauss's law we have

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$$-\frac{1}{A_1}\frac{\partial i}{\partial z} = + j\omega\left[\epsilon_0 \frac{\partial E_y^{sc}}{\partial y} + \epsilon_0 \frac{\partial E_z^{sc}}{\partial z}\right]$$

The horizontal i_d and vertical $J^i \Delta z$ components of displacement current are

$$i_d = j\omega\epsilon_0 A_1 E_z^{sc}$$

$$J^i \Delta z = j\omega\epsilon_0 A_1 \frac{\partial E_y^{sc}}{\partial y} \Delta z$$

The expression for $J^i \Delta z$ is not obvious, we only know it from previous analysis.

Now the node equation may be written. The convection current is assumed horizontal only, and the node equation yields

$$J^i = -\frac{\partial i}{\partial z} - j\omega\epsilon_0 A_1 \frac{\partial E_z^{sc}}{\partial z}$$

Pierce used the following physical arguments to arrive at his expression for J^i. This occurred in the text, on pages 9 to 11. We quote "We will assume that the electron beam is very narrow and very close to the circuit, so that the displacement current along the stream is negligible compared with that from the stream to the circuit." Thus only convection current is allowed along the beam, and only displacement current is allowed perpendicular to it; this gives a node equation of the form

$$i - J^i \Delta z = i + di = i + \frac{\partial i}{\partial z} \Delta z$$

or

$$J^i = -\frac{\partial i}{\partial z} \tag{A2}$$

Derivations for J^i may be found in the texts by Gittins (ref. 7), p. 20, Soohoo (ref. 8), p. 109, Atwater (ref. 9), p. 180, Everhart and Angelokas (ref. 10), p. 67, Chodorow and Susskind (ref. 6), p. 147, Gandi (ref. 4), p. 355, Liao (ref. 5), p. 226, among others.

With this form for J^i Pierce gives the BTC equation as

$$E_z^{CKT} = -\frac{\partial V}{\partial z} = \frac{-K\Gamma^2\Gamma_1}{\Gamma^2 - \Gamma_1^2} i \tag{A3}$$

which has become the standard and fundamental equation relating beam-slow wave coupling. Notice Pierce neglected the longitudinal displacement current and thereby neglected the effects of E_z^{sc} on the coupling process. We can now state equation (A3) relates the convection current to the circuit field in the limit of a negligible longitudinal space charge field E_z^{sc}.

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He develops the propagation constants in this limit in Chapter II of the text. Later, in Chapter VII, he takes up the case wherein "space-charge effects" are considered. In essence, in this chapter he reproduces the results of his second paper, but uses a completely different format for developing them. Basically, he gives an equivalent circuit model to interpret equation (A1). This circuit is shown in figure 2. Between the beam and circuit proper, a distributed capacity C_1 is inserted. These capacitors are to model the "space-charge" terms that appeared in equation (A1), and thereby modify the beam-to-circuit coupling. The first problem now becomes clear. Notice the first term of equation (A1) is developed by neglecting "space charge" effects (the z-component of the space-charge field) while the second term is assumed to wholly take the space-charge effects into account. While this is a reasonable engineering approximation, further scrutiny reveals a problem. Pierce assumes the impressed current J^i (developed by neglecting the longitudinal, or z-component of the space charge field) flows thru the coupling capacitors and develops a "space charge voltage" V^{sc} given by

$$V^{sc} = J^i \, \frac{1}{j\omega C_1}$$

Then one assumes the space charge field is related to this space charge voltage by

$$E_z{}^{sc} = - \frac{\partial V^{sc}}{\partial z}$$

which gives the space-charge field as

$$E_z{}^{sc} = \Gamma J^i \, \frac{1}{j\omega C_1} = \frac{\Gamma^2 i}{j\omega C_1}$$

Thus the total field of the system is exactly that given in the second paper if one forces

$$C_1 = - \frac{\epsilon_0 A}{i} \, (\Gamma^2 i)$$

While the modified transmission line was only supposed to act as a mental aid in this heuristic approach, (quoting Pierce, "This circuit is intuitively so appealing that it was originally thought of by guess and justified later.") a consistency problem does however exist. Assuming J^i given by equation (A2) flows through the coupling capacitors C_1 is unjustified; this is because this form can only be developed when the longitudinal space-charge field $E_z{}^{sc}$ is neglected. In other words one neglects $E_z{}^{sc}$ to obtain J^i. Then one subsequently uses J^i so developed to now determine $E_z{}^{sc}$; this is not in the true spirit of an engineering approximation.

This is the best place to consider the development of the "space-charge" field as presented by Pierce in the second paper. Recall the new term with the physical interpretation of reflecting the inter-bunch forces was given as

$$E_z{}^{sc} = \frac{ji}{\omega \epsilon_0 A}$$

We develop this term in the following way. Starting with Maxwell's second equation we define the total current $\bar{I}_T$

$$\bar{I}_T = \nabla \times \bar{H} = \bar{J} + j\omega\epsilon_0\bar{E}$$

See the text by Chodorow and Susskind (ref. 6), p. 146 for more discussion of this point. Simply put, the divergence of the total current is identically zero

$$\nabla \cdot \bar{I}_T = 0 = \nabla \cdot \bar{J} + j\omega\epsilon_0 \ \nabla \cdot \bar{E}$$

For a strictly one dimensional model we have

$$I_T(t) = (\nabla \times \bar{H})_z = \left[\frac{\partial H_y(z)}{\partial x} - \frac{\partial H_x(z)}{\partial y} \right] = J_z + \epsilon_0 \frac{\partial}{\partial t} E_z{}^{sc}$$

For a one dimensional model wherein only variation with z is permitted, the terms in the bracket are identically zero. (No variation with x or y is possible.) Then the total current in this case is identically zero. Using this fact we find

$$0 = \frac{i}{A_1} + j\omega\epsilon_0 E_z{}^{sc}$$

or

$$E_z{}^{sc} = \frac{ji}{\omega\epsilon_0 A_1}$$

which is Pierce's correction term. An actual physical circuit is never infinite in lateral extent, so in a practical sense, $\bar{I}_T$ is never zero. One must determine $\bar{I}_T$ by measurement or other constraint imposed on the system, at points "external" to the region where the one dimensional approximation is being invoked. The similarity of the space charge correction term to a capacitor was in fact mentioned in the original paper (see the footnote on p. 114). The similarity may be developed with the following arguments. Start with the expression for the space charge field

$$E_z{}^{sc} = \frac{ji}{\omega\epsilon_0 A_1}$$

Assume a space-charge voltage may be defined as

$$E_z{}^{sc} = \Gamma V^{sc}$$

Then use the relation connecting J^i and i.

$$V^{sc} = \frac{J^i}{j\omega C_1}$$

This may be interpreted as Ohm's law for a distributed capacitor C_1

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Finally, the capacitor C_1 was replaced by the space charge parameter Q via the definition

$$Q \triangleq \frac{\beta_e}{2\omega C_1 K}$$

Notice the subtle changes that occurred in the two journal articles and the text. In the first article the term $j/m_1\beta$ was just the summation of terms that represented nonsynchronous and cut-off modes. In the second article the term was expanded to include the new correction term $ji/\omega\epsilon A$. In the text, the nonsynchronous and cut-off modes were abandoned and just the correction term was added to the basic BTC equation. That is, the text basically writes

$$\frac{j}{m_1\beta} = E_z{}^{sc} = \frac{ji}{\omega\epsilon A} = -\frac{j2KQ\Gamma^2 i}{\beta_e}$$

However, in the text the space charge term $ji/\omega\epsilon A$ is expressed in terms of the space charge parameter Q. Notice that the definition of Q underwent some revision from the first article to the text. The "space-charge parameter" or the "space-charge effect" was a difficult term and/or concept for Pierce to incorporate into the theory. As a matter of fact Gittins (ref. 7) made the remark that Q is not an easily understood parameter; To quote him, "a complicated and confusing parameter." It has been stated that C_1 and thus Q, may be negative quantities for certain tube conditions (ref. 15, p. 131).

