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ABSTRACT

The fundamental control synthesis issue of establishing a priori convergence rates of approximation schemes for feedback controllers for a class of distributed parameter systems is addressed within the context of hereditary systems. Specifically, a factorization approach is presented for deriving approximations to the optimal feedback gains for the linear regulator-quadratic cost problem associated with time-varying functional differential equations with control delays. The approach is based on a discretization of the state penalty which leads to a simple structure for the feedback control law. General properties of the Volterra factors of Hilbert-Schmidt operators are then used to obtain convergence results for the controls, trajectories and feedback kernels. Two algorithms are derived from the basic approximation scheme, including a fast algorithm, in the time-invariant case. A numerical example is also considered.

CONTENTS

1. INTRODUCTION. This report represents a first application of the control synthesis technique based on factorization methods developed in [24]. The main significance of the results contained herein stems from our ability to obtain a detailed analysis of a priori convergence rates for numerical approximation of feedback controllers for a class of distributed parameter systems. Although in the past 15 years much work has been done in developing approximation schemes for various distributed controllers, the important issue of convergence rates has been largely neglected. This is due primarily to the fact that it is a difficult problem, and control researchers have been content with the first order assessment regarding whether or not their approximation scheme converges at all (in certain correct senses). In this report convergence rates are established in the context of a challenging class of problems that include control delay terms. The underlying methods are of particular importance since they more tightly connect the modeling and control synthesis problems, and hence, have application to other distributed parameter systems.

The particular problem we address is the computation of the optimal feedback gain for the finite time linear regulator quadratic cost problem for systems governed by retarded functional differential equations (RFDE) with control delays. Feedback control laws for these systems have been previously derived for both the finite and infinite time problems in several articles under various hypotheses (see for example [6], [13], [14], [17], [24], [25]). A common approach in several of these articles is to relate the RFDE with control delays to an evolution equation in an appropriate state space. The feedback control law then arises in the familiar form as a solution to an operator Riccati equation. However, when point delays appear in the control, an unbounded input operator results and this operator then appears in the quadratic term of the Riccati equation. The presence of this unbounded term would appear to complicate any analysis of approximation schemes for the feedback gains.

The approach in [24] utilizing factorization theory [11], [22], allows us to circumvent these difficulties. The interest here is to further pursue the approach of [24] to the problem of approximating the optimal feedback kernel and deriving convergence estimates for the approximations, and the resulting controls and trajectories. (We note some other applications of factorization include for example, filtering and smoothing of nonstationary processes over a finite interval [15], [16], inverse problems in the spectral theory of differential operators [8], [18], solutions to two point boundary value problems [21], and solutions to Fredholm equations of the first and second kinds, [11].) Although this specific problem has not to this author's knowledge been treated in the literature, several articles (e.g., [6], [10], [19]) have considered the approximation problem without control delays.

The approach in each of these articles involves expressing the RFDE as an evolution equation in the state space $R^N xL_2$, and then approximating the resulting dynamical system. Delfour [5] discretizes in both the spatial and time variables, while Kunisch [19] and Gibson [10] discretize in only the spatial variable. Delfour considers the time-varying problem and obtains weak convergence of the solutions to the approximating Riccati equations. In [10] and [19] the open-loop semigroup is first approximated by discretizing the history space, and then the approximation theory of [9] is used for subsequent convergence analysis. This analysis is based on exploiting the relationship developed in [9] between the open-loop semigroup and the Riccati equation defining the feedback law. Open loop semigroup approximations have been derived in [2], [3]. By careful considerations involving the adjoint semigroup together with properties deduced from the finite dimensionality of the control space, Gibson was able to demonstrate strong convergence of the approximating Riccati operators. When coupled with the finite dimensionality of the input space, this leads to uniform convergence of the feedback operators. This is a significant result with respect to control implementation, since with uniform convergence the optimality of the approximating feedback law is independent of the state.

The convergence analysis presented in [10] and [19] depends heavily on the fact that the control map has finite rank. This condition does not hold in the control delay problem and straightforward extensions of these convergence results to the control delay case are not apparent. However, the form of the feedback kernel derived in [24] remains amenable to approximation and convergence analysis regardless of the presence of control delays. The reason for this is that control delays do not present any complications in the open loop formulation of the optimal control law and that the relationship between open loop and closed loop is somewhat transparent in the approach of [24] - the feedback kernel is derived from linear operators involving the fundamental matrix of the RFDE and the solution to a certain factorization problem associated with the fundamental matrix. Convergence questions for approximations to the feedback gain then reduce to corresponding questions regarding convergence of solutions to related factorization problems. (We note that the integral Riccati equations of [9] are also equivalent to certain factorization problems [23], so that the convergence analysis in [10] and [19] can also be performed within a factorization context.)

It will be shown that if the cost on the state in the regulator problem is a discrete sum with no integral term, then the associated factorization problem is solved by matrix inversion, and the exact feedback kernel can be defined in terms of the fundamental matrix solution, quadrature, and the solutions to finite dimensional linear equations. (This form of the solution generalizes a result of Manitius [20] for the problem with terminal state penalty and no control delays.) Thus an approximation scheme for the problem containing an integral state penalty term can be developed by approximating this term by quadrature and solving exactly for the feedback kernel of the resulting discretized state cost problem. Using this approach together with factorization arguments we will be able to establish $0(1/n)$ L_∞ - convergence for the approximate feedback kernels in the time-varying control delay case.

This result is (analytically) somewhat sharper than Gibson's in that the L_∞ convergence of the kernels applies on the square as well as the diagonal, and also that a priori rates can be provided. The principal reason this sharper result can be obtained is that we exploit the fact that only a computable piece of the semigroup - that part contributed by the fundamental matrix solution - is required to define the feedback kernel. Thus it is never necessary to consider the more difficult problem of approximating the entire semigroup. We now briefly outline the organization of the report.

In Section 2 the necessary mathematical preliminaries are developed and discussed. Because the approach does not follow along a Riccati synthesis of the solution, we will recapitulate in this section portions of the discussion in [24] relevant to the present application - particularly certain aspects of the Volterra factorization and how they apply to the RFDE control problem.

Section 3 contains the L_∞ - convergence results for the feedback kernels, controls, and trajectories. Instead of considering specific quadrature schemes approximating the state cost, all the results are proved with respect to a sequence of Borel measures satisfying certain convergence hypotheses. The key tool of this section is a factorization lemma which asserts that the factorization problem is well-posed (in an appropriate sense) in the space of integral operators with essentially bounded kernels.

In Section 4, the explicit form of the optimal feedback kernel associated with discrete state cost is derived. The resulting approximation scheme developed from the cost discretizations is then used as an analytical tool to obtain further results regarding the feedback kernel. For example, using essentially matrix manipulations, a Wiener-Hopf integral equation for the optimal feedback kernel is derived which is shown to be the control delay generalization of the Wiener-Hopf equation Manitius [20] had previously derived for the feedback kernel via a maximum principle.

In Section 5 two algorithms representing implementation of the basic approximation scheme of Section 4 are derived. In the time-invariant case a fast algorithm is derived by exploiting the near Toeplitz structure of the system of equations that defines the feedback kernel. A simple numerical example is also presented.

2. PRELIMINARIES. Let $[-r, T]$ denote a closed and bounded interval in the real line with $r \geq 0$ and $T > 0$, and let Σ denote the class of Borel subsets of $[-r, T]$. For an arbitrary Banach space Y, $|y|$ will denote the norm of an element $y \in Y$, $B(Y, Z)$ will denote the space of bounded linear maps from Y into another Banach space Z, and for brevity we write $B(Y)$ for $B(Y,Y)$. Subscripts will sometimes be attached to the norm of an element to remove any ambiguities that might arise due to the fact that several different topologies will be used in the report. The notation A^* (respectively A') will be used to denote the adjoint (transpose) of an operator (matrix).

In the sequel the Banach space of continuous functions $C([-r, T], R^N)$ will be denoted X, the Hilbert space $L_2([-r, T], R^M)$ will be denoted U, and H will denote the Hilbert space $L_2([-r,T], R^N)$. Now define the resolution of the identity $E: \Sigma \longrightarrow B(U)$ by multiplication by the characteristic function, i.e. $[E(\omega)u](t) = \chi(\omega)(t)u(t)$ $(\chi(\omega)(t) = 0$ if $t \notin \omega$, $\chi(\omega)(t)=1$ if $t \in \omega)$, and let P^t denote the family of projections $E([-r, t])$. The complementary family,

$I - P^t$, will be denoted P_t. Note that P^t is strongly continuous, i.e. $t \longrightarrow P^t u$ is continuous for each $u \in U$.

In this section we shall review some of the results in [24] pertaining to the linear regulator problem with dynamics

$$\dot{x}(t) = \int_{-r}^{0} d_\theta \;\; \eta(t,\theta)\; x(t+\theta) + (BP_0 u)(t) \qquad t \geq 0 \tag{2.1}$$

$$x(t) = \phi(t) \qquad\qquad t \in [-r,0],$$

and quadratic cost functional,

$$J(u,x) = \int_{-r}^{T} \langle x(s),\; Q(s)\; x(s)\rangle \; d\mu(s) + \int_{-r}^{T} |u(s)|^2 ds. \tag{2.2}$$

In (2.1), we assume that $\phi(t)$ is continuous, $x(\cdot) \in X$, $u \in U$, $B \in B(U,H)$ and P_0 is the projection $E([0, T])$. (It would not affect subsequent convergence analysis of the feedback kernels to allow an arbitrary projection P_{t_0}, $t_0 \in [-r, T]$, in place of P_0. The choice $t_0 = 0$ reflects the problem formulation in which control policies cannot be implemented until time $t = 0$.) The only constraint we impose on B at this time is that it be causal, i.e., for each $t \in [-r,T]$, if $u_1 = u_2$ a.e. on $[0,t]$ then $(Bu_1)(s) = (Bu_2)(s)$ for a.e. $s \leq t$. The matrix valued function η is assumed measurable on $R \times R$ and is normalized so that $\eta(t,\theta) = 0$ for $\theta \geq 0$ and $\eta(t,\theta) = \eta(t,-r)$ for $\theta \leq -r$. It is further assumed that $\eta(t,\cdot)$ is left continuous for each t and there exists a function $m \in L_1(0,T)$ such that

$$|\mathrm{Var}\; \eta(t,\cdot)| \leq m(t) \tag{2.3}$$

where $|\cdot|$ denotes any matrix norm. In the cost (2.2), μ denotes an arbitrary positive regular Borel measure on $[-r,T]$, and $Q(\cdot)$ is Borel measurable with $Q(s) \geq 0$ μ - a.e. s and is μ - essentially bounded.

It is convenient at this time to introduce some operators associated with the optimization problem (2.1) - (2.2). We define:

$$L \varepsilon B(X); \quad Lx : t \longrightarrow \begin{cases} 0 & t \varepsilon [-r,0] \\ \int_0^t \int_{-r}^0 d_\theta \, \eta(s,\theta) \, x(s+\theta) ds & t \geq 0, \end{cases} \tag{2.4}$$

$$F \varepsilon B(U,X); \; Fu : t \longrightarrow \int_{-r}^t F(t,s) \, u(s) ds, \tag{2.5}$$

and the adjoint-like $F^{\#}$,

$$F^{\#} \varepsilon B(X,U); \quad F^{\#}x : t \longrightarrow \int_t^T F'(s,t) Q(s) \, x(s) \, d\mu(s), \tag{2.6}$$

where $F(t,s) = [P_0 B^* Y'(t,\cdot)]'(s)$ and $Y(t,s)$ is the fundamental matrix solution of the homogeneous problem (see [12]). $Y(\cdot,\cdot)$ satisfies the Volterra equation

$$Y(t,s) = \begin{cases} I - \int_s^t Y(t,\sigma) \eta(\sigma, s-\sigma) d\sigma & s \leq t \\ 0 & s > t, \end{cases} \tag{2.7}$$

and the solution to (2.1) can be realized as

$$x(t) = Y(t,0) \, \phi(0) + \int_{-r}^0 d_\beta \int_0^t Y(t,\sigma) \eta(\sigma, \beta - \sigma) d\sigma \, \phi(\beta) \tag{2.8}$$

$$+ \int_0^t Y(t,\sigma) (BP_0 u)(\sigma) d\sigma.$$

Also, $\sup |Y(t,s)| < \infty$, $Y(t,s)$ is absolutely continuous for $t \geq s$ for each s, and $Y(t,\cdot)$ is of bounded variation for each t. We note from the definition of $F(t,s)$ that $F(t,s) = 0$ for $s < 0$. Hence, $FP_0 = F$ and $P_0F^{\#} = F^{\#}$.

Using a completing the squares argument, the open loop control law for (2.1) - (2.2) can be easily derived in terms of the operators defined above.

Theorem 2.1. The optimal control $\hat{u}$ for the regulator problem (2.1) - (2.2) is

$$\hat{u} = M\tilde{\phi}$$

where $M \varepsilon B(X,U)$, $\phi \varepsilon X$,

$$M = -(I + F^{\#}F)^{-1}F^{\#}(I - L)^{-1},$$

$$\tilde{\phi}(t) = \begin{cases} \phi(0) & t \geq 0 \\ \phi(t) & t \varepsilon [-r,0]. \end{cases}$$

Proof [24].

In [24] the feedback control law for (2.1) - (2.2) was derived from the open loop control above using a factorization approach. In an effort to motivate this approach and make the report somewhat more self-contained, we will briefly retrace some of the steps in [24] leading to the feedback law.

