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PRECONDITIONING MATRICES FOR CHEBYSHEV DERIVATIVE OPERATORS

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ABSTRACT:

The problem of preconditioning the matrices arising from pseudo-spectral Chebyshev approximations of first order operators is considered in both one and two dimensions. In one dimension a preconditioner represented by a full matirx which leads to preconditioned eigenvalues that are real, positive and lie between 1 and $\pi/2$, is already available. Since there are cases in which it is not computationally convenient to work with such a preconditioner, we study a large number of preconditioners which are "more sparse" (in particular three and four diagonal matrices). The eigenvalues of such preconditioned matrices are compared. In particular, the analysis is carried out for the quantity $\max|\lambda i|/\min|\lambda i|$, where λ_i are the preconditioned eigenvalues.

We apply the results to the problem of finding the steady state solution to an equation of the type $u_t = u_x + f$, where Chebyshev collocation is used for the spatial variable and time discretization is performed by the Richardson method.

In two dimensions different preconditioners are proposed for the matrix which arises from the pseudo-spectral discretization of the steady state problem in the square

$$A = \{(x,y,): -1 \leq x \leq 1, -1 \leq y \leq 1\}$$

$$U_x + U_y = f$$

$$U(x, y, 0) = U_0$$

with boundary conditions at $x = 1$ and $y = 1$. Results are given for the CPU time and the number of iterations using a Richardson iteration method for the unpreconditioned and preconditioned cases.

1. INTRODUCTION

To obtain the pseudo-spectral or collocation approximation let P_N be an interpolation operator. Let f(x) be a sufficiently smooth function defined in [-1,1] where f(x)=0 at the appropriate boundaries which yields a well-posed problem for (1.1). Then $P_N f$ is the interpolation of f at the collocation points x_j, i.e.

$$P_N f(x_j) = f(x_j) \quad \text{and} \quad P_N f \epsilon B_N. \quad j = 0,\ldots,N$$

To obtain a Chebyshev Gauss-Lobatto pseudo-spectal approximation in the interval [-1,1] we choose $x_j = \cos j\pi/N$ $(j = 0,\ldots,N)$, which when $j \neq 0,N$ are the extrema of the N^{th} order Chebyshev polynomials $T_N(x) = \cos(N\cos^{-1}x)$. In order to construct the interpolant of f(x) at x, we define the polynomials

$$g_j(x) = \frac{(1-x^2)T_N'(x)(-1)^{j+1}}{\bar{c}_j N^2(x - x_j)} \qquad (j = 0,\ldots,N)$$

$$\bar{c}_0 = \bar{c}_N = 2, \quad \bar{c}_j = 1 \quad (1 \leq j \leq N-1).$$

One can easily see that $g_j(x_k) = \delta_{jk}$.

The N^{th} degree interpolation polynomial $P_N f$ to f is given by

$$P_N f(x) = \sum_{j=0}^{N} f(x_j) g_j(x) \qquad x \epsilon \mathbb{R} \tag{1.8}$$

We must now be able to express derivatives of $P_N f$ in terms of f at the collocation points x_j. Differentiating (1.8) we obtain

$$\frac{d^n P_N f(x)}{dx^n} = \sum_{j=0}^{N} f(xj) \frac{d^n}{dx^n} g_j(x) \tag{1.9}$$

so that

$$\frac{d^n P_N f(x_k)}{dx^n} = \sum_{j=0}^{N} f(x_j) \, (D_n)_{kj} \tag{1.10}$$

where

$$(1.11) \qquad (D_n)_{jk} = \frac{d^n}{dx^n} g_k(x)\Big|_{x = x_j}$$

The pseudo-spectral Chebyshev derivative operator can be represented by the $N \times N$ matrix $S_n = \{s_{ij}\}$,

where

$$s_{jk} = \frac{\bar{c}_j(-1)^{j+k}}{\bar{c}_k(x_j - x_k)} \qquad (k \neq j)$$

$$s_{jj} = \frac{-x_j}{2(1 - x_j^2)}, \quad s_{00} = \frac{2N^2 + 1}{6} = -s_{NN}$$

In particular the Chebyshev pseudo-spectral approximation for $u_t = u_x$, $u(x,0) = u_0(x)$ is given by $u_N = \sum_{k=0}^{N} u(x_j,t)g_j(x)$ and

