NONLINEAR FILTER DESIGN

L. R. Hunt* and Paul Whitney

ABSTRACT

Early work on the topic of nonlinear input-output system modeling is due to Volterra. Moreover, Wiener's research on nonlinear system identification is quite well known. Recent papers by Koh and Powers on Volterra filtering and nonlinear system identification and Boyd, Tang and Chua on measuring Volterra kernels have demonstrated the feasibility of obtaining "good" estimates of the first few kernels (especially the first and second) of a Volterra series. However, the higher order terms seem difficult to obtain for a variety of reasons.

The purpose of this paper is to introduce a new technique for identifying nonlinear systems, and we begin with a single input - single output system. Assuming the system is initially at rest, we calculate the first kernel (first convolution integral in the continuous case or first convolution sum in the discrete case). We then obtain a controllable and observable linear realization in a particular canonical form. We probe the actual nonlinear system with an appropriate input (or inputs) and determine the output (or outputs). For the linear system we compute the input that produces the same output. In the difference between the inputs to the nonlinear and linear systems, we find basic information about the nonlinear system. There is an interesting class of nonlinear systems for which this type of identification scheme should prove to be acccurate.

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1. Introduction

Assume we have a system that can be probed with inputs to produce corresponding outputs, which we record. We investigate the problem of identifying a nonlinear mathematical model of this physical system. For simplicity we take a deterministic approach (even though statistical techniques can be useful in our method) because we are interested in discovering information about the inherent mathematical structure in the systems. Moreover, we shall consider only the continuous time case for single-input single-output models.

Most nonlinear system modeling techniques (including the work of Wiener [1]) depend on the research of Volterra [2] and the mathematics intrinsic in the Volterra series and Volterra kernels. Recent excellent references for Volterra and Wiener theories are Fliess, Lamnabhi, and Lamnabhi-Lagarrigue [3] and Schetzen [4], respectively. Interesting nonlinear mathematical modeling results can be found in Boyd, Tang, and Chua [5] for the continuous time case and Koh and Powers [6] for the discrete time case. All of the above research depends on the Lie algebra structure inherent in a nonlinear system. For an examination of this Lie algebra structure in realization theory we suggest a paper of Crouch [7].

Recent papers in the area of nonlinear control have demonstrated a remarkable fact; many physical systems have mathematical models that are feedback equivalent to linear systems. References for the equivalence problem (in which the output equation is ignored) are [8], [9], [10], [11], [12], [13], [14], [15], [16]. Applications are in the following various areas (many of the models are multi-input, multi-output)

i) aircraft control.

ii) robotics.

iii) satellite control.

iv) chemical engineering.

v) electric motor control.

Authors have also investigated the problem of determining conditions under which a nonlinear system with output is feedback equivalent to a linear system with linear output (see [18] and [19]).

The above facts suggest the following approach to nonlinear modeling. Instead of trying to identify Volterra kernels, why don't we attempt to identify the nonlinear feedback that pushes our system toward a linear system? If a system is feedback equivalent to a linear system with linear output, we should obtain good results. If a system is not feedback equivalent, then we should obtain an "approximation" that is at least as good as the linear model obtained through standard techniques. However, the fact that so many physical systems have feedback equivalent (to a linear system) models is extremely encouraging.

Our technique for identifying single-input, single-output nonlinear systems which are initially at rest is described as follows:

1) Assume that the linear part of the system has been found and realized in controllable canonical form. Since we desire a minimum realization here, we suppose that our state space realization is also observable.

2) probe the nonlinear system with an input u producing an output y. Invert the linear part of the system to find an input u_L producing the same output y.

3) read the states $x = (x_1, x_2, \ldots, x_n)$ of the linear system or attach a Luenberger observer (or Kalman filter in the presence of noise).

4) determine functions $\alpha(x)$ and $\beta(x)$ so that

$$u_L = \alpha(x) + \beta(x)u$$

for all inputs u, or more realistically, for an interesting finite subclass of probes u.