To sketch how the factorization ideas arise in the feedback control synthesis, we will consider for convenience the case μ = Lebesgue measure, $Q(\cdot) = I$, $U = H$, and B = the identity operator. Using (2.7) we can then write (2.1) - (2.2) as

$$\min \langle x,x \rangle + \langle u,u \rangle \qquad (2.9)$$

subject to

$$x = f + Tu \tag{2.10}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on U (we can also take X = U in this formulation), $T \in B(U)$ is the Hilbert-Schmidt operator

$$Tu: t \longrightarrow \int_0^t Y(t,s)u(s)ds,$$

and f is the forcing term in (2.8) resulting from the initial condition ϕ,

$$f(t) = Y(t,0)+\phi\ (0)\ \int_{-r}^{0} d_\beta \int_0^t Y(t,\sigma)\ \eta\ (\sigma,\ \beta-\sigma)d\sigma\ \phi\ (\beta).$$

Standard arguments then give the optimal control solution $\hat{u}$ as

$$\hat{u} = -(I + T^*T)^{-1}T^*f.$$

If we now consider (2.9) - (2.10) with the modification that $u \in P_tU$, $t \geq 0$, and replace f by an arbitrary forcing term f_t, the solution to this modified problem is realized as

$$\hat{u}_t = -(I + P_tT^*TP_t)^{-1}P_tT^*f_t.$$

Using a principle of optimality (see [24]) it can be shown that the choice $f_t = P_t[TP^t\hat{u} + f]$ in the modified problem leads to $\hat{u}_t = P_t\hat{u}$ for each t. Thus, given a partition of $[-r,T]$, $-r = t_o < t_1 < \cdots t_n = T$, it follows that

$$\hat{u} = \sum E(\omega_i)\ \{I + P_{t_i}T^*TP_{t_i}\}^{-1}P_{t_i}T^*f_{t_i}, \tag{2.11}$$

$\omega_i = [t_i,\ t_{i+1}]$. Next, examining the term $P_{t_i}T^*f_{t_i}$, it is evident that its values are determined by $P_{t_i}\ f_{t_i}$. And for $s \geq t_i$,

$$f_{t_i}(s) = Y(s,0)\phi(0) + \int_{-r}^{0} d_\beta \{\int_{0}^{s} Y(s,\alpha)\eta(\alpha,\beta-\alpha)d\alpha\}\phi(\beta)$$

$$+\int_{0}^{t} Y(s,\alpha)\hat{u}(\alpha).$$

But the above is recognized as the solution to

$$\dot{x}(s) = \int_{-r}^{0} d_\theta \eta(s,\theta)x(s+\theta), \qquad s \geq t$$

$$x(s) = \hat{x}(s) \qquad t-r \leq s \leq t,$$

where $\hat{x}(s)$ is the optimal trajectory. The variation of constants formula then implies for $s \geq t_i$,

$$f_{t_i}(s) = Y(s,t_i)\hat{x}(t_i) + \int_{t_i-r}^{t_i} d_\beta \{\int_{t_i}^{s} Y(s,\alpha)\eta(\alpha,\beta-\alpha)d\alpha\}\hat{x}(\beta).$$

Thus the optimal control in (2.11) at time $t \in [t_i,t_{i+1}]$ only uses the state information $\hat{x}(s)$, $s \in [t_i-r,t_i]$. This control law is now "open-loop" just on each subinterval $[t_i, t_{i+1}]$. The limit as the mesh of the partitions in (2.11) tends to zero would hopefully produce the feedback solution. This is indeed the case (cf [24]), and although we will not go into all of the specifics regarding the limiting procedure, we will briefly discuss the very much related notion of the projection integral ([11]).

Using the same notations as before, let $G: [-r,T] \longrightarrow B(U)$. Assume that G is strongly continuous, i.e. $t \longrightarrow G(t)u$ is continuous for each $u \in U$. Now let $K \in B(U)$ be a Hilbert-Schmidt operator and consider Riemann sums of the form

$$\sum E(\omega_i)KG(t_i'),\ -r = t_0 < \cdots < t_n = T,$$

$\omega_i = [t_i, t_{i+1}]$, and $t_i' \in \omega_i$. These sums can be shown to converge in the operator norm as the mesh of the partitions tends to zero [11]. This limit is represented by the projection integral

$$\int dEKG(t), \tag{2.12}$$

and it can further be shown [11] that

$$\left| \int dEKG(t) \right|_{HS} \leq |K|_{HS} \sup |G(t)|. \tag{2.13}$$

Here $|\cdot|_{HS}$ denotes the Hilbert-Schmidt norm.

Two important projection integrals on the space $\mathcal{K}$ of Hilbert-Schmidt maps in B(U) are obtained from the selections $G_+(t) = P^t$ and $G_-(t) = P_t$ in (2.12). The resulting mappings $p_\pm$,

$$p_\pm(K) = \int dEKG_\pm(t)$$

are bounded projections on $\mathcal{K}$. If $K \in R(p_\pm)$ we say that K is causal (anticausal). The elements of $R(p_\pm)$ are quasinilpotents. In the sequel we shall also write $K_\pm$ for $p_\pm(K)$. Note also that P^tU and P_tU are invariant subspaces of $p_\mp(K)$, respectively.

We note that in the space U a Hilbert-Schmidt map K is necessarily an integral operator with kernel, say K(t,s). In this case $p_\pm(K)$ are simply the Volterra operators

$$p_+(K)u:\ t \longrightarrow \int_{-r}^{t} K(t,s)u(s)ds,$$

$$p_-(K)u:\ t \longrightarrow \int_{t}^{T} K(t,s)u(s)ds.$$

With this bit of background we can now state the basic factorization results that will be used in the sequel.

Theorem 2.2. (Gohberg-Krein). Let $K \varepsilon \mathscr{K}$. Then there exist unique operators $X_{\pm} \varepsilon R(p_{\pm})$ such that

$$I + K = (I + X_-)(I + X_+) \tag{2.14}$$

if and only if $(I + P_tKP_t)$ is invertible for each $t \varepsilon [-r,T]$. Furthermore, $W_- = (I + X_-)^{-1} - I$ is given by the projection integral

$$W_- = -\int dEKG(t)$$

with

$$G(t) = P_t(I + P_tKP_t)^{-1}.$$

The decomposition of (I + K) into the product in (2.14) is called the Volterra or special (right) factorization of I + K, and we will sometimes refer to $X_{\pm}$ as the causal (anticausal) factor of K. Note that uniqueness of the factorization implies that $X_- = (X_+)^*$ when K is self-adjoint.

Two results that will be useful in subsequent convergence analysis are stated as corollaries below.

Corollary 2.3. Suppose $K \varepsilon \mathscr{K}$ is self-adjoint with

$$\sup_t |(I + P_tKP_t)^{-1}| \leq k.$$

Then I + K has the factorization (2.14) and $W_{\pm} = (I + X_{\pm})^{-1} - I$ satisfy

$$|W_{\pm}|_{HS} \leq k|K|_{HS}.$$

Corollary 2.4. Let $K_i \varepsilon \mathscr{K}$, $i = 1,2$ be self-adjoint with

$$\sup_t |(I + P_tK_iP_t)^{-1}| \leq k_i.$$

Let $W_i^* = (I + X_i^*)^{-1} - I$ where X_i^* denotes the anticausal factor of K_i. Then

$$|W_1^* - W_2^*|_{HS} \leq \min_{i \neq j} k_i \; |K_1 - K_2|_{HS} \; (1 + k_j \; |K_j|_{HS})$$

Proof. Let $G_i(t) = P_t(I + P_tK_iP_t)^{-1}$. Then

$$|W_1^* - W_2^*|_{HS} \leq \left| \int dEK_1(G_1 - G_2) \right|_{HS} + \left| \int dE(K_1 - K_2)G_2 \right|_{HS}$$

$$\leq |K_1|_{HS} \; |K_1 - K_2|_{HS} \; k_1 \; k_2 + |K_1 - K_2|_{HS} \; k_2$$

$$= k_2 |K_1 - K_2|_{HS} \; (1 + |K_1|_{HS} \; k_1)$$

Symmetry of the argument with respect to K_1 and K_2 gives the result.//

Returning now to the control solution in (2.11), it can be shown that the Riemann sums

$$\sum E(\omega_i) \; (I + P_{t_i} \; T^*TP_{t_i})^{-1} P_{t_i} \; T^*$$

converge to the projection integral

$$\int dE(I + X^*)^{-1}T^*,$$

where X is the causal factor of T^*T, i.e.

$$I + T^*T = (I + X^*)(I + X).$$

Now given an element $h \in L_2([-r,T],U)$ it is further possible (see [23]) to attach meaning to the expression

$$-\int dE(I + X^*)^{-1}T^*h \tag{2.15}$$

as a limit of sums

$$\sum_{i=1}^{n} E(\omega_{i,n})(I + X^*)^{-1}T^*h_{i,n}$$

where the sequence of simple functions

$$h_n = \sum \chi(\omega_{i,n})(t)h_{i,n}$$

converges in $L_2([-r,T],U)$ to h. The expression (2.15) is precisely $\hat{u}$ when h(t) is chosen via the principle of optimality, $h(t) = P_t[TP^t\hat{u} + f]$ (cf (2.11)).

Now we proceed to define the feedback solution to (2.1) - (2.2). This follows from a verification of hypotheses and the evaluation of (2.15), with h chosen above. Complete details can be found in [24].

Define the Hilbert space $H_\mu = L_2([-r,T], R^N; \mu)$ as the space of μ - square integrable functions on [-r,T] with values in R^N. It is evident that the map $Q: H_\mu \longrightarrow H_\mu$ defined by $(Qx)(t) = Q(t)x(t)$ is bounded and that $F^\#$ has the representation $F^\# = F^*Q$ where F^* is the adjoint of F considered as a mapping in $B(U,H_\mu)$. Hence, $F^\#F = F^*QF \geq 0$. In [24] it is verified that $F^\#F$ is Hilbert-Schmidt, so that Theorem 2.2 implies that $(I + F^\#F)$ has the factorization

$$I + F^\#F = (I + X^*)(I + X) \tag{2.16}$$

with X causal.

Now let $W^* = (I + X^*)^{-1} - I$. Since W^* is Hilbert-Schmidt it has a matrix kernel $W\_(t,s)$. Next define the M x N matrix valued function $P(t,\alpha)$ on [-r,T] x [-r,T] by

$$P(t,\alpha) = \int_{\alpha}^{T} K(t,s)Y(s,\alpha)d\mu(s), \ \alpha \geq t \tag{2.17}$$

where

$$K(t,s) = F'(s,t)Q(s) + \int_{t}^{s} W\_(t,\sigma) \ F'(s,\sigma)Q(s)d\sigma \tag{2.18}$$

and $Y(t,s)$ defined as in (2.7). The function $P(t,\alpha)$ provides the feedback solution to the regulator problem. This is made precise in the following.

Theorem 2.5. The optimal feedback control for (2.1) - (2.2) is given by

$$\hat{u}(t) = -P(t,t)\ \hat{x}(t) - \int_{t}^{\min(T,t+r)} P(t,\alpha) \int_{t-r}^{t} d_{\beta}\ \eta(\alpha,\beta-\alpha)\ \hat{x}(\beta)d\alpha$$

$$- \int_{t}^{T} P(t,\alpha)\ (BP^{t}\hat{u})(\alpha)d\alpha$$

where $\hat{x}$ denotes the optimal trajectory, and for each t, P^t denotes the projection on U, $(P^t u)(s) = \chi[-r,t](s)u(s)$. Furthermore, $P(t,\alpha)$ is square integrable (Lebesgue measure) on both the diagonal and the square $[-r,T] \times [-r,T]$.

3. <u>CONVERGENCE RESULTS</u>. The specific optimization problem we shall be considering is the following:

$$\min_{u,x} J(u,x) = \langle x(T), Q_0\, x(T)\rangle + \int_{-r}^{T} \langle x(s), Q(s)\, x(s)\rangle + |u(s)|^2 ds \tag{3.1}$$

subject to the constraint

$$\dot{x}(t) = \int_{-r}^{0} d_\theta\, \eta(t,\theta)\, x(t+\theta) + (BP_0u)(t) \qquad t \geq 0 \tag{3.2}$$

$$x(t) = \phi(t) \qquad t \in [-r,0], \tag{3.3}$$

where $\phi(\cdot) \in C([-r,0], R^N)$ and

$$(Bu)(t) = \sum_{i=0}^{k} \chi[r_i - r, T](t) B_i(t) u(t - r_i) + \int_{t-r}^{t} B(t,\theta) u(\theta) d\theta. \tag{3.4}$$

The assumptions on $\eta(\cdot,\cdot)$ are the same as in the preceding section. We shall assume that $0 = r_0 > -r_1 > \ldots > -r_k = -r$ and $\sup_t |B_i(t)| = b_i < \infty$, $\sup_{t,\theta} |B(t,\theta)| = b < \infty$. In the cost (3.1) we impose continuity on $Q(s)$.

Interpreting these assumptions in the context of Section 2, we have $\mu = \lambda + \delta$ where λ denotes Lebesgue measure, δ is the Dirac measure with support on $\{T\}$, and $Q(\cdot)$ is uniformly continuous on $[-r,T)$ with $Q(T) = Q_0$.

Now consider the following sequence $\{J_n\}$ of approximations to the cost $J(u,x)$:

$$J_n(u,x) = \int_{-r}^{T} \langle x(s), Q(s)\, x(s)\rangle\, d\mu_n(s) \tag{3.5}$$

$$+ \int_{-r}^{T} |u(s)|^2 ds,$$

where μ_n is a sequence of positive regular Borel measures such that:

H1. $\mu_n\{T\} = 1$ for all n,

H2. Given $\varepsilon > 0$ there exists m such that $n \geq m$ implies $|\mu_n[a,b) - |b-a|| < \varepsilon$ for all $a < b$, $a,b \in [-r,T]$.

In this section we will discuss the convergence properties of the solutions and feedback laws corresponding to the cost approximations above. Henceforth we refer to the optimization problem with cost (3.1) as problem $\mathscr{P}$, and the problem with cost (3.5) as problem $\mathscr{P}_n$. Unless otherwise noted, subscripts appearing on operators, functions, etc. (e.g. $F_n^{\#}$) will indicate that these terms are associated with problem $\mathscr{P}_n$.

We begin with the following simple result.

Lemma 3.1. Let μ be defined as above and let μ_n satisfy H1 and H2. Then $\mu_n - \delta \longrightarrow \lambda$ (Lebesgue measure) in the w^* topology of $C^*(-r,T)$.

