NASA Technical Memorandum 89123

VALIDATION OF THE SURE PROGRAM - PHASE 1

(NASA-TM-89123) VALIDATION OF THE SURE PROGRAM, PHASE 1 (NASA) 49 p Avail: NTIS HC A03/MF A01 CSCL 09B
N87-24937
G3/65 Unclas 0080331

Kelly J. Dotson

May 1987

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

INTRODUCTION

Due to the complexity of fault-tolerant computer systems, automated tools such as ARIES (Automated Reliability Interactive Estimation System), CARE III (Computer-Aided Reliability Estimation), and HARP (Hybrid Automated Reliability Predictor), ect., are being used in the reliability analysis and development of fault-tolerant computer architectures (ref. 1 and 2). The Semi-Markov Unreliability Range Evaluator program, SURE, is one of the latest reliability analysis tools to be introduced (ref. 3). The SURE program provides an alternative approach to the difficult task of solving convolution integrals traditionally used to determine the reliability of a system modeled by a semi-Markov model. The SURE program implements mathematics developed by White (ref. 4) and Lee (ref. 5) for analytically specifying lower and upper bounds on the death-state probabilities of a semi-Markov model.

If tools such as SURE are to be used in the development of highly reliable computer systems, the tools themselves must produce reliable outputs. How does one know that the bounds given by SURE actually envelop the exact unreliability for any given model of a system? White (ref. 4) and Lee (ref. 5) both provide mathematical proofs for their lower and upper bounds on the death-state probabilities of a semi-Markov model. Given that these proofs are correct, the question then becomes, are these mathematical bounds correctly implemented in the SURE program? To answer this question, i.e. validate SURE, three major studies of SURE are being conducted: (1) comparison of SURE's bounds to exact analytic solutions for simple semi-Markov models, (2) comparison of SURE's bounds to estimates from other reliability analysis tools (ARIES, MARK, and CARE III) for more complex models, and (3) analysis of the mathematical bounds themselves.

This paper describes the results of the first phase in the effort to validate SURE version 4.3. During this first phase, SURE's bounds (those developed by White) were compared to exact solutions analytically derived for simple semi-Markov models. To verify correct implementation of the bounds, fifteen simple semi-Markov models were constructed. Each model was solved directly for the individual death-state probabilities, and these exact death-state probabilities were compared to the bounds given by SURE. Due to the difficulty in solving even simple models by hand, the state size of the models was limited.

COMPUTATION OF DEATH-STATE PROBABILITIES FOR SEMI-MARKOV MODELS

In a Markov-type model of a fault-tolerant system, the unreliability of the system is equivalent to the sum of all of the death-state probabilities in the model. Each death-state probability can be determined analytically by summing the probabilities of traversing every path through the model that leads to that death state. The exact death-state probability is determined mathematically by solving a series of convolution integrals. As the model gets larger, more convolution integrals need to be solved. Due to the difficulty in solving even a small number of convolution integrals, the models in this study were limited to five states.

To calculate each death-state probability, each path to that death state can be analyzed transition by transition. In a pure Markov model, the transitions between states are exponentially distributed. However, in a semi-Markov model, transitions between states can be described by distributions other than the exponential. For this work, transition rates were limited to distributions that are mathematically tractable, namely the exponential, uniform and impulse distributions.

In the SURE program, the transition rates are labeled either slow or fast. Slow transition rates correspond to fault arrivals in a computer system and are assumed to be exponentially distributed. Fast rates describe the system's response to faults and can be characterized by any distribution. The slow transitions are denoted in the models by a greek character representing the rate, and the general transitions are denoted by a capital letter that represents a particular distribution. The following three path classifications based on the state transitions rates are used in SURE: (1) slow on path, slow off path, (2) fast on path, arbitrary off path, and (3) slow on path, fast off path. (Note: the transition on the path being analyzed is called the on-path transition and the remaining transitions are referred to as off-path.) Figures 1, 2, and 3 are examples of these path steps as shown in The SURE Reliability Analysis Program.

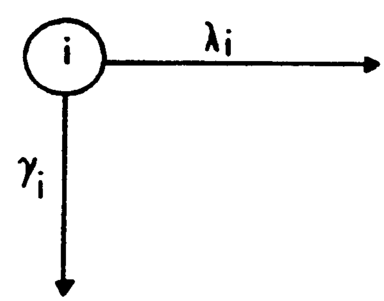


FIGURE 1

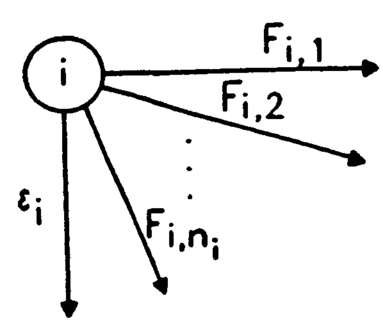


FIGURE 2

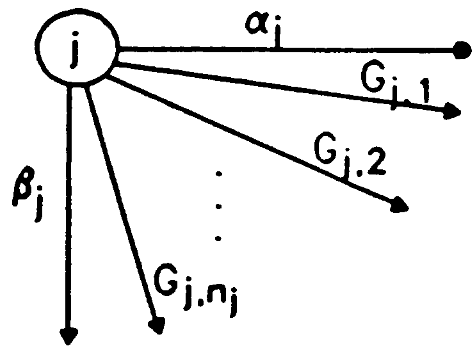


FIGURE 3

The unreliability of the modeled system at time T is the sum of the death-state probabilities at T. The probability of being in a death state at the end of each of the above path classifications, denoted $P_i(T)$ where i = 1, 2, or 3 for each classification, are defined as follows:

$$P_1(T) = \int_0^T \int_0^{T-x1} \cdots \int_0^{T-x1-\ldots-xn} \lambda_1 e^{-(\lambda 1+\gamma 1)x1} \lambda_2 e^{-(\lambda 2+\gamma 2)x2} \ldots \lambda_k e^{-(\lambda k+\gamma k)xk} dx_k \cdots dx_1$$

$$P_2(T) = \int_0^T \int_0^{T-x1} \cdots \int_0^{T-x1-\ldots-xn} e^{-\varepsilon 1x1}[1-F_{1,2}(x_1)]\cdots[1-F_{1B1}(x_1)] \cdots e^{-\varepsilon m}[1-F_{m,2}(x_m)]\cdots[1-F_{m,Bm}(x_m)]dF_{m,1}(x_m)\cdots dF_{1,1}(x_1)$$

$$P_3(T) = \int_0^T \int_0^{T-x1} \cdots \int_0^{T-x1-\ldots-xn} \alpha_1 e^{-(\alpha 1+\beta 1)y1}[1-G_{1,2}(y_1)]\cdots[1-G_{1,c1}(y_1)] \cdots \alpha_n e^{-(\alpha n+\beta n)yn}[1-G_{n,1}(y_n)]\cdots[1-G_{n,cn}(y_n)]dy_n\cdots dy_1$$

To compute the death-state probability from a path that contains a combination of classifications, the integrals are simply combined as in White's Synthetic Bounds (ref. 4) p. 7-8. Using these formulas, one can calculate the unreliability of a simple semi-Markov model.

In order to use SURE to find the unreliability of a given model, the model must be described using a simple language that enumerates all of the transitions of the model. In SURE, the slow transitions are simply specified by giving the exponential rate, and the fast transitions are specified by giving the mean, μ, and standard deviation, σ, of that transition's distribution and specifying the transition probability ρ. These statistics are defined independently of any competing exponential transitions. Given the general path step in Figure 4, the following formulas define the mean, variance, and transition probability for a general fast transition F_1:

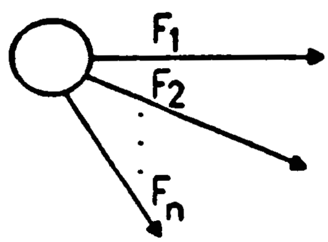


FIGURE 4

$$\mu(F_1) = \rho^{-1}(F_1) \int_0^\infty t[1 - F_2(t)][1 - F_3(t)] \cdots [1 - F_n(t)]dF_1(t)$$

$$\sigma^2(F_1) = \left[\rho^{-1}(F_1) \int_0^\infty t^2[1 - F_2(t)][1 - F_3(t)] \cdots [1 - F_n(t)]dF_1(t)\right] - \mu^2(F_1)$$

$$\rho(F_1) = \int_0^\infty [1 - F_2(t)[1 - F_3(t)] \cdots [1 - F_n(t)]dF_1(t)$$

The following is a collection of fifteen simple test cases that compare the exact solution for the death-state probabilities in each test model to the bounds given by SURE. For each test case, the model is given along with the equations for the exact death-state probabilities. Equations for μ, σ^2, and ρ are given for models with non-exponential transitions. A table that compares the exact solutions to the SURE bounds for a range of parameter values follows each model. For all cases, the mission-time default of 10 hours was used.

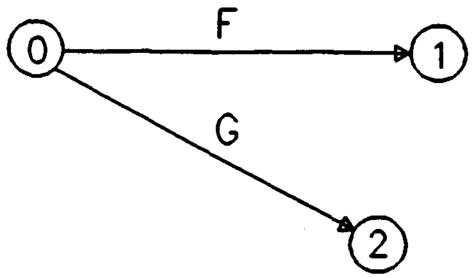
A relative error estimate is also given for the bounds. Here,

$$\text{relative error} = \frac{|\text{SURE bound furthest from exact solution} - \text{exact solution}|}{\text{exact solution}}$$

This error estimate gives a measure of the tightness of the bounds. The unreliability estimate given by the SURE bounds is more precisely expressed when the bounds are tight. A small relative error is indicative of tight bounds. Correspondingly, a large relative error indicates a wide spread in the bounds. The acceptable degree of tightness in the bounds depends on the needs of the user and the intended application.

A hand calculator was used to obtain numerical values from the analytic solutions for specific values of the model parameters. In cases where the calculator had insufficient precision to correctly compute the analytic solutions, a program called MARK (ref. 6) was used to determine the death-state probabilities. MARK uses a combination of Padé approximations, scaling, and squaring techniques to compute a matrix exponential needed to determine the death-state probabilities. The solutions obtained by MARK are indicated throughout the paper by an @. Since the SURE bounds are given in six significant digits, the exact solutions were also rounded off to six significant digits. All warning and error messages that were output by the SURE program when each case was run are noted at the bottom of the appropriate table and are explained in the discussion of the results.