Assuming that we obtain the linear part of a nonlinear system, the states $x = (x_1, x_2, \ldots, x_n)$, and the functions $\alpha(x)$ and $\beta(x)$, we show that the above technique yields excellent results for those systems having nonlinear mathematical models which are feedback equivalent to controllable and observable linear systems having linear output.

In section 2 of this paper we consider the Volterra series approach, several examples, and prove a result concerning our identification technique. Section 3 contains a discussion of future directions and related problems.

2. An Identification Technique

We start with a state space representation of a nonlinear system and derive a Volterra series expansion via the formulas found in [3]. Our single-input, single-output nonlinear system is

$$\dot{x}(t) = f(x) + ug(x)u$$

(2.1) $$y(t) = h(x)$$

where f and g are real analytic vector fields on $\mathbb{R}^n$, h is a real analytic function, u is the control or input and f and h both vanish at the origin. If dh denotes the gradient of h and $\langle \cdot,\cdot \rangle$ denotes the duality of one forms and vector fields, then the Lie derivative of h with respect to f is

$$L_f h = \langle dh, f \rangle .$$

Similarly, we can define

$$L_g h,\ L_f^2 h, = L_f(L_f h),\ L_g^2 h,\ L_g L_f h,\ L_f L_g h, \text{ etc.}$$

The Volterra series for the system starting at the origin at time $t = 0$ is

$$
\begin{aligned}
y(t,u) &= \int_0^t w_1(t,\tau_1)u(\tau_1)d\tau_1 \\
&+ \int_0^t \int_0^{\tau_2} w_2(t,\tau_2,\tau_1)u(\tau_2)u(\tau_1)d\tau_2 d\tau_1 + \cdots \\
&+ \int_0^t \int_0^{\tau_s} \cdots \int_0^{\tau_2} w_s(t,\tau_s,\ldots,\tau_1)u(\tau_s)\ldots u(\tau_1)d\tau_s \ldots d\tau_1 + \cdots .
\end{aligned}
\tag{2.2}
$$

Here

$$
\begin{aligned}
w_1(t,\tau_1) &= \sum_{\nu_0,\nu_1,\geq 0} L_f^{\nu_0} L_g L_f^{\nu_1} h(0) \frac{(t-\tau_1)^{\nu_1} \tau_1^{\nu_0}}{\nu_1! \nu_0!} \\
w_2(t,\tau_2,\tau_1) &= \sum_{\nu_0,\nu_1,\nu_2 \geq 0} L_f^{\nu_0} L_g L_f^{\nu_1} L_g L_f^{\nu_2} h(0) \frac{(t-\tau_2)^{\nu_2} (\tau_2-\tau_1)^{\nu_1} \tau_1^{\nu_0}}{\nu_2! \nu_1! \nu_0!} \\
&\vdots \\
w_s(t,\tau_s,\ldots,\tau_1) &= \sum_{\nu_0,\ldots,\nu_s \geq 0} L_f^{\nu_0} L_g L_f^{\nu_1} \ldots L_g L_f^{\nu_s} h(0) \frac{(t-\tau_s)^{\nu_s} \ldots \tau_1^{\nu_0}}{\nu_s! \ldots \nu_0!} \\
&\vdots
\end{aligned}
\tag{2.3}
$$

Since $f(0) = 0$, we take all ν_0 in the above summations to be 0.

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Example 2.1. We consider the 2-dimensional nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1^2 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(2.4)

$$y = h(x) = x_1.$$

Computing some Lie derivatives we find $L_gL_fh(0) = 1$ in $w_1(t,\tau_1)$, $L_gL_fL_gL_f^3h(0) = 2$ in $w_2(t,\tau_2,\tau_1)$ and $L_gL_fL_gL_f^3L_gL_f^3h(0) = 4$ in $w_3(t,\tau_3,\tau_2,\tau_1)$. In fact we can find at least one nonzero term in each of $w_1, w_2, \ldots, w_s, \ldots$ giving us an infinite number of nonzero Volterra kernels corresponding to an infinite dimensional Lie algebra structure associated with system (2.4).

