NASA Contractor Report 178354

ICASE REPORT NO. 87-55

ICASE

A HYBRID PERTURBATION GALERKIN TECHNIQUE WITH APPLICATIONS TO SLENDER BODY THEORY

James Geer

Carl Andersen

(NASA-CR-178354) A HYBRID PERTURBATION GALERKIN TECHNIQUE WITH APPLICATIONS TO SLENDER BODY THEORY Final Report (NASA) 58 p Avail: NTIS HC A04/MF A01 CSCL 12A

N87-28366

Unclas

G3/64 0097597

Contract No. NAS1-18107
August 1987

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

A HYBRID PERTURBATION GALERKIN TECHNIQUE WITH APPLICATIONS TO SLENDER BODY THEORY

James F. Geer
Dept. of Systems Science
Watson School of Engineering,
Applied Science, and Technology
S.U.N.Y.
Binghamton, New York 13901

and

Carl M. Andersen
Dept. of Mathematics
College of William and Mary
Williamsburg, VA 23185

ABSTRACT

A two-step hybrid perturbation-Galerkin method to solve a variety of applied mathematics problems which involve a small parameter is presented. The method consists of: (1) the use of a regular or singular perturbation method to determine the asymptotic expansion of the solution in terms of the small parameter; (2) construction of an approximate solution in the form of a sum of the perturbation coefficient functions multiplied by (unknown) amplitudes (gauge functions); and (3) the use of the classical Bubnov-Galerkin method to determine these amplitudes. This hybrid method has the potential of overcoming some of the drawbacks of the perturbation method and the Bubnov-Galerkin method when they are applied by themselves, while combining some of the good features of both. The proposed method is applied to some singular perturbation problems in slender body theory. The results obtained from the hybrid method are compared with approximate solutions obtained by other methods, and the degree of applicability of the hybrid method to broader problem areas is discussed.

Research for the first author was supported under the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while he was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

Research for the second author was partially funded through NASA Cooperative Agreeement NCCI-40.

1. INTRODUCTION

A two-step hybrid analysis technique, which combines perturbation techniques with the Galerkin method, promises to be useful in the analysis of a wide variety of differential equations type problems. The hybrid technique was apparently first studied by Ahmed K. Noor and collaborators in conjunction with the finite element analysis of geometrically nonlinear problems in structural mechanics [22-39].

The Galerkin method [7] has, of course, been known and used for a long time. But a principal problem associated with its successful application lies in the choice of appropriate basis functions. It was proposed by Nagy [19] and developed by others [1,2,20] that the expense of analysis of certain nonlinear structures problems can be sharply reduced by selecting a relatively small number of linear buckling modes as trial functions for use in a Galerkin technique. Then the set of solutions corresponding to a continuously varying parameter, such as a load factor, can be approximately determined, at least over some limited range of loading, with a relatively small number of nonlinear discretized equations, in place of the much larger number of nonlinear finite element equations needed for modeling even relatively simple geometric configurations.

In a series of papers [22-39], Noor and his collaborators have shown for a variety of structural mechanics problems that the terms in a Taylor series expansion of the solution of a parameterized system of discretized equations can be particularly effective as Galerkin trial functions (or basis vectors). In addition, it has been repeatedly demonstrated that the "reduced-basis" solutions can be useful for significantly larger values of the expansion parameter than the truncated Taylor series solutions on which they are based. Noor and collaborators [29,24] have also applied the same general principles but without discretization to some thermal analysis and structures problems. A treatment of the reduced basis method from a mathematical point of view is given by Fink and Rheinbolt [6].

Some general observations about the technique are the following. First, in many perturbation problems, much effort has to be expended to compute each additional term (analytically) in a perturbation expansion. Through the use of the proposed hybrid method, the known perturbation terms can be exploited more fully. Secondly, another way of viewing the technique is to recognize that in many perturbation expansions the functional form of the higher order terms can be well approximated by a linear combination of the lower order terms. Thus, much of the effect of the higher terms may be included by applying the reduced basis technique to a small number of lower order terms. Finally, preliminary unpublished investigations indicate that, while the use of a Taylor series expansion is frequently limited by a finite radius of convergence, the proposed hybrid method can sometimes yield good results even well outside the radius of convergence.

It is our belief that the junction of perturbation and Galerkin techniques can be useful in a wide variety of application areas, and in these applications the hybrid technique will give better approximations than the perturbation method alone. In this paper, we will present applications of the technique, independent of finite element or finite difference methods, and in an area well apart from structural mechanics. In particular, in the next section, we shall describe the method in more detail and then apply it to a simple two point boundary value problem in Section 3. In Sections 4 and 5, we shall apply the method to two singular perturbation problems in slender body theory. In Section 6, we make some observations about our method and indicate some areas for further study.

2. DESCRIPTION OF THE METHOD

The method we wish to describe is a two-step hybrid analysis technique. It is based upon the successive use of a straightforward perturbation expansion method and the classical Bubnov-Galerkin approximation technique. In the perturbation method, the solution to a particular problem involving a small parameter is developed in terms of a series of unknown "perturbation" functions with preassigned coefficients, i.e., gauge functions. The perturbation functions are usually determined by solving a recursive set of differential equations which are, in general, simpler than the original governing differential equation. By contrast, in the Bubnov-Galerkin technique one seeks an approximate solution to the problem in the form of a linear combination of specified (known) coordinate functions with unknown coefficients. The coefficients are determined by demanding that the residual formed by substituting the trial solution into the governing differential equations is orthogonal to each of the coordinate functions.

While the perturbation and Galerkin methods are useful and have been successful in providing approximate solutions to a wide variety of nonlinear (and otherwise difficult) problems, each has certain drawbacks. The perturbation method has at least two major drawbacks. First, as the number of terms in the perturbation expansion increases, the mathematical complexity of the equations which determine the unknown functions increases rapidly. Thus, in most practical applications, the perturbation series is limited to only a few terms. A second drawback to the perturbation method is the requirement of restricting the perturbation parameter to small values in order to obtain solutions of acceptable accuracy. (These drawbacks of the perturbation method have been recognized and several modifications or extensions have been proposed, see e.g., Van Dyke [42] and Andersen and Geer [3].) The main shortcoming of the Bubnov-Galerkin method is the difficulty, from a practical point of view, of selecting good coordinate functions.

To illustrate the general ideas of the hybrid (or "reduced basis") method, suppose we are seeking (an approximation to) the solution U to the problem

$$L(U,\varepsilon) = 0 \tag{2.1}$$

where L is some differential (or integral) operator and ε is a small parameter. Here (2.1) holds in some domain D, and, in addition, U must satisfy certain conditions on the boundary of D. Now the application of the hybrid perturbation/Galerkin method can be divided into the following two distinct steps: (1) generation of the coordinate functions by a straightforward perturbation expansion of U and (2) computation of the amplitudes of these coordinate functions by using the Bubnov-Galerkin method.

To describe this idea in more detail, suppose that the solution to (2.1) can be expanded by a perturbation technique into a series of the form

$$U = \sum_{j=0}^{n-1} u_j \gamma_j(\varepsilon) + O(\gamma_n(c)) \tag{2.2}$$

where each u_j is independent of ε and $\{\gamma_j(\varepsilon)\}$ is an appropriate asymptotic sequence of gauge functions. The equations to determine the u_j are obtained by substituting (2.2) into (2.1) and setting the coefficient of γ_j equal to zero, for $j = 0,1,\cdots,n-1$. In a similar manner, the boundary conditions for each u_j are determined by using (2.2) in the boundary condition for U.

The perturbation functions u_j are now chosen as coordinate functions for the Bubnov-Galerkin technique; and an approximation $\tilde{U}$ for U is sought in the form

$$\tilde{U} = \sum_{j=0}^{n-1} u_j\delta_j \tag{2.3}$$

where the (unknown) parameters $\delta_j = \delta_j(\varepsilon)$ represent the amplitudes of the coordinate functions u_j. To determine these parameters, we apply the Bubnov-Galerkin technique to the governing equation (2.1). Thus, we substitute (2.3) into (2.1) and demand that the residual be orthogonal to the n coordinate functions over the domain D, i.e.,

$$\int_D L\left(\sum_{j=0}^{n-1} u_j\delta_j,\varepsilon\right)u_k\,dx = 0, \quad k = 0,1,\cdots,n-1. \tag{2.4}$$

Equations (2.4) represent a set of n equations for the n unknown amplitudes. While (2.4) must, in general, be solved numerically, solving it is much simpler than numerically solving (2.1). In particular, for a fixed value of ε, the solution to (2.4) is a point in n-dimensional space, where n is reasonably small, while the solution of (2.1) is a continuous function.

We should note that this particular choice of coordinate functions overcomes the main drawback of the Bubnov-Galerkin method. By the way they are constructed, the perturbation coordinate functions are (under certain assumptions) elements of a set of functions which span the space of solutions in a neighborhood of their point of generation. Thus, they should fully characterize the solution U in that neighborhood. Also, in many applications, the functions u_j are determined by solving a set of linear equations, even though the original operator L may be nonlinear. The first property is necessary for the convergence of the Bubnov-Galerkin method, while the second property enhances the effectiveness of the proposed hybrid method for solving nonlinear problems.

Another important property of the proposed method is that the coordinate functions, i.e., the perturbation functions, do not need to come from a regular perturbation expansion. In fact, all that is needed is a formal asymptotic expansion of the solution to (2.1) for small values of ε in the form of (2.2), where the $\{\gamma_j(\varepsilon)\}$ are a set of appropriate gauge functions, e.g., expressions which involve $\log(\varepsilon)$ or fractional powers of ε. Thus, the proposed method has the potential of being applied to singular as well as regular perturbation problems. In the following section, we shall illustrate our method using the regular perturbation expansion of a simple two point boundary value problem, while in Sections 4 and 5 we shall apply the method to some singular perturbation problems in slender body theory.

3. A SIMPLE EXAMPLE

To illustrate the ideas just discussed, we consider the following simple example. Consider the two-point boundary value problem:

$$U'' - \varepsilon U' = \varepsilon, \quad 0 < x < 1,$$

(3.1)

$$\text{with } U(0) = 0 \text{ and } U(1) = 0.$$

The exact solution to this problem is $U = (1 - e^{\varepsilon x})/(1 - e^{\varepsilon}) - x$, which exhibits boundary layer behavior around $x = 1$ for large values of ε. A regular perturbation expansion of U yields a series of the form

$$U = \sum_{j=1}^{n} u_j(x)\varepsilon^j + O(\varepsilon^{n+1}) \tag{3.2}$$

where each u_j is a polynomial in x of degree $(j+1)$ and vanishes at $x =$ 0 and 1. Following the ideas discussed above, we look for an approximate solution $\tilde{U}$ in the form

$$\tilde{U} = \sum_{j=1}^{n} u_j(x)\, \delta_j(\varepsilon) \tag{3.3}$$

where the amplitudes δ_j are to be determined. To determine them, we substitute (3.3) into (3.1) and apply the Bubnov-Galerkin criterion (2.4) to obtain the set of equations

$$\sum_{k=1}^{n} a_{j,k}(\varepsilon)\, \delta_k = b_j(\varepsilon), \quad j = 1,2,\cdots,n \tag{3.4}$$

where

$$a_{j,k}(\varepsilon) = \int_0^1 u_j(x)[u_k''(x) - \varepsilon u_k'(x)]dx, \quad b_j = \varepsilon \int_0^1 u_j(x)dx.$$

In this case, the equations for the δ_j are linear since the original problem is linear. For example, setting $n = 2$ in (3.4), we find

$$\delta_1 = \frac{\varepsilon}{1+\varepsilon^2/60}, \quad \delta_2 = \frac{\varepsilon^2}{1+\varepsilon^2/60}.$$

For small values of ε, we see that these expressions reduce to ε and ε^2, as they should. For $n = 4$, we find (by using the symbolic computation system MACSYMA [16]) that

$$\delta_1 = \frac{15120\varepsilon + 420\varepsilon^3}{15120 + 420\varepsilon^2 + \varepsilon^4}, \quad \delta_2 = \varepsilon\delta_1$$

$$\delta_3 = \frac{15120\varepsilon^3}{15120 + 420\varepsilon^2 + \varepsilon^4}, \quad \delta_4 = \varepsilon\delta_3.$$

(Note that the δ_j's remain finite as ε becomes large, which suggests that the approximate hybrid solution for this problem may be valid even for large values of ε.)