PROBLEM 1



$$F(t) = \begin{cases} t/a, & t \le a; \\ 1, & t > a; \end{cases} \qquad G(t) = \begin{cases} t/b, & t \le b \\ 1, & t > b \end{cases}$$

Assumption: $a \le b \le T$

Transition Description

$$\rho(F) = \int_0^\infty [1 - G(T)]dF(T) = \int_0^a (1 - t/b)(1/a)dt = 1 - a/(2b)$$

$$\mu(F) = \rho^{-1}(F)\int_0^\infty t[1 - G(t)]dF(t) = (2b)(2b - a)^{-1}\int_0^a t(1 - t/b)(1/a)dt$$
$$= (3ab - 2a^2)(6b - 3a)^{-1}$$

$$\sigma^2(F) = \rho^{-1}(F)\int_0^\infty t^2[1 - G(t)]dF(t) - \mu^2(F)$$
$$= (2b)(2b - a)^{-1}\int_0^a t^2(1 - t/b)(1/a)dt - \mu^2(F)$$
$$= (6a^2b^2 - 6a^3b + a^4)(72b^2 - 72ab + 18a^2)^{-1}$$

$$\rho(G) = \int_0^\infty [1 - F(t)]dG(t) = \int_0^a (1 - t/a)b^{-1}dt = a/(2b)$$

$$\mu(G) = \rho^{-1}(G)\int_0^\infty t[1 - F(t)]dG(t) = 2ba^{-1}\int_0^a tb^{-1}(1 - t/a)dt = a/3$$

$$\sigma^2(G) = \rho^{-1}(G)\int_0^\infty t^2[1 - F(t)]dG(t) - \mu^2(G)$$
$$= 2ba^{-1}\int_0^a t^2b^{-1}(1 - t/a)dt - \mu^2(G) = a^2/18$$

Death-State Probabilities

$$D_1(T) = \int_0^T [1 - G(t)]dF(t) = \int_0^a (1 - t/b)a^{-1}dt = 1 - a/(2b)$$

$$D_2(T) = \int_0^T [1 - F(t)]dG(t) = \int_0^a (1 - t/a)b^{-1}dt = a/(2b)$$

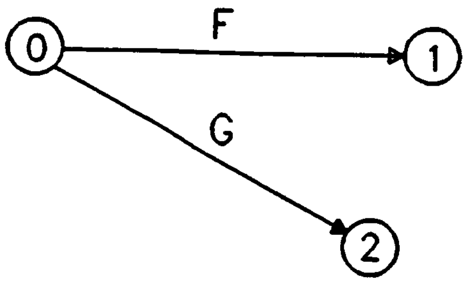
TABLE 1

a: uniform parameter for the (0,1) transition

b: uniform parameter for the (0,2) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
a= 1e-6	$D_1(T)$	9.50000e-01	(9.49999e-01, 9.50000e-01)	0.000
b= 1e-5	$D_2(T)$	5.00000e-02	(5.00000e-02, 5.00000e-02)	0.000
a= 1e-6	$D_1(T)$	9.99995e-01	(9.99994e-01, 9.99995e-01)	0.000
b= 1e-1	$D_2(T)$	5.00000e-06	(5.00000e-06, 5.00000e-06)	0.000
a= 5e-8	$D_1(T)$	1.00000e+00	(1.00000e+00, 1.00000e+00)	0.000
b= 4e-8	$D_2(T)$	2.50000e-09	(2.50000e-09, 2.50000e-09)	0.000
a= 1e-3	$D_1(T)$	9.50000e-01	(9.49372e-01, 9.50000e-01)	0.066
b= 1e-2	$D_2(T)$	5.00000e-02	(4.99750e-02, 5.00000e-02)	0.050
a= 2e-2	$D_1(T)$	9.00000e-01	(8.88231e-01, 9.00000e-01)	1.308
b= 1e-1	$D_2(T)$	1.00000e-01	(9.90000e-02, 1.00000e-01)	1.000

PROBLEM 2



$F(t) = 0, \; t < a; \quad 1, \; t \geq a;$

$G(t) = t/b, \; t < b \quad 1, \; t \geq b$

Assumption: $a < b < T$

Transition Description

$$\rho(F) = \int_0^\infty [1 - G(t)]dF(t) = 1 - G(a) = 1 - a/b$$

$$\mu(F) = \rho^{-1}(F)\int_0^\infty t[1 - G(t)]dF(t) = b(b - a)^{-1}[a(1 - a/b)] = a$$

$$\sigma^2(F) = \rho^{-1}(F)\int_0^\infty t^2[1 - G(t)]dF(t) - \mu^2(F)$$

$$= b(b - a)^{-1}[a^2(1 - a/b)] - \mu^2(F) = 0$$

$$\rho(G) = \int_0^\infty [1 - F(t)]dG(t) = \int_0^a b^{-1}dt = a/b$$

$$\mu(G) = \rho^{-1}(G)\int_0^\infty t[1 - F(t)]dG(t) = ba^{-1}\int_0^a tb^{-1}dt = a/2$$

$$\sigma^2(G) = \rho^{-1}(G)\int_0^\infty t^2[1 - F(t)]dG(t) - \mu^2(G)$$

$$= ba^{-1}\int_0^a t^2b^{-1}dt - \mu^2(G) = a^2/12$$

Death-State Probabilities

$$D_1(T) = \int_0^T [1 - G(t)]dF(t) = 1 - G(a) = 1 - a/b$$

$$D_2(T) = \int_0^T [1 - F(t)]dG(t) = \int_0^a b^{-1}dt = a/b$$

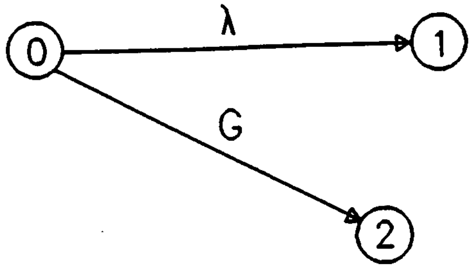
TABLE 2

a: impulse parameter for the (0,1) transition

b: uniform parameter for the (0,2) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
a= 1e-6	$D_1(T)$	9.00000e-01	(8.99999e-01, 9.00000e-01)	0.000
b= 1e-5	$D_2(T)$	1.00000e-01	(9.99999e-02, 1.00000e-01)	0.001
a= 1e-6	$D_1(T)$	9.99990e-01	(9.99989e-01, 9.99990e-01)	0.000
b= 1e-1	$D_2(T)$	1.00000e-05	(9.99999e-06, 1.00000e-05)	0.010
a= 1e-2	$D_1(T)$	9.00000e-01	(8.91000e-01, 9.00000e-01)	1.000
b= 1e-1	$D_2(T)$	1.00000e-01	(9.93333e-02, 1.00000e-01)	0.667
a= 5e-7	$D_1(T)$	9.80000e-01	(9.79999e-01, 9.80000e-01)	0.000
b= 2.5e-5	$D_2(T)$	2.00000e-02	(2.00000e-02, 2.00000e-02)	0.000
a= 5e-7	$D_1(T)$	9.99950e-01	(9.99950e-01, 9.99950e-01)	0.000
b= 1e-2	$D_2(T)$	5.00000e-05	(5.00000e-05, 5.00000e-05)	0.000

PROBLEM 3



$$F(t) = 1 - e^{-\lambda t},\ t > 0; \qquad G(t) = \begin{cases} 0, & t < \theta \\ 1, & t \geq \theta \end{cases}$$

Transition Description

$$\rho(G) = \int_0^\infty dG(t) = 1$$

$$\mu(G) = \rho^{-1}(G)\int_0^\infty t\,dG(t) = \theta$$

$$\sigma^2(G) = \rho^{-1}(G)\int_0^\infty t^2 dG(t) - \mu^2(G) = \theta^2 - \theta^2 = 0$$

Death-State Probabilities

$$D_1(T) = \int_0^T [1 - G(t)]dF(t) = \int_0^\theta \lambda e^{-\lambda t} dt = 1 - e^{-\lambda\theta}$$

$$D_2(T) = \int_0^T [1 - F(t)]dG(t) = 1 - (1 - e^{-\lambda\theta}) = e^{-\lambda\theta}$$

TABLE 3

λ: exponential parameter for the (0,1) transition
γ: impulse parameter for the (0,2) transition

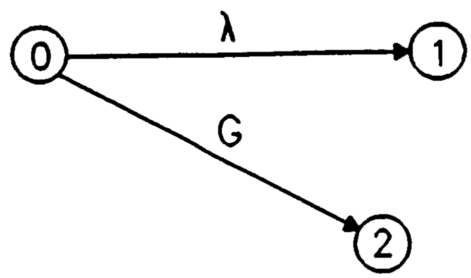
PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ= 1e-5	D_1(T)	4.99988e-05	(0.00000e+00, 5.00000e-05)+	100.000
γ= 5.0	D_2(T)	9.99950e-01	(0.00000e+00, 1.00000e+00)+	100.000
λ= 1e-5	D_1(T)	1.00000e-09	(9.90000e-10, 1.00000e-09)	1.000
γ= 1e-4	D_2(T)	1.00000e+00	(9.99900e-01, 1.00000e+00)	0.010
λ= 1e-2	D_1(T)	1.00000e-10	(9.99900e-11, 1.00000e-10)	0.010
γ= 1e-8	D_2(T)	1.00000e+00	(1.00000e+00, 1.00000e+00)	0.000
λ= 1e-1	D_1(T)	1.98013e-02	(1.08557e-02, 2.00000e-02)	45.177
γ= 2e-1	D_2(T)	9.80199e-01	(7.80000e-01, 1.00000e+00)	20.424
λ= 2e-2	D_1(T)	1.99980e-04	(1.79980e-04, 2.00000e-04)	10.001
γ= 1e-2	D_2(T)	9.99800e-01	(9.89800e-01, 1.00000e+00)	1.000
λ= 1e-4	D_1(T)	3.00000e-10	(2.99480e-10, 3.00000e-10)	0.173
γ= 3e-6	D_2(T)	1.00000e+00	(9.99997e-01, 1.00000e+00)	0.000
λ= 1e-1	D_1(T)	1.00000e-08	(9.99684e-09, 1.00000e-08)	0.032
γ= 1e-7	D_2(T)	1.00000e+00	(1.00000e+00, 1.00000e+00)	0.000
λ= 2e-3	D_1(T)	9.99500e-04	(2.92393e-04, 1.00000e-03)	70.746
γ= 5e-1	D_2(T)	9.99000e-01	(4.99000e-01, 1.00000e+00)	50.050
λ= 3e-4	D_1(T)	2.99955e-04	(0.00000e+00, 3.00000e-04)+	100.000
γ= 1.0	D_2(T)	9.99700e-01	(0.00000e+00, 1.00000e+00)+	100.000
λ= 4e-7	D_1(T)	4.00000e-10	(3.87351e-10, 4.00000e-10)	3.162
γ= 1e-3	D_2(T)	1.00000e+00	(9.99000e-01, 1.00000e+00)	0.100

\+ RECOVERY TOO SLOW

Note that for values of $\gamma > 0.1$, the relative error is large. The SURE program has difficulty handling general recovery transitions which are slow relative to the mission time.

Problem 4a will demonstrate the effect of using a slow exponential transition description when the exponential rate is actually fast. Problem 4b will, in contrast, show the effect of using means and standard deviations to describe slow exponential transition. Problem 4b uses the same model as in 4a except the (0, 1) transition is expressed as a general transition with mean and standard deviation. To demonstrate the problems associated with improper specification of an exponential transition, the same test cases containing a wide range of values for the exponential transition are given in Tables 4a and 4b.