We now see that all this work was essentially equivalent to decomposing the total field into circuit and space charge parts in the equation of motion, and attempting to eliminate the space charge term.

In summary, we may say Pierce had difficulty incorporating the inter-bunch forces into the analysis. The formula developed for the impressed current J^i and that for the space charge field $E_z{}^{sc}$, were incompatible, which made the theory internally inconsistent. The introduction of Q, a variable fit parameter, eased the problem somewhat and gave enough flexibility to match-up theory and experiments. The coupled mode ideas were being formed by writers at the time, but the definite decomposition of the field into circuit and space charge parts was never completely spelled out by Pierce. He never placed superscripts on electric field variables, rather the implication was that the field should be treated as a single entity with both circuit and space-charge properties. This omission causes much confusion in attempting to understand the theory. For example, in the old development for J^i one neglects the "longitudinal displacement current," which implies

$$j\omega\epsilon_0 A_1 E_z = 0 \qquad \Rightarrow E_z = 0$$

but later one obtains E_z in equation (A1). Thus one neglects E_z at one point, to permit finding it later. Another problem one might bring up is the notion that all flux lines terminating on the circuit is contradictory to the weak coupling assumption. These sorts of problems have tended to cast undue criticism on the entire theory. The self-consistent inclusion of $E_z{}^{sc}$ eliminates all of these problems.

A second method to develop J^i is as follows: Starting with Maxwell's second equation

$$\nabla \cdot (\nabla \times \bar{H}) = 0 = \nabla \cdot \bar{J} + j\omega\epsilon_0(\nabla \cdot \bar{E}^{CKT} + \nabla \cdot \bar{E}^{SC})$$

Assume the circuit field has zero divergence at all points not directly on the circuit. Using a rectangular coordinate system (fig. 1), we find

$$0 = \frac{1}{A_1} \frac{\partial i}{\partial z} + j\omega\epsilon_0 \frac{\partial E_y^{SC}}{\partial y} + j\omega\epsilon_0 \frac{\partial E_z^{SC}}{\partial z}$$

We may interpret the second term as J^i/A_1, since the units are amenable to such a choice. Then

$$J^i = - \frac{\partial i}{\partial z} - j\omega\epsilon_0 A_1 \frac{\partial E_z^{SC}}{\partial z}$$

This permits one to express J^i in the alternate form

$$J^i = j\omega\epsilon_0 A_1 \frac{\partial E_y^{SC}}{\partial y}$$

APPENDIX C

The reduction factor turns out to be an important parameter in TWT theory. The reasons for its introduction are as follows: the plasma (resonant) frequency for an infinite cloud of electrons neutralized by immobile ions is

$$\omega_p = (\bar{n}\rho_0/\epsilon_0)^{1/2}$$

In any finite, or real physical beam, the longitudinal electric field is decreased due to flux fringing outside of the beam. This fringing is further increased when the beam is enclosed in a conducting tube. With reference to figure C-1, we expand on the above remarks. The plasma frequency is the natural resonant frequency of the infinite plasma. When fringing occurs, the plasma becomes less "stiff," and the resonant frequency is somewhat less than ω_p. Part (a) of the figure represents a case wherein the beam diameter is sufficiently large such that the reduction in the resonant frequency is very small. Under this assumption, the resonant frequency ω_q is nearly ω_p, or the reduction factor is approximately unity. The reduction factor R is defined by

$$\omega_q = R\omega_p$$

Part (b) depicts the case for a small diameter beam in close proximity to a conducting tube. In this case the fringing flux and the inter-bunch flux are of the same order of magnitude. In this situation, the reduction factor is somewhere between zero and one. This flux distribution is assumed to be similar to that encountered in the TWT. The flux lines linking the beam to the circuit are modeled by the impressed current J^1, while those that are horizontal reflect the inter-bunch forces. Part (c) depicts the condition implied by Brillouin (ref. 39), see appendix G, and the special case of negligible "space-charge" effects, by Pierce. For this case, the inter-bunch forces are absent, and all lines from a bunch or anti-bunch terminate on the circuit. This is the limit of zero "space-charge" effects, and from the above reasoning, apparently the reduction factor is zero. This line of reasoning may be found in the texts (refs. 8 and 10), as well as a paper by Pierce (ref. 36).

Branch and Mihran (ref. 37) have calculated the reduction factor for various geometries, and presented the results in graphical form. The texts (refs. 12, 33, and 38) briefly discuss the reduction factor, and the procedure for its calculation.

APPENDIX D - "CONVENTIONAL" METHOD

This method differs from that of Pierce only in the way the "space-charge" term is written. Basically

$$E_z{}^{sc} = \frac{jR^2}{\omega\epsilon_0 A_1} i$$

and the equation of motion is written with the circuit and space charge fields explicitly separated. Pierce essentially did the same thing when he added the "space charge voltage" to the circuit voltage. Gittins (ref. 7) says this form is an alternative to the Q factor method of Pierce. He writes the above without the factor R, Beck (ref. 16) does the same. Thus Gittins, and others, were not quite clear on exactly what Pierce had done.

The dispersion relation is found using the same technique as that of Pierce, where for reference we write

$$I_0 = \rho_0 U_0 A_1 \qquad \rho_0 = -|e|n_0$$

$$V_0 = -\frac{U_0{}^2}{2|\bar{n}|} \qquad \bar{n} = -\frac{|e|}{m}$$

$$c^3 = \frac{KI_0}{4V_0} \qquad \omega_p{}^2 = \frac{\bar{n}\rho_0}{\epsilon_0}$$

The resulting dispersion was given as equation (20b) earlier. Comparing this to Pierce's form (eq. (20c)), leads to "matching-up" of terms to relate Q to R. The reduction factor has a reasonable physical interpretation, and the "match-up" was intended to relate Q (a somewhat nebulous parameter) to R. The first order match-up comes from simply equating the two terms, since they are supposed to represent the same physical effect.

Thus

$$-c^2 \Gamma^2 (4QC) = \frac{R^2 \omega_p{}^2}{u_0{}^2}$$

or

$$-4QC = \frac{\omega_q{}^2}{\Gamma^2 u_0{}^2}$$

Assume

$$\Gamma \simeq j \frac{\omega}{u_0} = j\beta_e \quad ,$$

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then

$$4QC^3 = \left(\frac{\omega_q}{\omega}\right)^2$$

which relates Q to the reduced plasma frequency ω_q. One problem apparent to the writer occurs when Q becomes negative, (a possible condition mentioned in ref. 15). Then, since C is necessarily positive, ω_q is imaginary. The "match-up" does not stop here, but has been carried one step further (ref. 11). The result is

$$\omega_q = \pm \frac{\omega\sqrt{4QC^3}}{1 \pm \sqrt{4QC^3}} \qquad (D1)$$

which is often quoted as the more exact relationship between Q and the reduced plasma frequency ω_q (ref. 12). Thus the "conventional" approach is just Pierce's form with R playing the same role as Q.