In Figures 1 and 2, we illustrate the comparison of the exact solution with our approximate solution and the perturbation solution for $n = 2$ and $n = 4$, respectively. For $n = 2$, the three solutions agree very well for values of ε up to about $\varepsilon = 1$. For larger values of ε, the perturbation solution begins to diverge from the exact solution, while $\tilde{U}$ still gives reasonable approximations up to about $\varepsilon = 5$. A similar comparison using four terms in the approximate solutions shows that our hybrid solution is consistently more accurate than the perturbation solution and gives reasonable results, even within the boundary layer, for values of ε up to about 10, where the perturbation solution is meaningless.

Some of the reasons why our hybrid method provides a good approximation in this case, as well as insights this example provides for the validity of our method, will be discussed in Section 6. The application of our hybrid method to several general classes of two point boundary value problems, as well as various extensions of the method, are currently under investigation and will be reported elsewhere.

4. POTENTIAL FLOW PAST A SLENDER BODY OF REVOLUTION

Slender body theory has been developed and applied largely in the context of classical fluid mechanics. For example, potential flow past an axially symmetric slender body has been studied extensively since the work of Munk [18]. This work is now a part of slender body theory, which is the theory of fields about a slender body. It is discussed in detail, along with relevant references, in various books, e.g., Thwaites [40] and Van Dyke [41], as well as the paper by Newman [21]. More recent work on the axially symmetric flow problem has been done by Moran [17], Landweber [15], and Handelsman and Keller [12]. Recently, these results have been extended and generalized by Geer [8]. In addition, the ideas of uniform slender body theory have been applied successfully to problems in other areas, such as Stokes flow past a slender body (Geer [9]), the electrostatic field about a dielectric slender body (Barshinger and Geer [4]), the scattering of an arbitrary acoustical wave by a slender body of revolution (Geer [10]), as well as a special case of electromagnetic scattering (Geer [11]).

To illustrate the applicability of our hybrid method to problems in this area, we shall consider two problems which involve a slender body of revolution. These problems will serve as model problems for the application of our method to more general problems in slender body theory. Typically, these problems are not regular perturbation problems, since the solutions contain terms involving $\log(\varepsilon)$ as well as powers of ε, where ε is the slenderness ratio of the body (see e.g., Geer [8]). In addition, the solutions to some of these problems involve series in inverse powers of $\log(\varepsilon)$, which converge (at best) very slowly, even for reasonably small values of ε. The specific problems we shall consider are: (1) potential flow past a slender body, which will be considered in this section and (2) the electrostatic potential around a perfectly conducting slender body (Section 5). Each of these problems contains features representative of larger classes of problems for which we believe the hybrid method will be of some significant value. For example, the flow problem is a good representative of a singular (as opposed to regular) perturbation problem involving a relatively rapidly converging series. Similar series occur in the study of certain Stokes flow problems (Geer [9]), acoustical scattering by a hard body (Geer [10]) or a perfectly conducting body (Geer [11]), the potential field about a dielectric body (Barshinger and Geer [4]), and certain problems involving oblate bodies (Homentcovschi [14]). The electrostatic problem involves a sequence of very slowly converging series. Similar slowly converging series also occur in investigations in several other problem areas, including certain other Stokes flow problems (Geer [9]), scattering by a soft body (Geer [10]), and problems involving the interaction of ships in shallow water (Davis and Geer [5]). For simplicity, we shall consider here only a simple example in each of these areas. These model problems will serve to illustrate the usefulness of our method, and it will also be clear how our method can be applied to more general problems in these areas.

In this section, we shall consider the problem of determining the potential flow around a slender body of revolution which is immersed in a uniform stream parallel to its axis (see e.g., Handelsman and Keller [12] or Geer [8], where this problem is generalized). We let the body be described by $r = (x^2 + y^2)^{1/2} = \varepsilon\sqrt{S(z)}$, for $0 \leq z \leq 1$ (see Figure 3). Here ε is the slenderness ratio of the body, defined as the ratio of the maximum radius of the body to its length, while $S(z)$ describes the particular shape of the body. We assume that $S(0) = 0 = S(1)$, while $S'(0) \neq 0 \neq S'(1)$, where the primes denote differentiation with respect to z. These last conditions ensure that the body has a rounded (i.e., not sharp) nose and tail. (See Geer [8] for a discussion of these conditions.) Then, following the ideas of Handelsman and Keller [12], we represent the part of the velocity potential Φ due to the presence of the body as the superposition of potentials due to point sources distributed along a portion of the axis of the body and lying entirely within the body. Thus, we write:

$$\Phi(r^2,z) = z - \int_\alpha^\beta \frac{1}{\sqrt{(z-\xi)^2+r^2}} f(\xi,\varepsilon)d\xi, \tag{4.1}$$

where $4\pi f$ is the unknown density of the source distribution and α and β, which determine the extent of the distribution within the body, satisfy $0 < \alpha < \beta < 1$ (see Figure 3). The expression (4.1) satisfies Laplace's equation outside the body and reduces to z at infinity. The condition that the normal component of the velocity vanish on the surface of the body, when used with (4.1) becomes

$$\frac{1}{2}\varepsilon\, S'(z) = \left(\frac{d}{dz}\right) \int_\alpha^\beta \frac{z-\xi}{\sqrt{(z-\xi)^2+\varepsilon^2 S(z)}}\, f(\xi,\varepsilon)d\xi, \tag{4.2}$$

which must hold for $0 \le z \le 1$. Equation (4.2) is an integral equation for the source density $f(z,\varepsilon)$, from which α and β can also be determined. In particular, Handelsman and Keller have shown that f can be determined as a series of the form

$$f(z,\varepsilon) = \sum_{i=1}^{\infty} \varepsilon^{2i} \sum_{j=0}^{i-1} (\log(\varepsilon^2))^j f_{i,j}(z) \tag{4.3}$$

where the coefficient functions $f_{i,j}(z)$ can be determined recursively. In particular, using some of the results of Handelsman and Keller, we can write

$$\begin{aligned} f(z,\varepsilon) &= \frac{\varepsilon^2}{4} S'(z) + \varepsilon^4 \log(\varepsilon^2) f_{2,1}(z) + \varepsilon^4 f_{2,0}(z) \\ &\quad + O(\varepsilon^6(\log(\varepsilon^2))^2), \\ f_{2,1}(z) &= \frac{1}{16} (S(z)S''(z))', \quad f_{2,0}(z) = \frac{1}{8}\frac{d}{dz} (L_1^{0,1} S'(z)), \end{aligned} \tag{4.4}$$

where $L_1^{0,1}$ is a certain linear operator, defined precisely by Handelsman and Keller (see also Geer [9], appendix B). The parameters α and β can be determined as power series in ε^2 and are given by

$$\begin{aligned} \alpha &= C_1(\tfrac{\varepsilon}{2})^2 - C_1C_2(\tfrac{\varepsilon}{2})^4 + C_1(C_1C_3 + 2C_2^2)(\tfrac{\varepsilon}{2})^6 + O(\varepsilon^8), \quad C_j = S^{(j)}(0)/j! \\ \beta &= 1 - d_1(\tfrac{\varepsilon}{2})^2 + d_1d_2(\tfrac{\varepsilon}{2})^4 - d_1(d_1d_3 + 2d_2^2)(\tfrac{\varepsilon}{2})^6 + O(\varepsilon^8), \quad d_j = (-1)^j S^{(j)}(1)/j!. \end{aligned} \tag{4.5}$$

To illustrate the application of the hybrid method to this problem, we shall treat α and β as known quantities (since they can be expressed as Taylor series in ε^2, with several coefficients having been determined), and concentrate on the behavior of f with ε. Following the ideas presented in Section 2, we look for an approximation $\tilde{f}$ to f in the form

(4.6) $$\tilde{f}(z,\varepsilon) = \sum_{i=0}^{n} \sum_{j=0}^{i-1} f_{i,j}(z)\, \delta_{ij}(\varepsilon)$$

where the coefficients $f_{i,j}(z)$ are those which appear in (4.3) above and the amplitudes $\delta_{i,j}(\varepsilon)$ must be determined. To determine them, we substitute (4.6) into (4.2), apply the orthogonality conditions, and obtain the conditions:

(4.7) $$b_{k,l} = \sum_{i=1}^{n} \sum_{j=0}^{i-1} c_{k,l,i,j}\, \delta_{i,j},$$

where

$$b_{k,l} = \frac{\varepsilon^2}{2} \int_0^1 S'(z) f_{k,l}(z)dz$$

$$c_{k,l,i,j} = \int_0^1 \{\frac{d}{dz} \int_\alpha^\beta \frac{(z-\xi)}{\sqrt{(z-\xi)^2+\varepsilon^2 s(z)}} f_{i,j}(\xi)d\xi\} f_{k,l}(z)dz.$$

Equations (4.7) are a system of linear equations for the unknown amplitudes δ_{ij}, which can be solved in a straightforward manner. As a special example, we consider the case when $n = 1$ and hence our approximate solution $\tilde{f}$ is of the form

$$\tilde{f}(z,\varepsilon) = \frac{1}{4} S'(z)\delta_{1,0},\ \delta_{1,0} = b_{1,0}/c_{1,0,1,0},$$

(4.8)

$$b_{1,0} = \frac{\varepsilon^2}{8} \int_0^1 (S'(z))^2 dz,\quad c_{1,0,1,0} = \frac{1}{16} \int_0^1 S'(z)\frac{d}{dz} \int_\alpha^\beta \frac{z-\xi}{\sqrt{(z-\xi)^2+\varepsilon^2 s(z)}} S'(\xi)d\xi dz.$$

From (4.8) we can easily show that

$$\delta_{1,0} = \varepsilon^2 + \gamma_1\varepsilon^4\log(\varepsilon^2) + \gamma_2\varepsilon^4 + O(\varepsilon^6(\log(\varepsilon^2))^2, \quad \text{as} \quad \varepsilon \to 0,$$

where γ_1 and γ_2 are certain constants. Hence, as $\varepsilon \to 0$, our approximate solution (4.8) reproduces the first term in the perturbation solution (4.4) exactly, while also "anticipating" some of the ε behavior of the higher order terms. For the special case when the body is a prolate spheroid, i.e., when $S(z) = 4z(1-z)$, the integrals appearing in (4.8) can be evaluated explicitly, yielding

$$\delta_{1,0} = \varepsilon^2[\sqrt{1 - 4\varepsilon^2} - 2\varepsilon^2 C(\varepsilon)]^{-1}$$

(4.9)

$$C(\varepsilon) = 2\log[\frac{1 + \sqrt{1-4\varepsilon^2}}{2\varepsilon}].$$

When (4.9) is inserted into (4.8), we find that (4.8) yields the exact solution to the integral equation (4.2) for this case. Thus, the hybrid method, for this special case, yields the exact solution using only the "information" contained in the first term of the perturbation solution. This point will be discussed in more detail in Section 6.