Proof. Choose $\varepsilon > 0$ and let $f \in C(-r,T)$. Let $\pi = \{t_i\}_{i=0}^{n}$ be a partition of $[-r,T]$ such that for $|t - s| < |\pi|$, $|f(t) - f(s)| < \varepsilon$. Then,

$$\left| \int_{-r}^{T} f \, d(\mu_n - \delta) - \sum_{i=0}^{n-1} f(t_i)\mu_n\,[t_i,t_{i+1}) \right| < \varepsilon\mu_n[-r,T],$$

and

$$\left| \int_{-r}^{T} f \, ds - \sum_{i=0}^{n-1} f(t_i)(t_{i+1} - t_i) \right| < \varepsilon(T+r).$$

By H2, $\sup |\mu_j[-r,T)| = k < \infty$. Now choose m such that $m' > m$ implies $|\mu_{m'}[a,b) - |a-b|| < \varepsilon/n|f|$. Then it follows that

$$|\sum_{i=0}^{n-1} f(t_i)\{\mu_{m'}[t_i,t_{i+1}) - (t_{i+1} - t_i)\}| < \varepsilon.$$

Hence,

$$|\int f\, d(\mu_{m'} - \delta) - \int f\, ds| < \varepsilon + 2k\varepsilon.$$

And the lemma is proved.//

Since by definition $F(t,s) = [P_0 B^* Y'(t,\cdot)]'(s)$ (cf(2.5)-(2.6)), from (3.4) it follows

$$F(t,s) = \sum_{i=0}^{k} f_i\,(t,s) \tag{3.6}$$

where

$$f_0(t,s) = Y(t,s)B_0(s) + \int_s^{\min(s+r),t} Y(t,\theta)B(\theta,s)d\theta,$$

and

$$f_i(t,s) = Y(t,s+r_i)B_i(s+r_i),\ i=1,\ldots k.$$

Now let $\gamma = \sup |Y(t,s)|$. Then using [12, p.149] and the bounds in (3.4) we have

$$\sup_{t,s} |f_0(t,s)| \leq \gamma(b_0 + br),\ \sup_{t,s} |f_i(t,s)| \leq b_i\gamma. \tag{3.7}$$

Also, for $t_2 > t_1 \geq s + r_i$,

$$|f_o(t_2,s) - f_o(t_1,s)| \leq \exp |m|_1 \int_{t_1}^{t_2} m(\sigma)d\sigma \{b_o + br\} \tag{3.8}$$

$$+ b\gamma(t_2 - t_1),$$

and

$$|f_i(t_2,s) - f_i(t_1,s)| \leq b_i \exp |m|_1 \int_{t_1}^{t_2} m(\sigma)d\sigma, \; i=1, \ldots k. \tag{3.9}$$

It is not difficult to show (see [24]) that $F^{\#} = F^{*}j$, $F^{\#}_n = F^{*}j_n$ where F^{*} is the B-space adjoint of F, i.e. $F^{*}: X^{*} \longrightarrow U$, and j, j_n are the mappings of X into X^{*},

$$j(x)y = \int \langle y(s), Q(s)\, x(s)\rangle \, d\mu(s),$$

$$j_n(x)y = \int \langle y(s), Q(s)\, x(s)\rangle \, d\mu_n(s).$$

Now it follows easily from definition and the estimates above that F is compact. Thus, using the w^{*}- convergence of $j_n(x) \rightarrow j(x)$ for each x (from Lemma 3.1), it can then be deduced that $F^{\#}_n \longrightarrow F^{\#}$ strongly. Consequently from the compactness of F it also follows that $F^{\#}_n F \longrightarrow F^{\#}F$ uniformly. Noting the form of the open loop control law (in Theorem 2.1), these general considerations are enough to demonstrate the L_2 - convergence of the approximate optimal controls and the uniform convergence of the corresponding optimal trajectories resulting from approximations based on $\mathscr{P}_n$. However, the major aim of this section is to produce the stronger L_∞ - convergence of the approximations for the feedback kernels as well as the controls, and this requires a somewhat more specific analysis.

Let Z denote the space $L_\infty([-r,T],R^M)$. From the definition of F(t,s) it is evident that $F^{\#}$ and $F^{\#}_n$ are also in B(X,Z). Our first result sharpens the convergence of $F^{\#}_n \longrightarrow F^{\#}$ discussed above. This result (and the method of proof) will form the basis for the L_∞ - convergence arguments later.

Lemma 3.2. $F_n^\# \longrightarrow F^\#$ strongly in $B(X,Z)$.

Proof. Let $x \in X$. By definition

$$[F^\# - F_n^\#]x: \quad t \longrightarrow \sum_{i=0}^{k} \int f_i'(s,t)Q(s)x(s)d(\mu-\mu_n)(s).$$

For each i define

$$\tilde{f}_i(s,t) = \begin{cases} f_i(s,t) & s > t + r_i \\ f_i(t+r_i,t) & s \le t+r_i. \end{cases}$$

Similarly define

$$Q_i(s,t) = \begin{cases} Q(s) & s > t+r_i \\ Q(t+r_i) & s \le t+r_i, \end{cases}$$

and

$$x_i(s,t) = \begin{cases} x(s) & s > t+r_i \\ x(t+r_i) & s \le t+r_i. \end{cases}$$

Considered as families of functions parameterized by t, $\{\tilde{f}_i(\cdot,t)\}$ is equicontinuous by virtue of (3.7)-(3.9), and $\{Q_i(\cdot,t)\}$ and $\{x_i(\cdot,t)\}$ are equicontinuous by virtue of the uniform continuity of $Q(\cdot)$ and $x(\cdot)$, respectively. Furthermore these families are clearly uniformly bounded. Hence the set

$$S = \{x_t \in C([-r,T], R^M): x_t(s) = f_i(s,t)Q_i(s,t)x_i(s,t), t \in [-r,T-r_i], \ i=0,1,\ldots k\}$$

is relatively compact in $C([-r,T],R^M)$. Now note that

$$|[F^{\#} - F_n^{\#}](x)(t)| \leq \sum_{i=0}^{k}\{| \int_{[-r,T]} f_i(s,t)Q_i(s,t)x_i(s,t)d(\mu-\mu_n)(s)|$$

$$+ | \int_{[-r,t+r_i]} \tilde{f}_i(s,t)Q_i(s,t)x_i(s,t)d(\mu-\mu_n)(s)|\}$$

Using the compactness of S it follows from Lemma 3.1 that the first integral above converges to zero uniformly with respect to t. And since the second integral has constant integrand for each t and i, uniform convergence is obtained here by using H2.//

Now let $H(t,s)$ and $H_n(t,s)$ denote the kernels of $F^{\#}F$ and $F_n^{\#}F$ respectively. Fubinis theorem implies

$$|H(t,s) - H_n(t,s)| = |\sum_{i,j} \int f_i'(\sigma,t)Q(\sigma)f_j(\sigma,s)d(\mu-\mu_n)(\sigma)|. \qquad (3.10)$$

Arguing as in the previous lemma we can show that $H_n(t,s)$ converges uniformly to $H(t,s)$. To see this note that for each fixed i and j

$$\int f_i'(\sigma,t)Q(\sigma)f_j(\sigma,s)d(\mu-\mu_n)(\sigma) = \begin{cases} \int_{[s+r_j,T]} f_i'(\sigma,t)Q(\sigma)f_j(\sigma,s)d(\mu-\mu_n)(\sigma) & s+r_j \geq t+r_i, \\ \int_{[t+r_i,T]} f_i'(\sigma,t)Q(\sigma)f_j(\sigma,s)d(\mu-\mu_n)(\sigma) & s+r_j < t+r_i. \end{cases}$$

Thus on the set $V = \{(s,t)\colon s + r_j \geq t + r_i\}$ we can write

$$\int f_i'(\sigma,t)Q(\sigma)f_j(\sigma,s)d(\mu-\mu_n)(\sigma) = \int_{[-r,T]} \tilde{f}_i'(\sigma,s,t)Q_j(\sigma,s)\tilde{f}_j(\sigma,s)d(\mu-\mu_n)(\sigma)$$

$$- \int_{[-r,s+r_j]} \tilde{f}_i'(\sigma,s,t)Q_j(\sigma,s)\tilde{f}_j(\sigma,s)d(\mu-\mu_n)(\sigma)$$

where $Q_j(\sigma,s)$ and $\tilde{f}_j(\sigma,s)$ are defined as in Lemma 3.2 and

$$\tilde{f}'_i(\sigma,s,t) = \begin{cases} f'_i(\sigma,t) & \sigma \geq s+r_j \\ f'_i(s+r_j,t) & \sigma < s+r_j . \end{cases}$$

Uniform convergence of the integrals parameterized by V follows in the manner of the proof of the lemma. Since this argument also holds for subsets of the form $\{(s,t): s + r_j < t + r_i\}$, we obtain the following result.

Lemma 3.3. With the notations above, $H_n(t,s) \longrightarrow H(t,s)$ uniformly on $[-r,T] \times [-r,T]$. In particular, $F_n^{\#}F \longrightarrow F^{\#}F$ in the Hilbert-Schmidt topology (in B(U)) and in the B(Z) topology.

To obtain L_∞ - convergence of the feedback kernels we will use a result analogous to the one above regarding the convergence of the kernels of the Volterra factors of $(I + F_n^{\#}F)$. Already we can use Corollary 2.4 and Lemma 3.3 to obtain Hilbert-Schmidt convergence of the factors (hence, L_2 - convergence of their kernels). But our ultimate interest is to demonstrate the stronger L_∞ - convergence of the kernels. We shall need the following two results which are of some interest in their own right.

Proposition 3.4. Let K be an integral operator on U with essentially bounded kernel K(t,s), $\operatorname{esssup}_{t,s} |K(t,s)| = \beta < \infty$. Suppose that

$$\sup_t |(I+P_tKP_t)^{-1}| = \alpha < \infty$$

so that (I + K) has the factorization

$$I + K = (I + X^*)(I + X) \tag{3.12}$$

with X causal (cf Theorem 2.2). Then $W = (I + X)^{-1} - I$ and W^* are integral operators with kernels $W_{\pm}(t,s)$ satisfying the bound

$$\operatorname*{esssup}_{t,s} |W_{\pm}(t,s)| < \rho(\alpha,\beta) \tag{3.13}$$

where

$$\rho(\alpha,\beta) = \beta[1 + \beta(T + r)(1 + \alpha\beta(T + r))^2].$$

Proof. First note that Corollary 2.3 implies $W = (I + X)^{-1} - I$ is Hilbert-Schmidt with $|W|_{HS} \leq \alpha|K|_{HS}$. Now the factorization (3.12) implies

$$(I + X^*) = (I + K)\ (I + W).$$

Substracting the identity from the above and applying the projection p_- (i.e. taking anticausal parts) results in for $\theta \geq t$,

$$X_-(t,\theta) = K(t,\theta) + \int_{\theta}^{T} K(t,\sigma)W_+(\sigma,\theta)d\sigma \quad \text{a.e. } t,\theta,$$

where $X_-(t,\theta)$ and $W_+(t,\theta)$ are the kernels of X^* and W respectively.

Hence for a.e. t,θ,

$$|X_-(t,\theta)| \leq \beta[\,1 + \int_{-r}^{T} |W_+(\sigma,\theta)|d\sigma].$$

Consequently,

$$\int_{-r}^{T} |X_-(t,\theta)|^2 d\theta \leq \beta^2 \int_{-r}^{T} [1 + \int_{-r}^{T} |W_+(\sigma,\theta)|d\sigma]^2 d\theta$$

$$= \beta^2 [(T + r) + 2 \int_{-r}^{T} \int_{-r}^{T} |W_+(\sigma,\theta)| d\sigma d\theta + \int_{-r}^{T} \int_{-r}^{T} |W_+(\sigma,\theta)| d\sigma \, ^2 d\theta$$

$$\leq \beta^2 (T + r)(1 + |W_+|_{HS})^2$$

$$\leq \beta^2 (T + r)(1 + a \, |K|_{HS})^2$$

$$\leq \beta^2 (T + r) \; [1 + a\beta(T + r)]^2 \qquad \text{a.e. } t.$$

Let $\tilde{a} = \beta^2 (T + r) \; [1 + a\beta(T + r)]^2$. Then since W^* satisfies the identity $W^* = -X^* - X^* W^*$, we have

$$|W\_(t,\theta)| \leq | \; X\_(t,\theta)| + \int_{-r}^{T} |X\_(t,\sigma) \; W\_(\sigma,\theta)| d\sigma \qquad \text{a.e. } t,\theta.$$

Hence,

$$\int_{-r}^{T} |W\_(t,\theta)|^2 d\theta \leq \int_{-r}^{T} |X\_(t,\theta)|^2 d\theta + 2 \int_{-r}^{T} |X\_(t,\theta)| \int_{-r}^{T} |X\_(t,s)| \; |W\_(s,\theta)| ds d\theta$$

$$+ \int_{-r}^{T} \left[\int_{-r}^{T} |X\_(t,s)| \; |W\_(s,\theta)| ds \right]^2 d\theta$$

$$\leq \tilde{a} + 2\tilde{a} \; |W\_|_{HS} + \tilde{a} \; |W\_|_{HS}^2$$

$$\leq \tilde{a} (\, 1 + a |K|_{HS})^2 \qquad \text{a.e. } t$$

Thus,

$$\operatorname*{esssup}_{t} \int_{-r}^{T} |W\_(t,\theta)|^2 d\theta \leq \beta^2 (T + r) \; [\; 1 + a\beta(T + r)]^4. \tag{3.14}$$

Now (3.12) also implies

$$I + X = (I + W^*)(I + K).$$

Subtracting the identity and applying p_- yields

$$W^* + K_- + [W^* K]_- = 0.$$

Hence,

$$|W_-(t,\theta)| \leq |K(t,\theta)| + \int_{-r}^{T} |W_-(t,s)||K(s,\theta)|ds.$$

$$\leq \beta\{1 + (T + r)^{1/2} \operatorname{ess\,sup}_t [\int_{-r}^{T} |W_-(t,s)|^2 ds]^{1/2}\}$$

The result follows from 3.14.//

As the result above may be regarded as the L_∞ analogue of Corollary 2.3, the next proposition is the L_∞ analogue of Corollary 2.4. The notations X, W, $X_\pm (t,s)$ and $W_\pm (t,s)$ will have the same meaning below as in the preceding proposition.