APPENDIX E

The basic beam-to-circuit coupling equation (BTC) is the cornerstone of the TLM. Augment the normal transmission line equations with the impressed current J^i.

$$\frac{dV}{dz} = -jXI$$

$$\frac{dI}{dz} = -jBV + j^i \tag{E1}$$

Eliminate I

$$\frac{d^2V}{dz^2} = -BXV - (jX)J^i$$

and define

$$\Gamma_1^2 = -BX \qquad \Gamma_1 = j\sqrt{BX}$$

to be the cold circuit forward propagation constant when $J^i = 0$. From transmission line theory

$$Z_0 = K = \sqrt{\frac{X}{B}} \qquad jX = K\Gamma_1$$

and

$$\frac{d^2V}{dz^2} = -\Gamma_1^2 V - K\Gamma_1 J^i$$

which yields the final result

$$V = \frac{-K\Gamma_1}{\Gamma^2 - \Gamma_1^2} J^i$$

APPENDIX F - EQUIVALENT TRANSMISSION LINE

Here we show how the transmission line approach can be somewhat supported from Maxwell's equations. Assume the structure is propagating a slow TM wave in the presence of nearby convection current, then the appropriate equations are

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu_o H_x$$

$$\frac{\partial H_x}{\partial z} = j\omega\epsilon_o E_y + J_y$$

Take $\partial/\partial z$ of the first, and use the second to find

$$\frac{\partial^2 E_z}{\partial z \partial y} - \frac{\partial^2 E_y}{\partial z^2} = \omega^2 \mu_o \epsilon_o E_y - j\omega\mu_o J_y$$

Assume no J_y component of current, which results in

$$\frac{\partial^2 E_y}{\partial z^2} + \frac{\omega^2}{c^2} E_y = \frac{\partial^2 E_z}{\partial z \partial y}$$

For variation as $e^{-\Gamma z}$ this becomes

$$\left(\Gamma^2 + \frac{\omega^2}{c^2}\right) E_y = \frac{\partial^2 E_z}{\partial z \partial y}$$

But $\Gamma^2 \simeq -\omega^2/v_\varphi^2$ and $v_\varphi^2 \ll c^2$ so the first term dominates

$$\frac{\partial^2 E_y}{\partial z^2} \doteq \frac{\partial^2 E_z}{\partial z \partial y}$$

and

$$\frac{\partial E_y}{\partial z} \doteq \frac{\partial E_z}{\partial y}$$

if the integration constant is assumed zero. The above is just the well known slow wave condition. Define

$$E_y = -\frac{\partial V}{\partial y} \tag{F1}$$

then

$$-\frac{\partial}{\partial z}\frac{\partial V}{\partial y} = \frac{\partial E_z}{\partial y} \rightarrow E_z = -\frac{\partial V}{\partial z} \tag{F2}$$

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which makes good physical sense. The line voltage is the integral of the transverse field, and $-\partial V/\partial z$ is the longitudinal field component. Pierce (ref. 3) backed off from equation (F2) on p. 112, we quote, "We can now, however, regard V not as a potential but merely as a convenient variable related to the field by (7.2)." His equation (7.2) is equation (F2). Thus the transmission line model for a slow TM wave does not violate Maxwell's equations; and it may be used with some small amount of confidence.

The BTC has been used extensively in the literature concerning traveling wave devices. The form with J^i is perfectly correct

$$E_z^{CKT} = \frac{-K\Gamma\Gamma_1}{\Gamma^2 - \Gamma_1^2} J^i$$

But the form wherein $J^i = -\partial i/\partial z$ has been used, neglects all inter-bunch flux.

$$E_z^{CKT} = \Gamma V = \frac{-K\Gamma^2\Gamma_1}{\Gamma^2 - \Gamma_1^2} i \tag{G2}$$

There exist several derivations of equation (G2) in the literature which seem quite general and rigorous. While we have no problem with them, we will show that all use basically the same argument as did Pierce. Namely the neglect of longitudinal space charge which means dropping $j\omega\epsilon_0 E_z^{SC}A$ or E_z^{SC} somewhere in the derivation.

One of the earliest developments of equation (G2) (in English) was given by Brillouin (ref. 39). With reference to figure G1, the charge in the beam q_n' induces a charge $-q_n'$ on the wall of section n, which calls for an additional q_n' on the capacity C_n. The fact that all induced charge developed by a segment of the stream is equal in magnitude to the charge in that segment, implies the segments do not share flux lines. Thus no inter-bunch flux lines are considered. This means E_z^{SC} and therefore $j\omega\epsilon_0 A E_z^{SC}$ (the longitudinal displacement current) is neglected. The resulting expression for the voltage on the line in terms of the charge density in the beam is

$$\frac{\partial^2 V}{\partial z^2} - L_0 C_0 \frac{\partial^2 V}{\partial t^2} = -L_0 \frac{\partial^2 \rho_\ell}{\partial t^2} \tag{G3}$$

Brillouin states very strongly that equation (G3) should be rigorously correct, and no adjustments (such as space-charge reduction factors) are needed. To show that the above is equivalent to equation (G2) we convert it into the same operator notation. For $e^{j\omega t - \Gamma z}$ variation, the above becomes

$$V\left[\frac{\omega^2}{v_\varphi^2} + \Gamma^2\right] = \frac{\omega}{v_\varphi^2} Z_0 \rho_\ell$$

where

$$v_\varphi = 1/\sqrt{L_0 C_0}$$

$$Z_0 = \sqrt{L_0/C_0}$$

Define

$$\Gamma_1 = j \frac{\omega}{v_\varphi}$$

$$K = Z_0$$

Then equation (G4) becomes

$$\left[\Gamma^2 - \Gamma_1^2\right] V = -j\Gamma_1 \omega K \rho_\ell$$

Using the continuity equation (here ρ_ℓ is in coul/m), notice that in all the differential equations the charge density is a linear density ρ_ℓ.

$$j\omega\rho_\ell = \Gamma i$$

we find

$$V = \frac{-K\Gamma\Gamma_1}{\Gamma^2 - \Gamma_1^2} i$$

which is indeed equation (G2).

Another derivation of equation (G2) may be found in the text by Hutter (ref. 11). On page 184 Hutter writes, "Several derivations, none too rigorous, can be given for this equation, and the one presented here first is that given by Bernier." The development by Bernier (in French) appeared in 1947. Hutter goes on, "If the displacement current within the beam is neglected, ...". The texts by Watson and Gewartowski (ref. 12), Kleen (ref. 13), and others (refs. 6 and 14), have derivations similar to that of Bernier. Watson and Gewartowski present a development that starts with the following equation (their eq. 10.1 - 24),

$$dp = -iE_z \, dz$$

Here dp is the incremental instantaneous power flow into the circuit from the length dz of the beam. The beam current is i. They refer to the text by Stratton to justify the above expression. Referring to Stratton, we find the relationship under consideration is Poynting's theorem.

$$\int_S (\bar{E} \times \bar{H}) \cdot \bar{n} ds + \int_V \bar{E} \cdot \bar{J} \, dV = - \int_V \left(\bar{E} \cdot \frac{\partial \bar{D}}{\partial t} + \bar{H} \cdot \frac{\partial \bar{B}}{\partial t} \right) dV$$

If we neglect $\bar{H} \cdot \partial\bar{B}/\partial t$ (which is valid for slow-wave systems), we find

$$\int_S (\bar{E} \times \bar{H}) \cdot \bar{n} ds = - \bar{E} \int \bar{J} \cdot \bar{n} ds - \int_V \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} \, dV$$

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or

$$dp = -Ei \, dz - \int_V \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} \, dV$$

Now if the displacement current along the beam is neglected, the dot product in the volume integral vanishes, (the displacement current is $\partial \bar{D}/\partial t$). Then equation (G5) results. The remaining calculation finally ends up at equation (G2). We have shown that the "standard" BTC coupling equation may be developed by alternate paths, yet all use the same assumption (the neglect of inter-bunch flux).

APPENDIX H - NUMERICAL LARGE SIGNAL CALCULATIONS

As early as 1953, the large signal behavior of the TWT has investigated numerically. This work was initiated by Nordsieck (ref. 29), and he mentioned in a footnote that this work had been done before he left Bell Laboratories in 1947. This paper clearly mentions the neglect of inter-bunch flux; (see the quote in the introduction).