PROBLEM 4a



$F(t) = 1 - e^{-\lambda t}, \quad t > 0 \qquad G(t) = \begin{cases} t/b, & t < b \\ 1, & t \geq b \end{cases}$

Assumption: $b < T$

Transition Descriptions

$$\rho(G) = \int_0^\infty dG(t) = 1$$

$$\mu(G) = \rho^{-1}(G)\int_0^\infty t\,dG(t) = \int_0^b tb^{-1}dt = b/2$$

$$\sigma^2(G) = \rho^{-1}(G)\int_0^\infty t^2 dG(t) - \mu^2(G) = \int_0^b t^2 b^{-1}dt - b^2/4 = b^2/12$$

Death-State Probabilities

$$D_1(T) = \int_0^T [1 - G(t)]dF(t) = \int_0^b (1 - t/b)\lambda e^{-\lambda t}dt = (\lambda b + e^{-\lambda b} - 1)/(\lambda b)$$

$$D_2(T) = \int_0^T [1 - F(t)]dG(t) = \int_0^b e^{-\lambda t}b^{-1}dt = (1 - e^{-\lambda b})/(\lambda b)$$

TABLE 4a

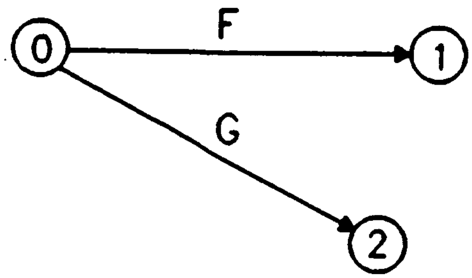
λ: exponential parameter for the (0,1) transition
b: uniform parameter for the (0,2) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ= 1e-6	$D_1(T)$	5.00000e-11	(4.95286e-11, 5.00000e-11)	0.943
b= 1e-4	$D_2(T)$	1.00000e+00	(9.99933e-01, 1.00000e+00)	0.007
λ= 1e-4	$D_1(T)$	5.00000e-09	(4.95286e-09, 5.00000e-09)	0.943
b= 1e-4	$D_2(T)$	1.00000e+00	(9.99933e-01, 1.00000e+00)	0.000
λ= 1e-5	$D_1(T)$	5.00000e-08	(4.52860e-08, 5.00000e-08)	9.428
b= 1e-2	$D_2(T)$	1.00000e+00	(9.93333e-01, 1.00000e+00)	0.667
λ= 1e-7	$D_1(T)$	5.00000e-11	(4.85093e-11, 5.00000e-11)	2.981
b= 1e-3	$D_2(T)$	1.00000e+00	(9.99333e-01, 1.00000e+00)	0.067
λ= 1e-2	$D_1(T)$	5.00000e-07	(4.95286e-07, 5.00000e-07)	0.943
b= 1e-4	$D_2(T)$	1.00000e+00	(9.99933e-01, 1.00000e+00)	0.007
λ= 1e+2	$D_1(T)$	9.00005e-02	(0.00000e+00, 1.00000e+00)!&	100.000
b= 1e-1	$D_2(T)$	9.99955e-01	(0.00000e+00, 1.00000e+00)	100.000
λ= 1e-4	$D_1(T)$	4.99998e-06	(3.50927e-06, 5.00000e-06)	29.814
b= 1e-1	$D_2(T)$	9.99995e-01	(9.33328e-01, 1.00000e+00)	6.667
λ= 1e-1	$D_1(T)$	9.90000e-01	(0.00000e+00, 1.00000e+00)+!δ	100.000
b= 1e+3	$D_2(T)$	1.00000e-02	(0.00000e+00, 1.00000e+00)	100.000
λ= 1e+2	$D_1(T)$	4.98337e-03	(4.93619e-03, 5.00000e-03)	0.947
b= 1e-4	$D_2(T)$	9.95017e-01	(9.94933e-01, 1.00000e+00)	0.501
λ= 1e+3	$D_1(T)$	9.99000e-01	(0.00000e+00, 1.00000e+00)!	100.000
b= 1.0	$D_2(T)$	1.00000e-03	(0.00000e+00, 1.00000e+00)	100.000
λ= 1e+5	$D_1(T)$	9.90000e-01	(0.00000e+00, 1.00000e+00)!&	100.000
b= 1e-3	$D_2(T)$	1.00000e-02	(0.00000e+00, 1.00000e+00)	100.000
λ= 1e+1	$D_1(T)$	4.99983e-05	(4.98493e-05, 5.00000e-05)	0.298
b= 1e-5	$D_2(T)$	9.99950e-01	(9.99943e-01, 1.00000e+00)	0.005
λ= 1e+4	$D_1(T)$	1.00000e+00	(0.00000e+00, 1.00000e+00)+!δ	100.000
b= 1e+3	$D_2(T)$	1.00000e-07	(0.00000e+00, 1.00000e+00)	100.000

\+ RECOVERY TOO SLOW
! RATE TOO FAST
& STANDARD DEVIATION TOO BIG
δ DELTA > TIME

For large values of λ(i.e. a fast exponential transition), the bounds separate except in cases where the competing recovery rate is very fast. These fast exponential rates should be expressed as general transitions with means and standard deviations.

PROBLEM 4b



$F(t) = 1 - e^{-\lambda t},\ \lambda > 0;\qquad G(t) = \begin{cases} t/b, & t < b \\ 1, & t \geq b \end{cases}$

Assumption: $b < T$

Transition Description

$$\rho(F) = \int_0^\infty [1 - G(t)]dF(t) = \int_0^b (1 - t/b)\lambda e^{-\lambda t}dt = (\lambda b - 1 + e^{-\lambda b})/(\lambda b)$$

$$\mu(F) = \rho^{-1}\int_0^\infty t[1-G(t)]dF(t) = \rho^{-1}\int_0^b t(1 - t/b)\lambda e^{-\lambda t}dt = \frac{\lambda b - 2 + \lambda b e^{-\lambda b} + 2e^{-\lambda b}}{\lambda(\lambda b - 1 + e^{-\lambda b})}$$

$$\sigma^2(F) = \rho^{-1}\int_0^\infty t^2[1 - G(t)]dF(t) - \mu^2(F) = \rho^{-1}\int_0^b t^2(1 - t/b)\lambda e^{-\lambda t}dt = \frac{2\lambda b - 6 + e^{-\lambda b}[\lambda^2 b^2 + 4\lambda b + 6]}{\lambda^2(\lambda b - 1 + e^{-\lambda b})} - \mu^2(F)$$

$$\rho(G) = \int_0^\infty [1 - F(t)]dG(t) = \int_0^\infty (1/b)e^{-\lambda t}dt = (1 - e^{-\lambda b})/(\lambda b)$$

$$\mu(G) = \rho^{-1}\int_0^\infty t[1 - F(t)]dG(t) = \rho^{-1}\int_0^b (t/b)e^{-\lambda t}dt = \frac{1 - \lambda b e^{-\lambda b} - e^{-\lambda b}}{\lambda(1 - e^{-\lambda b})}$$

$$\sigma^2(G) = \rho^{-1}\int_0^\infty t^2[1 - F(t)]dG(t) - \mu^2(G) = \rho^{-1}\int_0^b (t^2/b)e^{-\lambda t}dt - \mu^2(G) = \frac{2 - e^{-\lambda b}(\lambda^2 b^2 + 2\lambda b + 2)}{\lambda^2(1 - e^{-\lambda b})} - \mu^2(G)$$

Death-State Probabilities

$$D_1(T) = \int_0^T [1 - G(t)]dF(t) = \int_0^b (1 - t/b)\lambda e^{-\lambda t}dt = (\lambda b + e^{-\lambda b} - 1)/(\lambda b)$$

$$D_2(T) = \int_0^T [1 - F(t)]dG(t) = \int_0^b e^{-\lambda t}b^{-1}dt = (1 - e^{-\lambda b})/(\lambda b)$$

TABLE 4b

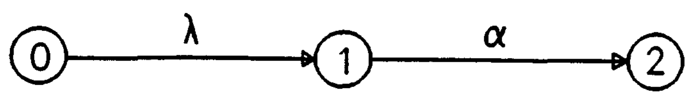
λ: exponential parameter for the (0,1) transition
b: uniform parameter for the (0,2) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ= 1e-6	$D_1(T)$	5.00000e-11	(4.99975e-11, 5.00000e-11)	0.005
b= 1e-4	$D_2(T)$	1.00000e+00	(9.99933e-01, 1.00000e+00)	0.007
λ= 1e-4	$D_1(T)$	5.00000e-09	(4.99975e-09, 5.00000e-09)	0.005
b= 1e-4	$D_2(T)$	1.00000e+00	(9.99933e-01, 1.00000e+00)	0.007
λ= 1e-5	$D_1(T)$	5.00000e-08	(4.97500e-08, 5.00000e-08)	0.500
b= 1e-2	$D_2(T)$	1.00000e+00	(9.93333e-01, 1.00000e+00)	0.667
λ= 1e-7	$D_1(T)$	5.00000e-11	(4.99750e-11, 5.00000e-11)	0.050
b= 1e-3	$D_2(T)$	1.00000e+00	(9.99333e-01, 1.00000e+00)	0.067
λ= 1e-2	$D_1(T)$	5.00000e-07	(4.99975e-07, 5.00000e-07)	0.005
b= 1e-4	$D_2(T)$	1.00000e+00	(9.99933e-01, 1.00000e+00)	0.007
λ= 1e+2	$D_1(T)$	9.00005e-01	(8.84248e-01, 9.00005e-01)	1.751
b= 1e-1	$D_2(T)$	9.99955e-02	(9.80001e-02, 9.99955e-02)	1.995
λ= 1e-4	$D_1(T)$	4.99998e-06	(4.83345e-06, 4.99998e-06)	3.331
b= 1e-1	$D_2(T)$	9.99995e-01	(9.33381e-01, 9.99995e-01)	6.661
λ= 1e-1	$D_1(T)$	9.90000e-01	(0.00000e+00, 9.90000e-01)+	100.000
b= 1e+3	$D_2(T)$	1.00000e-02	(0.00000e+00, 1.00000e-02)	100.000
λ= 1e+2	$D_1(T)$	4.98337e-03	(4.98313e-03, 4.98337e-03)	0.005
b= 1e-4	$D_2(T)$	9.95017e-01	(9.94950e-01, 9.95017e-01)	0.007
λ= 1e+3	$D_1(T)$	9.99000e-01	(9.97004e-01, 9.99000e-01)	0.200
b= 1.0	$D_2(T)$	1.00000e-03	(9.98000e-04, 1.00000e-03)	0.200
λ= 1e+5	$D_1(T)$	9.90000e-01	(9.89980e-01, 9.90000e-01)	0.002
b= 1e-3	$D_2(T)$	1.00000e-02	(9.99980e-03, 1.00000e-02)	0.002
λ= 1e+1	$D_1(T)$	4.99983e-05	(4.99982e-05, 4.99983e-05)	0.000
b= 1e-5	$D_2(T)$	9.99950e-01	(9.99943e-01, 9.99950e-01)	0.001
λ= 1e+4	$D_1(T)$	1.00000e+00	(9.99800e-01, 1.00000e+00)	0.020
b= 1e+3	$D_2(T)$	1.00000e-07	(9.99800e-08, 1.00000e-07)	0.020

+ RECOVERY TOO SLOW

The analytic solutions for the means and standard deviations were extremely numerically unstable for small values of λ and b. Consequently, Taylor series expansion was used to reduce the form of the statistics used in the input files for the cases where λ and b were small.