To obtain some idea of the accuracy of our approximate solution (4.6) for other body shapes, we have applied our method to a class of "dumbbell" shaped bodies, described by $S(z) = 4bz(1-z)\{1-bz(1-z)\}$, with $2 < b < 4$. In Figure 4, we illustrate our results for the representative case $b = 3$ and for $\varepsilon =$ 0.03, 0.07, and 0.11. In particular, we compare our hybrid solution (4.6) with the perturbation solution (4.3) and a solution to (4.2) obtained by purely numerical means (see the Appendix). As the figure clearly illustrates, the hybrid solution gives consistently better results than the perturbation solution alone, especially for the case $\varepsilon = 0.11$, which appears to be close to the radius of convergence of the perturbation solution. In addition, we have used our hybrid solution in (4.1) to calculate the (non-dimensional) pressure coefficient $C_p(z) = 1 - |\vec{\nabla}\Phi|^2$ on the surface of a slender body. In Figure 5, we have plotted C_p for the prolate spheroid with $\varepsilon = 0.3$, using our hybrid solution, the perturbation solution, and the exact solution. In Figure 6, we have made a similar comparison for our dumbbell shaped body with $b = 3$, which was also discussed by Wong, Liu, and Geer [43]. Figure 5 illustrates the facts that our hybrid method has reproduced the exact solution to the flow problem and that the perturbation solution is (slowly) converging to the correct result. Figure 6 again illustrates that the hybrid method produces consistently better results than the perturbation solution alone.

5. ELECTROSTATIC POTENTIAL FIELD ABOUT A SLENDER CONDUCTOR

We now consider the problem of determining the electrostatic potential field about a perfectly conducting slender body of revolution. For simplicity, we again consider only a simple version of this classical problem, namely, the case when there is no external field and the body is raised to unit potential. Thus, using the notation of the previous section, we represent the potential as

$$\Phi(r^2,z) = \int_\alpha^\beta \frac{1}{\sqrt{(z-\xi)^2+r^2}} f(z,\varepsilon)d\xi \tag{5.1}$$

where $4\pi f(z,\varepsilon)$ is again the (unknown) source strength density. The expression (5.1) satisfies Laplace's equation outside the body and vanishes at infinity. The boundary condition on the surface of the body leads to the condition

$$(5.2) \qquad \int_{\alpha}^{\beta} \frac{1}{\sqrt{(z-\xi)^2+\varepsilon^2 s(z)}} f(\xi,\varepsilon)d\xi = 1, \quad 0 \leq z \leq 1$$

which is an integral equation for $f(z,\varepsilon)$, from which α and β can also be determined. In particular, Handelsman and Keller [13] have shown that α and β are again given by the expressions (4.5), while f has an asymptotic expansion of the form

$$(5.3) \qquad f(z,\varepsilon) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} \varepsilon^{2m}(\log(\varepsilon^2))^{-j} f_{m,j}(z)$$

where the coefficient functions $f_{m,j}(z)$ can be determined recursively. In particular, using some of their results (see also Barshinger and Geer [4], where these results have been generalized to the case of a dielectric body) we can write

$$f_{0,1}(z) = -1,$$

$$f_{0,j+1}(z) = f_{0,j}(z)\log\{4z(1-z)/S(z)\}$$

$$+ \int_0^1 \{f_{0,j}(\xi) - f_{0,j}(z)\}/|\xi - z| \, d\xi, \; j = 1,2,\cdots$$

$$(5.4) \qquad f_{1,1}(z) = 0$$

$$f_{1,j+1}(z) = f_{1,j}(z)\log\{4z(1-z)/S(z)\}$$

$$+ \int_0^1 \{f_{1,j}(\xi) - f_{1,j}(z)\}/|\xi - z| \, d\xi$$

$$+ (1/4)S(z)f''_{0,j+1}(z) + Lf_{0,j}(z), \quad j = 1,2,\cdots,$$

where L is a certain linear operator defined explicitly by Handelsman and Keller [13] (see also Geer [9], Appendix B).

We note that the series (5.3) contains a sequence of (at best) slowly converging series involving inverse powers of $\log(\varepsilon^2)$. We have used the expressions (5.4) to compute some of the $f_{m,j}$ explicitly and then used these results in (5.3) to find approximations for the actual singularity density $f(z,\varepsilon)$. In Figure 7, we have plotted the approximation for f obtained from (5.3) using different numbers of terms. These approximations are compared with a solution to (5.2) obtained by purely numerical methods (see the Appendix). As can be seen from the figures, several terms are needed in the perturbation solution to obtain a good approximation, even for reasonably small values of ε.

We now apply our hybrid method to this problem and look for an approximate solution f in the form

(5.5) $$\tilde{f}(z,\varepsilon) = \sum_{m=0}^{n} \sum_{j=1}^{J_m} f_{m,j}(z)\delta_{m,j}(\varepsilon)$$

where the coefficients $f_{m,j}(z)$ are those which appear in (5.3) and the amplitudes $\delta_{m,j}$ are to be determined. To determine them, we substitute (5.5) into (5.2), apply the orthogonality conditions, and obtain a set of equations of the form (4.7), where

$$
(5.6)\qquad
\begin{aligned}
b_{k,l} &= \int_0^1 f_{k,l}(z)dz \\
c_{k,l,i,j} &= \int_0^1 \int_\alpha^\beta \frac{1}{\sqrt{(z-\xi)^2+\varepsilon^2 s(z)}} f_{i,j}(\xi)d\xi \, f_{k,l}(z)dz.
\end{aligned}
$$

Hence, the $\delta_{m,j}$ again satisfy a system of linear algebraic equations, which can be solved in a straightforward manner. In particular, for the special case when $n = 0$ and $J_0 = 1$ (i.e., f contains only one term), we find that

$$
(5.7)\qquad
\begin{aligned}
\delta_{0,1} &= -\{\int_0^1 \int_\alpha^\beta [(z-\xi)^2 + \varepsilon^2 s(z)]^{-1/2} d\xi \, dz\}^{-1} \\
&= (\log\varepsilon^2)^{-1} \\
&\quad + (\log\varepsilon^2)^{-2} \int_0^1 \log[\frac{4z(1-z)}{s(z)}]dz + O((\log\varepsilon^2)^{-3}) + O(\varepsilon^2(\log\varepsilon^2)^{-2}.
\end{aligned}
$$

Thus, $\delta_{0,1}$ reproduces the first term in the asymptotic expansion (5.3) exactly, while again "anticipating" some of the higher order ε behavior of the solution. For the special case when the body is a prolate spheroid with $S(z) = 4z(1-z)$, we find that the integral appearing in (5.7) can be evaluated explicitly to yield $\delta_{0,1} = \{C(\varepsilon)\}^{-1}$, where $C(\varepsilon)$ is defined in (4.9). When this result is inserted into (5.5), we find that (5.5) yields the <u>exact</u> solution to the integral equation (5.2). These points will be discussed more fully in Section 6.

In Figure 7, we have also plotted the approximation to $f(z,\varepsilon)$ obtained from our hybrid solution (5.5). As the figures clearly indicate, the hybrid method consistently produces better approximations than the perturbation solution alone. Some of the reasons for these better approximations will be discussed in the next section.

6. DISCUSSION AND OBSERVATIONS

We can now make some observations about the hybrid method and briefly discuss, at least heuristically, why it seems to work as well as it does.

If we return to equation 2.2, we note that the perturbation method seeks to find an approximate solution $\hat{U}$ to the true solution U of (2.1) in the form of a finite linear combination of the perturbation coordinate functions u_j with specified amplitudes (gauge functions) $\{\gamma_j(\varepsilon)\}$. Our hybrid method, on the other hand, seeks an approximate solution $\tilde{U}$ in the form of equation (2.3), which can be viewed as a linear combination of some of the perturbation coordinate functions with amplitudes $\{\delta_j\}$ which are not specified a priori. Instead, the $\{\delta_j\}$ are determined so that $\tilde{U}$ will be "the best" approximation of the form (2.3) using the Galerkin criterion (2.4). Thus, in a geometrical sense, the perturbation method seeks to approximate the shape of the true solution with a specific linear combination of the u_j, while the hybrid method seeks to approximate the shape of U with the "best possible" (in the Galerkin sense) linear combination of a specified subset of the $\{u_j\}$. Thus, the perturbation method ignores, for example, linear dependences among the $\{u_j\}$, while the hybrid method implicitly takes these dependences into account. An extreme case of this occurs in the problems in Sections 4 and 5 in the special case when the body is a prolate spheroid. In this case, all of the perturbation coordinate functions are constant multiples of the first function and hence are all linearly dependent (see, for example, Wong, Liu, and Geer [43]). The perturbation method thus produces an infinite number of terms in a formal asymptotic series, but each term has the same "shape" as the first term, with only the amplitudes of the terms differing. The hybrid method in essence "seeks out" (in the Galerkin sense described) the total dependence of the shape of the solution on the terms included in the approximation. Thus, in the case of the prolate spheroid, since the exact "shape" of the true solution is contained in the first term of the approximation, the hybrid method produces the exact solution. Only the amplitude of the term needs to be modified.

The hybrid method, then, would seem to work well when the perturbation coordinate functions $\{u_j\}$ exhibit a significant degree of linear dependence. To illustrate this dependence for the examples we have presented, in Figures 8-10 we have plotted several of the perturbation coordinate functions used in the hybrid approximation. In Figure 8, we have plotted the first 15 perturbation functions for the simple example discussed in Section 3 (with each function normalized so that its maximum absolute value is 1). We recall from their definition that each u_j is a polynomial of degree $j+1$ and hence they are all independent. However, as can be seen clearly from the figure, on the interval [0,1] these functions are highly dependent, with essentially only a few different "shapes" occurring among the first 15 functions. We have expressed each of these 15 funtions in terms of the 4 functions obtained by applying the Gram-Schmidt orthonormalization process to the first four functions u_1 to u_4. In Table I, we have presented the L_2-norm of the difference of these expressions and the original perturbation coordinate functions. The norms confirm that fact indicated by Figure 8 that essentially any linear combination of the first 15 coordinate functions can be well expressed in terms of just the first four u_j. Thus, we can say that the higher order perturbation coordinate functions add only a small amount of new information about the shape of the true solution. Presumably, these first four functions contain much of the "shape information" contained in the perturbation functions of order higher than fifteen as well. (We have examined several other model two point boundary value problems and have found conclusions similar to those just expressed. These examples, as well as other applications of our method, are currently under investigation and will be reported elsewhere.) These observations about the linear dependency of the perturbation functions help to account for the accuracy of our hybrid solution. If, on the other hand, the perturbation functions in a particular problem turn out to be only weakly dependent on the domain of interest, then we would not expect our hybrid solution to be much of an improvement over the perturbation solution.

In Figure 9, we have plotted the first four (normalized) perturbation functions used in the potential flow problem discussed in Section 4. In Table II, we have recorded certain L_2-norms indicating the accuracy of representing these functions, as well as the "exact" solutions for different values of ε, in terms of the first four coordinate functions. As the table clearly indicates, the fourth perturbation function adds very little "new" information about the shape of the solutions. In a similar manner, the first 10 perturbation functions used for the electrostatic problem discussed in Section 5 are presented in Figure 10, while Table III contains certain L_2-norms which indicate the degree of dependency among the different functions and the "exact" solutions. Again the degree of dependency that does exist helps to explain why the hybrid method produces considerably better accuracy than the perturbation method alone. (We have investigated several other slender body shapes in addition to the family of "peanut" shaped bodies presented here and have again found results concerning dependency similar to those indicated above.)