All of the numerical schemes (refs. 40 to 46) use equation (G3) as the fundamental relation between the voltage V induced on the helix by the convection current i and the corresponding linear rf charge density of the beam ρ_ℓ. Note all the partial differential equations use linear charge densities ρ_ℓ (coul/m), (or σ_ℓ) whereas in other parts of this paper we use volume densities. The text by Rowe (ref. 43), accepts equation (G3) as the starting point without question. He does, however, discuss the problem of properly taking space charge into account on p. 57 of the text. He says the charge density to be used in equation (G3) should be modified as

$$\frac{\partial^2 V}{\partial z^2} - L_o C_o \frac{\partial^2 V}{\partial t^2} = - L_o \frac{\partial^2}{\partial t^2} \rho_{m,n} \tag{H1}$$

where

$$\rho_{m,n} = \iint \psi(y)\rho(x,y,z,t) \, ds \tag{H2}$$

and $\psi(y)$ is a coupling function to be determined. Quoting Rowe on p. 57, "It is not necessary to assume all flux lines emanating from the stream terminate on the circuit. The quantity $\rho_{m,n}$ in equation (H1) is written in terms of space charge density in the stream as follows:," he then writes equation (H2). He does not carry the idea forward, since on the next page he replaces $\rho_{m,n}$ by ρ_ℓ with no additional comments. He briefly discusses this point again on pp. 218 and 226. Detweiler, a student and collaborator with Rowe, wrote a thesis (ref. 30) covering much of the material that is contained in Rowe's book. Detweiler and Rowe also published a comprehensive article on their work in 1971 (ref. 31). In the thesis the function $\psi(y)$ is introduced conceptually, but it is not determined in an exact manner. A differential equation for $\psi(y)$ is developed on p. 45 and then put into a different form (eq. (2.94) on p. 61). From pp. 62 to 66, work is done to approximate $\psi(y)$. Ultimately, an approximate expression (eq. (2.115) p. 66) for $\psi(y)$, is found. On this page Detweiler makes the statement quoted earlier in the introduction. This statement is very revealing. In essence, the effects of the inter-bunch forces are somehow determined in the limit wherein the bunches themselves are absent. The accuracy of further calculations is therefore severely compromised. As a matter of fact in their paper (ref. 31), most of the results are for the special case of negligible space-charge effects, (see assumption 2 on p. 58).

In the paper of 1971 (ref. 31), Detweiler and Rowe present the following coupling equation

$$\frac{\partial^2 V}{\partial t^2} - v_o^2 \frac{\partial^2 V}{\partial z^2} + 2\omega C d \frac{\partial V}{\partial t} = v_o Z_o \left[\frac{\partial^2 \sigma_\ell}{\partial t^2} + 2\omega C d \frac{\partial \sigma_\ell}{\partial t} \right] \tag{H3}$$

Notice this is the "standard" form augmented for cold-circuit loss embodied in the loss parameter d. The term $\sigma_\ell(z,t)$ is to represent a linear charge density induced onto the circuit by the electron stream. In the calculation of σ_ℓ, however, the beam density is approximated to be zero.

All of the other numerical schemes use the "standard" coupling equation. They differ, however in methods used to calculate the space charge field, as well as the procedure for numerically integrating the equations. Rowe and Detweiler model the beam as either charged rings or discs in a conducting tube and solve the resulting boundary value problem by the Green's function method. Tien (refs. 40 to 42) includes space charge in the equation of motion and assumes the inter-bunch forces can be approximated as

$$F^{SC} = qE^{SC} = \frac{q^2}{2\pi r_0 E_0} e^{-2(z' - z)r_0}$$

where the beam has been modeled as charged discs, and

r_0 = disc radius

z, z' = disc positions

Detweiler and Rowe find both the longitudinal and radial space charge field components. The longitudinal component appears to act only over a distance comparable to a wavelength, and the radial component is much weaker than the longitudinal one. They treat the total system as an initial-value problem with nine variables.

In summary, Rowe states that Brillouin's calculation does not include space-charge effects; and we concur with this. All of these numerical schemes are, however, internally inconsistent since they use the "standard" BTC coupling equation which neglects the existence of a longitudinal space-charge field.

Therefore the rather elaborate calculations found in their work are not warranted. We speculate the freedom to vary v_0, Z_0, C, and d in equation (H3) permitted them to match with experiments, even though the equation was more approximate than they had assumed. Some computer programs based on Detweiler's thesis have been used by NASA in the design of tubes for space applications, (private communication). For completeness we mention the BTC is not limited to the small signal approximation, and the equation of motion in these works is written in the Lagrangian form, rather than the Eulerian form where the velocity is treated as a field variable.

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APPENDIX I - USE OF BTC IN THE SOLID STATE

Back in 1966 Solymar and Ash (ref. 21) investigated the possibility of constructing a traveling wave device using a semiconductor in place of the beam. They used the standard BTC along with the equation of motion modified by the addition of terms to consider lattice scattering and carrier diffusion due to finite carrier temperature.

$$\frac{Dv}{Dt} = -|\bar{n}|\left(E_z^{CKT} + E_z^{sc}\right) - \nu v - v_T^2 \frac{\nabla \rho}{\rho}$$

where ν is the collision frequency, and v_T is the thermal velocity. Their dispersion equation was

$$\left[\Gamma^2 - \Gamma_1^2\right]\left\{(j\beta_e - \Gamma)\left[j\beta_e\left(1 - \frac{j\nu}{\omega}\right) - \Gamma\right] - \frac{v_T^2}{u_o^2}\Gamma^2 + \frac{\omega_p^2}{u_o^2}\right\} = -j2\beta_e\Gamma^2\Gamma_1 c^3$$

By a clever method they were able to show the acoustic wave amplifier (AWA) has a similar dispersion. By matching up terms on the right sides of the above and that for the AWA they were able to define an "effective" gain parameter C. Using values appropriate for the experimentally working AWA, they found the gain factor C to be about 10.5. They interpreted this to mean that somehow the collisions and/or diffusion provided a tighter coupling in the solid state. They predicted gain in the device using Si at 290 K of up to 85 dB/cm at 1 GHz. Surprisingly they did not give the value of C used in the calculation. We have indirectly determined they used C = 3.45, by reproducing their result.

Ettenberg and Nadan (refs. 22 to 24) also studied the device in the same manner and in one paper they used C = 0.22 (also not given, but determined indirectly), while in Nadan's thesis, the value C = 1.0 was used (see p. 115). Their final paper (ref. 23) predicted no gain until the frequency was near the collision frequency (about 1000 GHz). Again no value for C was given.

Recently Yu and Eshbach (ref. 28) re-investigated the device, again using the same technique. They followed the work of Kino and Reeder (ref. 20), (covered in the next appendix), and arbitrarily chose K = 10 Ω. After disappointing experimental results, they surmised the "reduction factor" was perhaps too large in the solid state (private communication).

Thus most investigators dealing with the solid state have apparently not worried about the neglect of inter-bunch flux, or were not aware that the BTC rested on that assumption. Also the rather insignificant effort to attempt to ascertain C (other than Solymar and Ash), seems to play down the very important role this fit parameter serves. In other words, the theory really cannot be used to predict gain in untested devices, since one has no way to judge the two fit parameters. The gain per wavelength turns out to be a sensitive function of the value of the fit parameters. Notice Solymar and Ash attempted to ascertain C, but they did not also match-up the space-charge parameters. It turns out the equivalent QC is complex, with magnitude around 10^7. Thus the validity of the match-up is highly questionable. The reason why the dispersion relations match-up is that the AWA may also be modeled as an equivalent transmission line with an impressed current. The physical mechanisms for gain in the TWT and AWA are, however, very different, and the match-up, though clever, is not justified. See reference 47 for more discussion on this point.

APPENDIX J - APPLICATION TO THE RAYLEIGH WAVE AMPLIFIER

Kino and Reeder (ref. 20) developed a coupled or normal mode theory for the Rayleigh wave amplifier. They used the original equations of Pierce, albeit in different notation. First they assume a potential field exists for the system in the form

$$\varphi = \varphi_a + \varphi_s$$

where φ_a is that due to the acoustic wave, and φ_s is due to the carriers in the semiconductor. The expression for φ_a is

$$\varphi_a(h) = -\frac{\omega\beta_a \rho_s' W Z_c(\beta h)}{\beta^2 - \beta_a^2}$$

which after appropriate notation change is

$$\varphi_a(h) = -\frac{K\Gamma\Gamma_1}{\Gamma^2 - \Gamma_1^2} i$$

or the familiar BTC. The expression for φ_s is

$$\varphi_s = \frac{\rho_s'}{\beta_e \varepsilon} M(\beta h)$$

where ρ_s' is an equivalent surface charge density at a height h above the piezoelectric. The factor $M(\beta h)$ is the space charge potential factor.

