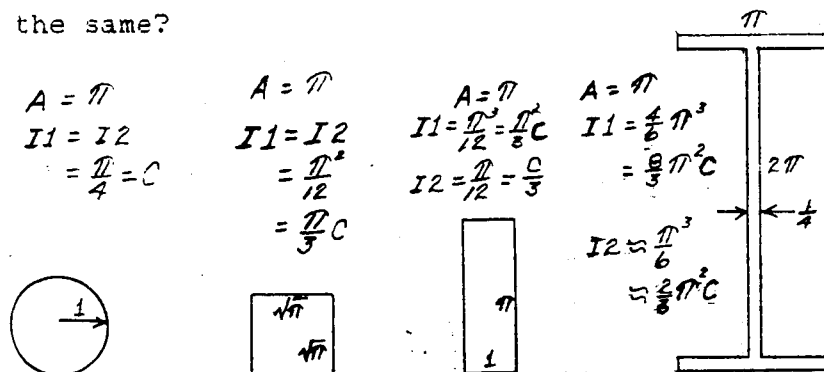# MASS MODELING FOR BARS

by

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BUTLER ANALYSES

## Summary

Methods of modeling mass for bars are surveyed. A method for extending John Archer's concept of consistent mass beyond just translational inertia effects to rotational inertia effects is included. Results are compared against detailed models. Recommendations are published for various types of modeling situations.

## Methods

Would you say that the inertia matrix for beams of these four sections would be the same if the area, material and length for all four were the same?

$A = \pi$
$I1 = I2$
$= \frac{\pi^2}{4} = C$

$A = \pi$
$I1 = I2$
$= \frac{\pi^2}{12}$
$= \frac{\pi}{3} C$

$A = \pi$
$I1 = \frac{\pi^3}{12} = \frac{\pi^2}{8}C$
$I2 = \frac{\pi^2}{12} = \frac{C}{3}$

$A = \pi$
$I1 = \frac{4}{6}\pi^3$
$= \frac{8}{3}\pi^2 C$

$I2 = \frac{\pi^3}{6}$
$= \frac{2}{8}\pi^2 C$

That is exactly what NASTRAN will give you if you ask for Coupled Mass on the PARAM card. NASTRAN looks at the area, the span and the density and gives you a coupled mass matrix based only on the amount of uniformly distributed mass between grid points. The COUPMASS routine does not concern itself with the sectional data that you include on your PBAR card other than area. It does compute moment-of-inertia terms based on the assumption that inertia effects are the same in plane 1 and plane 2. You may wonder why I bother to engage in such an inquiry. It all came about as I was investigating how to formulate the mass matrix for beams of variable cross-section

136

when modeled as equivalent prismatical beams.  I found that a con-
siderable correction was needed to balance the inertias in the two
principal planes after coupled mass formulation.  I was inclined to
base any such corrections as an extension of the logic which may
have already been applied to the uniform bar.  When I observed that
coupled mass formulation was insensitive to distribution over the
section, I thought that it would be worthwhile to try to extend the
theory to these additional terms.

That is the truth.  Now let's form a judgement.  Is this bad?
It will be shown below that coupled mass in spite of its insensitiv-
ity to differences in section is a definite improvement over the
default condition of breaking the total mass into two pieces and
lumping it at the two ends without first and second moments.  In
order to form a basis of judgement, let's look at the various ways
to model mass in bar elements.  NASTRAN only admits of prismatical
BAR elements.  That means a constant cross-section over its entire
length.  So this study will be confined to the modeling of mass for
prismatical beams.  The criterion for goodness of modeling will be
the match that a given model makes with the frequencies of free-free
beam modes using Timoshenko (1) as the arbiter for correctness.
Free-free modes were chosen, because these modes activate more of
the bar mass than other modes.  There are essentially two elastic
conditions that the user opts for in his modeling of bending--with
or without shear stiffness.  Because the shear contribution to elas-
tic bending always acts in series with the contribution from curva-
ture, the net effect of including shear stiffnesss in the bending
behavior of a model, is to lower the modal frequencies.  In reality,
there are 3 elastic conditions; because if the user includes a com-
plement of properties including longitudinal, bending and torsion,
but confines his freedoms to only translations, he is depriving his
model of some of the rotational contributions to bending, which act
in parallel with the transverse translational terms with the net
result that the modal frequencies will be lower than those contain-
ing a complete set of rotational freedoms.  All of the beams that
were modeled in this study were done <u>without shear deformation terms
in the stiffness matrix</u>.  Because of the importance of all of these
conditions, the descriptions of the various patterns of modeling

that are to be described, will be identified with both the stiffness and the mass modeling schemes.

Scheme 1.    SCALAR MASS

Translational DOF's  3 dof/GP
Stiffness properties  A, Il, I2
Non-zero rows & cols of KGG Matrix 1,2,3....7,8,9
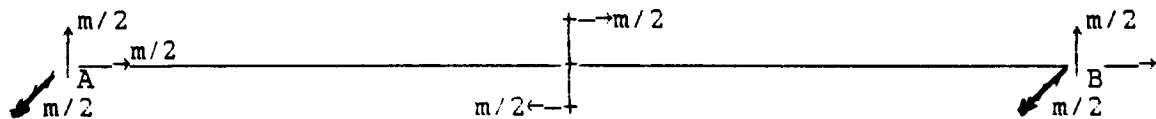Non-zero rows & cols of MGG Matrix 1,2,3....7,8,9



Figure  1

Scheme 1A.    SCALAR FINE MESH

Patterns the same as SCALAR except the mesh of GP's is fine.

Scheme 1B.    SCALAR 3 CONDENSE

Patterns the same as SCALAR FINE MESH except the fine mesh is condensed to a coarse mesh.

Scheme 1C.    SCALAR 5 CONDENSE

Translational & 2 Bending DOF's    5 dof/GP
Stiffness Properties  A, Il, I2
Non-zero rows & cols of KGG Matrix 1,2,3,..5,6,7,8,9,...,11,12
Non-zero rows & cols of MGG Matrix 1,2,3,......7,8,9........
Non-zero rows & cols of MAA Matrix 1,2,3,..5,6,7,8,9,...,11,12

Scheme 2.    LUMPED MASS

Translational & Rotational DOF's    6 dof/GP
Stiffness Properties  A, Il, I2, J
KGG Matrix  All rows & cols active
MGG Matrix  6x6 groups/GP. No coupling between GP's
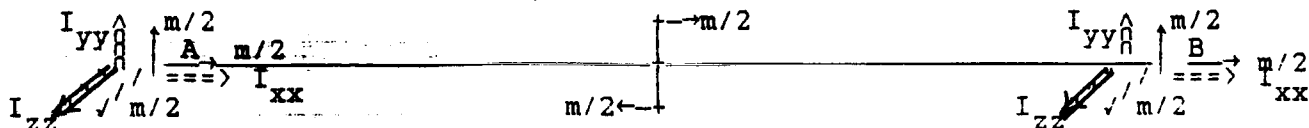
138

Figure   2

Scheme 2A.      LUMPED FINE MESH
     Pattern same as LUMPED MASS except the mesh of GP's is fine.


Scheme 2B.      LUMPED CONDENSED
     Pattern same as LUMPED FINE MESH except MAA matrix develops
     same coupling between GP's that KAA matrix has.


Scheme 3.       TRANSLATIONAL COUPLING
     No Torsional DOF's    5 dof/GP
     Stiffness Properties   A, I1, I2
     Non-zero rows & cols of KGG 1,2,3,..5,6,7,8,9,..11,12
     Non-zero rows & cols of MGG 1,2,3,..5,6,7,8,9,..11,12
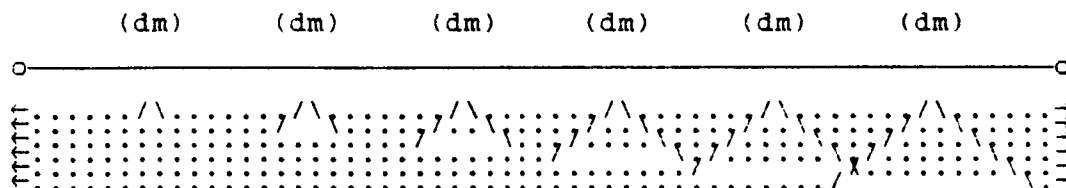



Figure  3

Scheme 3A.      TRANSLATIONAL COUPLING FINE MESH
     Pattern the same as TRANSLATIONAL  COUPLING  except  the  mesh  of
   GP's is fine.