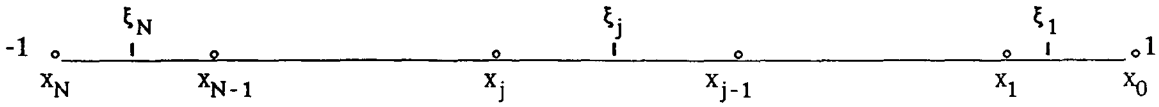
$$\frac{\partial u_N}{\partial t}(x_j,t) = \sum_{k=0}^{N} u(x_j,t)s_{kj} .$$

2. OUTLINE OF THE PROBLEM

In the interval [-1,1] let

$$\xi_j = \cos \frac{2j-1}{2N} \pi \qquad (j = 1,2,\dots,N)$$

ξ_j lies between x_j and x_{j-1}



The modes x_j and ξ_j have the following properties:

$$T_N(\xi_j) = 0 \quad j = 1,\dots,N, \text{ and } T_N(x_i) = (-1)^i \quad i = 0,\dots,N$$

Then consider the pseudo-spectral Chebyshev derivative operator with homogeneous boundary conditions at x = 1. This operator can be represented by the N × N matrix $S_N = \{s_{ij}\}$, where

$$S_N = \begin{cases} \dfrac{-2N^2 + 1}{6} & \text{if } i = j = N \\[2ex] \dfrac{\bar{c}_i(-1)^{j+i}}{\bar{c}_j\,(x_i - x_0)} & \text{if } i \neq j \\[2ex] \dfrac{-x_j}{2(1 - x_j^2)} & \text{if } i = j = 1,\dots,N-1 \end{cases}$$

The matrix S_N is full. The condition number $C(S_N)$ of S_N is large. We have the following result which was obtained by Daniele Funaro.

Lemma 1-1: The condition number of S_N increases at least like N^2.

Proof: Let $\| \cdot \|$ denote the norm in $\mathfrak{L}(\mathbb{R}^N, \mathbb{R}^N)$. Then the condition number of S_N is given by

(2.1) $$c(S_N) = \|S_N\| \, \|S_N^{-1}\| \, .$$

It is known that $\|S_N\| \geq \rho(S_N) \geq C_1N^2$, where $\rho(S_N)$ is the spectral radius of S_N and c_1 is a constant independent of N. On the other hand, we have

$$(2.2) \qquad \|S_N^{-1}\| = \sup_{\|\vec{\varphi}\|_{\mathbb{R}^N} = 1} \|S_N^{-1}\vec{\varphi}\|_{\mathbb{R}^N}.$$

Choose $\vec{\varphi}_0 = \left[\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}\right]$. Then $\|\vec{\varphi}_0\|_{\mathbb{R}^N} = 1$ and $(S_N^{-1}\vec{\varphi}_0)_j = \frac{1}{\sqrt{N}}(x_j - 1)$, for $j = 1, \ldots, N$. Furthermore,

$$(2.3) \qquad \|S_N^{-1}\vec{\varphi}_0\|_{\mathbb{R}^N} = \frac{1}{\pi}\sum_{j=1}^{N}(x_j - 1)^2 \frac{\pi}{N} \geq \frac{1}{\pi}\int_{-1}^{1}(x - 1)^2\, \omega dx = c_2$$

where c_2 does not depend on N.

This implies $\|S_N^{-1}\| \geq \|S_0^{-1}\vec{\varphi}_0\|_{\mathbb{R}^N} \geq c_2$. Finally, using (2.1), we get $C(S_N) \geq C_3N^2$. This proves the claim.

Although the condition number is particularly meaningful for numerical applications, its determination is generally very difficult. Another quantity which is meaningful for practical application is $\sigma(M) = \max_i |\lambda_i| / \min_i |\lambda_i|$ where M is an $N \times N$ matrix and λ_i $i = 1, \ldots, N$ are its eigenvalues. It can be shown empirically that $\sigma(S_N)$ behaves like N.

We are interested in finding a "preconditioner" for S_N. In particular we are concerned with finding a matrix R_N such that the quantitity $\sigma(R_N^{-1}S_N)$ is small. In general this does not imply that the corresponding condition number will be small (so the word "preconditioner" is not correct).

In [DF] S_N is preconditioned by $R_N = Z_ND_N$ where the $N \times N$ matrix $D_N = \{d_{ij}\}$ is defined by

$$d_{ii} = -1/(x_{i-1} - x_i) \qquad i = 1, \ldots, N$$

$$d_{ii-1} = 1/(x_{i-1} - x_i) \qquad i = 2, \ldots N$$

$$d_{ij} = 0 \qquad \text{otherwise}$$

Hence D is the upwind finite differences matrix relative to the grid x_i. $Z_N:P_{N-1} \to P_{N-1}$ is the operator which maps the values of a polynomial in P_{N-1} at the staggered grid points $\{\xi_1,\dots,\xi_N\}$ into the values at the mesh points $\{x_1,\dots,x_N\}$.