The total potential is then written as

$$\varphi(h) = \rho_s' \left[\frac{M(\beta h)}{\beta_e \varepsilon_0} - \frac{\omega\beta_a W Z_c(\beta h)}{\beta^2 - \beta_a^2} \right]$$

and after a notation change becomes

$$\varphi(h) = \left[\frac{M(\beta h)}{\omega W \varepsilon_0} - \frac{K\Gamma\Gamma_1}{\Gamma^2 - \Gamma_1^2} \right] i$$

Thus the first term is equivalent to $-j\Gamma/\omega C_1$ in Pierce's treatment. They relate the z and y components of the field by

$$E_z = jE_y$$

which is valid when $\rho = 0$. But this implies

$$E_y = \partial V/\partial z$$

which is at odds with the condition in Pierce's work.

They introduce a reduction factor R as

$$R = \frac{\varepsilon}{\varepsilon_0} M'(\beta_a h) \tanh (\beta_a d)$$

$$M'(\beta_a h) = M\left[1 + \left(\frac{\varepsilon_d}{\varepsilon_0} - 1\right)M\right]^{-1}$$

which plays the same role as the C_1 capacitors. The results are shown to agree somewhat with experiments, and we think this is the case due to the three essentially free fit parameters in the theory; namely σ the surface conductivity (a quantity almost impossible to determine experimentally), K_r^2 the effective electromechanical coupling constant, and h, the very small (on the order of 500 Å) gap between the semiconductor and the piezoelectric surface.

Thus the rather elegant theory presented for this device was based on several inconsistent assumptions that were not addressed by the authors.

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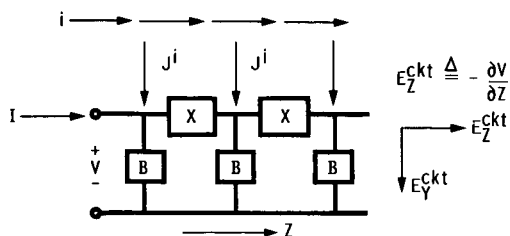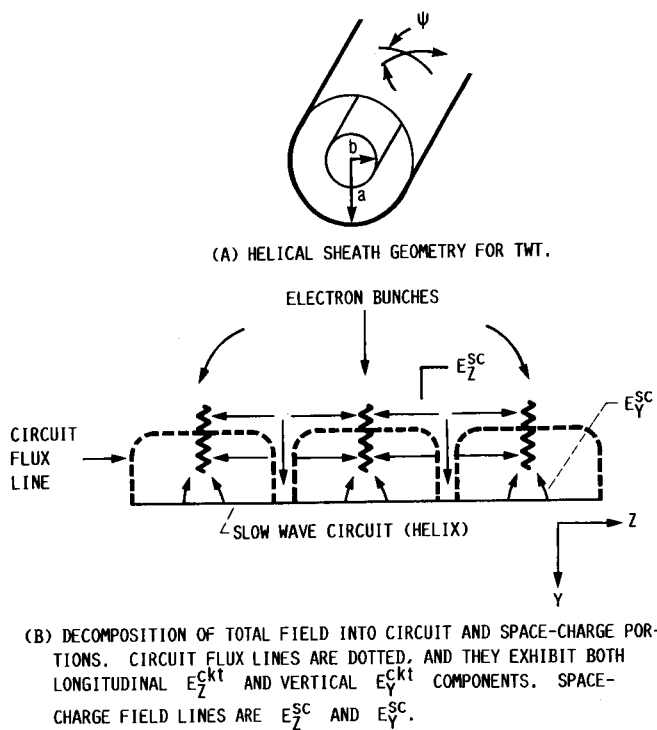   Phys., vol. 26, no. 5, May 1955, pp. 627-638.

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TABLE I. – THE SOLUTIONS FOR COMPLEX Γ and η FOR
SPECIFIC VALUES OF u_o, FOR THE EQUATION SET (31) – (34)
[Beam current $I_o > 0.$]

$u_o,$ cm/sec	$\Gamma_r,$ cm^{-1}	$\Gamma_i,$ cm^{-1}	$\eta_r,$ cm^{-1}	$\eta_i,$ cm^{-1}	u_o/v_φ
2.55×10^9	-0.142	15.36	93.6	96	0.954
2.60	-.2885	15.015	82.81	27.75	.973
2.65	-.364	14.765	63	22.35	.992
2.70	-.435	14.53	50.55	19	1.01
2.80	-.578	14.09	35.75	14.5	1.048
2.90	-.686	13.715	27.75	12.2	1.085
3.0	-.778	13.36	22.67	10.8	1.123
3.4	-.791	12.33	12.6	7.69	1.272
3.5	-.710	12.13	10.9	6.9	1.31

TABLE II. – A COMPARISON OF THE COEFFICIENTS OF QUARTIC POLYNOMIALS FOR THE "FIELD" (CHU AND JACKSON)
CALCULATION WITH THOSE OF THE TRANSMISSION LINE MODELS

Coefficient	a_4	a_3	a_2	a_1	a_o
Chu and Jackson	1	$-j2\beta_e$	$\eta^2 + k^2 - \beta_e^2 + \beta_p^2$	$-j2\beta_e(\eta^2 + k^2)$	$\beta_p^2 k^2 - \beta_e^2(\eta^2 + k^2)$
Pierce	$1 - 4QC^3$	$-j2\beta_e$	$\Gamma_1^2(4QC^3 - 1)$ $+ j2\beta_e\Gamma_1 C^3 - \beta_e^2$	$j2\beta_e\Gamma_1^2$	$\beta_e^2\Gamma_1^2$
Conventional	1	$-j2\beta_e$	$R^2\beta_p^2 - \beta_e^2 - \Gamma_1^2$ $+ j2\beta_e\Gamma_1 C^3$	$j2\beta_e\Gamma_1^2$	$\Gamma_1^2(\beta_e^2 - \beta_p^2 R^2)$
f–Factor	1	$-j2\beta_e$	$-j\omega\epsilon_o A_1 f\beta_p^2 - \beta_e^2 - \Gamma_1^2$ $+ j2\beta_e\Gamma_1 C^3(1 - R^2)$	$j2\beta_e\Gamma_1^2$	$\Gamma_1^2(\beta_e^2 + j\omega\epsilon_o A_1 f\beta_p^2)$

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(A) HELICAL SHEATH GEOMETRY FOR TWT.

ELECTRON BUNCHES

(B) DECOMPOSITION OF TOTAL FIELD INTO CIRCUIT AND SPACE-CHARGE POR-
TIONS. CIRCUIT FLUX LINES ARE DOTTED, AND THEY EXHIBIT BOTH
LONGITUDINAL E_Z^{ckt} AND VERTICAL E_Y^{ckt} COMPONENTS. SPACE-
CHARGE FIELD LINES ARE E_Z^{sc} AND E_Y^{sc}.

$$E_Z^{ckt} \triangleq - \frac{\partial V}{\partial Z}$$

(C) EQUIVALENT TRANSMISSION LINE COUPLED TO BEAM
VIA J^i. POSTULATED RELATION BETWEEN LINE
VOLTAGE V AND E_Z^{ckt} IS SHOWN.