PROBLEM 5



$F(t) = 1 - e^{-\lambda t}, \ t > 0; \qquad G(t) = 1 - e^{-\alpha t}, \ t > 0$

Death-State Probability

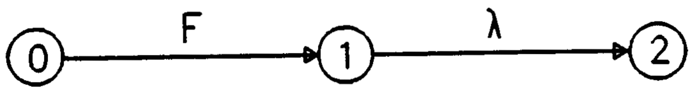
$$D_2(T) = \int_0^T \int_0^{T-x} \lambda e^{-\lambda x} \alpha e^{-\alpha y} \, dy dx = (-\lambda + \alpha - \alpha e^{-\lambda T} + \lambda e^{-\alpha T})/(\alpha - \lambda)$$

TABLE 5

λ: exponential parameter for the (0,1) transition
α: exponential parameter for the (1,2) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ= 1e-4 α= 1e-3	$D_2(T)$	4.98171e-06	(4.98167e-06, 5.00000e-06)	0.367
λ= 1e-6 α= 1e-1	$D_2(T)$	3.67878e-06	(3.67878e-06, 3.67878e-06)	0.000
λ= 1e-2 α= 1.0	$D_2(T)$	8.60233e-02	(8.60233e-02, 8.60233e-02)	0.000
λ= 1e-3 α= 1e-5	$D_2(T)$	4.98321e-07	(4.98317e-07, 5.00000e-07)	0.337
λ= 1.0 α= 1e-6	$D_2(T)$	9.00000e-06	(9.00000e-06, 9.00000e-06)	0.000
λ= 1e-1 α= 1e-7	$D_2(T)$	3.67879e-07	(3.67879e-07, 3.67879e-07)	0.000
λ= 2e-5 α= 3e-5	$D_2(T)$	2.99950e-08	(2.99950e-08, 3.00000e-08)	0.017
λ= 1e-2 α= 2e-2	$D_2(T)$	9.05592e-03	(9.00000e-03, 1.00000e-02)	10.425
λ= 8e-7 α= 8.5e-7	$D_2(T)$	3.39999e-11	(3.39998e-11, 3.40000e-11)	0.000
λ= 1e-7 α= 2e-2	$D_2(T)$	9.36537e-08	(9.33333e-08, 1.00000e-07)	6.776

PROBLEM 6



$F(t) = \begin{cases} 0, & t < a; \\ 1, & t \geq a \end{cases}$ $\qquad G(t) = 1 - e^{-\lambda t}, \; t > 0$

Transition Description

$$\rho(F) = \int_0^\infty dF(t) = 1$$

$$\mu(F) = \rho^{-1}(F)\int_0^\infty t\,dF(t) = a$$

$$\sigma^2(F) = \rho^{-1}(F)\int_0^\infty t^2\,dF(t) - \mu^2(F) = a^2 - a^2 = 0$$

Death-State Probability

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda e^{-\lambda y}\,dy\,dF(x) = 1 - e^{-\lambda(T-a)}$$

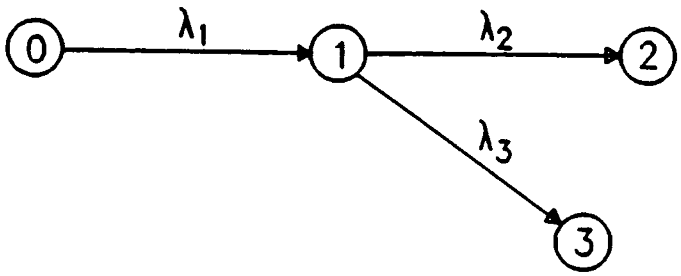
TABLE 6

a: impulse parameter for the (0,1) transition
λ: exponential parameter for the (1,2) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
a= 1e-4 λ= 1.0	$D_2(T)$	9.99955e-01	(9.99854e-01, 9.99955e-01)	0.010
a= 1.0 λ= 1e-2	$D_2(T)$	8.60688e-02	(0.00000e+00, 1.00000e-01)+	100.000
a= 1e-4 λ= 1e-4	$D_2(T)$	9.99490e-04	(9.98401e-04, 1.00000e-03)	0.109
a= 1e-8 λ= 1e-1	$D_2(T)$	6.32121e-01	(6.32117e-01, 6.32121e-01)	0.001
a= 1e-1 λ= 1e-8	$D_2(T)$	9.90000e-08	(8.71539e-08, 1.00000e-07)	11.966
a= 2e-5 λ= 1e-6	$D_2(T)$	9.99993e-06	(9.99528e-06, 1.00000e-05)	0.047
a= 4e-5 λ= 3e-2	$D_2(T)$	2.59181e-01	(2.59031e-01, 2.59182e-01)	0.058
a= 3e-7 λ= 2e-4	$D_2(T)$	1.99800e-03	(1.99789e-03, 2.00000e-03)	0.100
a= 1e-6 λ= 1e-5	$D_2(T)$	9.99950e-05	(9.99849e-05, 1.00000e-04)	0.010
a= 1.1e-3 λ= 1e-3	$D_2(T)$	9.94908e-03	(9.90626e-03, 1.00000e-02)	0.512
a= 3e-4 λ= 2e-3	$D_2(T)$	1.98007e-02	(1.97601e-02, 2.00000e-02)	1.006
a= 2e-4 λ= 1e-7	$D_2(T)$	9.99980e-07	(9.98386e-07, 1.00000e-06)	0.159

+ RECOVERY TOO SLOW

PROBLEM 7



$F(t) = 1 - e^{-\lambda 1 t}$; $t > 0$; $G(t) = 1 - e^{-\lambda 2 t}$; $t > 0$; $H(t) = 1 - e^{-\lambda 3 t}$ $t > 0$

Death-State Probabilities

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda 1 x} \lambda_2 e^{-(\lambda 2 + \lambda 3) y} \, dydx$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_3} \left\{ \frac{1}{\lambda_1} - \frac{e^{-\lambda 1 T}}{\lambda_1} - \frac{e^{-\lambda 1} - e^{-(\lambda 2 + \lambda 3) T}}{\lambda_2 + \lambda_3 - \lambda_1} \right\}$$

$$D_3(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda 1 x} \lambda_3 e^{-(\lambda 2 + \lambda 3) y} \, dydx$$

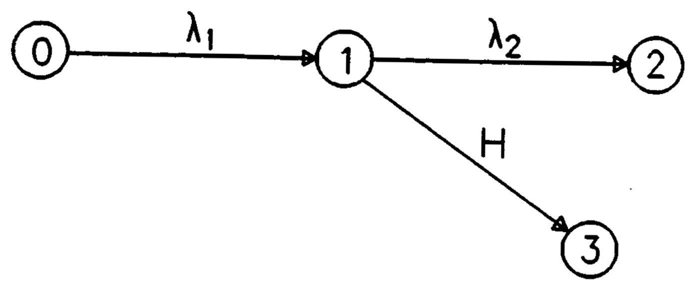
$$= \frac{\lambda_1 \lambda_3}{\lambda_2 + \lambda_3} \left\{ \frac{1}{\lambda_1} - \frac{e^{-\lambda 1 T}}{\lambda_1} - \frac{e^{-\lambda 1 T} - e^{-(\lambda 2 + \lambda 3) T}}{\lambda_2 + \lambda_3 - \lambda_1} \right\}$$

TABLE 7

λ_1 : exponential parameter for the (0,1) transition
λ_2 : exponential parameter for the (1,2) transition
λ_3 : exponential parameter for the (1,3) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1 = 1e-2				
λ_2 = 1e-3	D_2(T)	4.81958e-04	(4.81500e-04, 5.00000e-04)	3.743
λ_3 = 1e-4	D_3(T)	4.81958e-05	(4.81500e-05, 5.00000e-05)	3.743
λ_1 = 1e-7				
λ_2 = 1e-2	D_2(T)	4.83726e-08	(4.83316e-08, 5.00000e-08)	3.364
λ_3 = 1e-5	D_3(T)	4.83726e-11	(4.83317e-11, 5.00000e-11)	3.364
λ_1 = 1e-5				
λ_2 = 2e-5	D_2(T)	9.99733e-09	(9.99733e-09, 1.00000e-08)	0.027
λ_3 = 5e-5	D_3(T)	2.49933e-08	(2.49933e-08, 2.50000e-08)	0.027
λ_1 = 1e-6				
λ_2 = 1e-7	D_2(T)	4.83740e-12	(4.83332e-12, 5.00000e-12)	3.361
λ_3 = 1e-2	D_3(T)	4.83740e-07	(4.83332e-07, 5.00000e-07)	3.361
λ_1 = 1e-1				
λ_2 = 1e-7	D_2(T)	3.67879e-07	(3.67879e-07, 3.67879e-07)	0.000
λ_3 = 2e-7	D_3(T)	7.35758e-07	(7.35758e-07, 7.35758e-07)	0.000
λ_1 = 1e-2				
λ_2 = 5e-5	D_2(T)	2.41830e-05	(2.41625e-05, 2.50000e-05)	3.378
λ_3 = 3e-8	D_3(T)	1.45098e-08	(1.44975e-08, 1.50000e-08)	3.378

PROBLEM 8



$F(t) = 1 - e^{-\lambda_1 t},\ t > 0;\quad G(t) = 1 - e^{-\lambda_2 t},\ t > 0;\quad H(t) = \begin{cases} t/b, & t \le b \\ 1, & t > b \end{cases}$

Assumption: $b \le T$

Transition Description

$$\rho(H) = \int_0^\infty dH(t) = 1$$

$$\mu(H) = \rho^{-1}(H)\int_0^\infty t\,dH(t) = b^{-1}\int_0^b t\,dt = b/2$$

$$\sigma^2(H) = \rho^{-1}(H)\int_0^\infty t^2 dH(t) - \mu^2(H) = b^{-1}\int_0^b t^2 dt - b^2/4 = b^2/12$$

Death-State Probabilities

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y}[1 - H(y)]dydx$$
$$= \lambda_1 \lambda_2 \int_0^T e^{-\lambda_1 x} \int_0^b e^{-\lambda_2 y}(1 - y/b)dydx = [1 - (1 - e^{-\lambda_2 b})/(\lambda_2 b)](1 - e^{-\lambda_1 T})$$

$$D_3(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 y} dH(y)dx = \lambda_1 b^{-1}\int_0^T e^{-\lambda_1 x}\int_0^b e^{-\lambda_2 y}dydx$$
$$= (1 - e^{-\lambda_2 b})(1 - e^{-\lambda_1 T})/(\lambda_2 b)$$