Preconditioning by $Z_N D_N$ results in preconditioned eigenvalues that are real, positive and lie between 1 and $\pi/2$. The ratio $\sigma(D_N^{-1} Z_N^{-1} S_N)$ is bounded by $\pi/2$ (see [DF]). This is particularly interesting when the solution of the system

(2.4) $$S_n u + f = 0$$

has to be found. If an iterative method is used, iterating $M_N = (Z_N D_N)^{-1} S_N$ instead of S_N results in convergence in a few iterations. In the end of the computation the system $(Z_N D_N)u + f = 0$ has to be solved. So we require that the matrix $R_N = Z_N D_N$ can be inverted easily. Although Z is a full matrix, it can be inverted very inexpensively in N log N operations. Thus the matrix $R_N = Z_N D_N$ can be inverted very inexpensively. Nevertheless there are cases in which it is not computationally convenient to work with a full preconditioner.

In particular when using an implicit method to find the solution at time T > 0 of the equation

$$\begin{aligned} &u_t = S_N u + f, \\ (2.5) \qquad &u(x,0) = u_0(x), \\ &u(1,t) = 0 \quad 0 \leq t \leq T. \end{aligned}$$

If the implicit Euler method is used, iterates of the matrix $(I + \Delta t S_N)$ are considered. A good preconditioner for this matrix turns out to be the matrix $(I + \Delta t Z_N D_N)$. Unfortunately, due to the fact that Z_N is a full matrix, $(I + \Delta t Z_N D_N)$ cannot be inverted inexpensively. In this work we shall present a large number of preconditioners which can be applied to the situations illustrated above.

3. ANALYSIS IN ONE DIMENSION

In order to have the matrix $I + \Delta t\, Z_N D_N$ which can be easily inverted we substitute Z_N by some suitable matrix which has a simpler form and which has to be regarded as an approximation of the operator related to Z_N.

The first idea is the following. Take $\tilde{Z}_N = \{\tilde{z}_{ij}\}$ such that

$$
\begin{aligned}
\tilde{z}_{ii} &= .5 \\
\tilde{z}_{ii+1} &= .5 \\
\tilde{z}_{NN} &= 1.
\end{aligned}
\tag{3.1}
$$

Then $\tilde{Z}_N D_N$ is a tridiagonal matrix so tha $I + \Delta t\, \tilde{Z}_N D_N$ is also a tridiagonal matrix. It can be shown that the eigenvalues of $M_N = (\tilde{Z}_N D_N)^{-1} S_N$ take the following form.

$$\lambda_k = k \sum_{j=0}^{k-1} x_j \quad k = 1,\ldots,N$$

Thus, the eigenvalues of the preconditioned matrix are real and positive. Hence we have $\sigma(M_N) = N$. The choice of $\tilde{Z}_N$ as in (3.1) corresponds to shifting the values from the staggered mesh to the initial mesh by averaging the two neighbour values. Instead of this we can choose $\tilde{Z}_N = \{\tilde{Z}_{ij}\}$ corresponding to interpolation by first order polynomials. This leads to the following definition of the matrix $\tilde{Z}_N = \{\tilde{Z}_{ij}\}$

$$
\begin{aligned}
z_{ii} &= \frac{x_i - \xi_{i+1}}{\xi_i - \xi_{i+1}} \qquad i = 1,\ldots,N-1 \\
z_{ii+1} &= \frac{x_i - \xi_i}{\xi_{i+1} - \xi_i} \qquad i = 1,\ldots,N-1
\end{aligned}
\tag{3.2}
$$

We found emperically, that the eigenvalues of the preconditioned matrix are still real and positive, but the quantity $\sigma(M_N)$ is now worse than that of the previous case.

Another simple preconditioner which this time is not of the form $Z_N D_N$ is defined by $R_N = \{r_{ij}\}$ where

$$
(3.3)\qquad
\begin{array}{lll}
r_{ii} = 0 & & i = 1,\dots,N\text{-}1 \\
r_{NN} = 1/(x_{N-1} - x_N) & & \\
r_{i,i+1} = 1/(x_{i-1} - x_{i+1}) & & i = 1,\dots,N\text{-}1 \\
r_{i,i-1} = 1/(x_{i-1} - x_{i+1}) & & i = 2,\dots,N\text{-}1 \\
r_{N-1,N} = 1/(x_{N-1} - x_N) & &
\end{array}
$$

In this case we still get real and positive preconditioned eigenvalues. Namely, they take the form:

$$
\left\{ \lambda_k = \frac{k \sin \pi/N}{\sqrt{1 - x_k^2}} \quad k = 1,\dots,N\text{-}1 \right\} \cup \{\lambda_N\},
$$

where λ_N is approximately equal to 2.46. We have $\sigma(M_N) = N\text{-}1$. This is the best we tested using tridiagonal preconditioners. Up to now the improvements are poor so we have to consider better approximations of the matrix Z_N.