Suppose that we do not know system (2.4), but use linear analysis to identify the linear part of the system which is associated with the first Volterra kernel in (2.3). After realization in controllable canonical form we have the linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + u_L \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(2.5)

$$y = x_1 .$$

For an input u into the nonlinear system we obtain an output y(t), which we assume is smooth. Substituting y(t) into (2.5) and differentiating we have

$$\dot{y}(t) = \dot{x}_1 = x_2$$
$$\ddot{y}(t) = \dot{x}_2 = u_L .$$

Thus plugging u_L into the linear system yields the same output as substituting u into the nonlinear system. It is clear from (2.4) (if we knew such equations) that

$$y(t) = \dot{x}_1 = x_2$$
$$\ddot{y}(t) = \dot{x}_2 = x_1^2 + u \ .$$

and $u_L = x_1^2 + u$.

We do know u, u_L and $x_1 = y$ (the output of the linear system). If we recognize that $u_L - u = x_1^2$ for pairs of inputs u and u_L, then we have identified the nonlinear system in an extremely easy manner. For example, the input $u = 1 - \frac{t^4}{4}$ into the nonlinear system corresponds to the input $u_L = 1$ into the linear system for producing the identical output $\frac{t^2}{2}$. Then $u_L - u = 1 - (1 - \frac{t^4}{4}) = (\frac{t^2}{2})^2 = x_1^2$.

This illustrates the general procedure involving steps i) - iv) that we mentioned in the introduction.

The preceeding example is of mathematical interest. An example of a physical system for which our 4 step technique should prove useful is the single link manipulator with joint elasticity [17]. To conserve pages we shall not work out the details here.

In step 4 of our technique we are to determine functions $\alpha(x)$ and $\beta(x)$ so that $u_L = \alpha(x) + \beta(x)\, u$. In our examples we have the special case that $\beta(x) \equiv 1$ and then identify $\alpha(x)$.

It is well understood that for a controllable and observable linear system of dimension n, there are n^2 parameter that must be identified as is easily seen in a canonical form (either controllable or observable).

Let's consider the possibility of a canonical form for a nonlinear system which is feedback equivalent to a controllable and observable linear system of dimension n.

For feedback equivalence there are three types of operations involved:

a) state space coordinate changes

b) nonlinear feedback of the form $u + \alpha(x)$

c) space dependent input changes of the form $\beta(x)u$, where $\beta(x)$ is nonvanishing.

Most feedback equivalence results are local in nature, so we assume that all discussions in this paper hold for some open neighborhood of the origin in state space.

Suppose we consider the nonlinear system on $\mathbb{R}^n$

$$\text{(2.6)} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ a_1x_1+a_2x_2+\ldots+a_nx_n+\alpha(x) \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta(x) \end{bmatrix}$$

$$y = c_1x_1 + c_2x_2 + \ldots + c_nx_n = h$$

where $\alpha(x)$ and $\beta(x)$ are real analytic, $\beta(x)$ is nonvanishing with $\beta(0) = 1$, and $dh, dL_p h, \ldots, dL_f^{h-1}h$ are linearly independent. It is very easy to see that letting $u_L = \alpha(x) + \beta(x)u$, we have a controllable and observable linear system with controllable canonical form

$$
(2.7)\qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ a_1x_1+a_2x_2+\ldots+a_nx_n \end{bmatrix} + u_L \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

$$
y = c_1x_1 + c_2x_2 + \ldots + a_nx_n \ .
$$

Thus systems like (2.6) fit our technique for identifying nonlinear system which are feedback equivalent to linear systems having linear output.

Theorem 2.1. A nonlinear system (2.1) which is feedback equivalent to a controllable and observable n-dimensional linear system (with linear output) can be put in the form (2.6).