Scheme 3B.      TRANSLATIONAL COUPLING CONDENSED
     Pattern  the  same as TRANSLATIONAL FINE MESH except the fine mesh
   is condensed to a fine mesh.

Scheme 4.     TRANSLATIONAL & ROTATIONAL COUPLING

Pattern the same as TRANSLATIONAL MESH except that each term in the MGG matrix, excluding longitudinal dof's, consists of contributions from the two types of coupling.

$$\{\tfrac{dT}{dm}\} \qquad \{\tfrac{dT}{dm}\} \qquad \{\tfrac{dT}{dm}\} \qquad \{\tfrac{dT}{dm}\} \qquad \{\tfrac{dT}{dm}\} \qquad \{\tfrac{dT}{dm}\}$$
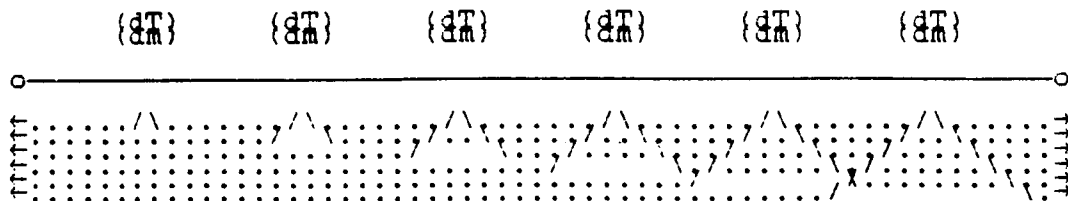
Figure 4

Scheme 4A.     TRANSLATIONAL & ROTATIONAL COUPLING FINE MESH

Pattern the same as TRANSLATIONAL & ROTATIONAL COUPLING except that the mesh of GP's is fine.

Scheme 4B.     TRANSLATIONAL & ROTATIONAL COUPLING CONDENSED

Pattern the same as TRANSLATIONAL & ROTATIONAL COUPLING FINE MESH except that the fine mesh is condensed to coarse.

The modeling schemes will be applied to one beam of circular cross-section and to another of rectangular cross-section. In order to maintain the results as comparable as possible, the same eigen-value extraction method was used wherever possible. GIVENS method was used for all runs except two, where INVPWR was substituted when GIVENS had difficulty. The results are shown in tabular form. But before the results are discussed, the subject of coupled mass will taken up. John Archer (2) made a huge contribution to our field with his consistent mass method of modeling mass in finite elements. He took the arbitrariness out of apportioning of mass into finite elements. He operated on the premise that any particle of mass located between the end points of a span had an influence in dynamics on both connecting points. He very cleverly said that the dynamic deformation in bending can be reasonably approximated by the static deformation in bending, under loading conditions that matched those

140

used during the determination of the stiffness terms. Thus the phrase consistent seemed appropriate. But the cleverest part of his scheme was his method of vectoring the interior contributions to the connecting points. He appealed to Maxwell's Reciprocity Law in the particular form popularly know in Civil Engineering circle as the Mueller-Breslau principle of influence lines. Graphically, here is how it works. Impose a boundary deformation that is used for the stiffness matrix for example a unit transverse deformation at end B and solve for the deformation at all interior points, while holding displacements at all other connecting degrees of freedom to zero.
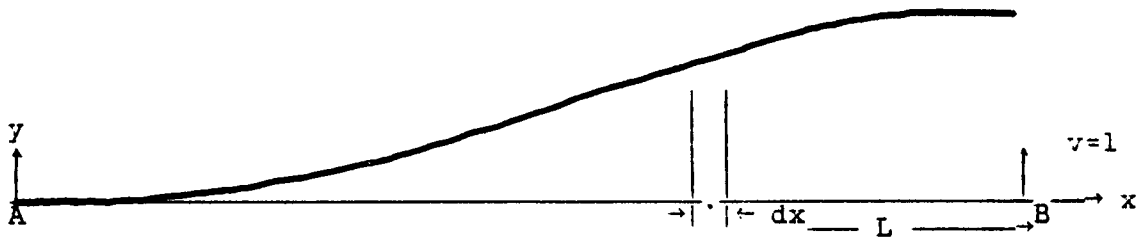


Figure 5

The non-dimensionalized equations for displacement and slope are

$$v(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 , \qquad \theta(x) = \frac{6}{L}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] .$$

Now assume that $v(x)$ represents the amplitude of the acceleration at any location $x$ along the beam.

$$Acc(x) = v(x) \ddot{q}.$$

The increment of mass at $x$ is $(\rho \, A \, dx)$ . So the approximate increment of force produced by this assumed acceleration is

$$dF = (\rho \, A \, dx) \, v(x) \ddot{q}$$

141

Because the cross-section is constant, the density can be written as a linear density; i.e.

$$(\rho\ A) = \lambda.$$

The approximate increment of force at $x$ due to transverse dynamic displacement at end B is

$$dF(B) = \lambda\ v(x)\ dx\ \ddot{q}.$$

To apportion $(\lambda v \ddot{q}\ dx)$ to ends A & B, John Archer applied the Mueller-Breslau principle of influence lines (3), which derives from the Maxwell Reciprocity Rule, which in turn derives from the Betti/Rayleigh Law. In order to emphasize a point that was not made explicit in John Archer's paper, I choose to go directly to the Maxwell Reciprocity Theorem as it applies in the special case of beams under concentrated loads. It says that if a beam is loaded with two different sets of loads and constraints, F1 and F2, and the response to these respective loads are u1 and u2, the work done by the first set of loads acting through the second set of displacements is equal to the work done by the second set of loads acting through the first set of displacements (4).

$$F1\ x\ u2 = F2\ x\ u1.$$

The beam systems to which we will apply this law are as follows. The inertia load $dP$ acting at $x$ on a beam that is clamped at both ends and displaced a unit amount at end B. Label the displacement curve $v(x)$ and the curve of slopes $\theta(x)$.



Figure 6

The other condition is for the beam to be displaced a unit amount at end A and all other displacements and rotations are held to zero. Label the displacement curve $\delta(x)$ and the curve of slopes $\phi(x)$.
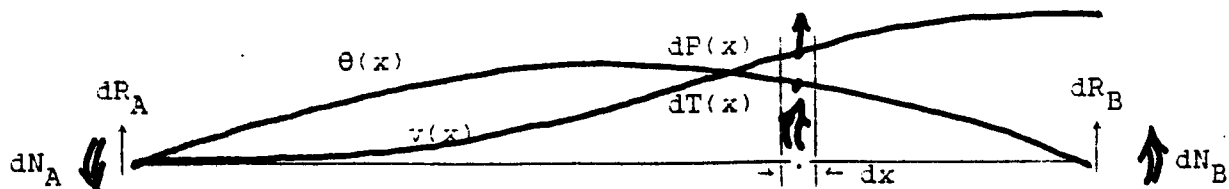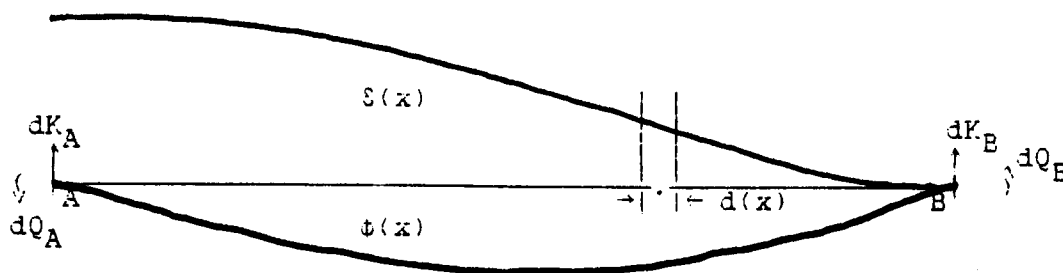


Figure 7

The nondimensionalized displacement and slope are calculated to be

$$\delta(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3, \qquad \phi(x) = -\frac{6}{L}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right].$$