We consider an approximation of the operator related to Z_N by interpolation with a polynomial of degree 2. One possible choice is the following: $\tilde{Z} = \{\tilde{z}_{ij}\}$ where

$$
(3.4)\qquad
\left\{
\begin{array}{ll}
\hat{z}_{11} = (x_1 = \xi_2)/(\xi_1 - \xi_2) & \\
\hat{z}_{12} = (x_1 - \xi_1)/(\xi_2 - \xi_1) & \\
\hat{z}_{i,i-1} = (x_i - \xi_i)(x_i - \xi_{i+1})/((\xi_{i-1} - \xi_i)(\xi_{i-1} - \xi_{i+1})) & i = 2,\dots,N\text{-}1 \\
\hat{z}_{ii} = (x_i - \xi_{i-1})(x_i - \xi_{i+1})/((\xi_i - \xi_{i-1})(\xi_i - \xi_{i+1})) & i = 2,\dots,N\text{-}1 \\
\hat{z}_{i,i+1} = (x_i - \xi_{i-1})(x_i - \xi_i)/((\xi_{i+1} - \xi_{i-1})(\xi_{i+1} - \xi_i)) & i = 2,\dots,N\text{-}1 \\
\hat{z}_{N,N-1} = (x_N - \xi_N)/(\xi_{N-1} - \xi_N) & \\
\hat{z}_{NN} = (x_N - \xi_{N-1})/(\xi_N - \xi_{N-1}) &
\end{array}
\right.
$$

Now $\hat{Z}$ is a three-diagonal matrix, hence $\hat{Z}D$ is a four-diagonal matrix. The numerical experiments performed up to $N = 32$ give the following results. The eigenvalues λ of

$\hat{M}_N = (Z_N D_N)^{-1} S_N$ are in general complex. The have positive real part greater than .98. Most of them are concentrated at 1. Figure 3.1 shows the behavior of the λ's for N = 12 and N = 20.

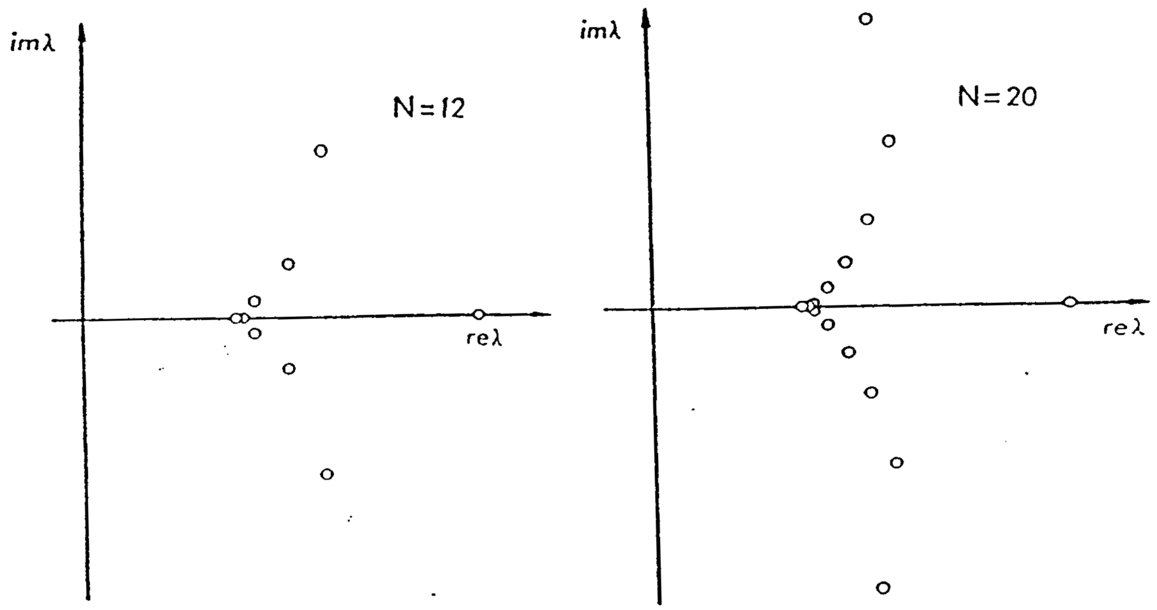


Figure 3.1 - Location of the preconditioned eigenvalues in the complex plane.