The ability of the hybrid method to pick out certain dependencies in higher order terms can be seen explicitly in another way. To illustrate this way for the problem considered in Section 5, we note that we can construct a hybrid approximate solution of the form (5.5) with only one term. When this is done, the single amplitude $\delta_{0,1}$ is given by (5.7). Using (5.4) and (5.7), we note that the hybrid solution can be expressed as

$$\tilde{f}(z,\varepsilon) = f_{0,1}(z)\delta_{0,1}(\varepsilon)$$

$$(6.1) \qquad = f_{0,1}(z)\{(\log(\varepsilon^2))^{-1} + (\int_0^1 f_{0,2}(z)f_{0,1}(z)dz)(\log(\varepsilon^2))^{-2}$$

$$+ O((\log(\varepsilon^2))^{-3})\}.$$

The first term in the second line of (6.1) is just the first term in the perturbation expansion (5.3). The second term, however, represents the component of the second perturbation function $f_{0,2}$ which is parallel to $f_{0,1}$. Similar observations hold for the other problems we have considered when the δ_j's are expanded for small values of ε. Based upon this discussion, there is another way of characterizing those problems for which the hybrid method should be helpful. If the perturbation method produces an approximate solution which seems to have about the right shape, but not the proper amplitude, then the hybrid method should be applied. This was very evident in the computation of the pressure coefficient for a prolate spheroid considered in Section 4 (see Figure 5). A more interesting example was considered by Davis and Geer [5], who used uniform slender body theory to investigate the interaction of ships maneuvering in shallow water. Their analysis produced expressions for the forces and torques acting on the ships which had about the right shape (when expressed as a function of the distance between the ships and compared with some experimental evidence), but not the proper amplitude. Thus, it would seem plausible to expect that the hybrid method should improve the accuracy of their results for this problem.

The convergence of the hybrid method as the number of terms in the expression increases has been investigated experimentally, but has not been addressed formally. For instance, for our simple example of Section 3, we have computed the relative L_2 error of both the perturbation and hybrid solutions for values of ε between 0 and 15, using either $n = 2$ or $n = 4$ terms in the approximation. The results are plotted in Figure 11. Clearly, for each value of ε, the error in the hybrid solution is diminishing as n increases from 2 to 4. Asymptotically, as ε approaches infinity, these errors approach 0.9258 ($n = 2$) and 0.9045 ($n = 4$). It is interesting to note that the perturbation solution has a finite radius of convergence $R = 2\pi$, since the exact solution has singularities at $\varepsilon = \pm 2\pi i$. (The denominator in the expressions for the deltas with $n = 4$ has a roots at $\varepsilon = \pm 6.306i$, which are good estimates of these singularities.) The hybrid solution provides a reasonable approximation for values of ε beyond the radius of convergence of the solution upon which it is based, and presumably this approximation will improve as n is increased. This point will be discussed at greater length elsewhere, when we apply our method to several other two point boundary value problems.

There are several ways in which the basic method we have presented can be refined or extended. For example, a set of functions other than the set $\{u_j\}$ can be used in the orthogonality conditions (2.4) to determine the amplitudes $\{\delta_j\}$. That is, we could demand that the residual formed by inserting (2.3) into (2.1) should be orthogonal to another set of functions, say $\{v_j\}$, which might be related, say, to the adjoint of the operator L. In addition, some other criterion, in place of (2.4), might be used to determine the $\{\delta_j\}$, e.g., a least squares criterion. For computational purposes, there may be some advantage to orthogonalizing and scaling the $\{u_j\}$ before applying the conditions (2.4). Also, there is no reason why the first n terms in the perturbation solution must be used as a basis for the hybrid solution. In fact, it is certainly conceivable that certain higher order perturbation functions might add more new shape information about the solution than some lower order terms. In the problem considered in Section 5, for example, we could consider forming a hybrid solution consisting of, for example, $f_{0,1}$, $f_{0,2}$, $f_{0,3}$, and $f_{1,2}$. The problem of choosing an "optimal" set of perturbation functions to use as a basis for the hybrid solution (both from a computational and theoretical point of view) is currently under investigation and progress on this point, as well as those mentioned above, will be reported as progress is made.

REFERENCES

1. Almroth, B. O., Sterlin, P., and Brogan, F. A., "Use of global functions for improvement in efficiency of nonlinear analysis," AIAA Nat. Conf. (1981), Paper No. 81-0575, pp. 286-292.

2. Almroth, B. O., Sterlin, P., and Brogan, F. A., "Automatic choice of global shape functions in structural analysis," AIAA J., Vol. 16 (1978), pp. 525-528.

3. Andersen, C. M. and Geer, J. F., "Power series expansions for the frequency and period of the limit cycle of the van der Pol equation," SIAM J. Appl. Math., Vol. 42 (1982), pp. 678-693.

4. Barshinger, R. N. and Geer, J. F., "The electrostatic field about a slender dielectric body of revolution," SIAM J. Appl. Math., to appear.

5. Davis, A. M. J. and Geer, J. F., "The application of uniform slender body theory to the motion of two ships in shallow water," J. Fluid Mech., Vol. 114 (1982), pp. 419-441.

6. Fink, J. P. and Rheinbolt, W. C., "On the error behavior of the reduced basis technique for nonlinear finite element approximations," ZAMM 63, (1983), pp. 21-28.

7. Galerkin, B. G., "Vestnik Inzhenerov," Tech., Vol. 19 (1915), pp. 897-908.

8. Geer, J. F., "Uniform asymptotic solutions for potential flow about a slender body of revolution," J. Fluid Mech., Vol. 67 (1975), pp. 817-827.

9. Geer, J. F. "Stokes flow past a slender body of revolution," J. Fluid Mech., Vol. 78 (1976), pp. 577-600.

10. Geer, J. F., "The scattering of a scalar wave by a slender body of revolution," SIAM J. Appl. Math., Vol. 34 (1978), pp. 348-370.

11. Geer, J. F., "Electromagnetic scattering by a perfectly conducting slender body of revolution: Axial incident plane wave," SIAM J. Appl. Math., Vol. 38 (1980), pp. 93-102.

12. Handelsman, R. and Keller, J. B., "Axially symmetric potential flow around a slender body," J. Fluid Mech., Vol. 28 (1967), pp. 131-147.

13. Handelsman, R. and Keller, J. B., "The electrostatic field around a slender conducting body of revolution," SIAM J. Appl. Math., Vol. 15 (1967), pp. 824-843.

14. Homentcovschi, D., "Uniform asymptotic solutions for the potential field about a thin oblate body of revolution," SIAM J. Appl. Math., Vol. 42 (1981), pp. 44-65.

15. Landweber, L., David W. Taylor Model Basin Report, No. 761 (1951).

16. MACSYMA Reference Manual, Version Eleven, MACSYMA Group, Symbolics Incorporated, 1985.

17. Moran, J. P., "Line source distributions and slender body theory," J. Fluid Mech., Vol. 17 (1962), pp. 285-297.

18. Munk, N. M., Rep. Nat. Adv. Comm. Aero., Washington, 184 (1924), pp. 268-282.

19. Nagy, D. A., "Modal representation of geometrically nonlinear behavior by the finite element method," Computers and Structures, Vol. 10 (1979), pp. 683-688.

20. Nagy, D. A. and Konig, M., "Geometrically nonlinear finite element behavior using buckling mode superposition," Comput. Methods Appl. Mech. Engrg., Vol. 19 (1979), pp. 447-484.

21. Newman, J. N., "Applications of slender body theory in ship hydrodynamics," Ann. Rev. Fluid Mech., Vol. 2 (1970), pp. 67-94.

22. Noor, A. K., "Recent advances in reduction problems for nonlinear problems," Computers & Structures, Vol. 13 (1981), pp. 31-44.

23. Noor, A. K., "Reduction method for the nonlinear analysis of symmetric anisotropic panels," Internat. J. Numer. Methods Engrng., Vol. 23 (1986), pp. 1329-1341; also in Finite Element Methods for Nonlinear Problems, edited by P. G. Bergan, K. J. Bathe, and W. Wunderlich, Springer-Verlag, 1986, pp. 389-407.

24. Noor, A. K., "Hybrid analytical technique for nonlinear analysis of structures," AIAA J., Vol. 23 (1985), pp. 938-946.

25. Noor, A. K., Andersen, C. M., and Peters, J. M., "Global-local approach for nonlinear shell analysis," Proc. 7th ASCE Conf. on Electronic Comp., Washington U., St. Louis, Missouri, August 1979, pp. 634-657.

26. Noor, A. K., Andersen, C. M., and Peters, J. M., "Reduced basis technique for collapse analysis of shells," AIAA J., Vol. 19 (1981), pp. 393-397.

27. Noor, A. K., Andersen, C. M., and Peters, J. M., "Two-stage Rayleigh-Ritz technique for nonlinear analysis of structures" in Innovative Numerical Analysis for the Applied Engineering Sciences, Univ. Virginia Press (1980), pp. 743-753.

28. Noor, A. K., Andersen, C. M., and Tanner, J. A., "Mixed models and reduction techniques for large-rotation nonlinear analysis of shells for revolution with application to tires," NASA TP-2343 (1984).

29. Noor, A. K. and Balch, C. D., "Hybrid perturbation/Bubnov-Galerkin technique for nonlinear thermal analysis," AIAA J., Vol. 22 (1984), pp. 287-294.

30. Noor, A. K., Balch, C. D., and Shibut, M. A., "Reduction methods for nonlinear steady-state thermal analysis," Internat. J. Numer. Methods Engrg., Vol. 20 (1984), pp. 1323-1348.

31. Noor, A. K. and Peters, J. M., "Bifurcation and post-buckling analysis of laminated composite plates via reduced basis technique," Comput. Methods Appl. Mech. Engrg., Vol. 29 (1981), pp. 271-295.

32. Noor, A. K. and Peters, J. M., "Elastic collapse analysis of shells via global-local approach," in Nonlinear Finite Element Analysis in Structured Mechanics, Walter Wunderlich, E. Stein, and K. J. Bathe (eds.), Springer-Verlag, New York, 1981, pp. 169-184.

33. Noor, A. K. and Peters, J. M., "Mixed models and reduced/selective integration displacement models for nonlinear analysis of curved beams," Internat. J. Numer. Methods Engrg., Vol. 19 (1981), pp. 1783-1803.

34. Noor, A. K. and Peters, J. M., "Multiple-parameter reduced basis technique for bifurcation and post-buckling analysis of composite plates," Internat. J. Numer. Methods Engrg., Vol. 19 (1983), pp. 1783-1803.

35. Noor, A. K. and Peters, J. M., "Nonlinear analysis via global-local mixed finite element approach," Internat. J. Numer. Methods Engrg., Vol. 15 (1980), pp. 1363-1380.

36. Noor, A. K. and Peters, J. M., "Recent advances in reduction methods for instability analysis of structures," Computers & Structures, Vol. 16 (1983), pp. 67-80.

37. Noor, A. K. and Peters, J. M., "Reduced basis technique for nonlinear analysis of structures," AIAA J., Vol. 18 (1980), pp. 455-462.

38. Noor, A. K. and Peters, J. M., "Tracing post-limit-point paths with reduced basis technique," Comput. Methods Appl. Mech. Engrg., Vol. 28 (1981), pp. 217-240.

39. Noor, A. K., Peters, J. M., and Andersen, C. M., "Mixed models and reduction techniques for large-rotation nonlinear problems," Comput. Methods Appl. Mech. Engrg., Vol. 44 (1984), pp. 67-89.

40. Thwaites, B. (Ed.), Incompressible Aerodynamics, Oxford University Press, London (1960).

41. Van Dyke, M., Perturbation Methods in Fluid Mechanics, Academic Press, New York (1964).

42. Van Dyke, M., "Analysis and improvement of perturbation series," Quart. J. Mech. Appl. Math., Vol. 27 (1974), pp. 423-450.