Proof: The state space coordinate changes $(\xi_1, \xi_2, \ldots, \xi_n)$ to move the dynamic equation

$$
\dot{x} = f(x) + ug(x)
$$

from (2.1) in the direction of Brunovsky canonical form must satisfy the partial differential equations

$$
\begin{aligned}
&\langle d\xi_1, (ad^k f, g)\rangle = 0, \quad k = 0,1,\ldots,n-2 \\
&\langle d\xi_2, (ad^k f, g)\rangle = 0, \quad k = 0,1,\ldots,n-3 \\
&\qquad \vdots \\
(2.8)\qquad &\langle d\xi_{n-1}, g\rangle = 0 \\
&\langle d\xi_1, f\rangle = \xi_2 \\
&\langle d\xi_2, f\rangle = \xi_3 \\
&\qquad \vdots \\
&\langle d\xi_{n-1}, f\rangle = \xi_n \ .
\end{aligned}
$$

Here $(ad^k f,g)$ is the usual Lie bracket notation (see [13]. Under such coordinate changes equations (2.1) become

$$(2.9)\quad \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_{n-1} \\ \dot{\xi}_n \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \xi_3 \\ \vdots \\ \xi_n \\ a_1\xi_1+a_2\xi_2+\ldots+a_n\xi_n+\tilde{\alpha}(\xi) \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{\beta}(x) \end{bmatrix}$$

$$y = h(\xi) \ .$$

The state space coordinate transformations $\xi = (\xi_1,\xi_2,\ldots,\xi_n)$ are not unique. It is shown in [18] that if system (2.1) is feedback equivalent to a nonlinear system with linear output, then at least one of these transformations gives us that linear output. Hence, with this choice, we can assume the output equation in (2.9) is $y = c_1\xi_1 + c_2\xi_2 +\ldots+ c_n\xi_n$.

If $\tilde{\beta}(0) = 1$, we have the desired form. Assume $\tilde{B}(0) = r$, $r \neq 1$, and let

$$(2.10)\quad \begin{aligned} x_1 &= \frac{1}{r}\xi_1 \\ x_2 &= \frac{1}{r}\xi_2 \\ &\vdots \\ x_n &= \frac{1}{r}\xi_n \end{aligned}$$

Then (2.9) becomes

$$(2.11)\quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ a_1x_1+a_2x_2+\ldots+a_nx_n+\alpha(x) \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta(x) \end{bmatrix}$$

$$y = c_1x_1+c_2x_2+\ldots+c_nx_n$$

with $\beta(0) = 1$. □

In example 2.1, if we attempt to identify the nonlinear system by our technique, the first step is to model the linear part of the system. Then we pursue the function $\alpha(x)$, which is a function of the states of the linear system.

However, consider the following diagram where NL denotes the nonlinear system, and L inverse is the inverse of the linear system (the linear part that we have identified).

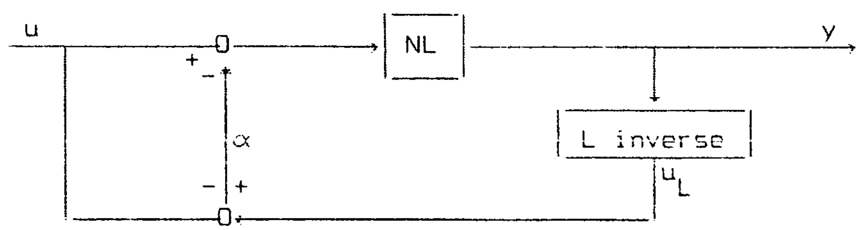
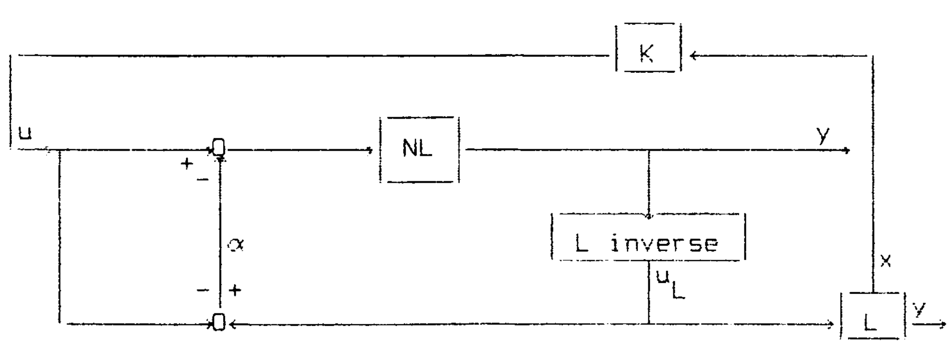