The corresponding $\sigma(M_N)$ is represented in Table 3.1 for various N. This time $\sigma(M_N)$ is bounded with respect to N.

N	$\sigma(M_N)$
8	2.602
16	2.724
24	2.758
32	2.770

Table 3.1 - Case of Four-diagonal preconditioner

Other possibilities for the tridiagonal matrix $\hat{Z}$ were tested. For example one possible choice, analogous to that of the matrix in (3.1), is to take $\hat{Z}_N = \{\hat{z}_{ij}\}$ such that

$$\begin{aligned} \hat{z}_{i,i-1} &= -1/8 \\ \hat{z}_{ii} &= 3/4 \\ \hat{z}_{i,i+1} &= 3/8 \end{aligned} \tag{3.5}$$

Among all the experiments, the matrix proposed in (3.4) gives the best results. Five-diagonal preconditioners were tested, through interpolation by third degree polynomials. The results do not imporve those corresponding to Table 3.1.

Now, if the system of linear equations (1.5) is solved, for example, by the implicit Richardson method, we are concerned with $\sigma(M_N)$, where $M_N = (I + \Delta t\ \hat{Z}_N D_N)^{-1}(I + \Delta t\ S_N)$ and $\hat{Z}_N$ is the matrix in (3.4). The graph of $\sigma(M_N)$ versus Δt is reported in Figure 2.2 for some values of N. So, in addition to the fast convergence of the iterative scheme, the preconditioning matrix can be inverted efficiently.