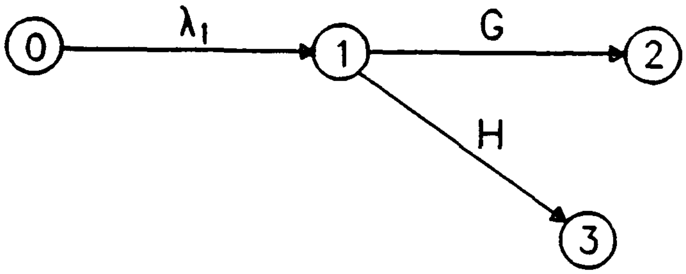
TABLE 8

λ_1 : exponential parameter for the (0,1) transition
λ_2 : exponential parameter for the (1,2) transition
b: uniform parameter for the (1,3) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1= 4e-3				
λ_2= 3e-3	$D_2(T)$	5.88160e-10	(5.86118e-10, 6.00000e-10)	2.013
b= 1e-5	$D_3(T)$	3.92107e-02	(3.91912e-02, 4.00000e-02)	2.013
λ_1= 5e-6				
λ_2= 3e-6	$D_2(T)$	7.49981e-14	(7.25994e-14, 7.50000e-14)	3.198
b= 1e-3	$D_3(T)$	4.99987e-05	(4.98537e-05, 5.00000e-05)	0.290
λ_1= 5e-4				
λ_2= 2e-3	$D_2(T)$	4.98752e-10	(4.93699e-10, 5.00000e-10)	1.013
b= 1e-4	$D_3(T)$	4.98752e-03	(4.98365e-03, 5.00000e-03)	0.250
λ_1= 3e-2				
λ_2= 2e-2	$D_2(T)$	2.59182e-09	(2.58922e-09, 2.59182e-09)	0.100
b= 1e-6	$D_3(T)$	2.59182e-01	(2.59166e-01, 2.59182e-01)	0.006
λ_1= 1e-2				
λ_2= 1e-5	$D_2(T)$	4.75813e-10	(4.59862e-10, 5.00000e-10)	5.083
b= 1e-3	$D_3(T)$	9.51626e-02	(9.47355e-02, 1.00000e-01)	5.083
λ_1= 3e-6				
λ_2= 2e-6	$D_2(T)$	2.99996e-17	(2.99691e-17, 3.00000e-17)	0.102
b= 1e-6	$D_3(T)$	2.99996e-05	(2.99974e-05, 3.00000e-05)	0.007
λ_1= 1e-3				
λ_2= 1e-2	$D_2(T)$	4.97508e-12	(4.97341e-12, 5.00000e-12)	0.501
b= 1e-7	$D_3(T)$	9.95017e-03	(9.94978e-03, 1.00000e-02)	0.501
λ_1= 2e-5				
λ_2= 2e-5	$D_2(T)$	3.99960e-14	(3.98148e-14, 4.00000e-14)	0.453
b= 2e-5	$D_3(T)$	1.99980e-04	(1.99914e-04, 2.00000e-04)	0.033
λ_1= 5e-5				
λ_2= 4e-5	$D_2(T)$	9.99749e-10	(6.85995e-10, 1.00000e-10)	31.383
b= 1e-1	$D_3(T)$	4.99874e-04	(4.56119e-04, 5.00000e-04)	8.753
λ_1= 2e-3				
λ_2= 1e-6	$D_2(T)$	9.90066e-10	(6.79455e-10, 1.00000e-09)	31.373
b= 1e-1	$D_3(T)$	1.98013e-02	(1.80709e-02, 2.00000e-02)	8.739
λ_1= 8e-2				
λ_2= 7e-2	$D_2(T)$	1.92730e-05	(1.86711e-05, 1.92735e-05)	3.123
b= 1e-3	$D_3(T)$	5.50652e-01	(5.49481e-01, 5.50671e-01)	0.213
λ_1= 4e-4				
λ_2= 3e-4	$D_2(T)$	5.98742e-07	(3.17732e-08, 6.00000e-07)	94.693
b= 1	$D_3(T)$	3.99141e-03	(1.23619e-03, 4.00000e-03)	69.029

Here, SURE's bounds separated in cases where the recovery rate was slow.

PROBLEM 9



$F(t) = 1 - e^{-\lambda t}$, $t > 0$; $G(t) = \begin{cases} 0, & t < a; \\ 1, & t \geq a; \end{cases}$ $H(t) = \begin{cases} t/b, & t \leq b \\ 1, & t > b \end{cases}$

Assumption: $a < b < T$

Transition Description

$$\rho(G) = \int_0^\infty [1 - H(t)]dG(t) = 1 - H(a) = (b - a)/b$$

$$\mu(G) = \rho^{-1}(G)\int_0^\infty t[1 - H(t)]dG(t) = b(b - a)^{-1}[a(1 - H(a))] = a$$

$$\sigma^2(G) = \rho^{-1}(G)\int_0^\infty t^2[1 - H(t)]dG(t) - \mu^2(G) = b(b - a)^{-1}[a^2(1 - H(a))] - a^2 = 0$$

$$\rho(H) = \int_0^\infty [1 - G(t)]dH(t) = \int_0^a (1/b)dt = a/b$$

$$\mu(H) = \rho^{-1}(H)\int_0^\infty t[1 - G(t)]dH(t) = (b/a)\int_0^a (t/b)dt = a/2$$

$$\sigma^2(H) = \rho^{-1}(H)\int_0^\infty t^2[1 - G(t)]dH(t) - \mu^2(H) = (b/a)\int_0^a (t^2/b)dt - a^2/4 = a^2/12$$

Death-State Probabilities

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda e^{-\lambda x}[1 - H(y)]dG(y)dx = \lambda\int_0^T e^{-\lambda x}(1 - a/b)dx$$
$$= (b - a)(1 - e^{-\lambda T})/b$$

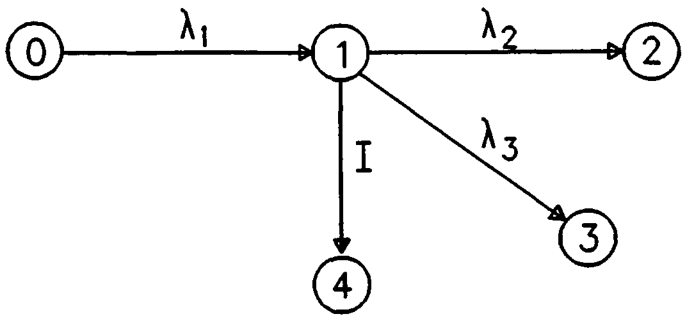
$$D_3(T) = \int_0^T \int_0^{T-x} \lambda e^{-\lambda x}[1 - G(y)]dH(y)dx = \lambda\int_0^T e^{-\lambda x}\int_0^a (1/b)dx = a(1 - e^{-\lambda T})/b$$

TABLE 9

λ: exponential parameter for the (0,1) transition
a: impulse parameter for the (1,2) transition
b: uniform parameter for the (1,3) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ= 1e-2				
a= 1e-7	$D_2(T)$	9.51531e-02	(9.49876e-02, 9.99900e-02)	5.083
b= 1e-3	$D_3(T)$	9.51626e-06	(9.49980e-06, 1.00000e-05)	5.083
λ= 1e-3				
a= 1e-3	$D_2(T)$	8.95515e-03	(8.91790e-03, 9.00000e-03)	0.501
b= 1e-2	$D_3(T)$	9.95017e-04	(9.92124e-04, 1.00000e-03)	0.501
λ= 1e-6				
a= 1e-2	$D_2(T)$	4.99998e-06	(4.90048e-06, 5.00000e-06)	1.990
b= 2e-2	$D_3(T)$	4.99998e-06	(4.93152e-06, 5.00000e-06)	1.369
λ= 1e-1				
a= 1e-6	$D_2(T)$	5.68909e-01	(5.68875e-01, 5.68909e-01)	0.006
b= 1e-5	$D_3(T)$	6.32121e-02	(6.32094e-02, 6.32121e-01)	0.004
λ= 1e-2				
a= 1e-5	$D_2(T)$	6.34417e-02	(6.33137e-02, 6.66667e-02)	5.083
b= 3e-5	$D_3(T)$	3.17209e-02	(3.16597e-02, 3.33333e-02)	5.083
λ= 1e-4				
a= 1e-7	$D_2(T)$	9.99499e-04	(9.99467e-04, 9.99999e-04)	0.050
b= 1e-1	$D_3(T)$	9.99500e-10	(9.99478e-10, 1.00000e-09)	0.050
λ= 1e-4				
a= 1e-6	$D_2(T)$	8.99550e-04	(8.99459e-04, 9.00000e-04)	0.050
b= 1e-5	$D_3(T)$	9.99500e-05	(9.99429e-05, 1.00000e-04)	0.050
λ= 1e-1				
a= 1e-3	$D_2(T)$	5.68909e-01	(5.67292e-01, 5.68909e-01)	0.284
b= 1e-2	$D_3(T)$	6.32121e-02	(6.30876e-02, 6.32121e-02)	0.197
λ= 1e-6				
a= 1e-7	$D_2(T)$	8.99996e-06	(8.99967e-06, 9.00000e-06)	0.003
b= 1e-6	$D_3(T)$	9.99995e-07	(9.99973e-07, 1.00000e-06)	0.002

PROBLEM 10



$F(t) = 1 - e^{-\lambda 1 t}$; $t > 0$; $G(t) = 1 - e^{-\lambda 2 t}$; $t > 0$; $H(t) = 1 - e^{-\lambda 3 t}$, $t > 0$; $I(t) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$

Assumption: $a < T$

Transition Description

$$\rho(I) = \int_0^{\infty} dI(t) = 1$$

$$\mu(I) = \rho^{-1}(I)\int_0^{\infty} t\,dI(t) = a$$

$$\sigma^2(I) = \rho^{-1}(I)\int_0^{\infty} t^2 dI(t) - \mu^2(I) = a^2 - a^2 = 0$$

Death-State Probabilities

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda 1 x} \lambda_2 e^{-(\lambda 2+\lambda 3)y}[1 - I(y)]dydx$$

$$= \lambda_1 \lambda_2 \int_0^T e^{-\lambda 1 x} \int_0^a e^{-(\lambda 2+\lambda 3)y} dydx = \lambda_2 (1 - e^{-(\lambda 2+\lambda 3)a})(1 - e^{-\lambda 1 T})/(\lambda_2+\lambda_3)$$

$$D_3(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda 1 x} \lambda_3 e^{-(\lambda 2+\lambda 3)y}[1 - I(y)]dydx$$

$$= \lambda_1 \lambda_3 \int_0^T e^{-\lambda 1 x} \int_0^a e^{-(\lambda 2+\lambda 3)y} dydx = \lambda_3 (1 - e^{-(\lambda 2+\lambda 3)a})(1 - e^{-\lambda 1 T})/(\lambda_2+\lambda_3)$$

$$D_4(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda 1} e^{-(\lambda 2+\lambda 3)y} dI(y)dx = \lambda_1 \int_0^T e^{-\lambda 1} e^{-(\lambda 2+\lambda 3)a} dx$$