43. Wong, Tin-Chee, Liu, C. H., and Geer, J. F., "Comparison of uniform perturbation and numerical solutions for some potential flows past slender bodies," Computers & Fluids, Vol. 13 (1985), pp. 271-283.

APPENDIX

In this appendix, we shall describe briefly the iterative method we used to solve equations (4.2) and (5.2) numerically. Using a result of Handelsman and Keller [12], we see that the right side of equation (4.2) approaches $2f(z,\varepsilon)$ as $\varepsilon \to 0$. Thus, if we add and subtract $2f(z,\varepsilon)$ on the right side of (4.2) and then define

$$\text{(A.1)} \qquad F(z,\varepsilon) = 4f(z,\varepsilon)/\varepsilon^2,$$

we find that (4.2) can be written as

$$\text{(A.2)} \qquad S'(z) = F(z,\varepsilon) + L(F(z,\varepsilon)),$$

where the linear operator L is defined by

$$\text{(A.3)} \qquad L(F(z)) = (1/2)\,\{(d/dz)\int_\alpha^\beta \frac{z-t}{\sqrt{(zt)^2+\varepsilon^2 s(z)}} F(t)\,dt - 2F(z)\}.$$

From Handelsman and Keller [12], it follows that $L(F(z)) = O(\varepsilon^2 F(z))$ as $\varepsilon \to 0$. This suggests that we could solve equation (A.2) by defining the sequence of approximations $\{F_k(z)\}$ by

$$\text{(A.4)} \qquad F_0(z) = S'(z)$$

$$\text{(A.5)} \qquad F_{k+1}(z) = L(F_k(z)),\ k = 0,1,2,\cdots .$$

The iterative scheme (A.3)-(A.5) was used to solve equation (4.2) numerically. As indicated in Section 4, the limits of integration α and β were treated as known quantities, using the Taylor series expansions obtained by Handelsman and Keller, as well as several more terms which we calculated using the symbolic computation system MACSYMA [16]. The terms in these series were used with the ratio and root tests to estimate the radius of convergence of the formal perturbation series, as mentioned in Section 4. We found that the scheme (A.3)-(A.5) converged rather quickly for $\alpha \leq z \leq \beta$, but converged rather slowly for $0 \leq z < \alpha$ or $\beta < z \leq 1$. Nevertheless, the scheme did produce (eventually) the correct solution to equation (4.2) on the entire interval $0 \leq z \leq 1$ for all the examples we tried.

Equation (5.2) can also be solved by iteration by first rewriting it as

$$1 = G(z,\varepsilon)f(z,\varepsilon) + M(f(z,\varepsilon)), \tag{A.6}$$

$$G(z,\varepsilon) = \int_\alpha^\beta \frac{1}{\sqrt{(z-t)^2+\varepsilon^2 s(z)}}\,dt = \log\left\{\frac{\beta-z+\sqrt{(\beta-z)^2+\varepsilon^2 s(z)}}{\alpha-z+\sqrt{(\alpha-z)^2+\varepsilon^2 s(z)}}\right\} \tag{A.7}$$

$$M(f(z,\varepsilon)) = \int_\alpha^\beta \frac{f(t,\varepsilon) - f(z,\varepsilon)}{\sqrt{(z-t)^2+\varepsilon^2 s(z)}}\,dt. \tag{A.8}$$

Using a result from Handelsman and Keller [13], we see that $G(z,\varepsilon) = O(\log(\varepsilon^2))$ and $M(F(z)) = O(F(z))$, as $\varepsilon \to 0$. These facts suggest that we define a sequence of approximations by

$$f_0(z,\varepsilon) = 1/G(z,\varepsilon), \tag{A.9}$$

(A.10) $$f_{k+1}(z,\varepsilon) = \{1 - M(f_k(z,\varepsilon))\}/G(z,\varepsilon), \quad k = 0,1,2,\cdots \quad .$$

We found that the scheme (A.9)-(A.10) converged to the solution of (5.2) for all the cases we tried, but with the same type of non-uniform convergence as noted above.

Table I

u(z)	n = 1	2	3	4
$u_j(z)$:				
j = 1	0	0	0	0
2	1.000	0	0	0
3	0.189	0.189	0	0
4	1.000	0.100	0.100	0
5	0.225	0.225	0.014	0.014
6	1.000	0.123	0.123	0.005
7	0.233	0.233	0.018	0.018
8	1.000	0.129	0.129	0.007
9	0.235	0.235	0.020	0.C20
10	1.000	0.130	0.130	0.008
11	0.235	0.235	0.020	0.020
12	1.000	0.130	0.130	0.008
13	0.235	0.235	0.020	0.020
14	1.000	0.131	0.131	0.008
15	0.235	0.235	0.020	0.020
U (ε = 2)	0.122	0.013	0.001	0.000
U (ε = 5)	0.269	0.067	0.014	0.003
U (ε = 10)	0.403	0.164	0.062	0.021

L_2-norms of the differences $u(x) - \sum_{j=1}^{n} a_j g_j(x)$ for the simple two-point boundary value problem of Section 3, where the functions $u(x)$ have been normalized and the functions $g_j(x)$ have been obtained from the perturbation functions u_1 to u_4 by the Gram-Schmidt process. The numbers appearing in the table are the quantities

$$(1 - \sum_{j=1}^{n} a_j^2)^{1/2}, \quad \text{where} \quad a_j = \int_0^1 u(z)g_j(z)dz.$$

Thus, the smaller the entry the better the function is approximated. The last three rows in the table correspond to the exact solution of the problem (3.1).

Table II

$f(z)$	n = 1	2	3	4
$f_{1,0}$	0	0	0	0
$f_{2,1}$	0.647	0	0	0
$f_{2,0}$	0.975	0.045	0	0
$f_{3,2}$	0.885	0.445	0.095	0
f (c = .03)	0.045	0.000	0.000	0.000
f (c = .07)	0.195	0.032	0.000	0.000
f (c = .11)	0.525	0.276	0.032	0.032

L_2-norms of the differences $f(z) - \sum_{j=1}^{n} a_j g_j(z)$, where the functions $f(z)$ have been normalized and the functions $g_j(z)$ have been obtained from the potential flow perturbation coefficient functions $f_{1,0}$, $f_{2,1}$, $f_{2,0}$, and $f_{3,2}$ by the Gram-Schmidt process. The numbers appearing in the table are the quantities

$$(1 - \sum_{j=1}^{n} a_j^2)^{1/2}, \quad \text{where} \quad a_j = \int_0^1 f(z) g_j(z) dz.$$

The last three rows in the table correspond to the solution of the integral equation (4.2) obtained numerically.

Table III

f(z)	n = 1	2	3	4
$f_{0,j}(z)$:				
j=1	0	0	0	0
2	0.823	0	0	0
3	0.985	0.032	0	0
4	0.990	0.089	0.000	0
5	0.991	0.158	0.000	0.000
6	0.991	0.245	0.000	0.000
7	0.992	0.349	0.032	0.000
8	0.993	0.468	0.055	0.000
9	0.994	0.592	0.084	0.000
10	0.995	0.706	0.126	0.000
11	0.997	0.799	0.179	0.000
12	0.998	0.868	0.239	0.000
13	0.998	0.915	0.302	0.000
14	0.999	0.946	0.370	0.045
15	0.999	0.965	0.439	0.071
f (ε = 0.3)	0.045	0.000	0.000	0.000
f (ε = .07)	0.195	0.032	0.000	0.000
f (ε = .11)	0.525	0.276	0.032	0.032

L_2-norms of the differences $f(z) - \sum_{j=1}^{n} a_j g_j(z)$, where the functions $f(z)$ have been normalized and the functions $g_j(z)$ have been obtained from the electrostatic potential perturbation functions $f_{0,1}$, $f_{0,2}$, $f_{0,3}$, and $f_{0,4}$ by the Gram-Schmidt process. The numbers appearing in the table are the quantities

$$(1 - \sum_{j=1}^{n} a_j^2), \quad \text{where} \quad a_j = \int_0^1 f(z) g_j(z)\, dz.$$

The last three rows in the table correspond to the solution of the integral equation (5.2) obtained numerically.

FIGURE CAPTIONS

1. Comparison of solutions to the two point boundary value problem (3.1): (o o o o o) exact solution, (- - - - -) perturbation solution, (_______) hybrid solution, using two terms in the approximate solutions for (a) $\varepsilon = 2$ and (b) $\varepsilon = 5$.

2. Comparison of solutions to the two point boundary value problem (3.1): (o o o o o) exact solution, (- - - - -) perturbation solution, (_______) hybrid solution, using four terms in the approximate solutions for (a) $\varepsilon = 5$ and (b) $\varepsilon = 10$.

3. A typical slender body of revolution, with an indication of the coordinate system and the region of distribution of singularities.

4. Comparison of approximate solutions to the integral equation (4.2) obtained numerically (o o o o o), using the perturbation solution (- - - - -), and using the hybrid solution (_______) for: (a) $\varepsilon = 0.03$; (b) $\varepsilon = 0.07$; (c) $\varepsilon = 0.11$. In (a), only one term was used in the perturbation and hybrid solutions, while in (b) and (c) three terms were used.

5. Pressure coefficient for a prolate spheroid with $\varepsilon = 0.3$ using the exact solution (o o o o o), the perturbation solution (- - - - -) with one, three, and six terms, and the hybrid solution (_______) with one term.

6. Pressure coefficient for a peanut shaped body of section 4 using the solution obtained numerically (o o o o o), the perturbation solution (- - - - -), and the hybrid solution (_______) for (a) $\varepsilon = 0.03$, (b) $\varepsilon = 0.07$, and (c) $\varepsilon = 0.11$. In (a), only one term was used in the perturbation and hybrid solutions, while in (b) and (c) three terms were used.

7. Comparison of approximate solutions to the integral equation (5.2) obtained numerically (o o o o o), using the perturbation solution (- - - - -), and using the hybrid solution (_______) for: (a) $\varepsilon = 0.03$; (b) $\varepsilon = 0.07$; and (c) $\varepsilon = 0.11$. In each case, four terms were used in the perturbation and hybrid solutions.

8. The first 15 perturbation coordinate functions $u_j(x)$ appearing in equation (3.2). The functions u_1 to u_3 are plotted with solid lines, while the remaining functions appear as dotted lines. Each function has been normalized so that its maximum absolute value is 1.

9. The first four potential flow perturbation coordinate functions $f_{i,j}(z)$ appearing in equation (4.3). The functions have been normalized so that each one is equal to 1 at $z = 0$.

10. The first 10 electrostatic potential perturbation coordinate functions $f_{0,j}(z)$ appearing in equation (5.3). The functions have been normalized so that each one is equal to 1 at $z = 0$.

11. Plot of the relative L_2 error between the exact solution to problem (3.1) and the perturbation solution (- - - - -) and the hybrid solution (______) using 2 and 4 terms in these approximate solutions.