Apply Maxwell's Law to the free bodies from A to x for this pair

$$dR_A \times \delta(0) + dP(x) \times \delta(x) + dN_A \times \phi(0) + dT(x) \times \phi(x)$$

$$= dK_A \times v(0) + dQ_A \times \theta(0).$$

Substitute the boundary values

$$dR_A \times 1 + dP(x) \times \delta(x) + 0 + dT(x) \times \phi(x) = 0 + 0$$

This collapses to

$$dR_A = -dP(x) \times \delta(x) - dT(x) \times \phi(x).$$

Here $dT(x)$ is approximated in terms of sectional property I to be

$$dT(x) = (I\ dx)\ \theta(x)\ \ddot{q}.$$

143

Archer specialized the situation by considering the moment of iner-
tia distribution I $\theta(x)$ dx to be negligible; thus the increment of
reaction at end A due to an increment of translational force at in-
termediate point x is

$$dR_A = -dP(x) \ x \ \delta(x)$$

Using the previously stated approximation for dP(x) as $\lambda$ dx v(x) $\ddot{q}$,
it becomes

$$dR_A = \lambda \ v(x) \ \delta(x) \ \ddot{q} \ dx.$$

If $dR_A$ is assumed to be made of an incremental inertia term $dM_A$ be-
ing accelerated through coordinate q, the equation can be written as

$$dR_A = dM_A \ \ddot{q} = \lambda \left[ x(x) \ \delta(x) \right] dx \ \ddot{q} \ .$$

The total inertial reaction from incremental contributions over the
whole beam is obtained by integrating over the length L.

$$M_A = \lambda \int_0^L v(x) \ \delta(x) \ dx \ .$$

So, for our example case, letting $\lambda L$ equal the total mass m of the
beam,

$$M_{A,B} = \lambda \int_0^L \left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right] \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] dx = 22 \frac{\lambda L^2}{420} = \frac{22L}{420} m.$$

In words, this term represents the amount of mass that couples to
the y translational dof at end A due to a dynamic displacement of
the y translational dof at end B. This is the pattern of analysis

that is used to develop the bending terms in the 12 x 12 coupled translational mass matrix. Turning to similar coupling for axial (longitudinal) deformation, the shape functions are modified from pure static deformation. No coupling is provided for torsion, because relationships between 2nd area moments in the KGG matrix and 2nd mass moments in the KGG matrix are highly variable for torsion. The total translational coupled mass for a bar is shown on page 8.2-22 of the Programmer's Manual which for convenience is duplicated below.

$$[M^e] = \frac{m}{420}\left[\begin{array}{cccccc|cccccc} 175 & 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 \\ & 156 & 0 & 0 & 0 & 22\ell & 0 & 54 & 0 & 0 & 0 & -13\ell \\ & & 156 & 0 & -22\ell & 0 & 0 & 0 & 54 & 0 & 13\ell & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 4\ell^2 & 0 & 0 & 0 & -13\ell & 0 & -3\ell^2 & 0 \\ & & & & & 4\ell^2 & 0 & 13\ell & 0 & 0 & 0 & -3\ell^2 \\ \hline & & \text{SYM} & & & & 175 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 156 & 0 & 0 & 0 & -22\ell \\ & & & & & & & & 156 & 0 & 22\ell & 0 \\ & & & & & & & & & 0 & 0 & 0 \\ & & & & & & & & & & 4\ell^2 & 0 \\ & & & & & & & & & & & 4\ell^2 \end{array}\right]$$

where

$$m = (\rho A + \mu)\ell \quad .$$

Figure 8

One of the main purposes of this paper is to explore the significance of the moment terms that Archer de-emphasized. I use this word advisedly, because Archer states his general formula in 3 dimensions, but he applies it to only translational inertia in one dimension. The general formula is

$$m_{ij} = \int_{Vol} m(x,y,z) \, \gamma_i(x,y,z) \, \gamma_j(x,y,z) \, d\,Vol.$$

Note the density "m" is written as varying in 3 dimensions and so are the two shape functions $\gamma_i$ and $\gamma_j$. I was encouraged on the basis of this generality to enquire as to the contributions from distributions of mass over the cross-section. Why bother to investigate these details? Well, I was provoked into this inquiry when I was concentrating on a related topic. I was wrestling with the problem of finding a logical analytical approach to the mass properties of non-prismatical beams when modeling with equivalent bars. I frequently referred to the mass matrix. I was startled by the mass matrix which was generated with the coupled mass option for a bar with a rectangular cross-section. The moments of inertia for bending about both transverse axes were the same. I checked the theoretical manual. The formula there did not discriminate by axis. I checked John Archer's paper and the formula he published matched that in the theoretical manual. I checked the Programmer's Manual for the algorithm actually used for the BAR element. It did not discriminate either. I went back to John Archer's paper and studied it very hard. Non-prismatical beams should be much more sensitive to transverse variations of mass, so I had the incentive to explore Archer's general equation further. But, the translational case brought such startling improvements over the lumping practices of the times of the early 60's that there was no immediate incentive to explore coupled mass properties for terms other than translational. I proceeded in the same spirit that Archer used. The shape function for the slope of the static deformation would be used as the dynamic approximation to the amplitude of the rotational acceleration at a point. This required that the slope functions as well as displacement functions for the solutions of the Bernoulli-Euler beam would both have to be catalogued. They are shown in Figure 9.

$$v_2 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \qquad v_8 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$\theta_2 = -\frac{6}{L}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] \qquad \theta_8 = \frac{6}{L}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]$$

$$v_3 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \qquad v_9 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$\theta_3 = -\frac{6}{L}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] \qquad \theta_9 = \frac{6}{L}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]$$

$$v_5 = -x + 2\frac{x^2}{L} - \frac{x^3}{L^2} \qquad v_{11} = \frac{x^2}{L} - \frac{x^3}{L^2}$$

$$\theta_5 = -1 + 4\frac{x}{L} - 3\left(\frac{x}{L}\right)^2 \qquad \theta_{11} = 2\frac{x}{L} - 3\left(\frac{x}{L}\right)^2$$

$$v_6 = x - 2\frac{x^2}{L} + \frac{x^3}{L^2} \qquad v_{12} = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

$$\theta_6 = 1 - 4\frac{x}{L} + 3\left(\frac{x}{L}\right)^2 \qquad \theta_{12} = -2\frac{x}{L} + 3\left(\frac{x}{L}\right)^2$$

Figure 9

Now we are ready to develop the rotational coupling inertia terms. The contribution to the end reactions due to an incremental moment

can be written according to the Maxwell equation for concentrated loads on a beam as

$$dR_i = dT_j(x) \times \theta_i(x) \qquad \text{where,}$$

$$dT_j = \rho \, I_j \, \theta_j \ddot{q} \, dx \qquad \text{and,}$$

$$dR_i = dM_{ij} \, \ddot{q} \qquad =====> \qquad dM_{ij} \, \ddot{q} = (\rho \, I_j \, \theta_j dx) \, \theta_i \ddot{q}.$$

Integrate all increments over the entire length of the beam with the appropriate slope functions taken from the listings in Figure 9, to obtain the expression for rotational coupling inertia at end j due to an acceleration of end i. Density and Sectional area moment are constant over the length. The matrix for all i and j then follows.

$$M_{ij} = \rho \, I_j \int_0^L \theta_i(x) \, \theta_j(x) \, dx.$$

$[M_{ij}] = \dfrac{\rho}{30}$

| | $x_A$ | $y_A$ | $z_A$ | $\theta_A$ | $\phi_A$ | $\psi_A$ | $x_B$ | $y_B$ | $z_B$ | $\theta_B$ | $\phi_B$ | $\psi_B$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_A$ | | | | | | | | | | | | |
| $y_A$ | | $\frac{36}{L}I_{yy}$ | | | | $3I_{zz}$ | | $-\frac{36}{L}I_{zz}$ | | | | $3I_{zz}$ |
| $z_A$ | | | $\frac{36}{L}I_{yy}$ | | $-3I_{yy}$ | | | | $-\frac{36}{L}I_{yy}$ | | $-3I_{yy}$ | |
| $\theta_A$ | | | | | | | | | | | | |
| $\phi_A$ | | | $-3I_{yy}$ | | $4I_{yy}L$ | | | | $3I_{yy}$ | | $-I_{yy}L$ | |
| $\psi_A$ | | $3I_{zz}$ | | | | $4I_{zz}L$ | | $-3I_{zz}$ | | | | $-I_{zz}L$ |
| $x_B$ | | | | | | | | | | | | |
| $y_B$ | | $-\frac{36}{L}I_{zz}$ | | | | $-3I_{zz}$ | | $\frac{36}{L}I_{zz}$ | | | | $-3I_{zz}$ |
| $z_B$ | | | $-\frac{36}{L}I_{yy}$ | | $3I_{yy}$ | | | | $\frac{36}{L}I_{yy}$ | | $3I_{yy}$ | |
| $\theta_B$ | | | | | | | | | | | | |
| $\phi_B$ | | | $-3I_{yy}$ | | $-I_{yy}L$ | | | | $3I_{yy}$ | | $4I_{yy}L$ | |
| $\psi_B$ | | $3I_{zz}$ | | | | $-I_{zz}L$ | | $-3I_{zz}$ | | | | $4I_{zz}L$ |