$$= e^{-(\lambda 2+\lambda 3)a}(1 - e^{-\lambda 1 T})$$

TABLE 10

λ_1 : exponential parameter for the (0,1) transition
λ_2 : exponential parameter for the (1,2) transition
λ_3 : exponential parameter for the (1,3) transition
a: impulse parameter for the (1,4) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1 = 5e-4				
λ_2 = 4e-4	D_2(T)	1.99501e-12	(1.99281e-12, 2.00000e-12)	0.250
λ_3 = 3e-4	D_3(T)	1.49626e-12	(1.49460e-12, 1.50000e-12)	0.250
a= 1e-6	D_4(T)	4.98752e-03	(4.98700e-03, 5.00000e-03)	0.250
λ_1 = 1e-3				
λ_2 = 1e-5	D_2(T)	9.95017e-14	(9.93906e-14, 1.00000e-13)	0.501
λ_3 = 1e-4	D_3(T)	9.95012e-13	(9.93906e-13, 1.00000e-12)	0.501
a= 1e-6	D_4(T)	9.95017e-03	(9.94900e-03, 1.00000e-02)	0.501
λ_1 = 3e-3				
λ_2 = 2e-3	D_2(T)	1.18218e-09	(1.17620e-09, 1.20000e-09)	1.507
λ_3 = 1e-5	D_3(T)	5.91089e-12	(5.88098e-12, 6.00000e-12)	1.508
a= 2e-5	D_4(T)	2.95545e-02	(2.95364e-02, 3.00000e-02)	1.507
λ_1 = 2e-2				
λ_2 = 2e-3	D_2(T)	3.62495e-05	(2.39169e-05, 4.00000e-05)	34.021
λ_3 = 4e-4	D_3(T)	7.24990e-06	(4.78339e-06, 8.00000e-06)	34.021
a= 1e-1	D_4(T)	1.81226e-01	(1.57386e-01, 2.00000e-01)	13.155
λ_1 = 5e-6				
λ_2 = 4e-6	D_2(T)	1.99994e-13	(1.93057e-13, 2.00000e-13)	3.469
λ_3 = 1e-2	D_3(T)	4.99985e-10	(4.82643e-10, 5.00000e-10)	3.469
a= 1e-3	D_4(T)	4.99983e-05	(4.97903e-05, 5.00000e-05)	0.416
λ_1 = 1e-1				
λ_2 = 1e-1	D_2(T)	3.16060e-08	(3.15824e-08, 3.16060e-08)	0.075
λ_3 = 2e-5	D_3(T)	6.32121e-12	(6.31648e-12, 6.32121e-12)	0.075
a= 5e-7	D_4(T)	6.32121e-01	(6.32094e-01, 6.32121e-01)	0.004
λ_1 = 3e-6				
λ_2 = 2e-6	D_2(T)	5.99991e-19	(5.99925e-19, 6.00000e-19)	0.011
λ_3 = 1e-2	D_3(T)	2.99700e-15	(2.99963e-15, 3.00000e-15)	0.010
a= 1e-8	D_4(T)	2.99996e-05	(2.99992e-05, 3.00000e-05)	0.001
λ_1 = 2e-2				
λ_2 = 1e-8	D_2(T)	1.81269e-12	(1.73818e-12, 2.00000e-12)	10.333
λ_3 = 4e-5	D_3(T)	7.25077e-09	(6.95271e-09, 8.00000e-09)	10.333
a= 1e-3	D_4(T)	1.81269e-01	(1.79314e-01, 2.00000e-01)	10.333
λ_1 = 4e-5				
λ_2 = 3e-5	D_2(T)	1.19976e-09	(1.19844e-09, 1.20000e-09)	0.110
λ_3 = 4e-5	D_3(T)	1.59967e-09	(1.59791e-09, 1.60000e-09)	0.110
a= 1e-6	D_4(T)	3.99917e-04	(3.99877e-04, 4.00000e-04)	0.010

PROBLEM 11

$$⓪ \xrightarrow{\lambda_1} ① \xrightarrow{\lambda_2} ② \xrightarrow{\lambda_3} ③$$

$F(t) = 1 - e^{-\lambda 1 t}$; $\quad$ $G(t) = 1 - e^{-\lambda 2 t}$; $\quad$ $H(t) = 1 - e^{-\lambda 3 t}$

$t > 0$; $\quad$ $t > 0$; $\quad$ $t > 0$

Death-State Probability

$$D_3(T) = \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda 1 x} \lambda_2 e^{-\lambda 2 y} \lambda_3 e^{-\lambda 3 z} \, dzdydx$$

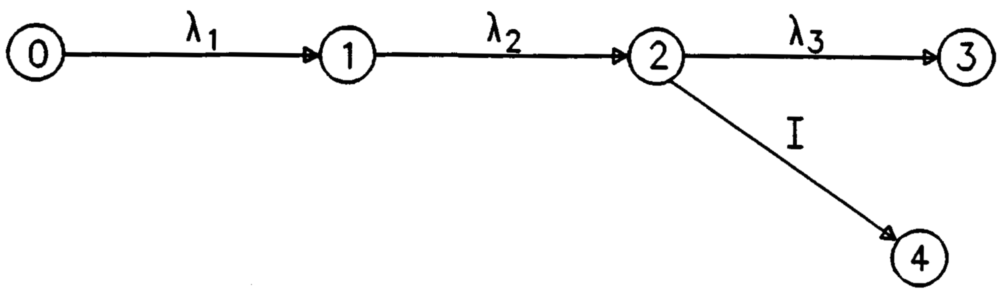
$$= 1 - e^{-\lambda 1 T} - \frac{\lambda_1 (e^{-\lambda 1 T} + e^{-\lambda 2 T})}{(-\lambda_1 + \lambda_2)} - \frac{\lambda_1 (1 - e^{-\lambda 2 T})}{(-\lambda_2 + \lambda_3)} + \frac{\lambda_1 \lambda_2 (1 - e^{-\lambda 3 T})}{\lambda_3 (-\lambda_2 + \lambda_3)}$$

TABLE 11

λ_1 : exponential parameter for the (0,1) transition
λ_2 : exponential parameter for the (1,2) transition
λ_3 : exponential parameter for the (2,3) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1 = 1e-4 λ_2 = 1e-5 λ_3 = 1e-6	D_3(T)	1.66629e-13	(1.66620e-13, 1.66667e-13)	0.023
λ_1 = 1e-1 λ_2 = 1e-2 λ_3 = 1e-3	D_3(T)	1.28398e-04	(1.28398e-04, 1.28398e-04)	0.000
λ_1 = 3e-4 λ_2 = 2e-4 λ_3 = 1e-4	D_3(T)	9.98501e-10	(9.98500e-10, 1.00000e-09)	0.150
λ_1 = 6e-6 λ_2 = 1e-2 λ_3 = 5e-4	D_3(T)	4.87126e-09	(4.86867e-09, 5.00000e-09)	2.643
λ_1 = 3e-1 λ_2 = 2e-1 λ_3 = 1e-1	D_3(T)	2.52580e-01	(2.52580e-01, 2.52580e-01)	0.000
λ_1 = 5e-2 λ_2 = 3e-2 λ_3 = 2e-2	D_3(T)	3.90668e-03	(3.90668e-03, 3.90668e-03)	0.000
λ_1 = 1e-7 λ_2 = 1e-4 λ_3 = 1e-1	D_3(T)	1.32086e-10	(1.32086e-10, 1.32086e-10)	0.000
λ_1 = 4e-2 λ_2 = 6e-4 λ_3 = 8e-6	D_3(T)	2.89949e-08	(2.89949e-08, 2.89949e-08)	0.000
λ_1 = 1e-3 λ_2 = 1e-6 λ_3 = 1e-7	D_3(T)	1.66264e-14	(1.66250e-14, 1.66667e-14)	0.242

PROBLEM 12



$F(t) = 1 - e^{-\lambda_1 t}$; $t > 0$; $G(t) = 1 - e^{-\lambda_2 t}$; $t > 0$; $H(t) = 1 - e^{-\lambda_3 t}$; $t > 0$; $I(t) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$

Assumption: $a < T$

Transition Description

$$\rho(I) = \int_0^\infty dI(t) = 1$$

$$\mu(I) = \rho^{-1}(I)\int_0^\infty t\,dI(t) = a$$

$$\sigma^2(I) = \rho^{-1}(I)\int_0^\infty t^2 dI(t) - \mu^2(I) = a^2 - a^2 = 0$$

Death-State Probabilities

$$D_3(T) = \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \lambda_3 e^{-\lambda_3 z}[1 - I(z)]dzdydx$$

$$= \lambda_1 \lambda_2 \lambda_3 \int_0^T e^{-\lambda_1 x} \int_0^{T-x} e^{-\lambda_2 y} \int_0^a e^{-\lambda_3 z} dzdydx$$

$$= (1 - e^{-\lambda_3 a})(1 - e^{-\lambda_1 T}) - \lambda_1(1 - e^{-\lambda_3 a})(e^{-\lambda_1 T} - e^{-\lambda_2 T})/(\lambda_2 - \lambda_1)$$

$$D_4(T) = \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} e^{-\lambda_3 z} dI(z)dydx$$

$$= \lambda_1 \lambda_2 \int_0^T e^{-\lambda_1} \int_0^{T-x} e^{-\lambda_2 y} e^{-\lambda_3 a} dydx$$

$$= e^{-\lambda_3 a}(1 - e^{-\lambda_1 T}) + \lambda_1 e^{-\lambda_3 a}(e^{-\lambda_2 T} - e^{-\lambda_1 T})/(\lambda_2 - \lambda_1)$$

TABLE 12

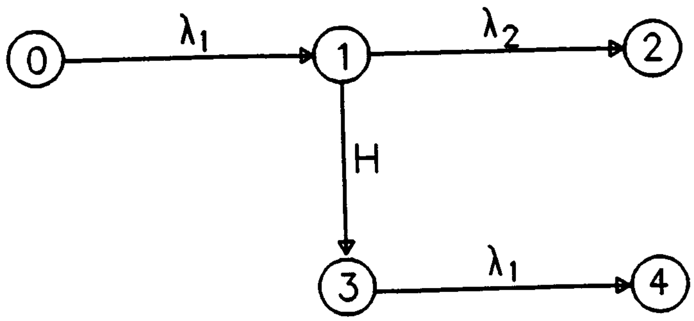
λ_1: exponential parameter for the (0,1) transition
λ_2: exponential parameter for the (1,2) transition
λ_3: exponential parameter for the (2,3) transition
a: impulse parameter for the (2,4) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1 = 5e-5				
λ_2 = 4e-5				
λ_3 = 3e-5	D_3(T)	2.99910e-19	(2.99796e-19, 3.00000e-19)	0.038
a= 1e-7	D_4(T)	9.99700e-08	(9.99637e-08, 1.00000e-07)	0.030
λ_1 = 3e-2				
λ_2 = 2e-2				
λ_3 = 1e-2	D_3(T)	2.54442e-10	(2.54141e-10, 2.54442e-10)	0.118
a= 1e-6	D_4(T)	2.54442e-02	(2.54395e-02, 2.54442e-02)	0.018
λ_1 = 1e-5				
λ_2 = 1e-3				
λ_3 = 1e-2	D_3(T)	4.98072e-10	(3.19326e-10, 5.00000e-10)	35.888
a= 1e-1	D_4(T)	4.97823e-07	(4.20146e-07, 5.00000e-07)	15.603
λ_1 = 4e-4				
λ_2 = 3e-4				
λ_3 = 1e-4	D_3(T)	5.98602e-12	(5.28031e-12, 6.00000e-12)	11.789
a= 1e-2	D_4(T)	5.98601e-06	(5.80834e-06, 6.00000e-06)	2.968
λ_1 = 1e-1				
λ_2 = 1e-4				
λ_3 = 1e-7	D_3(T)	3.67747e-16	(3.66385e-16, 3.67747e-16)	0.370
a= 1e-5	D_4(T)	3.67747e-04	(3.67544e-04, 3.67747e-04)	0.055
λ_1 = 6e-7				
λ_2 = 5e-7				
λ_3 = 1e-1	D_3(T)	1.49999e-16	(1.48202e-16, 1.50000e-16)	1.198
a= 1e-4	D_4(T)	1.50000e-11	(1.49683e-11, 1.50000e-11)	0.21
λ_1 = 3e-5				
λ_2 = 2e-6				
λ_3 = 4e-7	D_3(T)	1.19987e-15	(0.00000e+00, 1.20000e-15)+	100.000
a= 1.0	D_4(T)	2.99968e-09	(0.00000e+00, 3.00000e-09)	100.000
λ_1 = 5e-6				
λ_2 = 4e-6				
λ_3 = 5e-6	D_3(T)	1.99994e-20	(1.99514e-20, 2.00000e-20)	0.240
a= 4e-6	D_4(T)	9.99970e-10	(9.99566e-10, 1.00000e-09)	0.040

+ RECOVERY TOO SLOW

The bounds again tended to separate in cases where the recovery transition was slow with respect to the mission time.