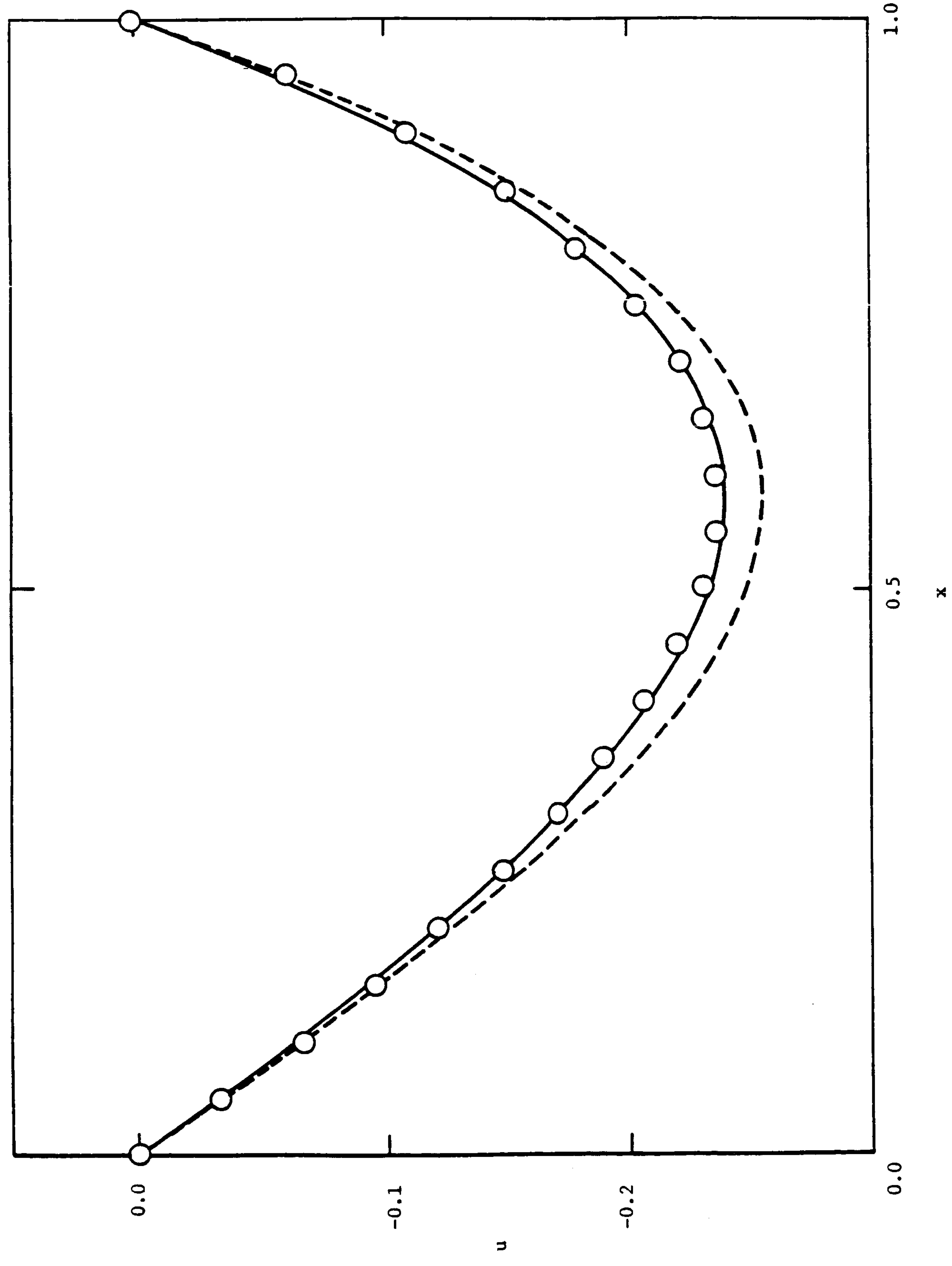


Figure 1(a)

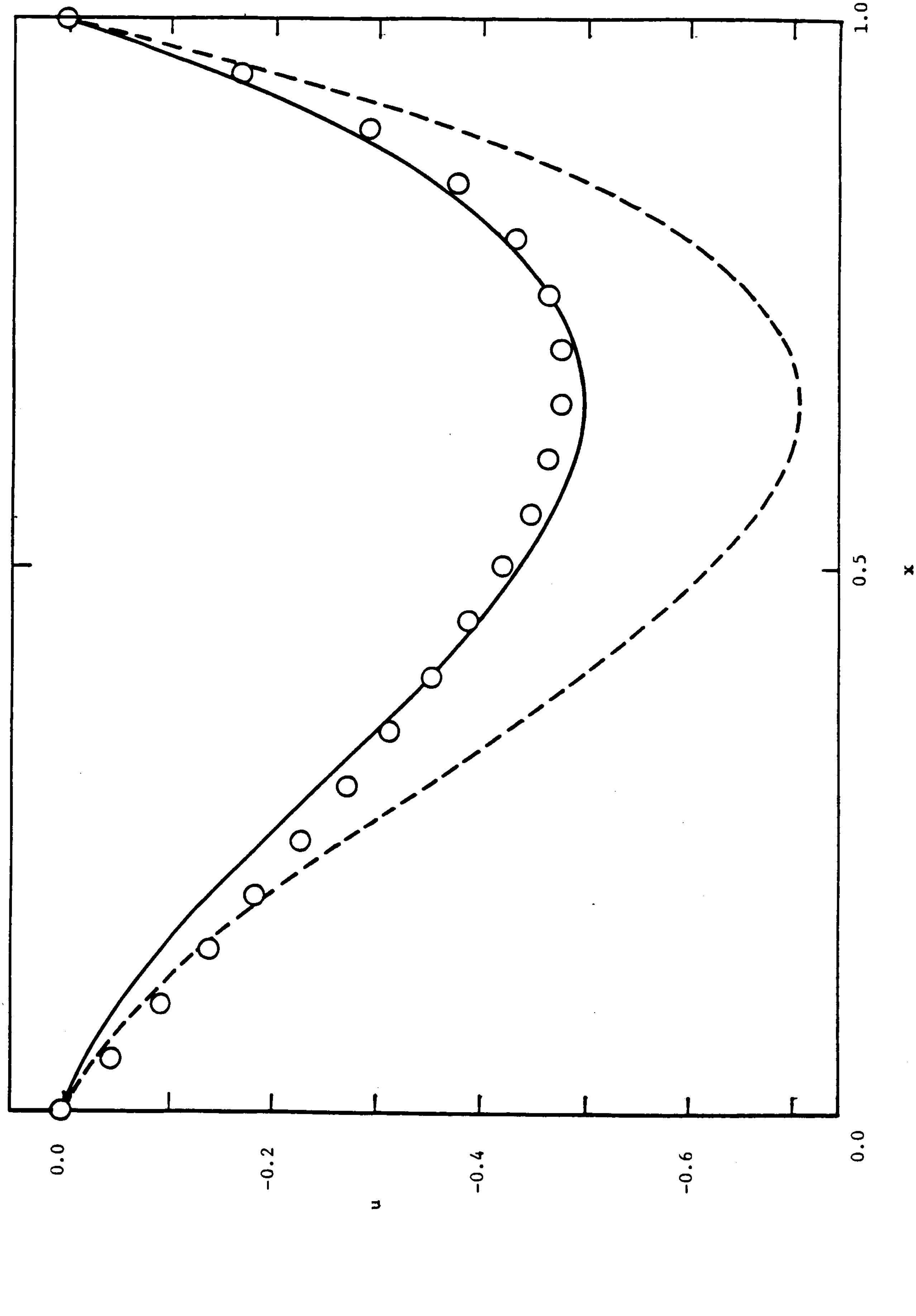


Figure 1(b)

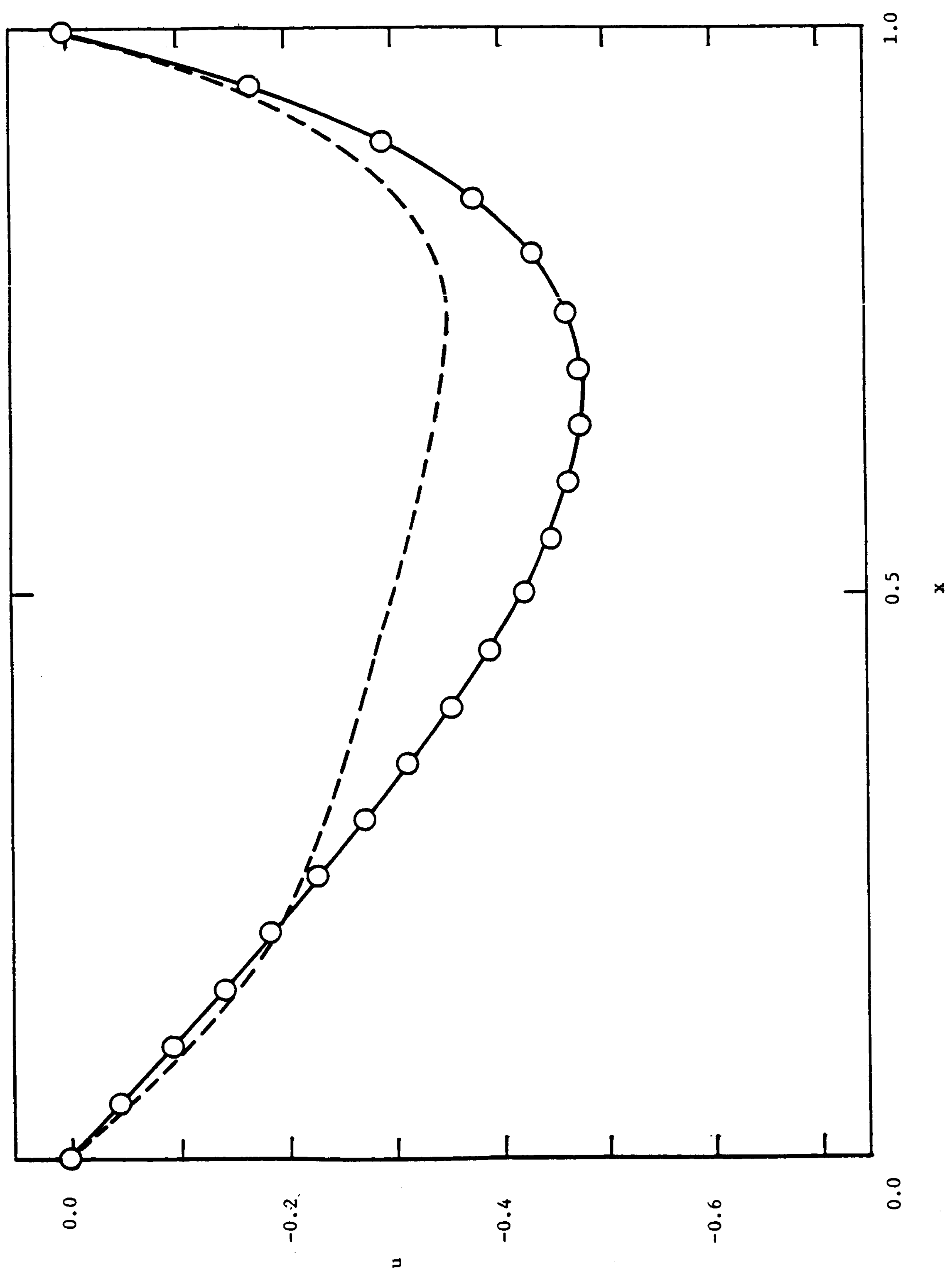


Figure 2(a)