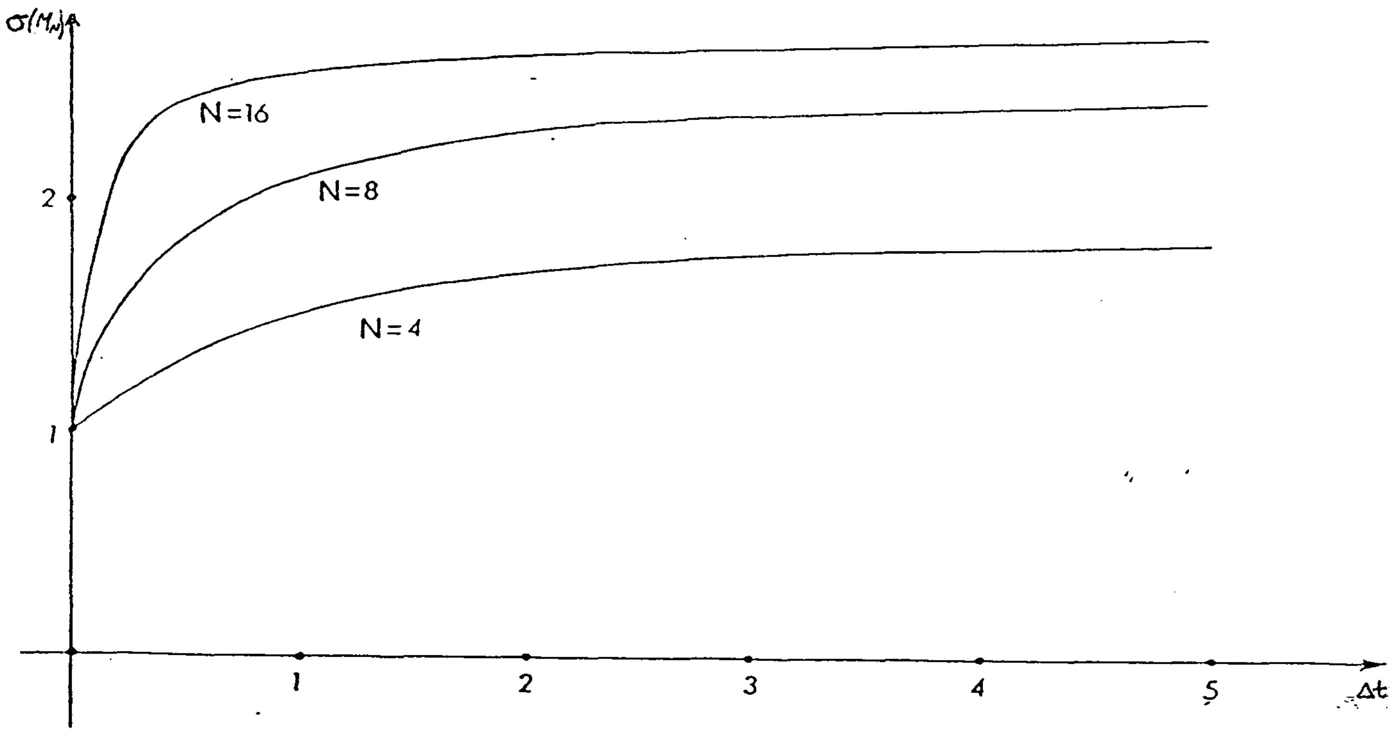


Figure 3.2

Now, if the steady state solution of problem (2.5) has to be found (which corresponds to the solution of problem (2.4), the explicit Richardson method can be used. This leads to the iterative scheme

$$u^{n+1} = (I + \Delta t S_N)u^n \ , \ n \in N \tag{3.6}$$

If λ denotes the general eigenvalue of S_N, the scheme is stable provided we choose Δt such that

$$0 < \Delta t < \inf_{\lambda} \left\{ \frac{-2\mathrm{Re}\lambda}{|\lambda|^2} \right\}. \tag{3.7}$$

Within the stability region we have $\rho(I + \Delta t S_N) < 1$ where ρ denotes the spectral radius.

In order to speed up the convergence we experimentally find Δt^* such that $\rho(I + \Delta t^* S_N)$ attains its minimum inside the interval of stability. In general Δt^* is not available. We use it here only to compare preconditioners. The same experiments are made for the preconditioned schemes, i.e.:

$$u^{n+1} = (I + \Delta t Z_N D_N)^{-1}(I + \Delta t S_N)u^n \tag{3.8}$$

$$u^{n+1} = (I + \Delta t \hat{Z}_N D_N)^{-1}(I + \Delta t S_N)u^n \tag{3.9}$$

where Z_N and $\hat{Z}_N$ are respectively, the full matrix proposed in [DF] and the tridiagonal matrix given by (3.4).

Both the schemes (3.8) and (3.9) converge to the same solution of (3.6). The results of these experiments are reported in Tables 3-2, 3-3, and 3-4

N	Maximum Δt for Stability	Δt^*	ρ Corresponding to Δt^*
8	.05461	.02724	.9817
16	.01836	.009120	.9919
24	.009232	.005244	.9939
32	.005138	.002521	.9966

Table 3-2 Minimum spectral radius of the unpreconditioned amplification matrix

N	Maximum Δt for Stability	Δt^*	ρ Corresponding to Δt^*
8	1.281	.7810	.2190
16	1.275	.7787	.2213
24	1.274	.7783	.2217
32	1.274	.7782	.2218

Table 3-3 Minimum spectral radius of the preconditioned matrix: case of Z_N.

N	Maximum Δt for Stability	Δt^*	ρ Corresponding to Δt^*
8	.7685	.5552	.4448
16	.6381	.3190	.7112
24	.4258	.2129	.8465
32	.3210	.1605	.9043

Table 3-4 Minimum spectral spectral radius of the preconditioned matrix: case of $\hat{Z}_N$

We also considered the second order Runge-Kutta scheme. This leads to the iterative scheme

$$(3.10) \qquad u^{n+1} = (I + \Delta t S_N + \frac{\Delta t^2}{2} S_N^2)u^n \quad , \quad n \in N$$

In this case the stability restriction on Δt is given by

$$(3.11) \qquad \Delta t^3|\lambda|^4 + 4\,\Delta t^2(\mathrm{Re}\lambda)|\lambda|^2 + 8\Delta t(\mathrm{Re}\lambda)^2 + 8\mathrm{Re}\lambda < 0.$$

for all eigenvalues, λ, of S_N. We obtained results that were qualitatively analogous to that of Tables 3-2, 3-3, and 3-4.

4. ANALYSIS IN TWO DIMENSIONS

In two dimensions we will consider the following steady state problem on the square $A = \{(x,y): -1 \leq x \leq 1, -1 \leq y \leq 1\}$ with homogeneous boundary conditions at $x = 1$, and $y = 1$.

$$(4.1) \qquad \begin{aligned} & u_x + u_y = f & \\ & u(x,y,0) = u_0 & (x,y)\in A \\ & u(1,y,t) = u(x,1,t) = 0. & t \geq 0 \end{aligned}$$