Proposition 3.5. Let K be as in the proposition above and let $\{K_n\}$ denote a sequence of integral operators on U with essentially bounded kernels $K_n(t,s)$ such that

$$\operatorname{ess\,sup}_{t,s} |K_n(t,s)| \leq \delta,$$

and

$$\lim_n \operatorname{ess\,sup}_{t,s} |K_n(t,s) - K(t,s)| = 0$$

Assume further that $K_n \geq 0$ for each n so that $I + K_n$ has the factorization

$$I + K_n = (I + X_n^*)(I + X_n) \quad \text{(with } X_n \text{ causal).}$$

Let $W_n(t,s)$ denote the kernel of the integral operator $(I + X_n)^{-1} - I$.

Then

$$\lim_n \operatorname{ess\,sup}_{t,s} |W_n(t,s) - W_+(t,s)| = 0.$$

Proof. First write $I + K_n = I + K + (K_n - K)$ so that

$$I + K_n = (I + X_-)(I + A_n)(I + X_+) \tag{3.15}$$

where $A_n = (I + X^*)^{-1}(K_n - K)(I + X)^{-1}$. By Proposition 3.4 $\operatorname{ess\,sup}_{t,s} |W_\pm(t,s)| \leq \alpha$ for some $\alpha < \infty$. Now since $I + K_n$ has the factorization, so does $I + A_n$. Specifically, $I + A_n = (I + Y_n^*)(I + Y_n)$ where $I + Y_n = (I + X_n)(I + W)$. It then follows from the identity

$$I + P_tA_nP_t = (I + P_tY_n^*P_t)(I + P_tY_nP_t)$$

that

$$\begin{aligned}\sup_t |(I + P_tA_nP_t)^{-1}| &\leq \sup_t |(I + P_tY_nP_t)^{-1}|^2 \\ &\leq (1 + |W_n|)^2 \sup|(I + P_tWP_t)^{-1}|^2 \\ &\leq [1 + |K_n|_{HS}]^2[\exp\{1/2(1 + |K|^2_{HS})\}]^2\end{aligned}$$

Here we have used Corollary 2.3 to obtain the first term in the product, and the fact that W is Hilbert-Schmidt and quasinilpotent together with [7, p. 1039] and Corollary 2.3 to obtain the second term. Now define $Z_n = (I + Y_n)^{-1} - I$ and let $Z_n(t,s)$ denote its kernel. Then since

$$\operatorname{ess\,sup}_{t,s} |A_n(t,s)| \leq \beta_n[1 + \alpha(T + r) + \alpha^2(T + r)^2]$$

where $\beta_n = \operatorname{ess\,sup}_{t,s} |K(t,s) - K_n(t,s)|$ and $\alpha \geq \operatorname{ess\,sup}_{t,s} |W_\pm(t,s)|$,

Proposition 3.4 implies

$$\operatorname{ess\,sup}_{t,s} Z_n(t,s) \leq \rho(\tilde{\alpha}_n, \tilde{\beta}_n)$$

with

$$\tilde{\beta}_n = \beta_n[1 + \alpha(T + r) + \alpha^2(T + r)^2]$$

and

$$\tilde{\alpha}_n = [1 + |K_n|_{HS}]^2[\exp\{1/2(1 + |K|^2_{HS})]^2.$$

Finally, note that

$$W_n = W + Z_n + WZ_n,$$

so that

$$\operatorname*{ess\,sup}_{t,s} |W_n(t,s) - W_+(t,s)| \leq \rho(\tilde{\alpha}_n, \tilde{\beta}_n)[1 + (T + r)\alpha].$$

But $\rho(\tilde{\alpha}_n, \tilde{\beta}_n) = O(\beta_n)$. This completes the proof.//

Before proceeding to the main result of the section we will make a digression to establish L_∞ convergence of the approximate control sequence and trajectories. Again we let Z denote the space $L_\infty([-r,T],R^M)$. For notational convenience we also introduce the subspace $X_0 \subset X$ of initial conditions,

$$X_0 = \{x \in X: \ x(t) = x(0) \text{ for } t \geq 0\}$$

and endow it with the subspace topology.

Theorem 3.6. Let $\mathscr{B}$ denote the unit ball in $C([-r,0],R^N)$. For $\phi \in \mathscr{B}$ let $\hat{u}_n(\phi)$ and $\hat{u}(\phi)$ denote the optimal controls for problems $\mathscr{P}_n$ and $\mathscr{P}$ respectively. Also let $\hat{x}_n(\phi)$ and $\hat{x}(\phi)$ denote the corresponding trajectories. Then, uniformly on $\mathscr{B}$,

(i) $\lim |\hat{u}_n(\phi) - \hat{u}(\phi)|_Z = 0,$
(ii) $\lim |\hat{x}_n(\phi) - \hat{x}(\phi)|_X = 0.$

Proof. Let $\phi \in \mathscr{B}$. Theorem 2.1 gives the optimal control law as

$$\hat{u}_n(\phi) = -(I + F_n^{\#}F)^{-1}F_n^{\#}(I - L)^{-1}\tilde{\phi}$$

where $\tilde{\phi} \in X_0$ is the extension of ϕ such that $\tilde{\phi}(t) = \phi(0)$ for $t \geq 0$.

Now $(I - L)^{-1}\tilde{\phi}$ is recognized as the solution to the homogeneous problem

$$\dot{x}(t) = \int_{-r}^{0} d_\theta\, \eta(t,\theta)x(t + \theta)$$

with initial condition ϕ. Denoting this solution $x(\phi)$, standard arguments (see [12]) give

$$|x(\phi)(t)| \leq |\phi| \exp |m|_1 \tag{3.16}$$

and

$$|x(\phi)(t_2) - x(\phi)(t_1)| \leq |\phi| \left[\int_{t_1}^{t_2} m(s)ds\right] \exp|m|_1. \tag{3.17}$$

Thus the set S,

$$S = \{x \in C([0,T], R^N): x(t) = [(I - L)^{-1}\tilde{\phi}](t) \text{ for } \tilde{\phi} \in X_0 \text{ with } |\tilde{\phi}| \leq 1\}$$

is relatively compact in $C([0,T],R^N)$. Now, the triangle inequality yields

$$\begin{aligned} |\hat{u}_n - \hat{u}|_Z \leq\ & |(I + F_n^{\#}F)^{-1}|_{B(Z)} \cdot |(F_n^{\#} - F^{\#})(I - L)^{-1}\tilde{\phi}|_Z \\ & + |(I + F_n^{\#}F)^{-1} - (I + F^{\#}F)^{-1}|_{B(Z)} \cdot |F^{\#}(I - L)^{-1}\tilde{\phi}|_Z \end{aligned} \tag{3.18}$$

Let $\delta \geq \operatorname{ess\,sup}_{t,s} H_n(t,s)$ for all n (cf (3.10) and Lemma 3.3).

Also let $W_n = (I + X_n)^{-1} - I$ where X_n is the causal factor of $F_n^{\#}F$, i.e.

$$(I + F_n^{\#}F) = (I + X_n^*)(I + X_n).$$

Thus,

$$(I+F_n^{\#}F)^{-1} = (I + W_n)(I + W_n^*).$$

Now let $W_n(t,s)$ denote the kernel of W_n. Then Proposition 3.4 implies $\operatorname{ess\,sup}_{t,s} |W_n(t,s)| \leq \rho(1,\delta)$ independently of n. Hence,

$$|(I + F_n^{\#}F)^{-1}|_{B(Z)} \leq [1 + \rho(1,\delta)(T + r)]^2, \tag{3.19}$$

and consequently

$$|(I+F_n^{\#}F)^{-1}-(I+F^{\#}F)^{-1}|_{B(Z)} \leq |(I+F_n^{\#}F)^{-1}|_{B(Z)} |F^{\#}{}_nF-F^{\#}F|_{B(Z)} |(I+F^{\#}F)^{-1}|_{B(Z)}$$

$$\leq [1 + \rho(1, \delta)(T + r)]^4 |F_n^{\#}F - F^{\#}F|_{B(Z)} \tag{3.20}$$

We also have from (3.16),

$$|F^{\#}(I - L)^{-1} \tilde{\phi}|_Z \leq |\phi| \ |F^{\#}|_{B(X,Z)} \cdot \exp |m|_1 \tag{3.21}$$

By Lemma 3.2, $F_n^{\#} \longrightarrow F^{\#}$ strongly in B(X, Z). Recall now that F(t,s) (the kernel of F) vanishes for $s < 0$. Then since S is relatively compact, it follows that $(F_n^{\#} - F^{\#})(I - L)^{-1} \longrightarrow 0$ uniformly in $B(X_0,Z)$. This result and Lemma 3.3 together with (3.19) - (3.21) inserted into (3.18) proves the first assertion of the theorem.

To prove the second part let $y_n(\phi) = \hat{x}_n(\phi) - \hat{x}(\phi)$. Then $y_n(\phi)$ satisfies (3.2) with zero initial condition and forcing term $B(\hat{u}_n - \hat{u})$. Therefore for some constant C,

$$|y_n(\phi)(t)| < C \operatorname{ess\,sup}_t |\hat{u}_n(\phi)(t) - \hat{u}(\phi)(t)|.$$

Thus (ii) follows from (i).//

Next we present the convergence properties of the sequence of approximating feedback kernels.

Theorem 3.7. Let $P_n(t,\alpha)$ and $P(t,\alpha)$ denote the feedback kernels of Theorem 2.5 associated with problems $\mathscr{P}_n$ and $\mathscr{P}$ respectively. Then,

$$\text{(i)}\ \lim_n \operatorname{ess\,sup}_t |P_n(t,t) - P(t,t)| = 0,$$
$$\text{(ii)}\ \lim_n \operatorname{ess\,sup}_{t,\alpha} |P_n(t,\alpha) - P(t,\alpha)| = 0.$$

Proof. From (2.17) - (2.18) we write

$$P_n(t,\alpha) = \sum_{i=0}^{k} \int_{\alpha}^{T} K_{n,i}(t,s)Y(s,\alpha)d\mu_n(s),$$

$$P(t,\alpha) = \sum_{i=0}^{k} \int_{\alpha}^{T} K_i(t,s)Y(s,\alpha)d\mu(s),$$

where

$$K_{n,i}(t,s) = f_i'(s,t)Q(s) + \int_t^{s-r_i} W_n(t,\theta)f_i'(s,\theta)Q(s)d\theta,$$

and

$$K_i(t,s) = f_i'(s,t)Q(s) + \int_t^{s-r_i} W(t,\theta)f_i'(s,\theta)Q(s)d\theta.$$

Thus,

$$|P_n(t,\alpha) - P(t,\alpha)| \leq \sum_{i=0}^{k} \{ | \int_{\alpha}^{T} [K_{n,i}(t,s) - K_i(t,s)] Y(s,\alpha) d\mu_n(s) | +$$

$$| \int_{\alpha}^{T} K_i(t,s) Y(s,\alpha) d(\mu - \mu_n)(s) | \} \qquad (3.22)$$

Now,

$$|K_{n,i}(t,s) - K_i(t,s)| \leq \int_{t}^{s-r_i} |W_n(t,\theta) - W(t,\theta)| \; |f_i'(s,\theta)| |Q(s)| d\theta$$

Lemma 3.3 and Proposition 3.5 imply that

$$\lim_{n} \operatorname{ess\,sup}_{t,\theta} |W_n(t,\theta) - W(t,\theta)| = 0.$$

And since $|f_i(s,\theta)|$ and $|Q(s)|$ are uniformly bounded, routine arguments yield a measurable set $\Omega \subset [-r,T]$ whose complement has zero Lebesgue measure such that

$$K_{n,i}(t,s) \longrightarrow K_i(t,s) \text{ uniformly on } \Omega \times [-r,T].$$

Using the uniform boundedness of $Y(s,\alpha)$ and the sequence of measures μ_n, it follows that the first integral in (3.22) tends to zero uniformly on $\Omega \times [-r,T]$. To prove convergence of the second integral in (3.22) we argue as in Lemma 3.2. Define the family of functions parameterized by α and β,

$\{\tilde{Y}(\cdot,\alpha,\beta)\}$,

$$\tilde{Y}(s,\alpha,\beta) = \begin{cases} Y(s,\alpha) & s \geq \beta \\ Y(\beta,\alpha) & s < \beta, \end{cases}$$

and the family of functions parameterized by t and γ, $\{\tilde{K}_i(t,\gamma,\cdot)\}$,

$$\tilde{K}_i(t,\gamma,s) = \begin{cases} K_i(t,s) & s \geq \gamma \\ \\ K_i(t,\gamma) & s < \gamma \end{cases}$$

It is straightforward to verify using the properties of $K_i(t,s)$ and $Y(s,\alpha)$ that the set

$$\{\tilde{K}_i(t,\rho,\cdot)\tilde{Y}(\cdot,\alpha,\rho):\ \rho = \max\ \{\alpha,t+r_i\}\}$$

is relatively compact in C ([-r,T], $R^{M\times N}$). Thus the argument in Lemma 3.2 applies here to demonstrate that

$$\lim_{n}\ \operatorname*{ess\ sup}_{t,\alpha}\ \left|\int_{\alpha}^{T} K_i(t,s)Y(s,\alpha)d(\mu_n-\mu)(s)\right| = 0.$$

And the theorem is proved.//

4. APPLICATIONS. In this section we begin by deriving the optimal feedback kernel associated with an arbitrary discrete state cost penalty. It will be evident that given the fundamental matrix Y(t,s), the feedback kernel in this case can be derived by quadrature and matrix inversion. This feedback structure when combined with the results of the preceding section leads to approximations to the optimal feedback kernel of problem $\mathscr{P}$ (recall (3.1)-(3.3)), a Wiener-Hopf characterization of this kernel, and a priori bounds on its magnitude.