PROBLEM 13



$F(t) = 1 - e^{-\lambda 1 t}$, $t > 0$; $G(t) = 1 - e^{-\lambda 2 t}$, $t > 0$; $H(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$

Assumption: $a < T$

Transition Description

$$\rho(H) = \int_0^\infty dH(t) = 1$$

$$\mu(H) = \rho^{-1}(H)\int_0^\infty t\,dH(t) = a$$

$$\sigma^2(H) = \rho^{-1}(H)\int_0^\infty t^2 dH(t) - \mu^2(H) = a^2 - a^2 = 0$$

Death-State Probabilities

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda 1 x} \lambda_2 e^{-\lambda 2 y} [1 - H(y)] dy dx = \lambda_1 \lambda_2 \int_0^T e^{-\lambda 1 x} \int_0^a e^{-\lambda 2 y} dy dx$$
$$= (1 - e^{-\lambda 2 a})(1 - e^{-\lambda 1 T})$$

$$D_4(T) = \int_0^T \int_0^{T-x} \int_0^{T-x} \lambda_1 e^{-\lambda 1 x} e^{-\lambda 1 y} \lambda_1 e^{-\lambda 1 z} dz dH(y) dx$$
$$= e^{-\lambda 2 a} - e^{-\lambda 2 a - \lambda 1 T} - \lambda_1 T e^{-\lambda 2 a - \lambda 1 (T-a)}$$

TABLE 13

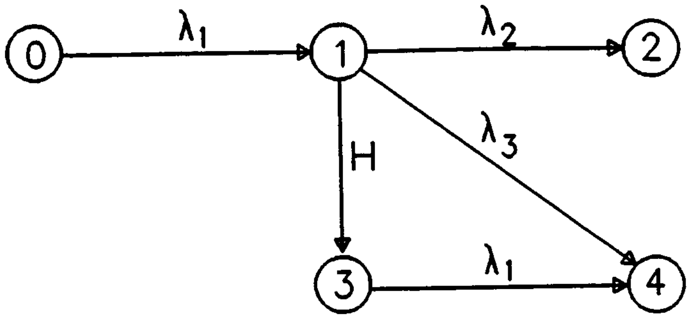
λ_1 : exponential parameter for the (0,1) and (3,4) transitions
λ_2 : exponential parameter for the (1,2) transition
a: impulse parameter for the (1,3) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1 = 1e-5				
λ_2 = 1e-4	$D_2(T)$	9.99950e-15	(9.98850e-15, 1.00000e-14)	0.110
a= 1e-6	$D_4(T)$	4.99967e-09	(4.99866e-09, 5.00000e-09)	0.020
λ_1 = 3e-4				
λ_2 = 2e-4	$D_2(T)$	5.99101e-14	(5.98892e-14, 6.00000e-14)	0.035
a= 1e-7	$D_4(T)$	4.49101e-06	(4.49072e-06, 4.50000e-06)	0.200
λ_1 = 2e-3				
λ_2 = 1e-2	$D_2(T)$	7.92053e-09	(7.86498e-09, 8.00000e-09)	1.003
a= 4e-5	$D_4(T)$	1.97352e-04	(1.97078e-04, 2.00000e-04)	1.342
λ_1 = 1e-2				
λ_2 = 1e-5	$D_2(T)$	9.51626e-10	(9.17202e-10, 1.00000e-09)	5.083
a= 1e-3	$D_4(T)$	4.67794e-03	(4.63361e-03, 5.00000e-03)	6.885
λ_1 = 5e-7				
λ_2 = 3e-7	$D_2(T)$	1.50000e-13	(9.93222e-14, 1.50000e-13)	33.78
a= 1e-1	$D_4(T)$	1.22500e-11	(1.05497e-11, 1.25000e-11)	13.880
λ_1 = 1e-1				
λ_2 = 1e-1	$D_2(T)$	6.32121e-08	(6.31452e-08, 6.32121e-08)	0.106
a= 1e-6	$D_4(T)$	2.64241e-01	(2.64204e-01, 2.64241e-01)	0.014
λ_1 = 3e-2				
λ_2 = 2e-2	$D_2(T)$	1.03652e-04	(8.79090e-05, 1.03673e-04)	12.050
a= 2e-2	$D_4(T)$	3.67882e-02	(3.52637e-02, 3.69363e-02)	4.144
λ_1 = 4e-4				
λ_2 = 3e-4	$D_2(T)$	5.98353e-06	(0.00000e+00, 6.00000e-06)+	100.000
a= 5.0	$D_4(T)$	2.65735e-09	(0.00000e+00, 8.00000e-06)	100.000
λ_1 = 3e-3				
λ_2 = 1e-3	$D_2(T)$	2.95397e-05	(0.00000e+00, 3.00000e-05)+	100.000
a= 1.0	$D_4(T)$	3.53276e-04	(0.00000e+00, 4.50000e-04)	100.000
λ_1 = 3e-5				
λ_2 = 2e-5	$D_2(T)$	2.99955e-10	(2.27676e-10, 3.00000e-10)	24.097
a= 5e-2	$D_4(T)$	4.45411e-08	(4.08515e-08, 4.50000e-08)	8.284

+ RECOVERY TOO SLOW

Note that the large relative error occurred in cases where the recovery rate was slow.

PROBLEM 14



$F(t) = 1 - e^{-\lambda 1 t}$; $t > 0$; $G(t) = 1 - e^{-\lambda 2 t}$; $t > 0$; $H(t) = 0,\ t < a;\ 1,\ t \geq a$; $I(t) = 1 - e^{-\lambda 3 t}$ $t > 0$

Assumption: $a < T$

Transition Description

$$\rho(H) = \int_0^\infty dH(t) = 1$$

$$\mu(H) = \rho^{-1}(H)\int_0^\infty t\,dH(t) = a$$

$$\sigma^2(H) = \rho^{-1}(H)\int_0^\infty t^2 dH(t) - \mu^2(H) = a^2 - a^2 = 0$$

Death-State Probabilities

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda 1 x} \lambda_2 e^{-(\lambda 2+\lambda 3)y}[1 - H(y)]dydx$$

$$= \lambda_1 \lambda_2 \int_0^T e^{-\lambda 1 x} \int_0^a e^{-(\lambda 2+\lambda 3)y} dydx = \lambda_2 (1 - e^{-(\lambda 2+\lambda 3)a})(1 - e^{-\lambda 1 T})/(\lambda_2+\lambda_3)$$

$$D_4(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda 1 x} \lambda_3 e^{-(\lambda 2(\lambda 3)y}[1 - H(y)]dydx$$

$$+ \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda 1 x} e^{-(\lambda 2+\lambda 3)y} \lambda_1 e^{-\lambda 1 z} dzdH(y)dx$$

$$= \lambda_3 (1 - e^{-(\lambda 2+\lambda 3)a})(1 - e^{-\lambda 1 T})/(\lambda_2+\lambda_3)$$

$$+ \lambda_1 e^{-(\lambda 2+\lambda 3)a}[(1 - e^{-\lambda 1 T})/\lambda_1 - Te^{-\lambda 1(T-a)}]$$

TABLE 14

λ_1: exponential parameter for the (0,1) and (3,4) transitions
λ_2: exponential parameter for the (1,2) transition
λ_3: exponential parameter for the (1,4) transition
a: impulse parameter for the (1,3) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1= 4e-3				
λ_2= 3e-3	D_2(T)	1.17632e-09	(1.17192e-09, 1.20000e-09)	2.013
λ_3= 2e-3				
a= 1e-5	D_4(T)	7.78982e-04	(7.78174e-04, 8.00001e-04)	2.698
λ_1= 3e-2				
λ_2= 2e-2	D_2(T)	5.18363e-08	(5.16584e-08, 5.18364e-08)	0.343
λ_3= 1e-1				
a= 1e-5	D_4(T)	3.69365e-02	(3.69151e-02, 3.69366e-02)	0.058
λ_1= 5e-5				
λ_2= 4e-5	D_2(T)	1.99949e-11	(1.93014e-11, 2.00000e-11)	3.468
λ_3= 1e-2				
a= 1e-3	D_4(T)	1.29931e-07	(1.28869e-07, 1.30000e-07)	0.817
λ_1= 1e-3				
λ_2= 1e-3	D_2(T)	4.97508e-11	(4.96277e-11, 5.00000e-11)	0.501
λ_3= 1e-6				
a= 5e-6	D_4(T)	4.96679e-05	(4.96443e-05, 5.00000e-05)	0.669
λ_1= 2e-3				
λ_2= 3e-6	D_2(T)	1.18808e-12	(1.18216e-12, 1.20000e-12)	1.003
λ_3= 1e-4				
a= 2e-5	D_4(T)	1.97353e-04	(1.97154e-04, 2.00000e-04)	1.342
λ_1= 1e-6				
λ_2= 1e-1	D_2(T)	3.99916e-09	(3.72249e-09, 4.00000e-09)	6.918
λ_3= 2e-3				
a= 4e-3	D_4(T)	1.29923e-10	(1.23601e-10, 1.30000e-10)	4.874

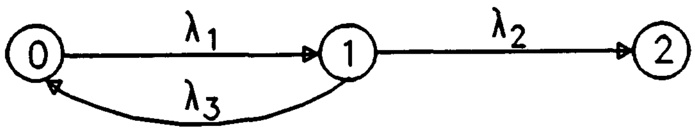
TABLE 14 (Continued)

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1 = 8e-6				
λ_2 = 7e-6	$D_2(T)$	2.79989e-15	(2.79300e-15, 2.80000e-15)	0.246
λ_3 = 1e-2				
a= 5e-6	$D_4(T)$	3.20383e-09	(3.20237e-09, 3.20400e-09)	0.046
λ_1 = 3e-2				
λ_2 = 2e-2	$D_2(T)$	5.13212e-03	(0.00000e+00, 5.18364e-03)+	100.000
λ_3 = 1e-5				
a= 1.0	$D_4(T)$	2.95728e-02	(0.00000e+00, 3.69389e-02)+	100.000
λ_1 = 1e-3				
λ_2 = 1e-2	$D_2(T)$	9.89564e-06	(6.53643e-06, 1.00000e-05)	33.946
λ_3 = 1e-1				
a= 1e-1	$D_4(T)$	1.47102e-04	(1.06778e-04, 1.50000e-04)	27.412
λ_1 = 6e-4				
λ_2 = 5e-4	$D_2(T)$	2.99102e-14	(2.99067e-14, 3.00000e-14)	0.300
λ_3 = 3e-2				
a= 1e-8	$D_4(T)$	1.79282e-05	(1.79276e-05, 1.80000e-05)	0.400
λ_1 = 1e-6				
λ_2 = 1e-5	$D_2(T)$	9.99994e-12	(6.62146e-12, 1.00000e-11)	33.785
λ_3 = 1e-8				
a= 1e-1	$D_4(T)$	4.90096e-11	(4.22053e-11, 5.00100e-11)	13.884

+ RECOVERY TOO SLOW

Again, large relative errors occurred when recovery times were relatively slow with respect to the mission time.