Figure 1

Here u and y are the input and output of the nonlinear system, respectively. The function α is the difference of u_L and u, but we have no need to identify it precisely. For example, suppose we are to stabilize the nonlinear system about the 0 equilibrium point. Choose a feedback matrix $K = [k_1, k_2, \dots, k_n]$ to stabilize the linear part of the system and apply the following diagram, where L denotes the linear system.



3. Related Problems

The following discussion involves possible computer-aided methods for implementing the modeling technique introduced in the first two sections of this paper.

Suppose we consider a physical system that we probe with a finite number of inputs and record the corresponding outputs. If we wish to model this system as an n^{th} order controllable and observable linear system, we first establish a criterion (usually least squares) for "closeness of match." We assume a model (e.g. controllable form)

$$(3.1) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-4} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ a_1x_1+a_2x_2+\ldots+a_nx_n \end{bmatrix} + u_L \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$y = c_1x_1+c_2x_2+\ldots+c_nx_n,$$

and compute the parameters $a_1,a_2,\ldots,a_n,c_1,c_2,\ldots,c_n$ giving us a "best match."

Thus we have approximated our system by a linear model in canonical form. Since those nonlinear systems that are feedback equivalent to a linear system with linear output contain the linear systems as a proper subclass, we use the identified linear model (3.1) as a first step in our more general process. We consider the "canonical form"

$$(3.2)\quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ a_1x_1+a_2x_2+\dots+a_nx_n+\alpha(x) \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta(x) \end{bmatrix}$$

$$y = c_1x_1+c_2x_2+\dots+c_nx_n,$$

where $a_1, a_2, \dots, a_n, c_1, c_2, \dots, c_n$ are known.

Invert the linear system (3.1) to obtain the finite number of corresponding inputs u_L having the same outputs y as the nonlinear system with inputs u. Knowing the states $x_1, x_2, \dots, x_n$ of the linear system, we set up a "closeness criterion" for computing $\alpha(x)$ and $\beta(x)$ in $u_L = \alpha(x)+\beta(x)u$. Perhaps an appropriate method here is to assume that $\alpha(x)$ and $\beta(x)$ are in some standard set of functions; e.g. polynomials of degree m,

$$\alpha(x) = \sum_{|k|\geq 1}^{m} \alpha_{(k)} x^{(k)}$$

$$\beta(x) = 1 + \sum_{|k|\geq 1}^{m} \beta_{(k)} x^{(k)}.$$

Here we have used a multi-index notation for the coefficients $\alpha(x)$ and $\beta(x)$, and $x = (x_1, x_2, \dots, x_n)$. For example we set

$$\alpha(x) = \alpha_1x_1+\alpha_2x_2+\dots+\alpha_nx_n+\alpha_{1,1}x_1^2+\alpha_{1,2}x_1x_2+\dots+\alpha_{1,n}x_1x_n+$$

$$\alpha_{2,2}x_2^2+\alpha_{2,3}x_2x_3+\dots+\alpha_{m,m,\dots,m}x_n^m.$$

Plans are to set up simulation studies as well as prove results concerning the above method.

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