Let ν denote a positive discrete measure on [-r,T] of the form

$$\int f d\nu = f(s_n) + \sum_{i=0}^{n-1} a_i f(s_i),\ \ s_i \varepsilon [-r,T],\ s_n = T.$$

Inserting this measure into (2.2) results in the cost

$$J(u,x) = \langle x(T), Q(T)\, x(T) \rangle + \sum_{i=0}^{n-1} a_i \langle x(s_i), Q(s_i)\, x(s_i) \rangle + \int_{-r}^{T} |u(s)|^2 ds \tag{4.1}$$

The optimal feedback kernel for this cost has the following semiseparable[1] structure.

Theorem 4.1. Define the matrix functions $G(t)$ and $\bar{Y}(t)$,

$$G(t) = [\sqrt{a_0}F'(s_0,t) \vdots --- \sqrt{a_{n-1}}F'(s_{n-1},t) \vdots F'(s_n,t)]'$$

$$\bar{Y}(t) = [\sqrt{a_0}Y'(s_0,t) \vdots --- \sqrt{a_{n-1}}Y'(s_{n-1},t) \vdots Y'(s_n,t)]'.$$

Also define the matrix

$$\tilde{Q} = \begin{bmatrix} Q(s_0) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & Q(s_n) \end{bmatrix}$$

Then the optimal feedback kernel $P(t,\alpha)$ for the problem with dynamics (2.1) and cost (4.1) can be expressed as

$$P(t,\alpha) = G'(t)\tilde{Q}(I + U(t))^{-1}\bar{Y}(\alpha) \tag{4.2}$$

where

$$U(t) = \int_{t}^{T} G(s)G'(s)\tilde{Q}ds. \tag{4.3}$$

Proof. It is straightforward to verify that the map $F^{\#}F$ resulting from the measure ν in (4.1) is an integral operator with separable kernel $G'(t)\tilde{Q}G(s)$. Now Theorem 2.5 implies

$$P(t,\alpha) = \sum_{s_i \geq \alpha} a_i K(t,s_i) Y(s_i,\alpha).$$

Here

$$K(t,s_i) = F'(s_i,t)Q(s_i) + \int_t^T W_-(t,\sigma)F'(s_i,\sigma)Q(s_i)d\sigma$$

and $W_-(t,\sigma)$ denotes the kernel of the anticausal operator W^* in the factorization

$$(I + F^{\#}F)^{-1} = (I + W)(I + W^*).$$

Using the fact that $F^{\#}F$ has a separable kernel, it can be verified [11,p.188] that

$$W_-(t,\sigma) = -G'(t)\tilde{Q}[I + U(t)]^{-1}G(\sigma)$$

with $U(t)$ defined as in (4.3). Thus we can write

$$P(t,\alpha) = \sum \{\sqrt{a}_i F'(s_i,t)Q(s_i) - G'(t)\tilde{Q}[I + U(t)]^{-1}$$

$$\cdot \int_t^T G(\sigma)\sqrt{a}_i F'(s_i,\sigma)Q(s_i)d\sigma\}\sqrt{a}_i Y(s_i,\alpha)$$

Noting the definitions of G, $\bar{Y}$ and U, it follows

$$P(t,\alpha) = [G'(t)\tilde{Q} - G'(t)\tilde{Q}[I + U(t)]^{-1}U(t)]\bar{Y}(\alpha)$$

$$= G'(t)\tilde{Q}[I + U(t)]^{-1}\bar{Y}(\alpha). //$$

Note that if in (4.1) we take for $i \neq n$, $a_i = 0$, and let the operator B denote a multiplication operator, i.e. $(Bu)(t) = B(t)u(t)$, then (4.2) reduces Manitius' result for terminal state penalty [20]

$$P(t,\alpha) = B'(t)Y'(T,t)\,Q(T)\,[I + \int_t^T Y(T,\sigma)B(\sigma)B'(\sigma)Y'(T,\sigma)Q(T)d\sigma]^{-1}Y(T,\alpha).$$

Thus Theorem 4.1 can be viewed as an extension of this result to problems with control delays and arbitrary discrete state penalty.

Now let μ_n be a sequence of discrete positive measures satisfying H1 and H2, and let $P_n(t,\alpha)$ denote the corresponding feedback kernels. The theorem implies each $P_n(t,\alpha)$ has the semiseparable form (4.2), while Theorem 3.7 implies the L_∞ convergence of $P_n(t,\alpha)$ to $P(t,\alpha)$(the optimal feedback kernel for problem $\mathscr{P}$) and also the L_∞ convergence of $P_n(t,t)$ to $P(t,t)$. Introducing subscripts in the obvious way, define for each n,

$$V_n(t) = [I + U_n(t)]^{-1} - I$$

so that the identity

$$V_n(t) = -U_n(t)[I + U_n(t)]^{-1}$$

holds. Multiplying by $G_n'(t)\tilde{Q}_n$ we obtain

$$\begin{aligned} G_n'(t)\tilde{Q}_n V_n(t) &= -G_n'(t)\tilde{Q}_n U_n(t)[I + U_n(t)]^{-1} \\ &= -G'_n(t)\tilde{Q}_n[I + U_n(t)]^{-1}U_n(t). \end{aligned}$$

Then using the definition of $U_n(t)$ and multiplying by $\bar{Y}_n(\alpha)$ it follows that

$$G_n'(t)\tilde{Q}_n V_n(t)\bar{Y}_n(\alpha) = \int_t^T G_n'(t)\tilde{Q}_n[I + U_n(t)]^{-1}G_n(\sigma)G_n'(\sigma)\tilde{Q}_n d\sigma \bar{Y}_n(\alpha) \tag{4.4}$$

From (4.2) and the definition of $V_n(t)$ we have for $\alpha \geq t$

$$P_n(t,\alpha) - G_n'(t)\tilde{Q}_n V_n(t)\bar{Y}_n(\alpha) = G_n'(t)\tilde{Q}_n\bar{Y}_n(\alpha).$$

And substituting (4.4) into the above

$$P_n(t,\alpha) = G_n'(t)\tilde{Q}_n\bar{Y}_n(\alpha) - \int_t^T G_n'(t)\tilde{Q}_n[I + U_n(t)]^{-1}G_n(\sigma)G_n'(\sigma)\tilde{Q}_n d\sigma \bar{Y}_n(\alpha) \tag{4.5}$$

Now let P_n denote the integral operator with kernel $P_n(t,\alpha)$. We then recognize $G_n'(t)\tilde{Q}_n[I + U_n(t)]^{-1}G_n(\sigma)$, with $\sigma \geq t$, as the kernel of the operator $[P_nB]_-$. Also $G_n'(t)\tilde{Q}_n\bar{Y}_n(\alpha)$ is recognized as the kernel of $F_n^{\#}Y$, where Y is the operator in B(H,X) defined

$$Yu: t \longrightarrow \int_0^t Y(t,s)u(s)ds. \tag{4.6}$$

Thus (4.5) represents the Wiener-Hopf equation

$$P_n = (F_n^{\#}Y)_- - [(P_nB)_-F_n^{\#}Y]_-.$$

Let P denote the operator with kernel $P(t,\alpha)$. Then since

$$\lim_n |P_n - P|_{HS} = 0 \text{ and } \lim_n |F_n^{\#}Y - F^{\#}Y|_{HS} = 0$$

(where the latter is essentially Lemma 3.3), by continuity of the projection p_- on the space of Hilbert-Schmidt maps we obtain

$$P = (F^{\#}Y)_- - [(PB)_-F^{\#}Y]_- \tag{4.7}$$

We formalize this discussion in the following.

Corollary 4.2. The Wiener-Hopf equation (4.7) has a unique Hilbert-Schmidt solution P. A version of the kernel of P is the optimal feedback kernel for the optimization problem $\mathcal{P}$ (3.1) - (3.2). Furthermore this version of the kernel can be approximated in the L_∞ topology on the diagonal as well as the square by the semiseparable kernels $P_n(t,\alpha)$.

Proof. We only need to prove the uniqueness assertion. Suppose there exist two Hilbert-Schmidt solutions of (4.7) and let δ denote their difference. Then we obtain

$$\delta + [(\delta B)_- F^\# Y]_- = 0$$

Clearly it is sufficient to show that $(\delta B)_- = 0$. Define

$$\tilde{\delta} = -[(\delta B)_- F^\# Y]$$

so that $\tilde{\delta}_- = \delta$. Then since B is causal it follows that $(\tilde{\delta} B)_- = (\delta B)_-$ and

$$\tilde{\delta} + (\tilde{\delta} B)_- F^\# Y = 0.$$

Multiplying by B and noting that $YB = F$ (cf (3.6) and (4.6))

$$(\tilde{\delta} B)_- + [(\tilde{\delta} B)_- F^\# F]_- = 0.$$

We will show that zero is the only solution to the equation

$$X + [XF^\# F]_- = 0, \tag{4.8}$$

thus proving $(\delta B)_- = 0$, and the result. Now (4.8) is equivalent to

$$[X(I + F^\# F)]_- = 0$$

with X anticausal. Since $F^\# F \geq 0$, there exists a causal Hilbert-Schmidt map V such that

$$I + F^\# F = (I + V^*)(I + V).$$

Thus X solves (4.8) if and only if Z solves

$$Z + [ZV]_- = 0. \tag{4.9}$$

(These solutions are related by $X = Z(I + V^*)^{-1}$.) Next consider the mapping $\mathscr{L}$ on the space of Hilbert-Schmidt operators

$$\mathscr{L}(Z) = [ZV]_-.$$

Then (4.9) is equivalent to $(I + \mathscr{L})Z = 0$. Now by induction we find that

$$\mathscr{L}^n(Z) = [ZV^n]_-,$$

so that

$$|\mathscr{L}^n(Z)| \leq |Z|_{HS}|V^n|.$$

But V is quasinilpotent, hence so is $\mathscr{L}$. Thus the only solution to (4.9) is $Z = 0$, and the theorem is proved.//

We note that the existence portion of the corollary was proved in a different manner in [24].

When B is a multiplication operator, $(PB)_- = PB$, so that (4.7) becomes

$$P = (F^{\#}Y)_- - [PBF^{\#}Y]_-.$$

The kernel of $F^{\#}Y$ is easily computed to be of the form $B'(t)A(t,s)$ where

$$A(t,s) = \int_{\max(t,s)}^{T} Y'(\sigma,t)Q(\sigma)Y(\sigma,s)d\sigma + Y'(T,t)Q(T)Y(T,s).$$

Let A denote the operator with kernel $A(t,s)$ and note that $F^{\#}Y = B^*A$. Next consider the following modification of the Wiener-Hopf equation (4.7),

$$\Pi = A_- - [\Pi BB^*A]_- \tag{4.10}$$

Using the same techniques as in the proof of Corollary 4.2 it is possible to show that (4.10) has a unique Hilbert-Schmidt solution Π_0. Also note that the corollary implies $B^*\Pi_0 = P$. The Wiener-Hopf equation (4.10) is equivalent to the parameterized family of Fredholm equations Manitius [20] derived via a maximum principle for obtaining the feedback kernels:

$$\Pi(t,s) = A(t,s) - \int_t^T \Pi(t,\theta)B(\theta)B'(\theta)A(\theta,s)d\theta \qquad (4.11)$$

Corollary 4.2 extends this Wiener-Hopf characterization of the feedback kernel to problems with control delays, and simultaneously provides approximate solutions.

The special factorization has been previously exploited in solving Wiener-Hopf equations on finite intervals of the type (4.10)-(4.11) that arise in inverse problems in the spectral theory of differential operators [8],[18], and in the filtering and smoothing problems for nonstationary processes [15], [16]. The corollary is in a sense a solution finding the right Wiener-Hopf problem.

Now we return to the original problem (3.1) - (3.3) and consider a specific sequence of measures for generating approximations to the optimal feedback kernel.

Let $\{\mu_n\}$ denote the sequence of measures

$$\int f d\mu_n = f(T) + (T + r)/n \sum_{i=0}^{n-1} f(i(T + r)/n - r). \qquad (4.12)$$

This sequence is easily shown to satisfy H1 and H2. Now suppose the majorizing function $m(t)$ (cf (2.3)) is bounded and the weighting function $Q(\cdot)$ has a bounded derivative. Letting $P_n(t,\alpha)$ denote the feedback kernel corresponding to the cost with measure μ_n, it is straightforward (although tedious) to derive a constant C from the results in Section 3 such that

$$\begin{aligned} &\operatorname*{ess\,sup}_{t,\alpha} \; |P(t,\alpha) - P_n(t,\alpha)| \le C/n, \\ &\operatorname*{ess\,sup}_{t} \; |(P(t,t) - P_n(t,t)| \le C/n. \end{aligned} \tag{4.13}$$

We can also use the approximations $P_n(t,\alpha)$ to obtain an a priori bound on $|P(t,\alpha)|$ in the following way.

First observe that for each t

$$\tilde{Q}_n(I + U_n(t))^{-1} \le \tilde{Q}_n, \quad n = 1,2,\ldots$$

Thus,

$$|P_n(t,\alpha)| \le |G_n'(t)||\tilde{Q}_n||\bar{Y}_n(\alpha)|, \quad n = 1,2,\ldots$$

where all the norms above are operator (matrix) norms on the appropriate Euclidean spaces. Now for each α, $|\bar{Y}_n(\alpha)|$ is bounded by $|\bar{Y}_n(\alpha)|_{HS}$. (Here $|\cdot|_{HS}$ denotes the Hilbert-Schmidt matrix norm which is the square root of the sum of the squares of the matrix entries.) And

$$|\bar{Y}_n(\alpha)|_{HS}^2 = |Y(T,\alpha)|_{HS}^2 + \frac{(T + r)}{n}\sum_{i=0}^{n-1} |Y(s_i,\alpha)|_{HS}^2$$

$$\le (T + r + 1)\sup_s |Y(s,\alpha)|_{HS}^2.$$

A similar bound holds for $|G_n'(t)|$,

$$|G_n'(t)|^2 \le (T + r + 1)\sup_s |F'(s,t)|_{HS}^2.$$

Thus,

$$|P_n(t,\alpha)| \le (T + r + 1) \sup_s |Q(s)| \cdot \sup_s |F'(s,t)|_{HS} \cdot \sup_s |Y(s,\alpha)|_{HS}$$

independently of n. And since $P_n(t,\alpha) \longrightarrow P(t,\alpha)$ a.e. (on the square and diagonal), the same bound holds for $|P(t,\alpha)|$.