PROBLEM 15



$F(t) = 1 - e^{-\lambda 1 t}, \ t > 0; \quad G(t) = 1 - e^{-\lambda 2 t}, \ t > 0; \quad H(t) = 1 - e^{-\lambda 3 t}, \ t > 0$

Death-State Probability

NOTE: To obtain an exact expression for $D_2(T)$, the following set of differential equations were solved.

$P_0'(T) = -\lambda_1 P_0(T) + \lambda_3 P_1(T)$

$P_1'(T) = \lambda_1 P_0(T) - (\lambda_2 + \lambda_3)P_1(T)$

$P_2'(T) = \lambda_2 P_1(T)$

$$D_2(T) = \Big[[-(\lambda_1+\lambda_2+\lambda_3)[(\lambda_1+\lambda_2+\lambda_3)^2-4\lambda_1\lambda_2]^{1/2}+(\lambda_1+\lambda_2+\lambda_3)^2-4\lambda_1\lambda_2]^{-1} *$$

$$e^{[(\lambda 1+\lambda 2+\lambda 3)^2-4\lambda 1\lambda 2]^{1/2} T/2} -$$

$$[-(\lambda_1+\lambda_2+\lambda_3)[(\lambda_1+\lambda_2+\lambda_3)^2-4\lambda_1\lambda_2]^{1/2}-(\lambda_1+\lambda_2+\lambda_3)^2+4\lambda_1\lambda_2]^{-1} *$$

$$e^{-[(\lambda 1+\lambda 2+\lambda 3)^2-4\lambda 1\lambda 2]^{1/2} T/2} \Big] \ 2\lambda_1\lambda_2 e^{-(\lambda 1+\lambda 2+\lambda 3)T/2}$$

TABLE 15

λ_1 : exponential parameter for the (0,1) transition
λ_2 : exponential parameter for the (1,2) transition
λ_3 : exponential parameter for the (1,0) transition

PARAMETERS	DEATH STATES	ANALYTIC SOLUTIONS	SURE BOUNDS	% ERROR
λ_1 = 5e-4 λ_2 = 4e-4 λ_3 = 3e-4	D_2(T)	9.96009e-06	(9.96001e-06, 1.00000e-05)	0.401
λ_1 = 1e-3 λ_2 = 1e-5 λ_3 = 1e-7	D_2(T)	4.98321e-07	(4.98317e-07, 5.00000e-07)	0.337
λ_1 = 1e-7 λ_2 = 1e-6 λ_3 = 1e-5	D_2(T)	4.99982e-12	(4.99981e-12, 5.00000e-12)	0.004
λ_1 = 3e-4 λ_2 = 2e-4 λ_3 = 1e-1	D_2(T)	2.20417e-06@	(2.20417e-06, 2.20417e-06)	0.000
λ_1 = 1e-5 λ_2 = 1e-7 λ_3 = 1e-3	D_2(T)	4.98321e-11	(4.98317e-11, 5.00000e-11)	0.337
λ_1 = 8e-4 λ_2 = 4e-4 λ_3 = 3e-4	D_2(T)	1.59202e-05	(1.59200e-05, 1.60000e-05)	0.501
λ_1 = 1e-1 λ_2 = 1e-2 λ_3 = 1e-3	D_2(T)	3.54027e-02	(3.54027e-02, 3.54027e-02)	0.000
λ_1 = 2e-3 λ_2 = 3e-1 λ_3 = 2e-4	D_2(T)	1.35515e-02@	(1.35515e-02, 1.35515e-02)	0.000
λ_1 = 5e-2 λ_2 = 4e-2 λ_3 = 1e-7	D_2(T)	7.45224e-02	(7.45224e-02, 7.45224e-02)	0.000
λ_1 = 3e-3 λ_2 = 2e-3 λ_3 = 1e-6	D_2(T)	2.95046e-04	(2.94999e-04, 3.00000e-04)	1.679

@ Analytic solution obtained using MARK

DISCUSSION OF RESULTS

For all of the cases considered, the bounds given by the SURE program were true. That is, the exact unreliability for a given model was always enclosed in SURE's bounds. In a majority of the cases, the upper bound was much closer to the exact solution than the lower bound. The upper bound provided not only a good estimator of the death-state probability but a conservative estimate as well. For the mostpart, the bounds were also very tight. The relative error was less than 5% in 74% of the cases and even less than 1% in many of these cases. In cases where the relative error was less than or equal to 1%, the SURE bounds usually agreed to at least two decimal places.

There were, however, certain parameter values which caused the bounds to separate even to the point, in a few cases, where the bounds would not provide a useful estimate of unreliability. The cases where the bounds tended to separate were characterized by a large relative error. The wide bounds can largely be attributed to slow non-exponential transitions, i.e. slow recovery rates. For non-exponential rates less than or equal to 5, the bounds were often widely separated. In some of these cases, the SURE program output a warning flag - "RECOVERY TOO SLOW". This warning flag, however, only appeared in cases where the relative error was 100%. It is recommended that this error flag also be issued in less extreme cases.

Separation of the bounds also occurred in cases where a fast exponential rate was expressed as a slow transition. When a slow exponential rate was greater than or equal to 0.01, i.e., the fault arrival rate was fast, there tended to be a moderate separation of the bounds. Problems 4a and 4b demonstrated the effect of describing a fast exponential transition as a slow transition. To yield accurate bounds, fast exponential transitions must be specified as general transitions. The warning flags - "RATE TOO FAST", "STANDARD DEVIATION TOO BIG", and "DELTA > TIME" - were output when fast transitions were not correctly specified. These flags were also only output when the relative error was 100%. Again, these flags should be issued in less extreme cases.

As mentioned earlier, the user must decide based on the intended application whether the bounds given by SURE are tight enough to meet the requirements. One should keep in mind when using SURE that the mathematics implemented in the program were based on the concept of a fault-tolerant system with slow fault-arrival rates and very fast recovery rates. The fast fault-arrival rates and slow fault-recovery rates that induced the separated bounds are not considered typical of a fault-tolerant computer system.

CONCLUSIONS

Overall, the bounds given by the SURE program gave good estimates of the death-state probabilities for all models considered. Of particular importance, no cases were found in which the bounds did not enclose the exact unreliability, although there were a small number of cases in which the bounds were substantially separated. In most cases, the upper bound was a particularly good estimator of the death-state probability. For the simple models examined, the separation in the bounds could largely be attributed to fast fault-arrival rates and general transition rates which were relatively slow. Further work is underway in the validation effort. Testing is planned using larger models and comparing SURE's bounds to estimates given by other reliability analysis tools such as CARE III and HARP.

SYMBOLS

The greek characters, λ, α, β, γ, and ε, represent exponential transition rates in all of the semi-Markov models.

The capital letters, F, G, H, and I, represent general transition distributions in all of the semi-Markov models.

$D_i(T)$ - probability of being in death-state i at time T

$P_i(T)$ - probability of being in a death state at time T at the end of a class i-type path transition

$P_i'(T)$ - first derivative of $P_i(T)$

T - mission time

$\mu()$ - statistical mean of a distribution

$\rho()$ - transition probability for a distribution

$\sigma^2()$ - statistical variance of a distribution

@ - a solution was obtained from MARK

+ - a RECOVERY TOO SLOW message was issued by the SURE program

! - a RATE TOO FAST message was issued by the SURE program

& - a STANDARD DEVIATION TOO BIG message was issued by the SURE program

δ - a DELTA > TIME message was issued by the SURE program

REFERENCES

1. Geist, Robert M.; and Trivedi, Kishor S.: Ultrahigh Reliability Prediction Fault-Tolerant Computer Systems. IEEE Trans. Compt., C-32, no. 12, Dec. 1983, pp. 1118-1127.

2. Trivedi, Kishor; Dugan, Joanne Bechta; Geist, Robert; and Smotherman, Mark: Modeling Imperfect Coverage In Fault-Tolerant Systems. Proceedings of the Fourteenth International Conference on Fault-Tolerant Computing, pp. 77 82, 1984.

3. Butler, Ricky W.: The SURE Reliability Analysis Program. NASA TM-87593, 1986.

4. White, Allan L.: Synthetic Bounds for Semi-Markov Reliability Models. NASA CR-178008, 1985.

5. Lee, Larry D.: Reliability Bounds for Fault-Tolerant Systems With Competing Responses to Component Failures. NASA TP-2409, 1985.

6. Ward, Robert C.: Numerical Computation of the Matrix Exponential With Accuracy Estimate. SIAM J. Numerical Anal., Vol. 14, no. 4, Sept. 1977, pp. 600-610.

Standard Bibliographic Page

1. Report No. NASA TM-89123	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Validation of the SURE Program - Phase 1		5. Report Date May 1987
		6. Performing Organization Code
7. Author(s) Kelly J. Dotson		8. Performing Organization Report No.
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No. 505-66-21-01
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546		13. Type of Report and Period Covered Technical Memorandum
		14. Sponsoring Agency Code --
15. Supplementary Notes		

16. Abstract

This paper presents the results of the first phase in the validation of the SURE (Semi-Markov Unreliability Range Evaluator) program. The SURE program gives lower and upper bounds on the death-state probabilities of a semi-Markov model. With these bounds, the reliability of a semi-Markov model of a fault-tolerant computer system can be analyzed. For the first phase in the validation, fifteen semi-Markov models were solved analytically for the exact death-state probabilities and these solutions compared to the corresponding bounds given by SURE. In every case, the SURE bounds covered the exact solution. The bounds, however, had a tendency to separate in cases where the recovery rate was slow or the fault arrival rate was fast.

17. Key Words (Suggested by Authors(s)) semi-Markov model; death-state probability; SURE; reliability		18. Distribution Statement Unclassified - Unlimited; Subject Category 65	
19. Security Classif.(of this report) Unclassified	20. Security Classif.(of this page) Unclassified	21. No. of Pages 48	22. Price A03

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA Langley Form 63 (June 1985)