Figure 10

Notice that torsional and axial inertias are not addressed. This is consistent with Bernoulli-Euler theory which assumes no coupling between bending and torsion and between bending and axial. The intention of this development is to augment the mass matrix based on translational distributions; so it would not supplant the COUPMASS operation. It would be added to it. It is interesting to compare the matrix of translational coupling with rotational coupling term by term to gauge their relative importance. Importance will be based on comparing magnitudes of samplings of like terms, by using a rectangular cross-section to reflect the difference in roles about the two axes.

Let the height to breadth ratio $h/b = 3/1$. In order to put corresponding terms in the same dimensions, the various quantities will be written with rectangular factors substituted into the properties: i.e.

$$m = \rho AL = \rho bhL. \qquad I_{zz} = \frac{bh^3}{12}. \qquad I_{yy} = \frac{hb^3}{12}.$$

Term 2,2      Translational vs. Rotational

$$\frac{156}{420} m \qquad\qquad vs \quad \frac{36\rho}{30L} I_{zz}$$

$$\frac{156}{420} \rho bhL \qquad\qquad\qquad \frac{36\rho}{30L} \frac{bh^3}{12}$$

$$\frac{12}{60} (\rho bh) \frac{13L}{7} \qquad\qquad\qquad\qquad \frac{12}{60} (\rho bh) \frac{3h^2}{4\,L}$$

Simplify by factoring the common coefficient. Then compare on the basis of a short beam ($L/h = 4$) and a long beam ($L/h = 12$).

$$\frac{13L}{7} \quad vs \quad \frac{3h^2}{4\,L} \quad =====> \quad \frac{52}{21} \frac{L^2}{h^2}. \quad \text{Short } 39.6 \text{ vs } 1. \quad \text{Long } 356.5 \text{ vs } 1.$$

Term 2,6        Translational   vs   Rotational

$$\frac{22L}{420} m = (\rho bhL) \frac{22L}{420} \qquad vs \qquad \frac{3\rho}{30} I_{zz} = \frac{3\rho}{30} \frac{bh^3}{12}$$

$$\frac{11}{210} L^2 \quad vs \quad \frac{h^2}{120} \quad ====> \quad \frac{44}{7} \frac{L^2}{h^2} \quad vs \quad 1. \quad Short \; 100.6 : 1. \quad Long \; 905.1 : 1.$$

Term 3,3        Translational vs   Rotational

$$\frac{156}{420} m = \frac{16}{35} \rho bhL \qquad vs \qquad \frac{36\rho}{30L} I_{yy} = \frac{36\rho}{30L} \frac{hb^3}{12} = \rho bh \frac{b^2}{10L}$$

$$\frac{26}{7} \frac{L^2}{b^2} \; vs \; 1 . \qquad Short \; L/b = 12. \qquad Long \; L/b = 36.$$
$$\qquad\qquad\qquad\quad Short \; 534.8 : 1. \qquad Long \; 4813.7 : 1 .$$

Term 6,6        Translational   vs   Rotational

$$\frac{4L^2}{420} m = \frac{L^2}{105} (\rho bhL) \qquad vs \qquad \frac{4\rho L}{30} I_{zz} = \frac{4\rho L}{30} \frac{bh^3}{12} = (\rho bhL) \frac{h^2}{90}$$

$$\frac{6}{7} \frac{L^2}{h^2} \; vs \; 1. \qquad Short \; 13.7 : 1 . \qquad Long \; 123.4 : 1 .$$

The conclusions from these samplings is that for a typical
beam cross-section, translational inertia coupling is 2 orders of
magnitude more important than rotational inertia couplings for short
beams, and for long beams is 3 orders of magnitude more important.
I found these results amazing. I had pre-judged that when the dis-
tribution in one direction was markedly different from that in the
other direction one would find the influence of the rotational cou-
pling to be signifigant. One is comfortable with the idea that the
sectional properties in the stiffness terms are limiting the non-
dimensionalized deflections and thus are limiting the assumed trans-
lational accelerations. But the idea that the overhung moments of
the translational masses should be the overriding influence vis-a-

vis the sum of all distributed inertias being accelerated in rotation by the beam slope leaves one uncomfortable. This result adds further commendation to the contribution of John Archer for providing us with so advanced a tool so early in the development of finite elements.

Experimental Results

Results of modeling for bending, axial, and torsion will be treated separately. Bending is discussed first. The most accurate method for simulating mass properties in bending modes is translational coupling. The rule for its use was stated by Archer in his original paper. One must employ one grid point more than the number of nodes for the mode associated with the highest frequency of interest. This is borne out by the data compiled from the NASTRAN runs for translational coupling modeled respectively with 2, 3, 4, and 11 points.

### FREQUENCIES FOR FREE-FREE BENDING MODES MODELED WITH TRANSLATIONAL COUPLING
#### CIRCULAR CROSS-SECTION

| TIMOSHENKO MODE* FREQ | 11 GRID POINTS FREQ | RATIO | 4 GRID POINTS FREQ | RATIO | 3 GRID POINTS FREQ | RATIO | 2 GRID POINTS FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|
| T2 874.92 | 874.97 | 1.00005 | 879.49 | 1.005 | 876.89 | 1.0023 | 1049.3 | 1.1993 |
| T2 2411.68 | 2412.41 | 1.0003 | 2421.90 | 1.004 | 2744.40 | 1.1380 | 3584.2 | 1.3797 |
| T2. 4728.44 | 4732.45 | 1.00085 | 5465.33 | 1.156 | . | . | . | . |
| T2 7815.6 | 7834.57 | 1.00242 | . | . | . | . | . | . |

*For a circular beam the modes in both transverse directions are the same, they are both being represented here by the notation T2 only.

#### RECTANGULAR CROSS-SECTION

| TIMOSHENKO MODE FREQ | 11 GRID POINTS FREQ | RATIO | 4 GRID POINTS FREQ | RATIO | 3 GRID POINTS FREQ | RATIO | 2 GRID POINTS FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|
| T2 505.14 | 505.16 | 1.00005 | 507.77 | 1.005 | 506.27 | 1.0022 | 605.8 | 1.20 |
| T2 1392.39 | 1392.80 | 1.00030 | 1398.28 | 1.004 | 1584.48 | 1.1379 | 2069.3 | 1.48 |
| T3 1515.42 | 1515.49 | 1.00005 | 1523.32 | 1.005 | 1518.82 | 1.0022 | 1817.5 | 1.20 |
| T2 2729.97 | 2732.28 | 1.00085 | 3155.40 | 1.156 | 3962.00 | 1.451 | . | . |
| T3 4177.15 | 4178.40 | 1.00030 | 4194.85 | 1.004 | 4753.43 | 1.138 | 6207.9 | 1.486 |
| T2 4512.35 | 4523.28 | 1.00240 | 5361.27 | 1.188 | . | . | . | . |

In the two point model of the beam of circular section, the frequency for the fundamental mode, which has 2 nodes, computes to only 20% of the correct value. In the three point model, the frequency of the fundamental is correct to 0.2%, but the 2nd bending mode, which has three nodes, computes to only 14% of the correct

value. In the four point model, the fundamental and 2nd bending modes are accurate to within 0.5%, but the 3rd bending mode, which has four nodes, computes to only 16% of the correct value. In the 11 point model, all frequencies for bending modes through the 9th are accurate to within 0.5%, but the tenth mode, which has eleven nodes is accurate to only 15%.