The T- dependence (the length of the problem interval) in the bound for $|P(t,\alpha)|$ can be expressed differently by noting that

$$\lim_{n} |\overline{Y}_n(\alpha)|^2_{HS} = |Y(T,\alpha)|^2_{HS} + \int_{\alpha}^{T} |Y(s,\alpha)|^2_{HS} ds$$

And after using the analogous bound on $|G'_n(t)|$ we obtain

$$|P(t,\alpha)| \le \sup_{S} |Q(s)| \cdot \{|F(T,t)|^2_{HS} + \int_{t}^{T} |F'(s,t)|^2_{HS} ds\}^{1/2}$$

$$\cdot \{|Y(T,\alpha)|^2_{HS} + \int_{\alpha}^{T} |Y(s,\alpha)|^2_{HS} ds\}^{1/2}.$$

When the system (3.1) - (3.4) is time-invariant and stable, a bound independent of T can be established for $|P(t,\alpha)|$. To see this we take $Q(\cdot) = Q$ (a constant matrix) and $\eta(t,\theta) = \eta(\theta)$ where all the roots of the characteristic equation

$$\Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta)$$

are assumed to have negative real part. Also for simplicity we define the operator B as

$$(Bu)(t) = B_0 u(t) + \int_{t-r}^{t} B(t-\theta)u(\theta)d\theta$$

with $|B_0|$, $\sup_t |B(t)| < b$. From the assumptions on $\eta(\cdot)$ it can be shown [12] that $Y(t,\alpha) = Y(t-\alpha)$ with $|Y(t)| \leq c\exp(-\mu t)$ for some $c, \mu \geq 0$. Thus,

$$\int_\alpha^T |Y(s,\alpha)|^2_{HS} ds \leq \int_0^\infty |Y(s)|^2_{HS} ds < \infty .$$

Since the Laplace transform $\hat{Y}(\lambda)$ of $Y(t)$ satisfies $Y(\lambda) = \Delta^{-1}(\lambda)$, (see [12]), the inequality above together with Parseval's formula yields

$$\int_\alpha^T |Y(s,\alpha)|^2_{HS} \leq 2\pi \int_{-\infty}^{\infty} |\Delta^{-1}(i\lambda)|^2_{HS} d\lambda.$$

Next note that

$$|F'(s,t)|_{HS} \leq b|Y(s-t)|_{HS} + \int_t^{\min(s,t+r)} |Y(s-\theta)|_{HS} |B(\theta-t)|_{HS} d\theta$$

$$\leq b|Y(s-t)|_{HS} + G(s,t)$$

where

$$G(s,t) = \begin{cases} b\sqrt{r}\left[\int_0^{s-t} |Y(u)|^2_{HS} du\right]^{1/2}, & t \leq s \leq t+r \\ b\sqrt{r}\left[\int_t^{t+r} |Y(s-\theta)|^2_{HS} d\theta\right]^{1/2}, & s > t+r \end{cases}$$

Straightforward approximations then yield

$$\int_t^T |F'(s,t)|^2_{HS} ds \leq 2\pi b^2 \{1 + \sqrt{2}r\}^2 \int_{-\infty}^{\infty} |\Delta^{-1}(i\lambda)|^2_{HS} d\lambda.$$

Thus, independent of the length of the problem interval,

$$|P(t,\alpha)| \leq |Q| \; 2\pi b(1+\sqrt{2}r) \int_{-\infty}^{\infty} |\Delta^{-1}(i\lambda)|_{HS}^{2} d\lambda.$$

5. ALGORITHMS. The results that have been presented thus far have emphasized the connections between factorization and the feedback kernels for hereditary systems with control delays, and have not focused on any of the implementation issues concerning the approximation scheme that has been defined. In this Section we will examine in greater detail algorithms based on Theorem 4.1. Particular attention will be paid to time-invariant systems. First, a quick remark about these algorithms in general.

We note that it has already been observed that combining (4.2) with the discretizations (4.12) results in (most cases) an 0(1/n) L_∞ convergence of the feedback kernels[2]. This result, which is also valid for time varying systems with control delays, is sharper than the approximations for the feedback operators obtained in [10], [19]. Neither of these articles establishes a priori rates of convergence, nor do the convergence results that are established translate into L_∞ convergence of the kernels. The underlying reason why we are able to obtain the stronger convergence is that we do not approximate the entire semigroup (or evolution operator), but only that piece that is contributed by the fundamental matrix (which we must solve for as a separate computation). In the general time-varying case with discrete state cost at the nodes $\{s_n\}_{i=1}^{n}$, this amounts to solving the n+1 Volterra equations

$$Y(s_i,\sigma) = I - \int_{\sigma}^{s_i} Y(s_i,u)\eta(u,\sigma-u)\,du \tag{5.1}$$

In the time-invariant case these computations reduce to the single Volterra equation

$$Y(t) = I - \int_{0}^{t} Y(u)\eta(u-t)du. \tag{5.2}$$

Once we have the solutions (5.1) or (5.2), the feedback structure (4.2) is straightforward and can be computed from quadrature, matrix inversion and multiplication.

Of course we are not constrained to directly solving (5.1) or (5.2), and we can use other methods for obtaining the fundamental solution - e.g. state approximation methods [2], [3], the method of steps [26], [27] or other available methods for solving Volterra functional differential equations [28], [29], [30]. Having said this, we assume throughout this section that the functions Y(t,s) and F(t,s) have been computed. (A couple of issues associated with these computations will be addressed later.)

One straightforward implementation of (4.2) consists in defining the grid $\{s_i\}_{i=0}^{n}$ so that $s_i - s_{i-1} = \Delta = T/n$, taking $a_i = \Delta$, and replacing the integral in (4.3) by a first order Euler quadrature with nodes $\{s_i\}$. To see where this leads us, first write $P(t,\alpha)$ in the more symmetrical fashion

$$P(t,\alpha) = G'(t)\tilde{Q}^{1/2}[I + \underset{\sim}{U}(t)]^{-1}\tilde{Q}^{1/2}\bar{Y}(\alpha),$$

where

$$\underset{\sim}{U}(t) = \tilde{Q}^{1/2} \int_t^T G(s)\, G'(s)ds\tilde{Q}^{1/2}$$

Let $\hat{U}(s_i)$ denote the approximation to $\underset{\sim}{U}(s_i)$,

$$\hat{U}(s_i) = \sum_{j \geq i} \Delta\tilde{Q}^{1/2}G(s_j)G'(s_j)\tilde{Q}^{1/2},$$

and form the approximations $\{\hat{P}(s_i,s_j)\}_{j \geq i}$ to $\{P(s_i,s_j)\}_{j \geq i}$,

$$\hat{P}(s_i,s_j) = G'(s_i)\tilde{Q}^{1/2}[I + \hat{U}(s_i)]^{-1}\tilde{Q}^{1/2}\bar{Y}(s_j) \tag{5.3}$$

Now note that

$$[I + \hat{U}(s_{i-1})]^{-1} = [I + \hat{U}(s_i) + \Delta\, \tilde{Q}^{1/2}G(s_i)G'(s_i)\tilde{Q}^{1/2}]^{-1},$$

and that

$$\text{Rank}\ (\Delta\tilde{Q}^{1/2}G(s_i)G'(s_i)\tilde{Q}^{1/2}) \leq M.$$

(Recall that M = dimension of the input space.) Thus, using the matrix identity,

$$(X + YY')^{-1} = X^{-1} - X^{-1}Y\,(Y'X^{-1}Y + I)^{-1}Y'X^{-1}$$

for compatible matrices X and Y, it follows that $(I + \hat{U}(s_{i-1}))^{-1}$ can be updated from $(I + \hat{U}(s_i))^{-1}$ in about $2MN^2n^2$ operations. Further, exploiting the semiseparable structure of $P(\cdot,\cdot)$, a rough operation count shows that $\{\hat{P}(s_i,s_j)\}_{j \geq i}$ can be computed in approximately $3MN^2n^3$ operations when $M \leq N << n$.

Considering that there are more than $MNn^2/2$ values in the matrices $\{\hat{P}(s_i,s_j)\}_{j \geq i}$, this algorithm is fairly efficient. However, we will subsequently show that it is possible to do substantially better in the time-invariant case. (We will also provide a more complete analysis in this case.)

For the remainder of this section we consider the problem defined by the dynamics

$$\dot{x}(t) = \int_{-r}^{0} d\eta(\Theta)\, x(t+\Theta) + \sum_{i=0}^{k} B_i\, u(t-r_i) + \int_{t-r}^{t} B(t-\Theta)\, u(\Theta)d\Theta, \quad t \geq 0 \tag{5.4}$$

$$x(t) = \phi(t), \qquad t \in [-r,0]$$

$$u(t) = 0, \qquad t \in [-r,0]$$

and cost

$$J(u,x) = \int_{0}^{T} \langle x(s), Qx(s)\rangle + |u(s)|^2\, ds \tag{5.5}$$

The assumptions here are the same as in (3.1) - (3.4), except now everything is time-invariant. (Previously the lower limit in the integral defining the cost J was taken as -r. This served as a notational expediency in the preceding sections, which we dispense with in the present section.)

We will explicitly consider the discretizations of J,

$$J_n = \sum_{i=0}^{n-1} \Delta \langle x(s_i), Q\, x(s_i)\rangle + \int_0^T |u(s)|^2\, ds, \tag{5.6}$$

where $\{s_i\}_{i=0}^n$ is a regular partition of the interval $[0,T]$ with mesh $\Delta = s_{i+1} - s_i = T/n$. (Although the mesh points of the partition change with n, we will not double subscript the s_i. This will not lead to any confusion in the sequel.) We will also assume that the point delays in the control, r_i, $i=1,\dots k$, correspond to some subset of the $\{s_i\}$, $i = 0,\dots n$.

Again we let $P_n(\cdot,\cdot)$ denote the optimal feedback kernel for cost J_n and let $P(\cdot,\cdot)$ denote the optimal feedback kernel for cost J. In the time-invariant case Var $|\eta(\cdot)|$ and $Q(\cdot)$ are constant, so that (4.13) holds for the sequence $\{P_n(\cdot,\cdot)\}$. The particular algorithm which we will be developing is based on approximately $P_n(\cdot,\cdot)$ at the mesh points $\{s_i\}$; thus it is first necessary to prove that (4.13) actually holds everywhere. Before showing this we need some notations and a couple of simple observations.

For any matrix M, as before $|M|_{HS}$ denotes the Hilbert-Schmidt norm of M (the square root of the sum of the squares of its entries), and for specificity we write $|M|_2$ for the operator norm of M with respect to the corresponding Euclidean metrics. Note that $|M|_{HS} \geq |M|_2$.

The fundamental matrix solution $Y(t,\alpha)$ to (5.4) (cf (2.7)) has the form $Y(t,\alpha) = Y(t-\alpha)$. We set

$$\gamma_Y = \sup_{t\epsilon[0,t]} |Y(t)|_{HS}, \tag{5.7}$$

and using [12, p. 149] we note that

$$|Y(t) - Y(s)|_{HS} = O(|t-s|), \quad t,s \; \epsilon \; [0,T]\ . \tag{5.8}$$

Similarly $F(t,s)$ (cf (3.6)) is a difference kernel, $F(t,s) = F(t-s)$, and using (5.7) and (5.8) we have

$$\sup_{t\epsilon[0,t]} |F(t)|_{HS} = \gamma_F < \infty, \tag{5.9}$$

and

$$|F(t) - F(s)|_{HS} = O(|t-s|), \text{ when } t, s \epsilon (r_i, r_{i+1}) \text{ for some } i,\ 0 \leq i \leq k-1 \tag{5.10}$$

Lemma 5.1. Let $P(t,\alpha)$ denote the optimal feedback kernel for the problem (5.4) - (5.5). Then

$$|P(t,\alpha) - P(t',\alpha')| = O(|t-t'| + |\alpha-\alpha'|).$$

Proof. From Theorem 2.5 we have

$$P(t,\alpha) = \int_{\alpha}^{T} K(t,s)Y(s-\alpha)\, ds,$$

where

$$K(t,s) = F'(s-t)\, Q + \int_{t}^{s} W_-(t,\sigma)F'(s-\sigma)Q d\sigma$$

Since $\sup_{t,s} |K(t,s)| < \infty$ (cf Proposition 3.4 and (5.9)), it follows from (5.8) that

$$|P(t,\alpha) - P(t,\alpha')| < K_1 |\alpha-\alpha'|$$

for some constant K_1 independent of t, α, α'. Thus using (5.7) it remains to show the existence of a constant K_2 such that

$$\int |K(t_2,s) - K(t_1,s)|\, ds \le K_2\, |t_2-t_1|.$$

Now F(t) has only a finite number of jump discontinuities, so (5.9) and (5.10) imply

$$\int |F(s-t_2) - F(s-t_1)| ds = 0\, (|t_2-t_1|).$$

Hence, it is only necessary to verify that

$$\int |W_-(t_2,\sigma) - W_-(t_1,\sigma)|\, d\sigma = 0\, (|t_2-t_1|) \qquad (5.11)$$