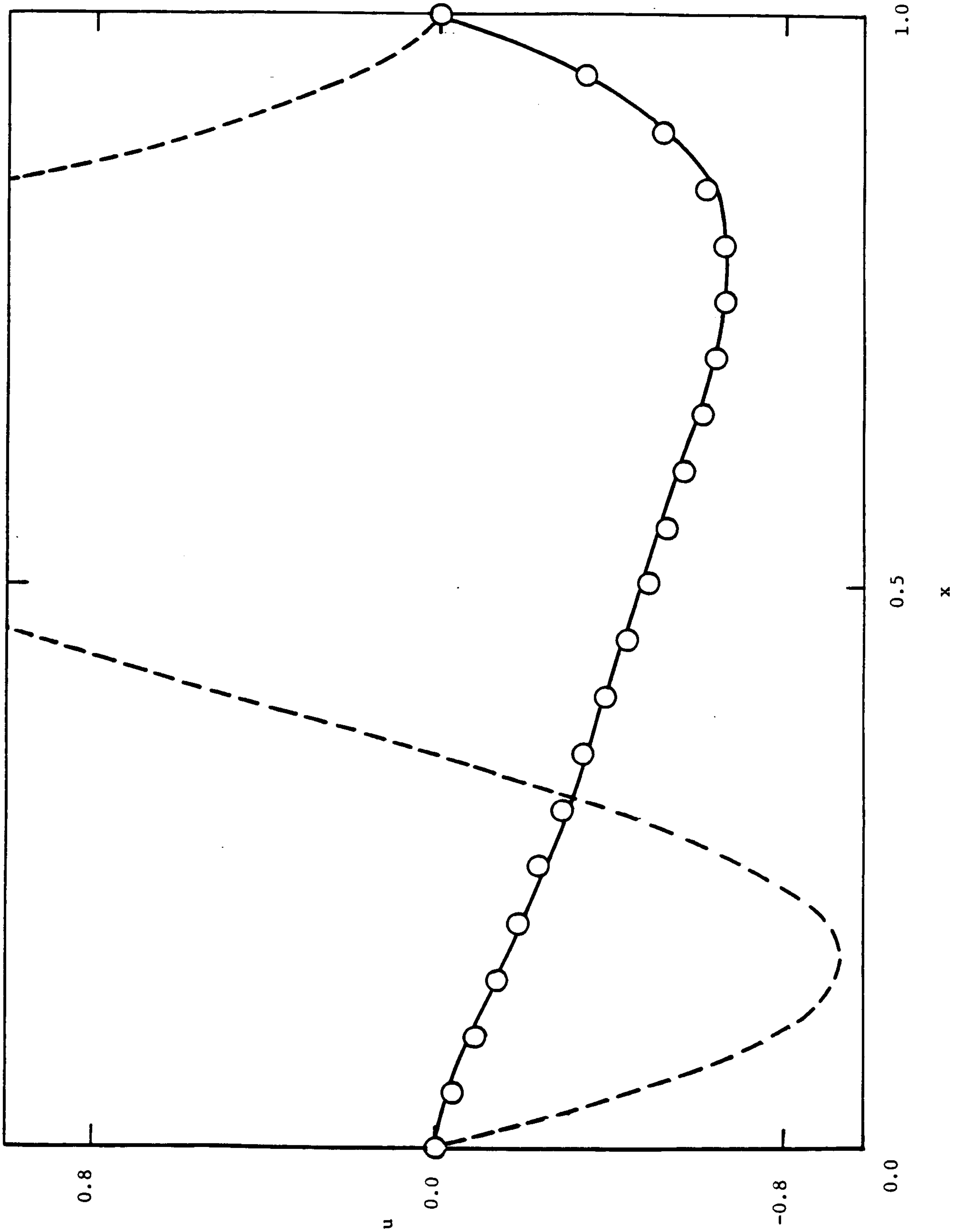


Figure 2(b)

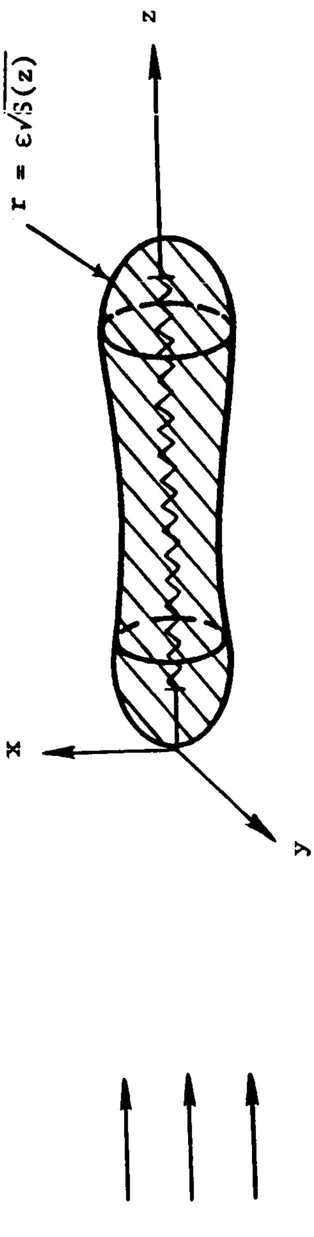


Figure 3

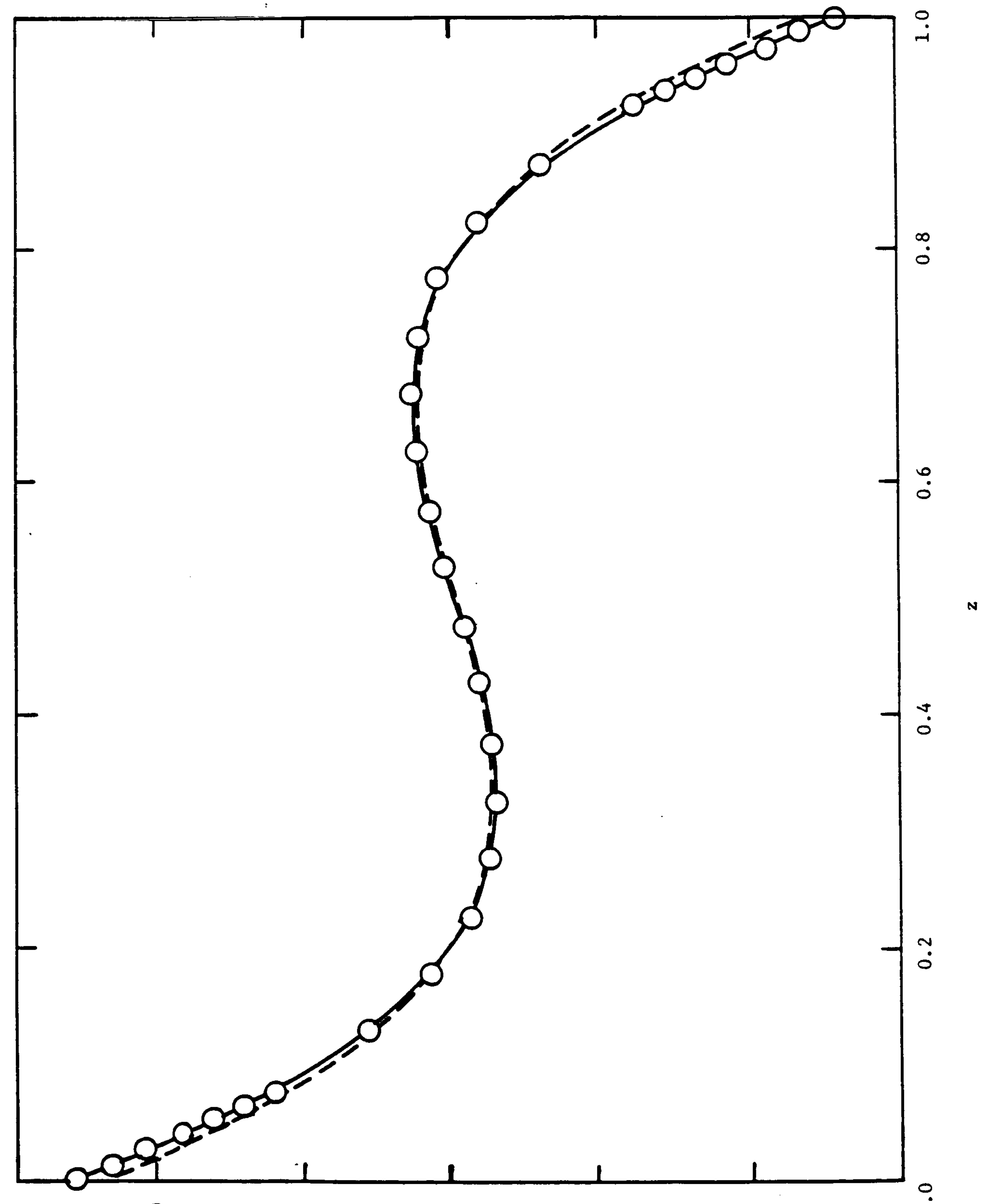


Figure 4(a)

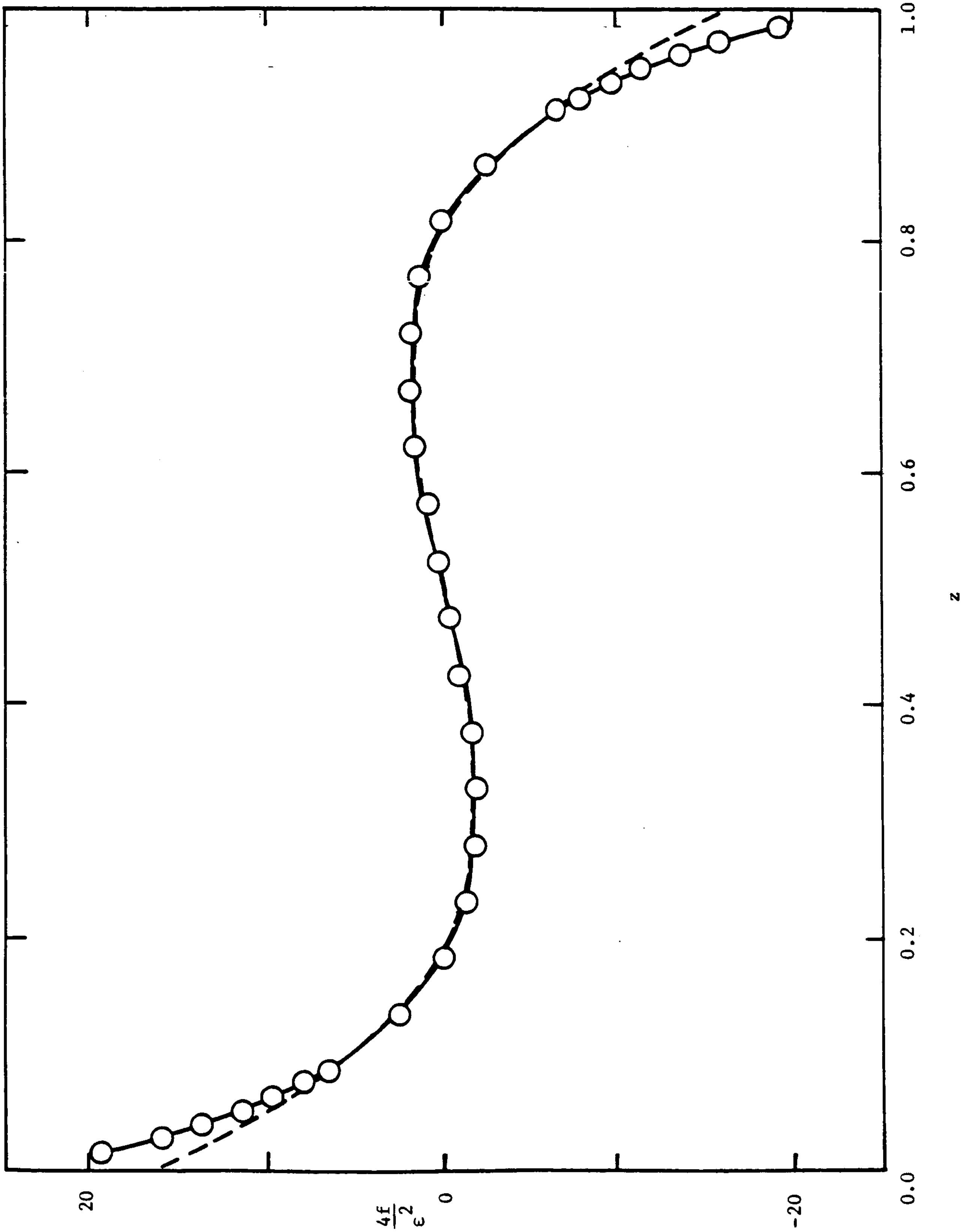


Figure 4(b)