A similar pattern of accuracy evolves for the rectangular cross-section as for the circular beam for any one type of bending mode. Following the T2 modes to higher and higher harmonics shows that the 2 point model is not adequate, but as soon as one more point is added, the fundamental for the 3 point model gives 0.2% accuracy. That same pattern appears in following the T3 modes. Likewise the pattern of accuracy for the 4 point model follows the circular results taken one type of mode at a time.

Does this complete the discussion of modeling of mass for beams? It would if the only consideration were accuracy. The cost in manpower is in calculating the sectional properties and in preparing the PBAR cards and a PARAM COUPMASS card. Computer time would be considerable. Because the density of coupled mass matrices of a pure series BAR model is 4% per 102 rows compared to 1% per 102 rows for the oft-used diagonal matrix or to 0.5% per 102 rows for the scalar mass created by default from PBAR cards. If condensation is used to reduce the order of the analysis set, the density of matrices is further increased. Computer cost could decline due to smaller order, but it could increase from being condensed to higher density; what the net may become will depend on the connectivity of the model (assuming that the model is other than a pure series beam). A logical next question deals with the effect that condensing has on the accuracy of the model. If one made all the right decisions for the beams in his model and then was confronted with the necessity to condense to small order, will condensing degrade the model--or to put it another way, what things must be held sacred against omitting so as to preserve a model's functional integrity. A series of runs were made in which the 11 point model with translational coupling were condensed to 2, 3, & 4 grid points.

## CONDENSATION EFFECT ON CIRCULAR BAR MODELED WITH TRANSLATIONAL MASS COUPLING

| TIMOSHENKO | 11 GRID POINTS | | 11 TO 4 GP's | | 11 TO 3 GP's | | 11 TO 2 GP's | |
|---|---|---|---|---|---|---|---|---|
| MODE FREQ | FREQ | RATIO | FREQ | RATIO | FREQ | RATIO | FREQ | RATIO |
| T2 874.92 | 874.97 | 1.00005 | 879.49 | 1.005 | 876.89 | 1.0023 | 1049.3 | 1.120 |
| T2 2411.68 | 2412.41 | 1.0003 | 2421.90 | 1.004 | 2744.40 | 1.138 | 2584.2 | 1.38 |
| T2 4728.44 | 4732.45 | 1.00085 | 5465.33 | 1.156 | . | . | . | . |
| T2 7815.6 | 7834.57 | 1.00242 | | | | | | |

## CONDENSATION OF RECTANGULAR BAR MODELED WITH TRANSLATIONAL MASS COUPLING

| TIMOSHENKO | 11 GRID POINTS | | 11 TO 4 GP's | | 11 TO 3 GP's | | 11 TO 2 GP's | |
|---|---|---|---|---|---|---|---|---|
| MODE FREQ | FREQ | RATIO | FREQ | RATIO | FREQ | RATIO | FREQ | RATIO |
| T2 505.14 | 505.16 | 1.00005 | 483.21 | 1.005 | 482.02 | 1.002 | 573.74 | 1.135 |
| T2 1392.39 | 1392.80 | 1.00030 | 1253.79 | 1.004 | 1408.13 | 1.140 | 1780.03 | 1.278 |
| T3 1515.42 | 1515.49 | 1.00005 | 1507.78 | 1.005 | 1503.41 | 1.002 | 1805.78 | 1.192 |
| T2 2729.97 | 2732.28 | 1.00085 | 2559.19 | 1.156 | 3045.98 | 1.450 | . | . |
| T3 4177.15 | 4178.40 | 1.00030 | 4070.93 | 1.004 | 4620.92 | 1.140 | . | . |
| T2 4512.35 | 4523.28 | 1.00240 | 3886.84 | 1.188 | 4316.92 | . | . | . |

Note that so long as enough degrees of freedom are retained for a mode in the beam of circular section, the frequency degrades only by a few tenths of a percent with condensation. Note also that the accuracy of the frequency degrades markedly (jumps of 20%) when condensation leaves insufficient dof's in the model. Exactly the same kinds of observations can also be for the beam of rectangular cross-section, if one views bending in a given direction as a class. Now we can generalize by saying that translational coupling gives precisely the same results out to five places for all modes in a given class. More importantly, we observe that for any bending class for any prismatical beam, the accuracy of a coarse mesh is exactly the same as that for starting with a fine mesh and condensing to a mesh of like coarseness. A single table can summarize the results for mass modeled with translational coupling for all beams.

## MODAL ACCURACY VS. MESH FOR BARS MODELED WITH TRANSLATIONAL MASS COUPLING

| MODE | 11 GP's | 4 GP's | 3 GP's | 2 GP's |
|---|---|---|---|---|
| 1ST | 1.00005 | 1.005 | 1.0023 | 1.120 |
| 2ND | 1.0003 | 1.004 | 1.138 | 1.380 |
| 3RD | 1.00085 | 1.156 | | |
| 4TH | 1.00242 | | | |

Turn now from the ideal to the more traditional ways to model mass. The lazy approach is to invest no more time to model mass than to make an entry for density (RHO) on the material card. In so

doing the user depends on all of NASTRAN default conditions to take over. The mass matrix will have only scalar terms on the diagonal positions of translation dof's. If he does not condense and uses GIVENS method for eigenvalue extraction, he must provide an ASET1 card specifying dof's 1,2, & 3 for all Grid Points in order to remove the singularities from the rotational dof's of the mass matrix. In this case it becomes extremely important to have a fine mesh. A series of runs were made with increasingly fine mesh from 2, 3, 4, 5, 6, and 11. These models included these properties on the PBAR card: A, I1, & I2. J was omitted. No DMI nor CONM2 cards were used. A value was inserted for RHO on the MAT1 card and ASET1 was set to 123 for all Grid Points.

The results of these runs are shown in the following table.

### FREQUENCIES FOR FREE-FREE BEND MODES MODELED WITH SCALAR MASS ONLY 3DOF/PT
### CIRCULAR CROSS-SECTION

| TIMOSHENKO | 11 GP'S | 6 GP'S | 5 GP'S | 4 GP'S | 3 GP'S | 2 GP'S |
|---|---|---|---|---|---|---|
| MODE FREQ | RATIO | RATIO | RATIO | RATIO | RATIO | RATIO |
| T2 874.92 | 0.97017 | 0.892 | 0.775 | 2.908 | 1.239 | 3.097 |
| T2 2411.68 | 0.95086 | 0.838 | 0.745 | . | . | . |
| T2 4728.44 | 0.93295 | 0.796 | . | . | . | . |
| T2 7815.6 | . | 0.734 | 0.555 | . | . | . |

### RECTANGULAR CROSS-SECTION

| TIMOSHENKO | 11 GRID POINTS | | 4 GRID POINTS | | 3 GRID POINTS | | 2 GRID POINTS | |
|---|---|---|---|---|---|---|---|---|
| MODE FREQ | FREQ | RATIO | FREQ | RATIO | FREQ | RATIO | FREQ | RATIO |
| T2 505.14 | 490.07 | 0.9702 | 1468.93 | 2.908 | 625.72 | 1.239 | 1564.26 | 3.097 |
| T2 1392.39 | 1248.10 | 0.8964 | 1534.08 | 1.102 | . | . | . | . |
| T3 1515.42 | 1500.06 | 0.9898 | 4406.79 | 2.909 | 1877.11 | 1.239 | . | . |
| T2 2729.97 | 2273.90 | 0.8329 | . | . | . | . | . | . |
| T3 4177.15 | 4054.33 | 0.9706 | . | . | . | . | . | . |
| T2 4512.35 | 3456.05 | 0.7659 | . | . | . | . | . | . |

For the models with only 2, 3, or 4 GP's the fundamental was not the lowest elastic mode. Some of the spurious modes had frequencies that one might expect in the range of the true modes; consequently one can be easily misled by coarse models with scalar mass. The frequencies of the true modes are off not by percentages but by factors as high as 3. But when the mesh is fine enough, frequencies accurate to within 5% are obtainable. Thus we do not sneer at this method. We say this can be quite useful for quick and dirty investigations of tentative designs especially in this day and age

of CAD preprocessors for generating the models where it is simple to specify a fine mesh. Scalar mass modeling can be characterized by the girl with the curl in the nursery rhymes. "When she was good, she was very, very good; and when she was bad she was awful." The rule of thumb for avoiding awful results is to have from 6 to 10 more GP's than the number of nodes in the highest mode of interest.