FIGURE 1. - BASIC GEOMETRY.

$$A_1 = \pi r_b^2$$
$$A_2 = 2\pi r_a \Delta Z$$

(A) BEAM OF RADIUS r_b INSIDE HELICALLY CONDUCTING SHEATH OF
LENGTH ΔZ. BOTH LONGITUDINAL CONVECTION CURRENT i AND
DISPLACEMENT CURRENT $j\omega\epsilon_0 E_Z^{sc}$ EXIST IN BEAM.
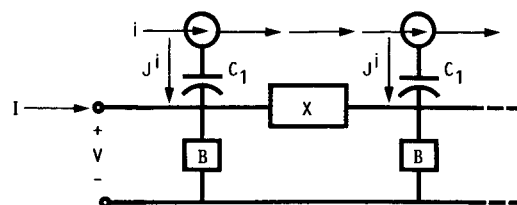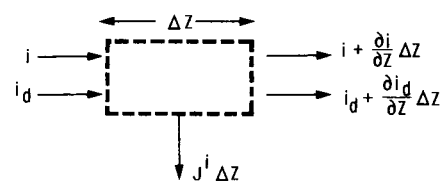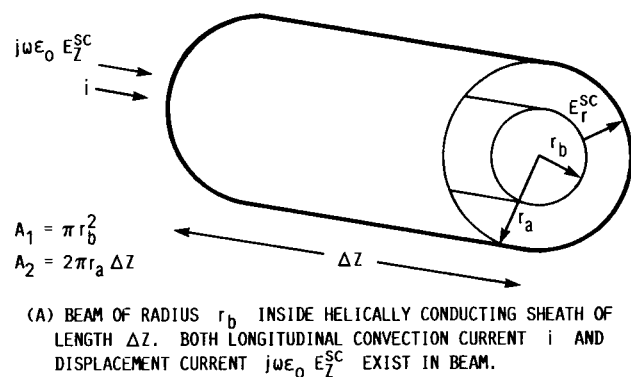
(B) SMALL VOLUME USED TO WRITE NODE EQUATION TO DEVELOP
J^i (SEE APPENDIX A). AT BOTTOM IS MODIFIED EQUIVA-
LENT CIRCUIT OF PIERCE USED TO INCORPORATE "SPACE-
CHARGE" EFFECTS (SEE APPENDIX A).

FIGURE 2. - TRANSMISSION LINE MODEL.

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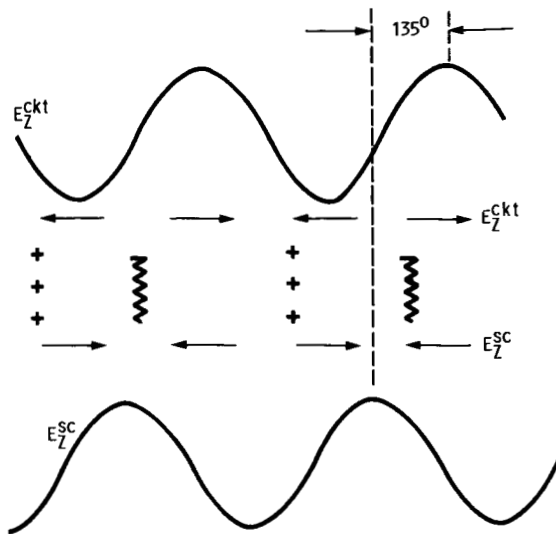
FIGURE 3. - POSSIBLE PHASE RELATION BETWEEN E_Z^{ckt} AND E_Z^{SC} DURING ENERGY TRANSFER FROM BUNCHED BEAM TO CIRCUIT. CASE SHOWN IS FOR $\angle g = 135^0$. INTUITIVELY ONE EXPECTS $\angle g$ TO LIE IN THE INTERVAL $(90^0, 180^0)$ DURING WAVE GROWTH.
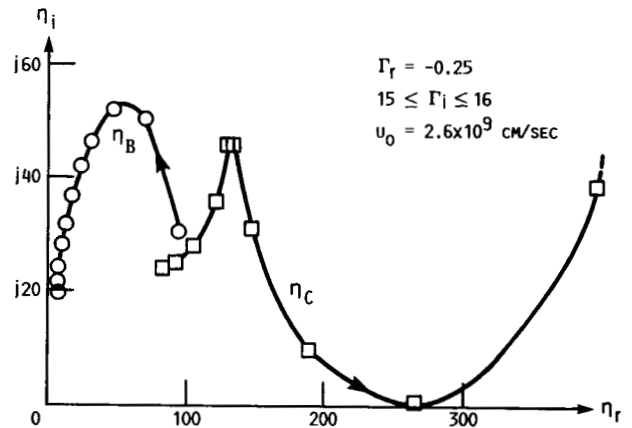
FIGURE 4. - PLOTS OF η_B (η_B IS η IN EQ. (31)) AND η_C (η IN EQ. (33)) AS Γ_i IS SWEPT OVER INTERVAL [15, 16] IN INCREMENTS OF 0.1.
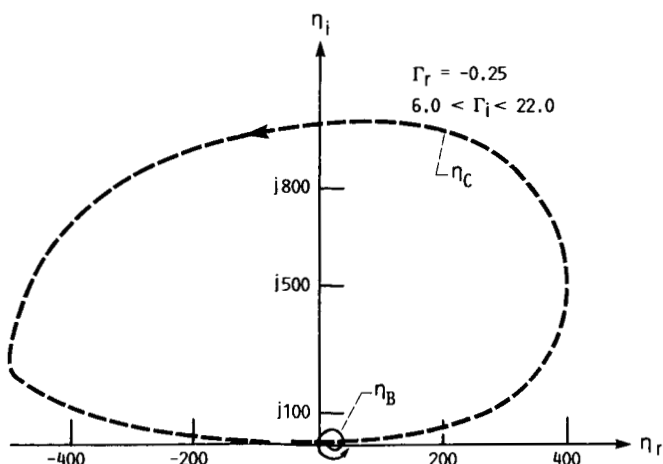
FIGURE 5. - GENERAL VARIATION OF η_B AND η_C AS Γ_i IS SWEPT OVER INTERVAL [6, 22]. IN GENERAL ONLY ONE SOLUTION IS FOUND FOR A SPECIFIC VALUE OF u_0.
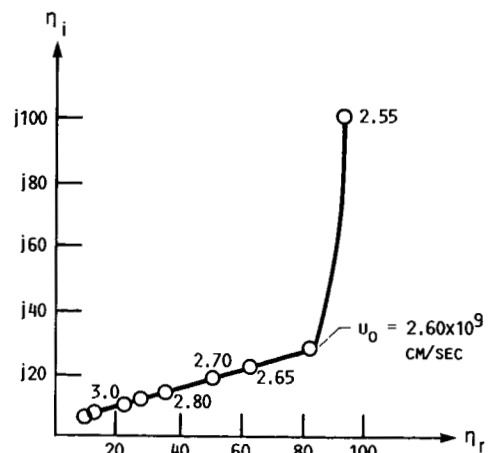
FIGURE 6. - VALUES OF η_r AND η_i THAT SATISFY EQUATIONS (31) TO (34) AS A FUNCTION OF THE DRIFT VELOCITY u_0.