To this end let W* denote the operator with kernel $W_-(t,\sigma)$ and let X* = $(I+W^*)^{-1}$-I. Denote the kernel of X* by $X_-(t,s)$. Also let W and X denote the adjoints of W* and X* respectively, with respective kernels $W_+(t,s)$ and $X_+(t,s)$. Now the factorization (recall Theorem 2.5)

$$I + F^{\#}F = (I+X^*)\,(I+X)$$

implies

$$X^* = [F^{\#}F]_- + [F^{\#}FW]_-, \tag{5.12}$$

where $F^{\#}F$ has kernel $H(t,s)$,

$$H(t,s) = \int_{\max(t,s)}^{T} F'(\sigma-t)\, QF(\sigma-s)d\sigma.$$

Note that

$$|H(t_2,s) - H(t_1,s)| = 0\,(|t_2-t_1| + |s_2-s_1|) \tag{5.13}$$

Now (5.12) is equivalent to

$$X_-(t,s) = H(t,s) + \int_s^T H(t,\Theta)\, W_+(\Theta,s)d\Theta, \text{ a.e. } s,t.$$

Because $H(t,s)$ is continuous, $X_{\pm}(t,s)$, $W_{\pm}(t,s)$ are also continuous [11], and the equation above holds pointwise. Thus (5.13) and the triangle inequality imply

$$|X_-(t_2,s) - X_-(t_1,s)| = 0\,(|t_2-t_1|)$$

independently of s. Then (5.11) follows from the estimate above and the resolvent identity,

$$W_-(t,s) + X_-(t,s) + \int_t^s X_-(t,\Theta)\, W_-(\Theta,s)d\Theta = 0. //$$

Proposition 5.2. $\sup_{t,\alpha\,\epsilon(0,T]} |P_n(t,\alpha) - P(t,\alpha)| = 0\,(1/n)$

Proof. Using the identity

$$A\,[I+BA]^{-1} = A^{1/2}\,[I+A^{1/2}BA^{1/2}]^{-1}A^{1/2}$$

for $A,B \geq 0$, write the feedback kernel (from Theorem 4.1) as,

$$P_n(t,\alpha) = \underset{\sim}{F}_n'(t)\,[I+\underset{\sim}{U}_n(t)]^{-1}\underset{\sim}{Y}_n(\alpha) \tag{5.14}$$

where

$$\underset{\sim}{F}(t) = \Delta^{1/2} [F'(s_o - t) Q^{1/2} \ldots F'(T-t) Q^{1/2}]' \tag{5.15}$$

$$\underset{\sim}{U}_n(t) = \int_t^T \underset{\sim}{F}_n(\sigma) \underset{\sim}{F}_n'(\sigma)\, d\sigma, \tag{5.16}$$

and

$$\underset{\sim}{Y}_n(\alpha) = \Delta^{1/2} \begin{bmatrix} Q^{1/2} Y(s_o - \alpha) \\ \vdots \\ Q^{1/2} Y(T-\alpha) \end{bmatrix} \tag{5.17}$$

Note the following bounds independent of n (recall (5.7) and (5.9)):

$$\sup_\alpha |\underset{\sim}{Y}_n(\alpha)|^2_{HS} \le T\, |Q^{1/2}|^2_{HS}\, \gamma^2_Y \tag{5.18}$$

$$\sup_t |\underset{\sim}{F}_n(t)|^2_{HS} \le T\, |Q^{1/2}|^2_{HS} \gamma^2_F \tag{5.19}$$

$$\sup_t |(I + \underset{\sim}{U}_n(t))^{-1}|_2 \le 1 \tag{5.20}$$

Now fix $(t_o, \alpha_o) \in (0,T] \times (0,T]$ with $t_o \le \alpha_o$. Then there exist indices i, j such that $(t_o, \alpha_o) \in (s_{i-1}, s_i] \times (s_{j-1}, s_j]$, $i \le j$. From (5.8) we obtain

$$|\underset{\sim}{Y}_n(s_j) - \underset{\sim}{Y}_n(\alpha_o)|_{HS} \le [\sum_{k \ge j}^{n} \Delta\, |Y(s_k - s_j) - Y(s_k - \alpha_o)|^2_{HS}]^{1/2}\, |Q^{1/2}|$$

$$= [\sum_{k \ge j}^{n} \Delta\, |Y(s_k - \alpha_o + \alpha_o - s_j) - Y(s_k - \alpha_o)\,|^2_{HS}]^{1/2}\, |Q^{1/2}|$$

$$= 0\,(1/n). \tag{5.21}$$

Similarly, expressing $F(\cdot)$ as a sum (as in (3.6)) and using (5.10) we obtain

$$|\underset{\sim}{F}_n(s_i) - \underset{\sim}{F}_n(t_o)|_{HS} = 0\,(1/n) \tag{5.22}$$

Next observe that

$$|(I+\underset{\sim}{U}_n(t_o))^{-1} - (I+\underset{\sim}{U}_n(s_i))^{-1}|_2 \le |(I+\underset{\sim}{U}_n(t_o))^{-1}|_2\, |\underset{\sim}{U}_n(t_o) - \underset{\sim}{U}_n(s_i)|_{HS}\, |(I+\underset{\sim}{U}_n(s_i))^{-1}|_2$$

$$\le |\underset{\sim}{U}_n(t_o) - \underset{\sim}{U}(s_i)|_{HS} \tag{5.23}$$

But from (5.16) and (5.19) it follows that

$$|\underset{\sim}{U}_n(t_o) - \underset{\sim}{U}_n(s_i)|_{HS} = 0\,(1/n) \tag{5.24}$$

Putting (5.21) - (5.24) together with the bounds (5.18) - (5.20), and using the triangle inequality we get

$$|P_n(t_o,\alpha_o) - P_n(s_i, s_j)|_2 = 0\,(1/n) \tag{5.25}$$

Then using the fact that (4.13) holds for a. e. t, α, and that the estimate (5.25) holds for any $(t_o, \alpha_o) \varepsilon (s_{i-1}, s_i] \times (s_{j-1}, s_j]$, Lemma 5.1 and the triangle inequality imply

$$|P_n(t,\alpha) - P(t,\alpha)|_2 = 0\,(1/n)$$

for all $t, \alpha \varepsilon (0,T] \times (0,T]$. //

Since the idea behind the algorithm defined in Theorems 5.3 and 5.4 is based on a relatively simple observation that gets somewhat obscured in the notation and proofs of the theorems, it is worthwhile here to briefly remark on this motivating idea.

If we return to the approximate gain defined in (5.3), it turns out that in the time-invariant case each $\hat{U}(s_i)$ is a principal minor of $\hat{U}(0)$. Thus what we would like to do is invert $I + \hat{U}(0)$ via a recursion in which all of the principal minors of $I + \hat{U}(0)$ are also inverted. Now by inspection $\hat{U}(0)$ can be identified with the covariance matrix of the random process $\{\Delta^{1/2}\xi_i\}$,

$$\xi_i = \Delta \sum_{j=o} Q^{1/2} F((i-j)\Delta)\omega_j,$$

where $E(\omega_i \omega_j') = \delta_{ij} I$. In the signal processing literature (see for example [31]) it is shown that processes of this type admit "fast" filter implementations due to their "near" Toeplitz covariance matrix structure. This is precisely the property we exploit.

Theorem 5.3. Define the symmetric $N(n+1) \times N(n+1)$ matrix $\hat{U}^n$ with $N \times N$ block entries $\hat{U}^n_{ij}$ where (for $0 \le j \le i \le n$)

$$\hat{U}^n_{ij} = \sum_{\sigma=0}^{j} \Delta^2 Q^{1/2} F((i-j)\Delta + \sigma\Delta) F'(\sigma\Delta) Q^{1/2}$$

Let Z_n denote the matrix on $R^{N(n+1)}$,

$$Z_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & & 0 \\ \vdots & & \ddots & \\ 0 & & 1 & 0 \end{bmatrix},$$

and for each $p = 0,1,\ldots n$, let $\Pi_{n,p}$ denote the projection

$$\Pi_{n,p}(x_1,\ldots x_{N(n+1)})' = (x_1 \ldots x_{N(p+1)}, 0,\ldots 0)'$$

For $p, q = 0, 1,\ldots n$ define (recall $\Delta = T/n$)

$$\hat{P}_n(\Delta p, \Delta q) = \hat{X}'_{n,n-p} Z_n^{N(n+1-p)} \underset{\sim}{Y}_n(\Delta q)$$

where $\underset{\sim}{Y}_n$ is defined in (5.17) and

$$\hat{X}_{n,p} = [I + \Pi_{n,p} \hat{U}^n \Pi_{n,p}]^{-1} \begin{bmatrix} \Delta^{1/2} Q^{1/2} F(0) \\ \vdots \\ \Delta^{1/2} Q^{1/2} F(\Delta p) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then

$$\max_{\substack{p \le q \\ p,q \in \{0,1,\dots n\}}} |\hat{P}_n(\Delta p, \Delta q) - P(\Delta p, \Delta q)|_2 = O(1/n)$$

Proof. Recall the representation (5.14) for the optimal feedback kernel for cost J_n,

$$P_n(t,\alpha) = \underset{\sim}{F}_n'(t)\,[I + \underset{\sim}{U}_n(t)]^{-1}\underset{\sim}{Y}_n(\alpha).$$

Thus we can write

$$P_n(\Delta p, \Delta q) = X'(\Delta p)\,\underset{\sim}{Y}_n(\Delta q) \tag{5.26}$$

where

$$X(\Delta p) = [I + \underset{\sim}{U}_n(\Delta p)]^{-1} \begin{bmatrix} 0 \\ \vdots \\ \Delta^{1/2} Q^{1/2} F(0) \\ \vdots \\ \Delta^{1/2} Q^{1/2} F((n-p)\Delta) \end{bmatrix} \tag{5.27}$$

Note that $\underset{\sim}{U}_n(\Delta p)$ is an $N(n+1) \times N(n+1)$ symmetric matrix with NxN block entires $\underset{\sim}{U}^n_{i,j}(\Delta p)$, $i,j = 0, 1, \dots n$ where (for $0 \le j \le i$)

$$\underset{\sim}{U}^n_{i,j}(\Delta p) = \begin{cases} \Delta Q^{1/2} \int_0^{(j-p)\Delta} F((i-j)\Delta + \sigma)\, F'(\sigma)\, d\sigma\, Q^{1/2}, & j \ge p \\ 0 & \text{otherwise} \end{cases} \tag{5.28}$$

For each $p = 0,1,\dots n$, let

$$X_{n,p} = [I + \Pi_{n,p}\,\underset{\sim}{U}_n(0)\,\Pi_{n,p}]^{-1} \begin{bmatrix} \Delta^{1/2} Q^{1/2} F(0) \\ \vdots \\ \Delta^{1/2} Q^{1/2} F(\Delta p) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5.29}$$

Observing that

$$\underset{\sim}{U}^n_{i+n-p,\,j+n-p}(\Delta(n-p)) = \underset{\sim}{U}_{i,j}(0);\ i,j = 0, 1, \dots p,$$

it follows from (5.27) and (5.29) that

$$X(\Delta(n-p)) = Z_n^{(n-p)N} X_{n,p}$$

Therefore (using (5.26)),

$$P_n(\Delta p,\Delta q) = X'_{n,n-p} Z_n^{N(n+1-p)} \underset{\sim}{Y}_n(\Delta q).$$

Now,

$$|P_n(\Delta p,\Delta q) - P_n(\Delta p,\Delta q)|_2 \le T^{1/2}|Q^{1/2}|_2 \gamma_Y |X_{n,n-p} - \hat{X}_{n,n-p}|_2$$

(Here we have used (5.18).) But using (5.29) and the definitions of X and $\hat{X}$,

$$|X_{n,n-p} - \hat{X}_{n,n-p}|_2 \le T^{1/2}|Q^{1/2}|\gamma_F|\hat{U}^n - U_n(0))|_{HS}.$$

And,

$$|U^n - U_n(0)|_{HS} = [\sum_{i,j=0} |\hat{U}^n_{ij} - \underset{\sim}{U}^n_{ij}(0)|^2_{HS}]^{1/2} \qquad (5.30)$$

It follows from (5.26) that

$$|\hat{U}^n_{ij} - \underset{\sim}{U}^n_{ij}(0)|_{HS} = 0(\Delta^2)$$

independent of i,j. Thus the right side of (5.30) is $0(\Delta)$. Hence,

$$\max_{p,q} |P_n(\Delta p,\Delta q) - \hat{P}_n(\Delta p,\Delta q)|_2 = 0(\Delta).$$

The result follows from the estimate above and Proposition 5.2.//

Using the extended LWR algorithm [31] we will next show that $\{\hat{X}_{n,p}\}^n_{p=0}$ can be computed in a total of $0(n^2)$ operations. The theorem above represents the approximate gain in the form

$$\hat{P}_n(\Delta p,\Delta q) = X'_{n,n-p} Z_n^{N(n+1-p)} \underset{\sim}{Y}_n(\Delta q).$$

Now fixing p and letting q vary, these products can be viewed as a convolution. Since convolutions have fast implementations in 0(nlogn) operations, it will follow then that $\{\hat{P}(\Delta p,\Delta q)\}_{q \ge p}$ can be computed in $0(n^2\log n)$ operations.

Theorem 5.4. $\{\hat{X}_{n,p}\}_{p=0}^{n}$ can be computed in $0(n^2)$ operations.

Proof. By definition $\hat{X}_{n,p}$ satisfies

$$[I + \Pi_{n,p} \hat{U}^n \Pi_{n,p}]\hat{X}_{n,p} = \Pi_{n,p} \underset{\sim}{F}_n(0) \tag{5.31}$$

We will exhibit a fast recursion for solving (5.31).