But there is a remarkably good side to Scalar modeling, as well. This comes about from condensing. A series of runs were made in which the 11 point Scalar model was condensed to 2, 3, & 4 GP's. The one added feature however is that the ASET dof's were set to 5 instead of 3. Only torsion was omitted. In the Guyan reduction the stiffness matrix acts as a template for the condensation of mass. If the stiffness matrix has terms corresponding to rotational degrees of freedom, i.e. terms based on I1 and I2, it will cause coupling in the mass matrix such that terms in the rotational positions will also appear.

## EFFECT OF CONDENSATION ON SCALAR MASS MODELS RETAINING 5 DOF/GP
### CIRCULAR CROSS-SECTIONS

| TIMOSHENKO MODE FREQ | 11 GRID POINTS FREQ | RATIO | 11 - 4 POINTS FREQ | RATIO | 11 - 3 POINTS FREQ | RATIO | 11 - 2 POINTS FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|
| T2 874.92 | 848.83 | 0.97017 | 852.78 | 0.975 | 850.53 | 0.9721 | 1001.0 | 1.144 |
| T2 2411.68 | 2293.17 | 0.95086 | 2301.19 | 0.954 | 2579.81 | 1.07 | 3192.9 | 1.324 |
| T2 4728.44 | 4411.44 | 0.93295 | 5039.87 | 1.066 | . | . | . | . |
| T2 7815.6 | . | . | . | . | . | . | . | . |

### RECTANGULAR CROSS-SECTIONS

| TIMOSHENKO MODE FREQ | 11 GRID POINTS FREQ | RATIO | 11 - 4 POINTS FREQ | RATIO | 11 - 3 POINTS FREQ | RATIO | 11 - 2 POINTS FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|
| T2 505.14 | 490.07 | 0.9702 | 492.35 | 0.975 | 491.05 | 0.972 | 577.93 | 1.144 |
| T2 1392.39 | 1248.10 | 0.8964 | 1328.59 | 0.954 | 1489.45 | 1.07 | 1843.41 | 1.324 |
| T3 1515.42 | 1500.06 | 0.9898 | 1477.05 | 0.975 | 1473.16 | 0.972 | 1733.79 | 1.144 |
| T2 2729.97 | 2273.90 | 0.8329 | 2909.77 | 1.066 | 3368.46 | 1.234 | . | . |
| T3 4177.15 | 4054.33 | 0.9706 | 3985.77 | 0.954 | 4468.36 | 1.07 | . | . |
| T2 4512.35 | 3456.05 | 0.7659 | 4858.28 | 1.077 | . | . | . | . |

In condensing to two dof's the frequency for the fundamental is still not good, missing the correct value by 14%, but this is a distinct improvement over the single span ratio of 3. In fact it ranks with other single span models with initially more complete

mass inputs. Condensing to three points gives good results for the fundamental and a fair 10% ratio for the circular and a better 7% for the two rectangular 2nd bending modes. Condensing to four points give acceptable results out through the 4th bending mode. A happy observation can be made here, to wit: if the initial modeling was made fine enough to give good results in a mode, that accuracy is maintained during condensation so long as the final mesh retains at least one grid point more than the number of nodes in the highest mode of interest.

The scheme that was considered to be sophisticated compared to scalar modeling before consistent mass was available for use was the lumping into 6x6 matrices at each point. The scheme is to take an arbitrary amount of mass surrounding the point and assign it and all of its distribution properties to just one point. For instance in a single span beam, one-half of the mass would be consigned to each end point and its center of gravity would be designated to be inboard from each end by one quarter of the length. It is incumbent on the user to supply 8 pieces of information about the mass for each grid point. Total mass, location of the center of gravity with respect to the parent grid point, and moments and products of inertia with respect to the center of gravity. With symmetrical beams such as the circular beam being run in these examples, this amounts to the three diagonal terms of moments of inertia, mass, and x-offset of the center of gravity. In our case the lumped mass matrix would be assembled internally in NASTRAN to look as follows.

$$
\begin{bmatrix}
m/2 & 0 & 0 & | & 0 & 0 & 0 \\
0 & m/2 & 0 & | & 0 & 0 & \frac{m}{2}\bar{x} \\
0 & 0 & m/2 & | & 0 & -\frac{m}{2}\bar{x} & 0 \\
\hline
0 & 0 & 0 & | & I_{xx} & 0 & 0 \\
0 & 0 & -\frac{m}{2}\bar{x} & | & 0 & I_{yy} & 0 \\
0 & \frac{m}{2}\bar{x} & 0 & | & 0 & 0 & I_{zz}
\end{bmatrix}
$$

Data is supplied on a CONM2 card and NASTRAN assembles it into the local matrix as shown above. A series of runs were made in which lumped matrices were supplied for a single span bar, double span, triple span, and ten-span bars. No mass was formed with PBAR's, but only with CONM2's. The results are in the following table.

FREQUENCIES FOR FREE-FREE MODES MODELED WITH LUMPED MASS
CIRCULAR CROSS-SECTION

| TIMOSHENKO MODE FREQ | 11 GRID POINTS FREQ | RATIO | 4 GRID POINTS FREQ | RATIO | 3 GRID POINTS FREQ | RATIO | 2 GRID POINTS FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|
| T2 874.92 | 857.45 | 0.9800 | 787.50 | 0.900 | 764.14 | 0.8734 | 533.9 | 0.6102 |
| T2 2411.68 | 2292.15 | 0.9504 | 1842.91 | 0.764 | 1510.17 | 0.6262 | SPUR. | . |
| T2 4728.44 | 4312.48 | 0.9120 | 2130.72 | 0.4506 | . | . | . | . |
| T2 7815.6 | . | . | . | . | . | . | . | . |

RECTANGULAR CROSS-SECTION

| TIMOSHENKO MODE FREQ | 11 GRID POINTS FREQ | RATIO | 4 GRID POINTS FREQ | RATIO | 3 GRID POINTS FREQ | RATIO | 2 GRID POINTS FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|
| T2 505.14 | 480.95 | 0.9521 | 442.67 | 0.876 | 431.03 | 0.853 | 299.66 | 0.593 |
| T2 1392.39 | 1248.10 | 0.8964 | 1038.64 | 0.746 | 826.21 | 0.593 | 303.01 | 0.218 |
| T3 1515.42 | 1500.06 | 0.9898 | 1376.55 | 0.908 | 1323.52 | 0.873 | 933.90 | 0.616 |
| T2 2729.97 | 2273.90 | 0.8329 | 1686.38 | 0.618 | 1101.49 | 0.403 | . | . |
| T3 4177.15 | 4054.33 | 0.9706 | 3216.82 | 0.770 | 2634.75 | 0.631 | . | . |
| T2 4512.35 | 3456.05 | 0.766 | 1819.99 | 0.403 | 1113.13 | 0.247 | . | . |

Note that even with an 11 point model the best accuracy that can be obtained is for the fundamental frequency and it varies from 1% for the stiffer rectangular to 2% for the circular and 5% for the softer rectangular. The 2nd bending frequency is accurate to 3% for the stiffer rectangular to 5% for the circular and only 10% for the softer rectangular. The 3rd bending frequency is within 9% for the circular but is off by 17% for the softer rectangular. The 4th bending is off by 14% and 23%. The 2 point models introduce spurious modes and miss the fundamental by 40%. The 3 point models give no spurious modes but are off the mark by 13% on the fundamental, by 38% on the frequency for 2nd bending and by 55% on that for 3rd bending. The 4 point models give only fair results for the fundamental frequency—within 10%—and poor for the rest.

Now the question is, there are times when the nature of the structure is such that lumped mass is the only practical thing available to the analyst, so can the situation improve with condensation?