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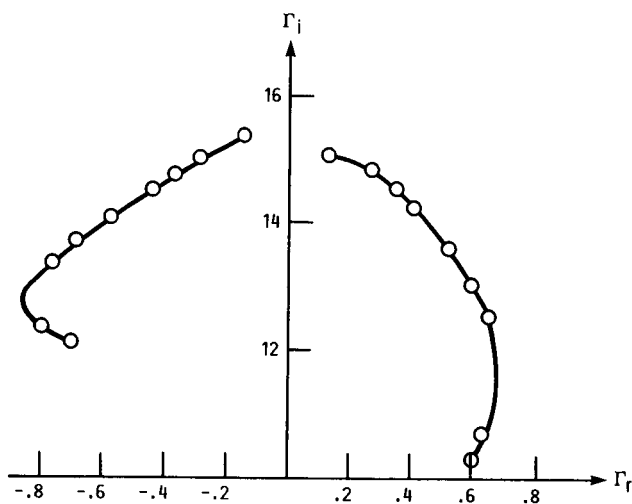
(A) LOCI OF TWO OF FOUR COMPLEX ROOTS OF EQUATION (31) AS u_0 IS VARIED. VALUES OF u_0 ARE LISTED IN TABLE I.
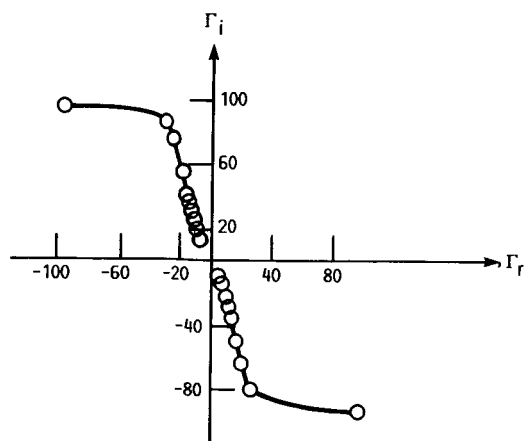
(B) REMAINING ROOTS OF EQUATION (31) WITH THE SAME RANGE OF VARIATION FOR u_0.

FIGURE 7. - TRAJECTORY OF CHU ROOTS.

	Γ_1, cm^{-1}
----- (UPPER)	14.52
- - - (LOWER)	12.17
——— "FIELD" SOLUTION	
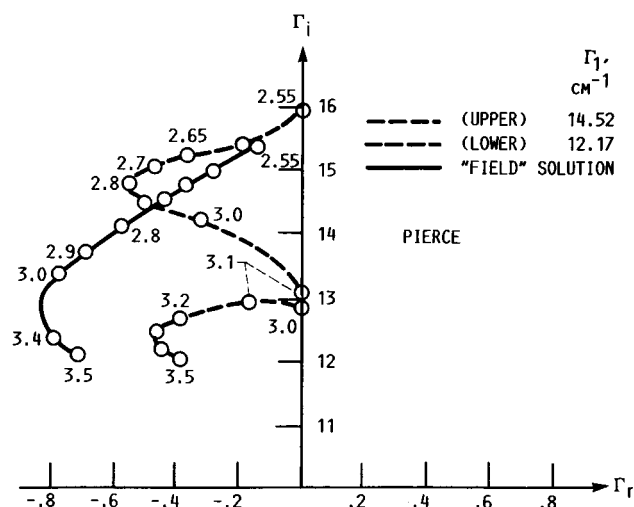
PIERCE

FIGURE 8. - LOCI OF GROWING ROOT USING "PIERCE" FORMULATION WITH C AND QC AS FIT PARAMETERS. VALUES OF u_0 AT SELECTED POINTS ARE INDICATED TO AID COMPARISON.

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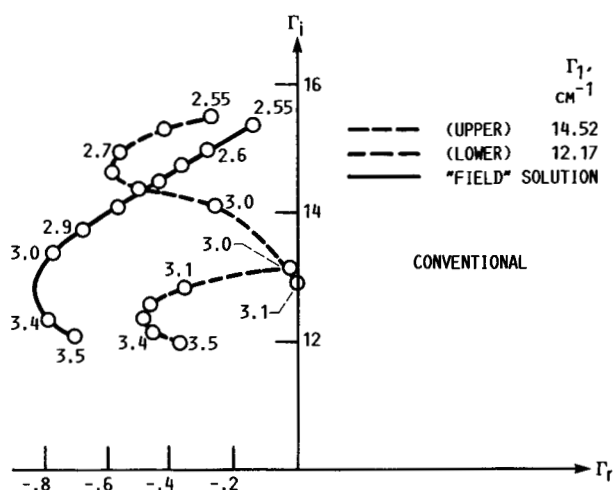
FIGURE 9. - GROWING ROOT LOCI FOR "CONVENTIONAL" APPROACH.
VARIATION IS SIMILAR TO "PIERCE" FORMULATION. HERE FIT
PARAMETERS ARE C AND R.
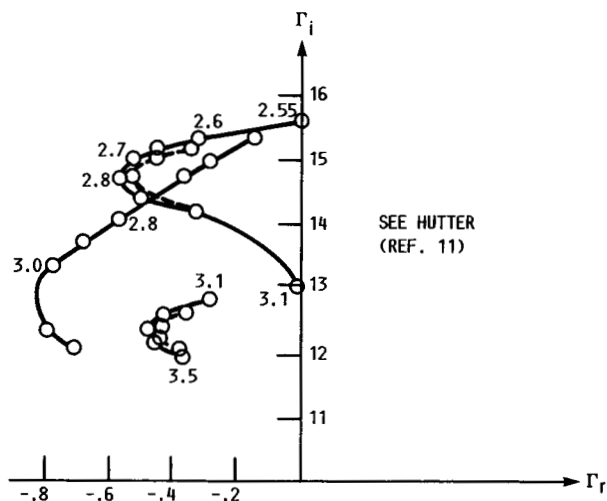
FIGURE 10. - GROWING ROOT LOCI USING MORE INVOLVED FORMULA-
TION AS FOUND IN REFERENCE 11. SOLID CURVES ARE FOR R_+,
WHILE DOTTED ONES ARE FOR R_-. UPPER AND LOWER SETS ARE
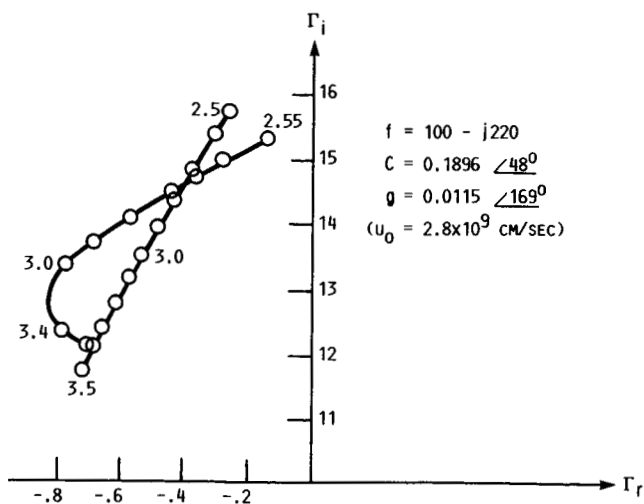FOR $|\Gamma_1| = 14.52$ CM^{-1} AND 12.27 CM^{-1}, RESPECTIVELY.

$$f = 100 - j220$$
$$C = 0.1896 \ \angle 48^0$$
$$g = 0.0115 \ \angle 169^0$$
$$(u_0 = 2.8 \times 10^9 \text{ CM/SEC})$$