The essential idea in obtaining a pseudo-spectral approximation to (4.1) is the same as it was in 1-dimension. That is, to approximate spacial derivatives by constructing a global interpolant through discrete points. To obtain the Chebyshev pseudo-spectral approximation we take as these points $x_j = y_j = \cos \pi j/N$ for $j = 1,2,...,N$. This means that we must interpolate at the N^2 points (x_i,y_j) for $i,j = 1,2,...,N$. Consequently, the Chebyshev derivative operator for this problem can be represented by the $N^2 \times N^2$ matrix $S_N^{(2)} + P^t S_N^{(2)} P$, where $S_N^{(2)}$ is a block diagonal matrix whose blocks are each equal to S_N, and P is a permutation matrix. If one orders the N^2 points (x_i,y_j) by rows then $S_N^{(2)}$ corresponds to the derivative in the x-direction and P is constructed so that $P^t S_N^{(2)} P$ corresponds to the derivative in the y-direction. Without preconditioning $S_N^{(2)} + P^t S_N^{(2)} P$ is ill-conditioned.

As we saw in section 3 ZD is a good preconditioner for S_N. Thus, a natural approach to finding a preconditioner for $S_N^{(2)} + P^t S_N^{(2)} P$ is to try $Z^{(2)}D^{(2)} + P^t Z^{(2)} D^{(2)} P$, where $Z^{(2)}$ and $D^{(2)}$ are $N^2 \times N^2$ block diagonal matrices whose blocks are the $N \times N$ matrices Z and D, respectively.

To analyze the behavior of the eigenvalues of the preconditioning matrix, we define λ as the generic eigenvalue of the preconditioned matrix, $\rho_N = \max_i |\lambda_i| / \min_i |\lambda_i| \, (i = 1,2,...N^2)$, σ_N is the maximum σ such that $\mathrm{Re}\,\lambda \geq \sigma$, and r_N is the minimum r such that $|\lambda - 1| \leq r$. In particular, r_N and σ_N give us an idea of the location of the eigenvalues. (See figure 4-1 below.)

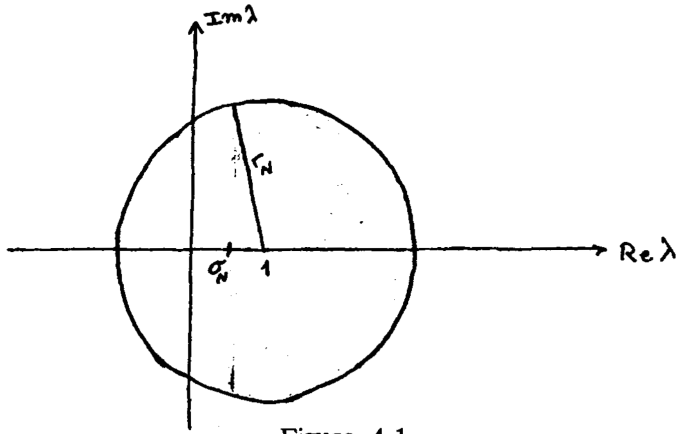


Figure 4-1

Numerical experiments performed for N = 4,6,8 are summarized in Table 4-1

N	ρ_N	r_N	σ_N
4	1.530	.530	1.000
6	1.552	.589	1.000
8	1.560	.622	1.000

Table 4-1

Although the eigenvalues of this preconditioned matrix, $(Z^{(2)}D^{(2)} + P^tZ^{(2)}D^{(2)}P)^{-1}(S_N^{(2)} + P^t S_N^{(2)}P)$, are well behaved the matrix is full and thus difficult to invert. Another approach to constructing a preconditioner is to substitute in $Z^{(2)}D^{(2)} + P^t Z^{(2)}D^{(2)}P$ the tridiagonal matrix $\hat{Z}$ defined by (2.4) in place of Z. We will denote this new $N^2 \times N^{(2)}$ matrix by $\hat{Z}^{(2)}D^{(2)} + P^t \hat{Z}^{(2)}D^{(2)}P$. This matrix represents a finite difference scheme depending on seven points as illustrated by the stencil in Figure 4-2.

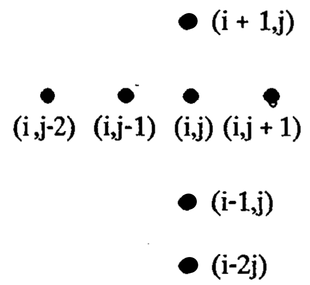


Figure 4-2

Results similar to those presented in Table 4-1 are prsented in Table 4-2 for $Z^{(2)}D^{(2)} + P^t Z^{(2)}D^{(2)}P$.