A simple calculation shows that

$$\hat{U}^n_{i+1,j+1} - \hat{U}^n_{i,j} = \Delta^2 Q^{1/2} F((i+1)\Delta)F'((j+1)\Delta)Q^{1/2}$$

Therefore the matrix $\delta(\hat{U}^n)$ with block entries $\hat{U}^n_{i+1,\, j+1} - \hat{U}^n_{i,j}$, $i,j = 0,1,\ldots n-1$, can be written

$$\delta(\hat{U}^n) = \Delta^2 \begin{bmatrix} Q^{1/2}F(\Delta) \\ \vdots \\ Q^{1/2}F(n\Delta) \end{bmatrix} [F'(\Delta)Q^{1/2} \;\ldots\; F'(n\Delta)Q^{1/2}]$$

Consequently, rank $(\delta(\hat{U}^n)) \leq N << n$. Furthermore (and importantly [31]) $\delta(\hat{U}^n)$ has a factorization of the form

$$\delta(\hat{U}^n) = D\, \Sigma\, D'$$

with D an nNxN matrix and Σ an NxN signature matrix. In fact, this factorization is easily done by inspection:

$$D = \Delta \left[\begin{array}{c|c} Q^{1/2}F(\Delta) & \overbrace{\quad 0 \quad}^{N-M} \\ \vdots & \\ Q^{1/2}F(n\Delta) & \end{array}\right], \qquad \Sigma = I.$$

Therefore the LWR algorithm [31, p.655] can be used to recursively compute the solutions M_p to

$$[I + \Pi_{n,p} \hat{U}^n \Pi_{n,p}] M_p = \begin{bmatrix} \hat{U}^n_{0,p+1} \\ \vdots \\ \hat{U}^n_{p,p+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.32}$$

$p = 0, 1,\ldots n$ in a total of $0\,(N^3n^2)$ operations. With these solutions in hand we can argue as in the scalar Toeplitz case (see for example [32]) to recursively solve the system (5.31). These details are supplied below.

So now consider solving the problem

$$[I + \Pi_{n,p} \hat{U}^n \Pi_{n,p}] \hat{X}_{n,p} = \Pi_{n,p} \underset{\sim}{F}_n(0), \tag{5.33}$$

given $\hat{X}_{n,p-1}$ and M_{p-1} (from (5.32)).

Let F_p denote the $N(p+1) \times M$ matrix composed of the first $N(p+1)$ rows of $\underset{\sim}{F}_n(0)$, and let f_p denote the NxM matrix composed of the $Np+1$ through $N(p+1)$ rows of $\underset{\sim}{F}_n(0)$. Thus we can write

$$F_{p+1} = \begin{bmatrix} F_p \\ f_p \end{bmatrix}$$

Let R_p denote the $N(p+1) \times N(p+1)$ matrix composed of the northwest corner of $I + \hat{U}^n$, and let r_p denote the NxN southeast corner of R_p. Also let $\underset{\sim}{M}_p \in R^{N(p+1)xN}$ denote the matrix consisting of the first $N(p+1)$ rows of M_p, and similarly let N_p denote the $N(p+1)xM$ matrix consisting of the first $N(p+1)$ rows of $\hat{X}_{n,p}$.

Now define

$$u_{p+1} = \begin{bmatrix} \hat{U}^n_{o,p+1} \\ \vdots \\ \hat{U}^n_{p,p+1} \end{bmatrix},$$

and consider the equation

$$\begin{bmatrix} R_p & u_{p+1} \\ u'_{p+1} & r_{p+1} \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} = \begin{bmatrix} F_p \\ f_p \end{bmatrix} \tag{5.34}$$

Note that

$$\begin{bmatrix} \mu \\ \nu \end{bmatrix} = N_{p+1}. \tag{5.35}$$

From the top of the equations in (5.34) we obtain

$$\mu = R_p^{-1} F_p - R_p^{-1} u_{p+1} \nu = N_p - \underset{\sim}{M}_p \nu. \tag{5.36}$$

Substituting this into the bottom equation, we get

$$[r_{p+1} - u'_{p+1} \underset{\sim}{M}_p] \nu = f_p - u_{p+1} N_p$$

Noting that

$$0 < \begin{bmatrix} I & 0 \\ -\underset{\sim}{M}'_p & I_{NxN} \end{bmatrix} \begin{bmatrix} R_p & u_{p+1} \\ u'_{p+1} & r_{p+1} \end{bmatrix} \begin{bmatrix} I & -\underset{\sim}{M}_p \\ 0 & I_{NxN} \end{bmatrix}$$

$$= \begin{bmatrix} R_p & 0 \\ 0 & r_{p+1} - \underset{\sim}{M}_p' u_{p+1} \end{bmatrix}$$

(I_{NxN} = NxN identity matrix), it follows

$$v = [r_{p+1} - u_{p+1}' \underset{\sim}{M}_p]^{-1} \{f_p - u_{p+1}' N_p\} \tag{5.37}$$

Thus, given M_p and N_p, N_{p+1} can be computed from (5.36) and (5.37) in $0(N^2 p)$ operations (p >> N). Hence, in particular $\{\hat{X}_{n,p}\}_{p=0}^{n}$ in (5.31) can be computed in a recursive manner in a total of $0(n^2)$ operations.//

In [33] infinite dimensional Chandrasekhar equations are derived for time-invariant hereditary systems without control delay. A fast algorithm based on approximating the Chandrasekhar equations is obtained and is shown to possess the same convergence properties as reported in [10]. Although there are no direct connections between the algorithm we developed here and the one in [33], there are some general connections between Chandrasekhar equations and the inversion of near Toeplitz systems [31].

We note that stability of the algorithms of this section with respect to the data Y(t), F(t) is easily demonstrated. For suppose $Y(\cdot)$ and $F(\cdot)$ were replaced by $Y_\varepsilon(\cdot)$ and $F_\varepsilon(\cdot)$ with

$$\sup_{t \in [0,T]} |Y(t) - Y_\varepsilon(t)|, |F(t) - F_\varepsilon(t)| < \varepsilon.$$

Using the obvious notation, we obtain the corresponding error estimates,

$$\sup_t |\underset{\sim}{F}_n - \underset{\sim}{F}_n^\varepsilon|, \sup_t |\underset{\sim}{Y}_n - \underset{\sim}{Y}_n^\varepsilon| = 0(\varepsilon),$$

and

$$\max |\hat{U}_{ij}^n - \hat{U}_{ij}^{n,\varepsilon}|_{HS} = 0(\varepsilon)\Delta,$$

all independent of n. Substituting these errors into the appropriate places in Theorem 5.3, it follows that an $O(\varepsilon)$ perturbation in the data $Y(t)$, $F(t)$ yields an $O(\varepsilon)$ perturbation in the estimate of the feedback kernel (all of these estimates in the sup norm).

Another feature to note is that although the implementation defined by Theorem 5.3 and Theorem 5.4 is recursive backward in time for the computation of the feedback kernel (just as one would suspect - e.g. the Riccati equation is solved backward in time), in terms of the algorithm's utilization of the fundamental matrix $Y(t)$, it is actually forward in time. Thus any approximating scheme for the computation of $Y(t)$ can be readily incorporated into the algorithm. This remark is also true for the implementation of Theorem 4.1 introduced in the beginning of this section.

One final remark concerning the algorithm is that in the event that the point delays $\{r_i\}_{i=1}^{k}$ do not correspond to a subset of the nodes $\{s_i\}_{i=0}^{n}$, the estimate in (5.22) is only $O(1/\sqrt{n})$. Thus the estimate in Theorem 5.3 would also be $O(1/\sqrt{n})$. Of course Theorem 4.1 has the flexibility to place nodes anywhere, and it may be possible to recover the stronger convergence by considering algorithms arising from different discretization strategies.

We conclude this section with the following simple scalar example:

$$\min J(u,x) = \int_0^2 |x(t)|^2 + |u(t)|^2 dt$$

subject to the constraint

$$\dot{x}(t) = x(t) + x(t-1) + u(t)$$

Since we are seeking the optimal feedback kernel $P(t,\alpha)$, it is not necessary to prescribe an initial condition above.

The algorithm described in the beginning of the section based on the approximation (5.3) and the recursive inversion of $I + \hat{U}(s_i)$ via the "matrix inversion lemma", was programmed using a few lines of Fortran code. Because the fundamental solution Y(t) to the differential equation is easily derived on the interval [0,2], the exact solution was used in the algorithm. (Recall that an $O(\varepsilon)$ error in the approximation of $Y(\cdot)$ results in an $O(\varepsilon)$ error in $P(t,\alpha)$.)

Discretizations with mesh width Δ = .025, .01, .005, were considered. The results for this problem coincided fairly well with the theory. We observed essentially linear (uniform) convergence of the feedback kernels as predicted. Tables 5.1 - 5.3 contain these results. Table 5.4 contains values of the kernel obtained via the Riccati equation approach using linear spline approximations of the history space. The gain computed using these approximations appeared to have converged to two (and in some instances three) significant figures. The author is indebted to Professor J. S. Gibson for providing these values for comparison.

Table 5.1
Δ = .025

	t			
P(t,α)	0.0	0.5	1.0	1.5
α t+0.0	2.7881	2.4237	1.7011	0.7462
t+0.1	2.3699	2.0632	1.4473	0.5799
t+0.2	2.0210	1.7566	1.2222	0.4252
t+0.3	1.7237	1.4972	1.0213	0.2789
t+0.4	1.4712	1.2790	0.8406	0.1381
t+0.5	1.2576	1.0972	0.6765	0.0000
t+0.6	1.0780	0.9411	0.5258	0.0000
t+0.7	0.9280	0.8037	0.3855	0.0000
t+0.8	0.8040	0.6822	0.2528	0.0000
t+0.9	0.7029	0.5472	0.1252	0.0000
t+1.0	0.6221	0.4775	0.0000	0.0000

Table 5.2
Δ = .01

	t			
P(t,α)	0.0	0.5	1.0	1.5
α t+0.0	2.7300	2.3951	1.6943	0.7521
t+0.1	2.3279	2.0378	1.4110	0.5845
t+0.2	1.9844	1.7341	1.2165	0.4285
t+0.3	1.6919	1.4770	1.0162	0.2810
t+0.4	1.4335	1.2609	0.8361	0.1391
t+0.5	1.2335	1.0808	0.6727	0.0000
t+0.6	1.0570	0.9270	9.5228	0.0000
t+0.7	0.9096	0.7916	0.3832	0.0000
t+0.8	0.7878	0.6721	0.2513	0.0000
t+0.9	0.6886	0.5659	0.1244	0.0000
t+1.0	0.6093	0.4710	0.0000	0.0000

Table 5.3
$\Delta = .005$

P(t,α)	0.0	0.5	1.0	1.5
t+0.0	2.7140	2.3855	1.6919	0.7541
t+0.1	2.3140	2.0293	1.4388	0.5859
t+0.2	1.9724	1.7265	1.2145	0.4295
t+0.3	1.6814	1.4703	1.0144	0.2817
α t+0.4	1.4344	1.2549	0.8346	0.1394
t+0.5	1.2556	1.0753	0.6714	0.0000
t+0.6	1.0500	0.9223	0.5217	0.0000
t+0.7	0.9036	0.7876	0.3824	0.0000
t+0.8	0.7825	0.6887	0.2508	0.0000
t+0.9	0.6839	0.5631	0.1241	0.0000
t+1.0	0.6051	0.4688	0.0000	0.0000

Table 5.4
Spline Approximation

P(t,α)	0.0	0.5	1.0	1.5
t+0.0	2.6915	2.3742	1.6860	0.7491
t+0.1	2.2986	2.0177	1.4355	0.5894
t+0.2	1.9639	1.7215	1.2165	0.4366
t+0.3	1.6749	1.4685	1.0150	0.2776
α t+0.4	1.4243	1.2489	0.8284	0.1257
t+0.5	1.2114	1.0605	0.6658	0.0217
t+0.6	1.0443	0.9234	0.5230	0.0005
t+0.7	0.8994	0.7862	0.3881	0.0032
t+0.8	0.7725	0.6620	0.2404	0.0005
t+0.9	0.6786	0.5567	0.1079	0.0004
t+1.0	0.6013	0.4690	0.0323	0.0000

CONCLUDING REMARKS. Our focus has been on the control problem for the RFDE on a finite interval. A natural question which arises is whether the approach can be adapted to treat the infinite time problem. Several aspects of the analysis can probably be extended to this case. (Some of these extensions appear nontrivial however). For example, although the factorization results and projection integrals discussed in Section 2 are based on Hilbert-Schmidt assumptions (which are not valid for problems on the semi-infinite interval), for stable time-invariant systems the infinite-time factorization counterpart to (2.16) is the classical Wiener-Hopf factorization. Thus we expect that the optimal control law can at least be formulated in the same manner as in Theorem 2.5 for the infinite-time problem. Any approximation scheme devised thereafter would necessarily have to consider approximating solutions to a Wiener-Hopf equation. In spirit such an approach might make contact with some work of Davis [4].

Fundamental to the analysis of this report was the exploitation of two properties of the factorization approach of [24]. First, that the entire semigroup never needs approximation, and second that connections between the open loop system and feedback kernel are fairly transparent via the factorization. These properties are not unique to the RFDE control problem, and as a final comment we remark that the general approach should lead to approximate feedback laws in other settings as well.

FOOTNOTES

1. The kernel $K(t,s)$ of an integral operator K is semiseparable if there exist matrix functions $H_i(t)$ and $G_i(s)$, $i = 1,2$, such that $K(t,s) = H_1(t)G_1(s)$ for $s < t$, and $K(t,s) = H_2(t)G_2(s)$ for $s \geq t$. The kernel is separable if we can choose $H_1 = H_2$ and $G_1 = G_2$. In this case the associated operator K has finite rank.

2. It is unfortunate that although this convergence is obtained using a first-order quadrature scheme, it is not a priori evident that employing a higher order scheme would result in improved convergence. The stumbling block is that the convergence analysis we have used is based on properties of the fundamental matrix, which is generally only absolutely continuous, and thus precludes any straightforward extensions.

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16. Abstract

The fundamental control synthesis issue of establishing a priori convergence rates of approximation schemes for feedback controllers for a class of distributed parameter systems is addressed within the context of hereditary systems. Specifically, a factorization approach is presented for deriving approximations to the optimal feedback gains for the linear regulator-quadratic cost problem associated with time-varying functional differential equations with control delays. The approach is based on a discretization of the state penalty which leads to a simple structure for the feedback control law. General properties of the Volterra factors of Hilbert-Schmidt operators are then used to obtain convergence results for the controls, trajectories and feedback kernels. Two algorithms are derived from the basic approximation scheme, including a fast algorithm, in the time-invariant case. A numerical example is also considered.

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