FIGURE 11. - GROWING ROOT LOCUS FOR f-FACTOR METHOD. HERE f IS
CHOSEN TO BE $100 - j220$. GAIN PARAMETER $C = 0.1896 \ \angle 48^0$ AND
FACTOR g IS $0.0115 \ \angle 169^0$ WHEN $u_0 = 2.8 \times 10^9$ CM/SEC.
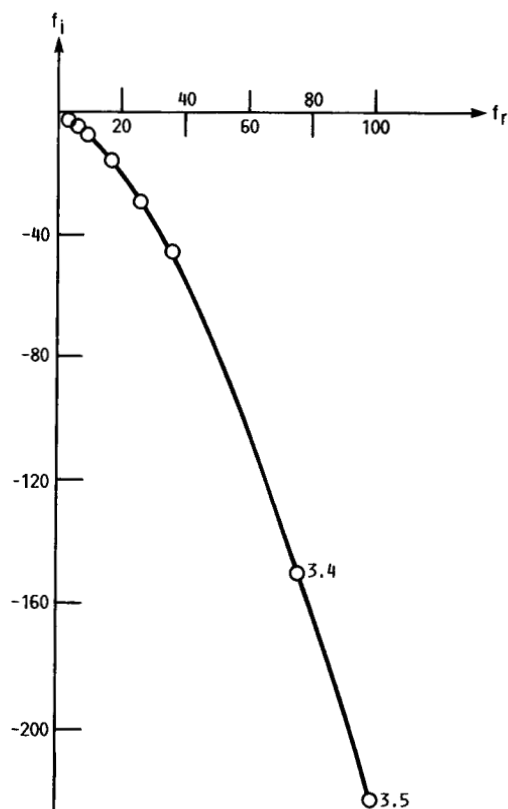
FIGURE 12. - VARIATION OF COMPLEX $f = f_r + jf_i$
VERSUS u_0 TO TRACK GROWING ROOT FROM FIELD
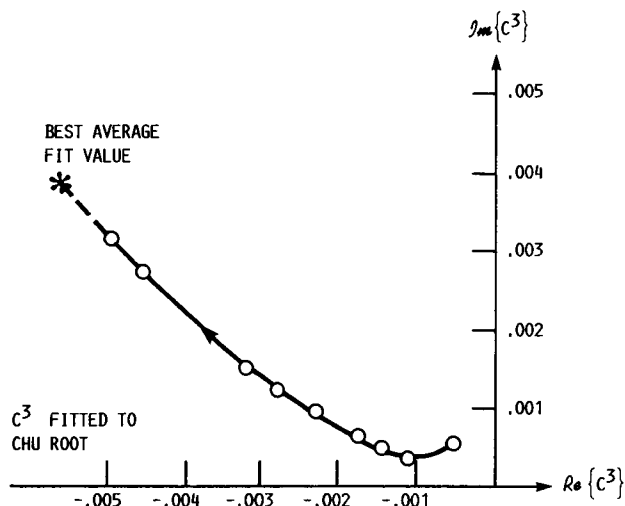SOLUTION.

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FIGURE 13. - TRAJECTORY OF COMPLEX GAIN PARAMETER c^3
AS u_0 VARIES. THIS CAUSES ROOT TO TRACK FIELD
SOLUTION ROOT. AT LEFT THE BEST AVERAGE VALUE
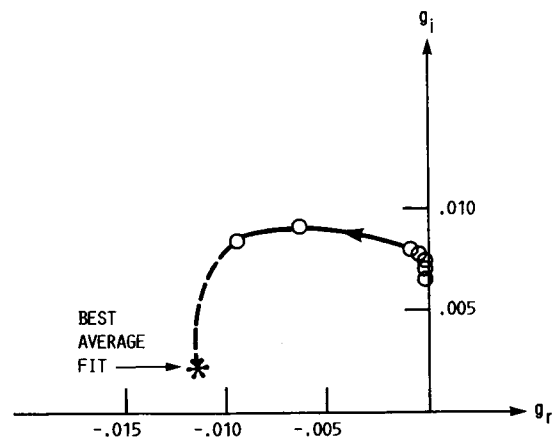WHICH PROVIDES A REASONABLE TRACKING FOR RANGE
OF u_0.

FIGURE 14. - VARIATION OF g-FACTOR WITH u_0. THIS IS
OBTAINED FROM VARIATION OF f FROM EQUATION (30).
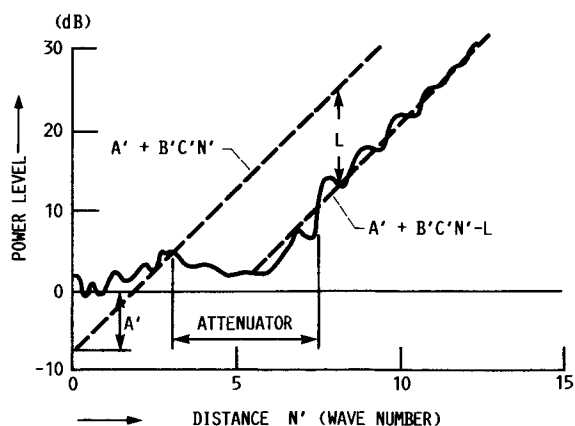PHASE ANGLE IS IN SECOND QUADRANT AS WOULD BE
EXPECTED.

FIGURE 15. - SKETCH OF FIGURE IN REFERENCE 36 SHOWING
MEASURED SIGNAL LEVEL AS FUNCTION OF DISTANCE ALONG
TUBE. STRAIGHT LINE FIT TO DATA YIELD TWO FIT
PARAMETERS. THESE ARE ESSENTIALLY JUST SLOPE (GAIN
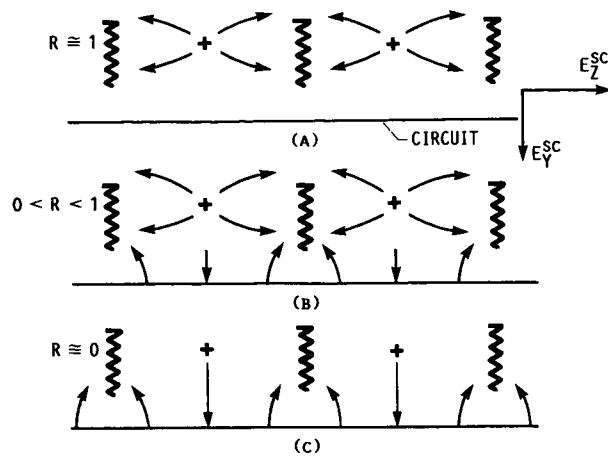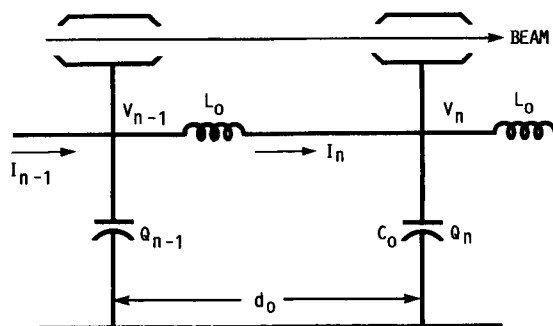PARAMETER) AND INTERCEPT (SPACE-CHARGE PARAMETER).

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FIGURE C-1. - PHYSICAL MODEL UNDERLYING SPACE-
CHARGE REDUCTION FACTOR APPROACH.

$$I_n - I_{n+1} = \frac{\partial q_n}{\partial t}$$

$$L \frac{\partial I_n}{\partial t} = V_{n-1} - V_n \qquad L = L_0 d_0$$

$$V_n = \frac{Q_n}{C} \qquad C = C_0 d_0$$

$$Q_n = q_n + q'_n = q_n + \rho_n d_0$$

FIGURE G1. - EQUIVALENT CIRCUIT USED BY BOTH BRILLOUIN AND
ROWE TO DEVELOP EQUATIONS FOR TWTs TO BE USED IN LARGE
SIGNAL CASE.

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National Aeronautics and
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15. Supplementary Notes

16. Abstract

We show how the complete field of the electron beam may be incorporated into the transmission line model theory of the traveling wave tube (TWT). The fact that the longitudinal component of the field due to the bunched beam is not used when formulating the beam-to-circuit coupling equation is not well-known. The fundamental partial differential equation for the traveling wave field is developed and compared with the older (now standard) ones. The new equation can be solved numerically using the same algorithms, but now the coefficients can be updated continuously as the calculation proceeds down the tube. The coefficients in the older equation are primarily derived from preliminary measurements and some trial and error. The newer coefficients can be found by a recursive method, since each has a well defined physical interpretation and can be calculated once a reasonable first trial solution is postulated. We compare the results of the new expression with those of the older forms, as well as to a field theory model to show the ease in which a reasonable fit to the field prediction is obtained. A complete summary of the existing transmission line modeling of the TWT is given to explain the somewhat vague ideas and techniques in the general area of drifting carrier-traveling circuit wave interactions. The basic assumptions and inconsistencies of the existing theory and areas of confusion in the general literature are examined and hopefully cleared up.

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