N	ρ_N	r_N	σ_N
4	2.206	.980	.897
6	5.026	1.453	.488
8	8.494	1.602	.306

Table 4-2

Although ρ_N corresponding to $\hat{Z}^{(2)}D^{(2)} + P^t \hat{Z}^{(2)}D^{(2)}P$ increases more quickly than ρ_N corresponding to $Z^{(2)}D^{(2)} + P^tZ^{(2)}D^{(2)}P$, the matrix can be inverted more efficiently. This is because $\hat{Z}^{(2)}D^{(2)} + P^t \hat{Z}^{(2)}D^{(2)}P$ is a banded matrix (with N lower codiagonals and 2N upper codiagonals).

Another preconditioner that we considered was of the form

$$Z^{(2)}P^tZ^{(2)}P(D^{(2)} + P^tD^{(2)}P).$$

As in the 1-dimensional case, if the steady-state solution is to be found, the explicit Richardson method can be used. We experimentally find Δt^* such that $\rho(I + \Delta t^* W_N)$ attains its minimum inside the region of stability. The same experiments are made for the preconditioned matrices. The results of the experiments are reported in Tables 3-3, 3-4, and 3-5.

N	Maximum Δt for Stability	Δt^*	ρ Corresponding to Δt^*
4	.1509563	.07547812	.9248009
6	.05228855	.02614427	.9715761
8	.0273053	.01365251	.9817108
10	.01944351	.009721750	.9810511

Table 4-3 Minimum spectral radius of the unpreconditioned amplification matrix

N	Maximum Δt for Stability	Δt^*	ρ Corresponding to Δt^*
4	.9879928	.7172827	.6693591
6	.9094622	.6584506	.8027519
8	.7200746	.4954113	.8715243
10	.6038381	.3623029	.8995660

Table 4-4 Minimum spectral radius of the preconditioned matrix:
Case of $Z_N P^t Z_N P(D_N + P^t D_N P)$

N	Maxiumum Δt for Stability	Δt^*	ρ Corresponding to Δt^*
4	1.009613	.6946138	.3763489
6	.815170	.6798409	.6681257
8	.7685055	.6870439	37895085

Table 4-5 Minimum spectral radius of the preconditioned matrix:

Case of $\hat{Z}_N D_N + P^t \hat{Z}_N D_N P$

We also considered the second order Runge-Kutta scheme, and we obtained results that were qualitatively analogous to those of Tables 4-3, 4-4, and 4-5.

We applied the Richardson schemes in the unpreconditioned version and in the preconditioned versions using the preconditioners $\hat{Z}_N D_N + P^t \hat{Z}_N \hat{D}_N P$ and $Z_N P^t Z_N P(D_N + P^t D_N P)$, to find the solution of the model problem

$$u_t = u_x + u_y - \alpha\sin(\alpha(x+1)) + \alpha\sin(\alpha(y+1))$$

$$u(x,y,0) = \sin(y-1)\sin(x-1)$$

$$u(-1,y,t) = u(x,-1,t) = 0$$

of the type in (4.1). We used the optimal Δt^* listed in tables 4-3, 4-4, and 4-5. Using the L_2 norm we considered the scheme to converge when the exact error stabilized. We also calculated global CPU times. The experiments were performed on the IBM 3081 and the results are reported in Tables 4-6,4-7, and 4-8.

N	No. of Iteration for Convergence	Error	CPU Time for Convergence (in seconds)
4	122	$.12405 \times 10^{-1}$	.21
6	400	$.104182 \times 10^{-3}$	1.33
8	900	$.47112 \times 10^{-6}$	5.82

Table 4-6 Unpreconditional Euler

N	No. of Interation for Convergence	Error	CPU Time for Convergence	CPU Time for Construction of Preconditioner and Finding LU Decomposition of Preconditioner
4	11	$.1246626 \times 10^{-1}$	.03	.00
6	34	$.10400 \times 10^{-3}$	.26	.04
8	69	$.4709596 \times 10^{-6}$	1.17	.11

Table 4-7 Preconditioned Richardson Case of $\hat{Z}_N D + P^t \hat{Z}_N DP$

N	No. of Interations for Convergence	Error	CPU Time for Convergence	CPU Time for Construction of Precondtioner and Finding LU Decomposition of Preconditioner
4	32	$.12468 \times 10^{-1}$	.09	.00
6	173	$.104255 \times 10^{-3}$	1.10	.00
8	263	$.4709613 \times 10^{-6}$	3.31	.00

Table 4-8 Precondition Richardson Case of $Z_N P^t Z_N P(D + P^t DP)$.

[DF] D. Funaro, "A preconditioning matrix for the Chebyshev differencing operator", Istituto Di Analisi Numerica del Consiglio Nazionale Delle Ricerche Corso C. Alberto, 5-27100 PAVIA (Italy) N. 509.