# EFFECT OF CONDENSATION ON BARS MODELED WITH LUMPED MASS
## CIRCULAR CROSS-SECTION

| TIMOSHENKO MODE | FREQ | 11 GRID POINTS FREQ | RATIO | 11 TO 4 GP's FREQ | RATIO | 11 TO 3 GP's FREQ | RATIO | 11 TO 2 GP's FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|---|
| T2 | 874.92 | 857.45 | 0.9800 | 861.76 | 0.985 | 859.36 | 0.982 | 1029.6 | 1.1769 |
| T2 | 2411.68 | 2292.15 | 0.9504 | 2301.71 | 0.954 | 2604.86 | 1.080 | 3386.86 | 1.4044 |
| T2 | 4728.44 | 4312.48 | 0.9120 | 4929.05 | 1.042 | 6192.49 | 1.296 | . | . |
| T2 | 7815.6 | . | . | 7843.33 | 1.004 | 9071.65 | 1.161 | . | . |

## RECTANGULAR CROSS-SECTION

| TIMOSHENKO MODE | FREQ | 11 GRID POINTS FREQ | RATIO | 11 TO 4 GP's FREQ | RATIO | 11 TO 3 GP's FREQ | RATIO | 11 TO 2 GP's FREQ | RATIC |
|---|---|---|---|---|---|---|---|---|---|
| T2 | 505.14 | 480.95 | 0.9521 | 483.21 | 0.956 | 482.02 | 0.954 | 573.74 | 1.135 |
| T2 | 1392.39 | 1248.10 | 0.8964 | 1253.79 | 0.900 | 1408.13 | 1.011 | 1780.03 | 1.278 |
| T3 | 1515.42 | 1500.06 | 0.9898 | 1507.78 | 0.995 | 1503.41 | 0.992 | 1805.78 | 1.192 |
| T2 | 2729.97 | 2273.90 | 0.8329 | 2559.19 | 0.937 | 3045.98 | 1.116 | . | . |
| T3 | 4177.15 | 4054.33 | 0.9706 | 4070.93 | 0.975 | 4620.92 | 1.106 | 6080.22 | 1.456 |
| T2 | 4512.35 | 3456.05 | 0.766 | 3886.84 | 0.862 | 4316.93 | 0.957 | . | . |

Condensation improves the performance of lumped mass modeling by a fraction of a percent, so long as there are sufficient dof's for a nodal pattern. In the higher modes the overshooting of the frequency from too few dof's tends to have a beneficial effect to counteract the overconcentration of mass. For instance in the 11 to 4 point condensation the 3rd and 4th modes are within 4% and 0.4% respectively. The fundamental can be found to within 1% when sufficient dof's are retained. Once again the stiffer modes perform better than the softer ones. The three point model shows consistency in good behavior into harmonics higher than expected.

When I started this project, I thought that what I'm now about to present should be saved till last because it was supposed to be the climax. Well, it is more appropriate to characterize what I'm about to present as re-enforcing the adage "If it ain't broke, don't fix it." I knew that translational coupling was good, but I never looked closely at how it compared with theory. Consequently, when I discovered that when Archer applied the Maxwell Reciprocity to the beam, he omitted the rotational terms, I thought I could contribute to the effectiveness of the coupled mass approach. Here are the "complete" results. A series of runs were made in which the mass matrix was composed of the sum of contributions from translational coupling and rotational coupling. Timoshenko inferred that refinements would be beneficial to the higher modes when rotary inertia

was included, so it was in the higher modes that I had expected to see proof of this notion. The models were composed of 11 , 4, 3, and 2 points. Results are tabulated below.

FREQUENCIES FOR FREE-FREE MODES MODELED WITH TRANS & ROTN COUPLING
CIRCULAR CROSS-SECTION

| TIMOSHENKO MODE FREQ | 11 GRID POINTS FREQ | RATIO | 4 GRID POINTS FREQ | RATIO | 3 GRID POINTS FREQ | RATIO | 2 GRID POINTS FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|
| T2 874.92 | 861.74 | 0.9849 | 866.04 | .990 | 863.59 | .987 | 1030.2 | 1.177 |
| T2 2411.68 | 2334.10 | 0.9678 | 2343.07 | 0.972 | 2649.22 | 1.098 | 3409.83 | 1.414 |
| T2 4728.44 | 4477.47 | 0.9469 | 5144.51 | 1.088 | 6306.74 | 1.334 | . | . |
| T2 7815.6 | 7216.31 | 0.9233 | 8494.81 | 1.088 | 9791.12 | 1.253 | . | . |
| T 11675.78 | 10483.74 | 0.8970 | . | . | . | . | | |

RECTANGULAR CROSS-SECTION

| TIMOSHENKO MODE FREQ | 11 GRID POINTS FREQ | RATIO | 4 GRID POINTS FREQ | RATIO | 3 GRID POINTS FREQ | RATIO | 2 GRID POINTS FREQ | RATIO |
|---|---|---|---|---|---|---|---|---|
| T2 505.14 | 483.20 | 0.9566 | 485.47 | 0.961 | 484.21 | 0.959 | 574.39 | 1.137 |
| T2 1392.39 | 1268.73 | 0.9119 | 1273.50 | 0.915 | 1434.58 | 1.030 | 1805.32 | 1.297 |
| T3 1515.42 | 1507.54 | 0.9948 | 1515.44 | 1.000 | 1511.03 | 0.997 | 1806.25 | 1.192 |
| T2 2729.97 | 2350.22 | 0.861 | 2680.70 | 0.982 | 3186.08 | 1.398 | . | . |
| T3 4177.15 | 4130.85 | 0.9898 | 4147.84 | 0.993 | 4696.57 | 1.124 | . | . |
| T2 4512.35 | 3651.26 | 0.8092 | 4259.93 | 0.944 | 4791.08 | 1.062 | . | . |

Instead of producing an improvement, the added terms from rotational coupling depressed the frequencies such that none came closer than 2 orders of magnitude of those with only translational coupling. The higher modes, instead of being improved with order number got worse. The frequency of the fifth bending mode came no closer than 10%. The two point combination was 2% better than the two point translational coupling model. Neither was good. The excellent performance of the translational 3 and 4 point models of holding to within a fraction of a percent was violated by adding rotational coupling by several percentage points. A more appropriate test seemed to be one involving a beam of unequal moments of inertia such as a beam with a rectangular cross-section. Even here the translational alone matched within a few tenths of a percent while the combination with rotational coupling was off by 1% in the stiffer direction and by 4% in the limber direction. Condensing gave the same results as the coarser models. So it can be concluded that rotational coupling of mass is not only not helpful, it is harmful to the coupling method of modeling mass.

Axial Modes

    There are only two methods available for modeling mass in the longitudinal axis of the beam. Lumping or coupling. Axial lumping puts all terms on the diagonal. Lumping by default to the PBAR card puts the center of gravity of the lumped mass at the Grid Point while lumping using CONM2 input allows the user to assign the center of gravity to a logical position. But it will be shown in the example runs later that shifting the center of gravity along the centroidal axis has no effect on axial modes. Only the fundamental axial frequency will be compared against the two methods of modeling. Runs were made with 11, 6, 5, 4, 3, and 2 points along the beam. Their results are shown below.

AXIAL MODES OF CIRCULAR BEAM.
FUNDAMENTAL AXIAL MODE FREQUENCY FROM TIMOSHENKO 4914.57

| | LUMPED | | | COUPLED | |
|---|---|---|---|---|---|
| # GP | FREQ | RATIO | · | FREQ | RATIO |
| 11 | 4894.06 | 0.9959 | | 4914.16 | 0.99998 |
| 6 | 4869.39 | 0.9908 | | 4912.64 | 0.9996 |
| 5 | 4833.82 | 0.9836 | | 4978.61 | 1.0131* |
| 4 | 4546.16 | 0.925 | | 4797.99 | 0.976 |
| 3 | 4424.39 | 0.9003 | | 4846.67 | 0.9862 |
| 2 | 3128.52 | 0.6366 | | 3831.63 | 0.7797 |

* This anomaly seems to have been caused by modeling 3 points close together in the middle, instead of spacing the points evenly.

    One can reason that axial modes for prismatical beams should be independent of shape of the cross-section, so long as Poisson effects are not taken into account, thus it will suffice to study axial mass modeling with a circular section. Comparing the fine mesh models for scalar, lumping, or translational coupling indicates that just breaking the longitudinal mass into small pieces and distributing them amongst elastic elements has a beneficial effect. All three fine mesh models yield results within 0.5% of The fundamental frequency. So the question to be answered is "How fine is fine?" Starting at the extreme of a single span puts all the mass at the ends with all of the elastic material in between. The frequency comes to only 64% of correct. A coupled mass model of a single span puts 5 parts out of 12 parts of the mass at either end and puts 2 parts into the coupling between the ends. The improves the frequency calculation to 78% accuracy. A double span lumped model

(3 GP) puts half the mass in the middle and one quarter at either end. The frequency calculation improves to 90%. A coupled mass model of a two span beam puts 5 parts in 24 at either end and 10 parts in 24 at the middle while 4 parts in 24 do the coupling. This brings the frequency to 98.5% of actual. Going next to a 3 span lumped model, $\frac{1}{6}$ of the mass is put at either end while $\frac{1}{3}$ of the mass is put at the two middle points. The frequency for this 4 point model is 92.5% accurate. This is now compared with a 3 span coupled model wherein 5 parts out of 36 are put at the end points; 10 parts out of 36 are assigned to each of the middle points and the remaining 6 parts are assigned to coupling between points. Here we lose ground a little achieving only 97.6% accuracy.

If one wants to achieve the same accuracy of staying within 0.5% in axial modeling as with bending, where the rule for meshes is 1 greater than the modal nodes, one must supply 5 more Grid Points than nodes in an axial mode for the coupled option and 7 more Grid Points than nodes in an axial mode for the lumped option.

The next question to ask is whether, if one modeled with the ideal number, one can preserve the accuracy if he were to condense to a coarser mesh? Data was gathered by condensing the 11 point models for lumped and coupled options to meshes of 2, 3, and 4 points.

EFFECT OF CONDENSATION ON AXIAL MODES OF CIRCULAR BEAM.
FUNDAMENTAL AXIAL MODE FREQUENCY FROM TIMOSHENKO 4914.57

| COND AMT. | LUMPED FREQ | RATIO | COUPLED FREQ | RATIO |
|-----------|-------------|--------|--------------|-------|
| 11-2 | 5365.34 | 1.0917 | 5391.85 | 1.097 |
| 11-3 | 5365.34 | 1.0917 | 5391.85 | 1.097 |
| 11-4 | 5048.91 | 1.0273 | 5070.99 | 1.032 |

Condensing the lumped model to 2 points degrades the axial fundamental frequency to 9% high. Strange to say adding one more point does not improve the accuracy one bit. Similarly condensing the coupled model to 2 and 3 points degrades the frequency to 10% high. A fourth point does improve the accuracy for both models to just 3% high. Condensing has a more drastic effect on the axial

mode than on the bending modes. Modeling with coupling has the advantage over lumped modeling here in that one does not need to model a fine mesh first to achieve 3% accuracy.

Torsion

The field is much more narrow when it comes to modeling the torsional mass in beams. No coupling algorithm is available, so all modeling is done by lumping. All of the data gathered from the runs on torsion can be presented in one table.

FUNDAMENTAL TORSIONAL MODE OF BEAMS WITH AND WITHOUT CONDENSATION
TIMOSHENKO CIRCULAR FN 3009.36   RECTANGULAR FN 1787.58

| GP | UNCONDENSED | | CONDENSED | | UNCONDENSED | | CONDENSED | |
|---|---|---|---|---|---|---|---|---|
| | FREQ | RATIO | FREQ | RATIO | FREQ | RATIO | FREQ | RATIO |
| 11 | 3035.19 | 1.009 | . | . | 1705.11 | 0.967 | . | . |
| 4 | 2819.39 | 0.937 | 3327.46 | 1.38 | 1583.89 | 0.889 | 1759.06 | 0.987 |
| 3 | 2743.90 | 0.912 | 3327.46 | 1.38 | 1541.47 | 0.865 | 1869.31 | 1.049 |
| 2 | 1940.23 | 0.645 | 3327.46 | 1.38 | 1089.98 | 0.612 | 1869.31 | 1.049 |

A fine mesh model gives accuracy to within 1% for circular and 3% for rectangular and it degrades slowly with $\frac{1}{3}$ mesh to 94% and 90% with $\frac{1}{4}$ mesh to 91% and 87%, but a single span gives a poor 64% and 61%. Condensing from an eleven point model to a coarser one has a uniformly degrading effect of 38% for circular, but has a uniformly beneficial effect for rectangular. Introducing the stiffness matrix into the condensation for rectangular sections has a different effect than circular sections in torsion because it couples according to a stiffer pattern than that through which the mass actually acts.

Grid Point Weight Generator

It might be well to examine the relationship of the mass properties computed by the GPWG module versus the character of the matrices MGG . GPWG computes the total mass of a structure, then employs a user specified reference point to locate the center of gravity and for computing the moments of inertia of the distribution of scalar mass throughout the structure. GPWG completely ignores any moments of inertia that are supplied to the individual grid points

of the MGG matrix. Therefore GPWG will report moments of inertia for a model which has absolutely no rotational degrees of freedom. GPWG gives information about the mass in a structure when viewed as a rigid body, while MGG indicates how localized mass is modeled amongst the elastic elements of a model. It is a good idea in a study such as this one to include the GPWG in every run to ensure that the rigid body properties of every model is exactly alike to give assurance that the same structure is being treated in every run.

## Summary of Beam Findings

If all we want to model is a beam or a beam-like structure we can form a few guide lines that can serve us well. It is when we get to complex structures in which beams make up only a part that the rules are less clear cut. Start with just beam-like structures. We will separate the decision making into three parts. Even before anything else, start with the back of an envelope and a reference like Den Hartog (4) or Timoshenko. Determine the frequency range of interest. Next decide what accuracy would be suitable for the task at hand. Estimate the average properties of the beam-like structure. Use those averages in the formulas, which Den Hartog has so logically published, to find out how many harmonics in bending, in torsion, and in longitudinal fall into the frequency range of interest. Decide on the level of detail that you want to invest and the degree of accuracy you want to achieve in your analysis. If a structure is in the early stage of design, many elements may be sketchy so that the approach may be to use a generous number of gird points in a model with primitive properties, then condense it. Then for modeling the mass one would use the rules for a scalar model. For bending, condense to a number of grid points equal to one more than the number of nodes in the highest harmonic. For axial modes condense to a grid point count equal to the number of nodes in the highest harmonic plus 7. For torsion, try to have as many grid points in the original model to represent the highest torsional mode without condensing, because they are liable to be degraded.

If the object is to certify a design, one generally has a

generous number of points in his model in order to recover stresses. In such cases, condensation will probably result in the retention of a mesh of points that will appear as a fine mesh to the vibrational modes. It is well however that the highest torsional mode and the highest axial mode be represented by a margin of about ten grid points.

One is advised to limit the use of CONM2 modeling of mass to those situations where there is not other alternative. For example, a non-prismatical beam, or mounted equipment, or a non-native portion of structure. If CONM2 elements are used with a well defined elastic model, it would be advisable to condense it by several factors to improve the mass modeling.

Summary for General Mass Modeling

One should still be guided by the expected harmonic nodal pattern as to the minimum number of gird points to assign to a model. For complicated structures this is not easy to estimate. As a start one could isolate individual pieces such as beams, plates, shells, or solids. Idealize each piece into classical closed form solution types such as free-free, pin-clamped, etc. Consult reliable sources for the modes and frequencies of these classical individuals, such as Den Hartog or Leissa (5). Determine the largest nodal pattern in the assembly, then try to extrapolate the mutual stiffening effect after these pieces are joined for the influence on the nodal pattern. You will probably have an adequate number based on the estimates from individual pieces. After the vibration analysis has been run, examine the nodal pattern of the highest harmonics of each class to see if you are close to a change in sign at every grid point. If so, you may have provided too coarse a mesh for harmonics that couldn't be found. If there are several grid points between nodes of the highest harmonics, you can be assured of having provided a good margin for that mode. Be especially careful in using lumped modeling to guard against too few grid points to avoid spurious modes.

Conclusion

The modeling of mass for BAR elements has been reviewed based on the accuracy to which various schemes predict the frequencies of modes with free-free boundary conditions only. The stiffness of BARS did not include the elasticity due to shear deformation. Findings confirmed that modeling by translational coupling will give almost perfect results for bending by following the guide line of one more grid point than nodes of the highest mode. Condensation has only a slight degrading effect on frequency prediction of bending modes when modeling mass with translational coupling. Condensation has an immensely beneficial effect on the bending modes when modeling mass with the scalar option or with CONM2 elements. Condensation can be tolerated for prediction of axial modes if the margin of retained grid points to nodes in the highest harmonic is generous. Condensation has a uniformly degrading effect on torsional modes. The ground work has been laid for extending the study of mass modeling from exclusively prismatical beams to non-prismatical beams. It was found that mass modeling by translational coupling cannot be improved upon by including rotary inertia coupling.

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