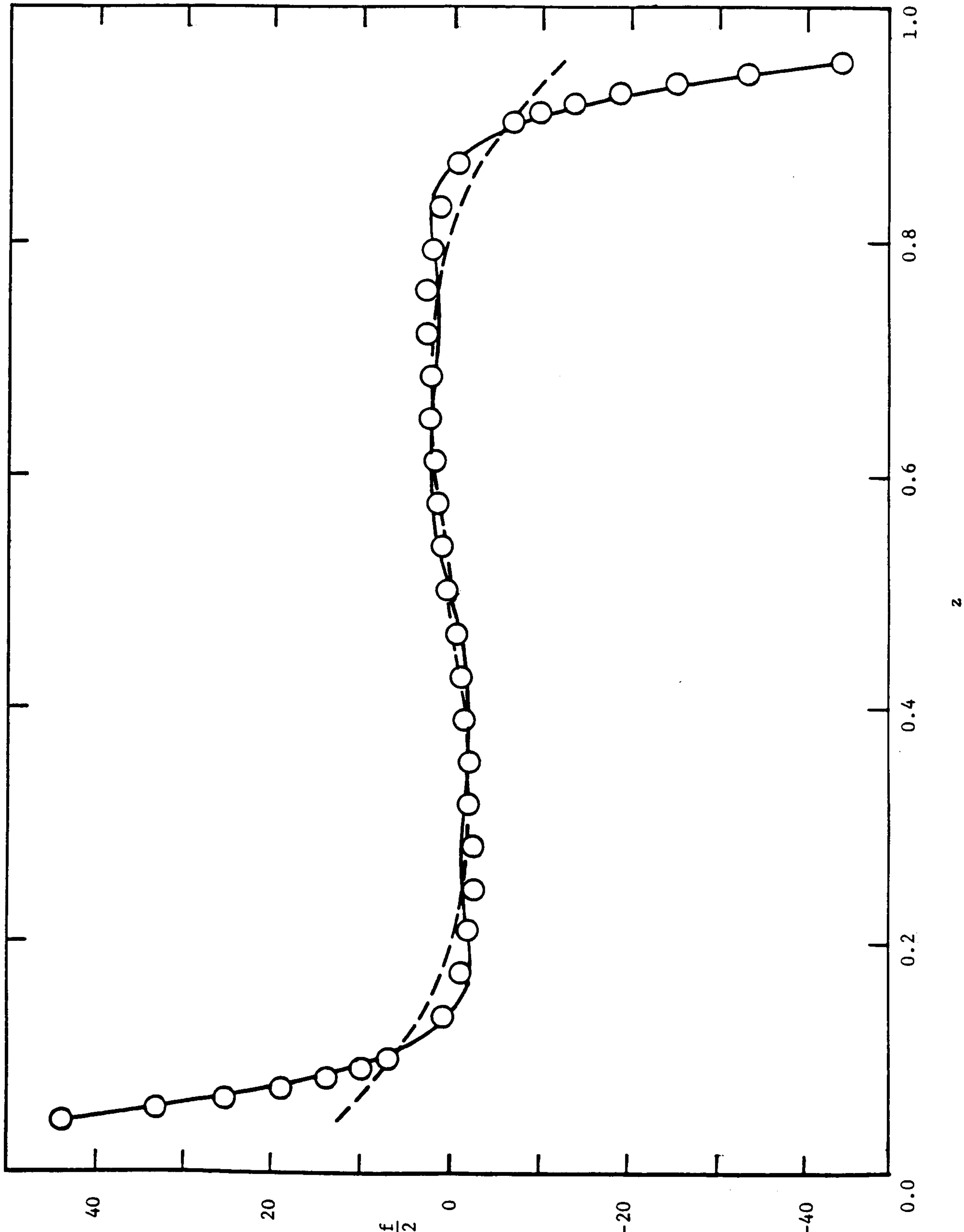


Figure 4(c)

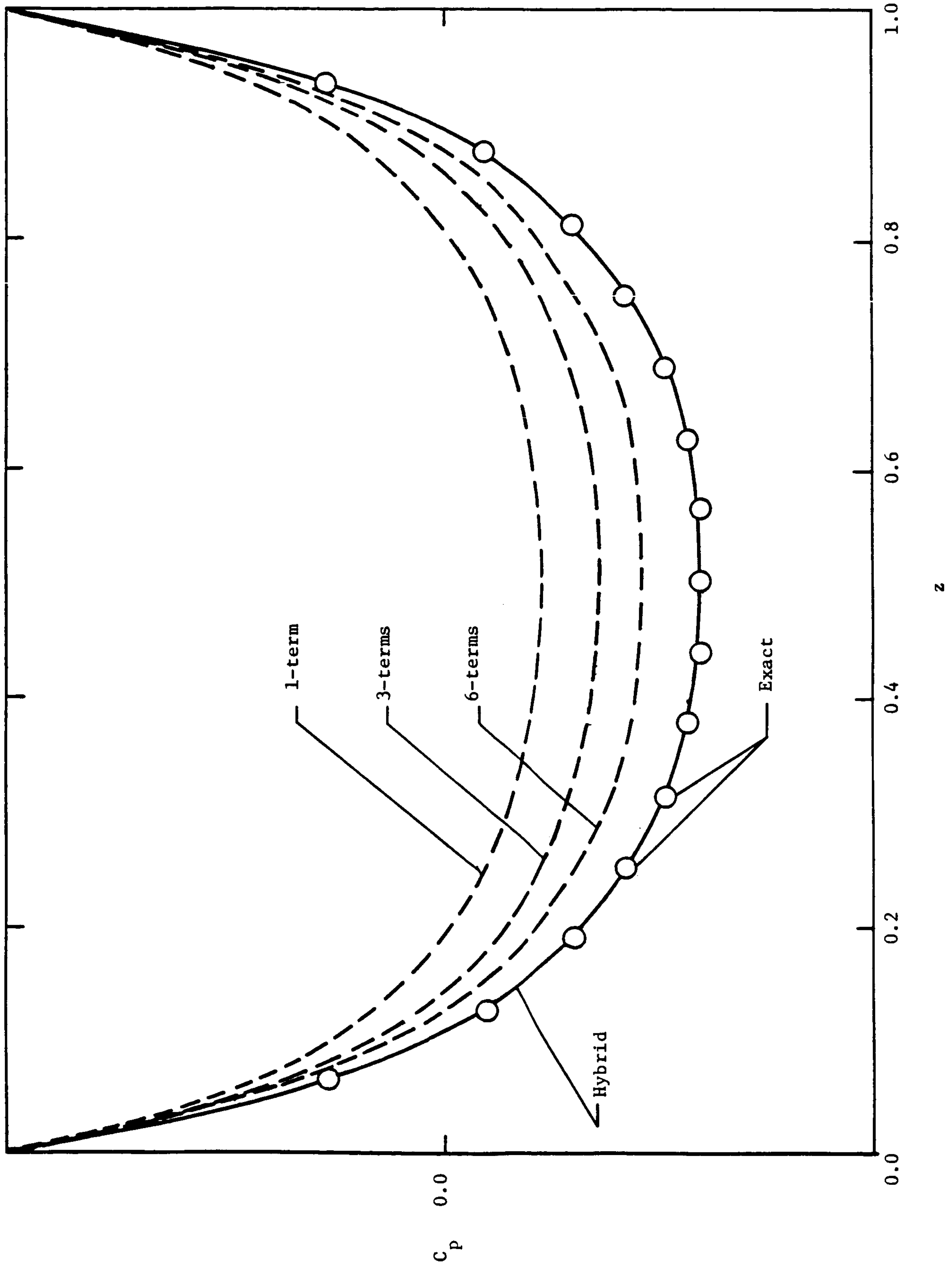


Figure 5

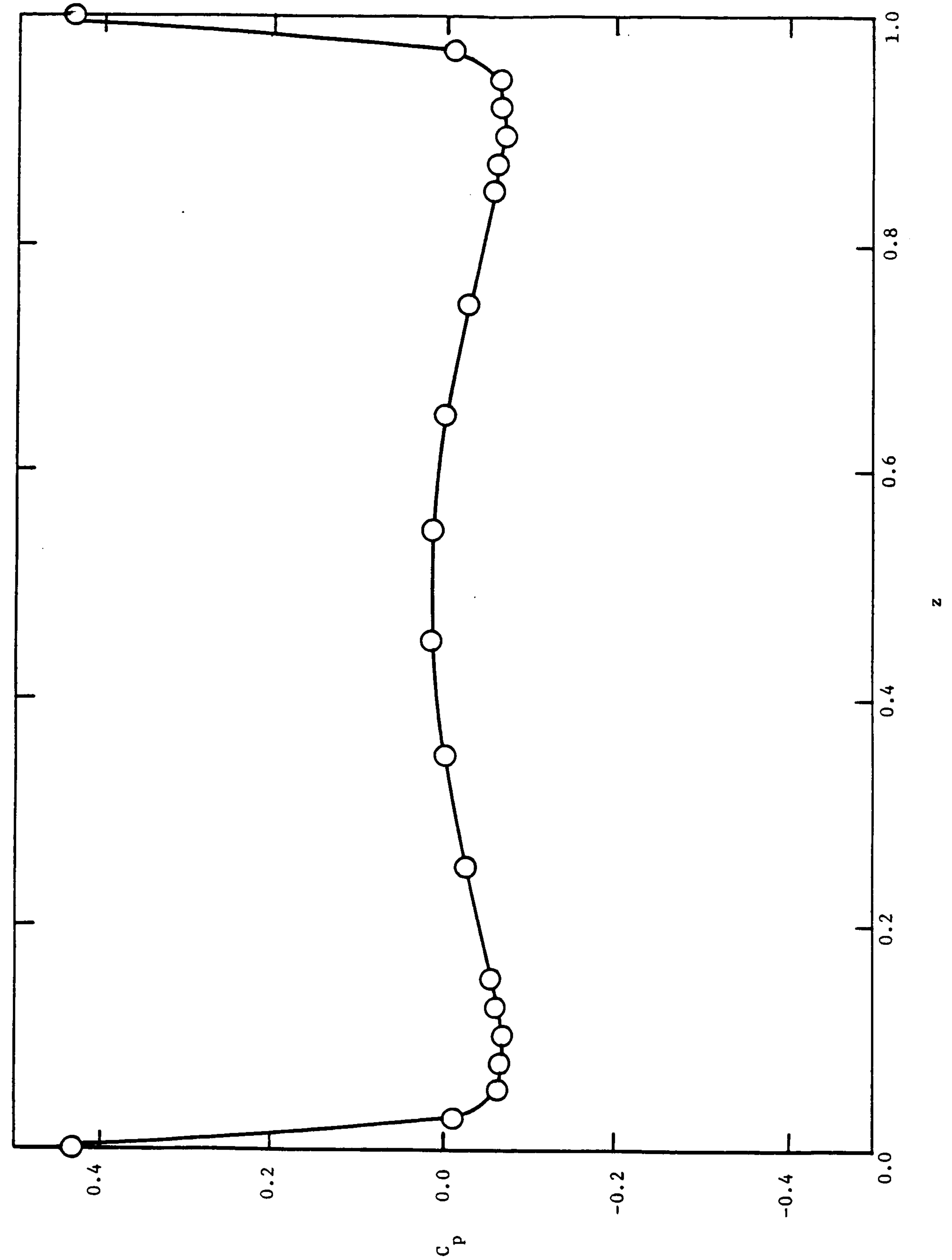


Figure 6(a)

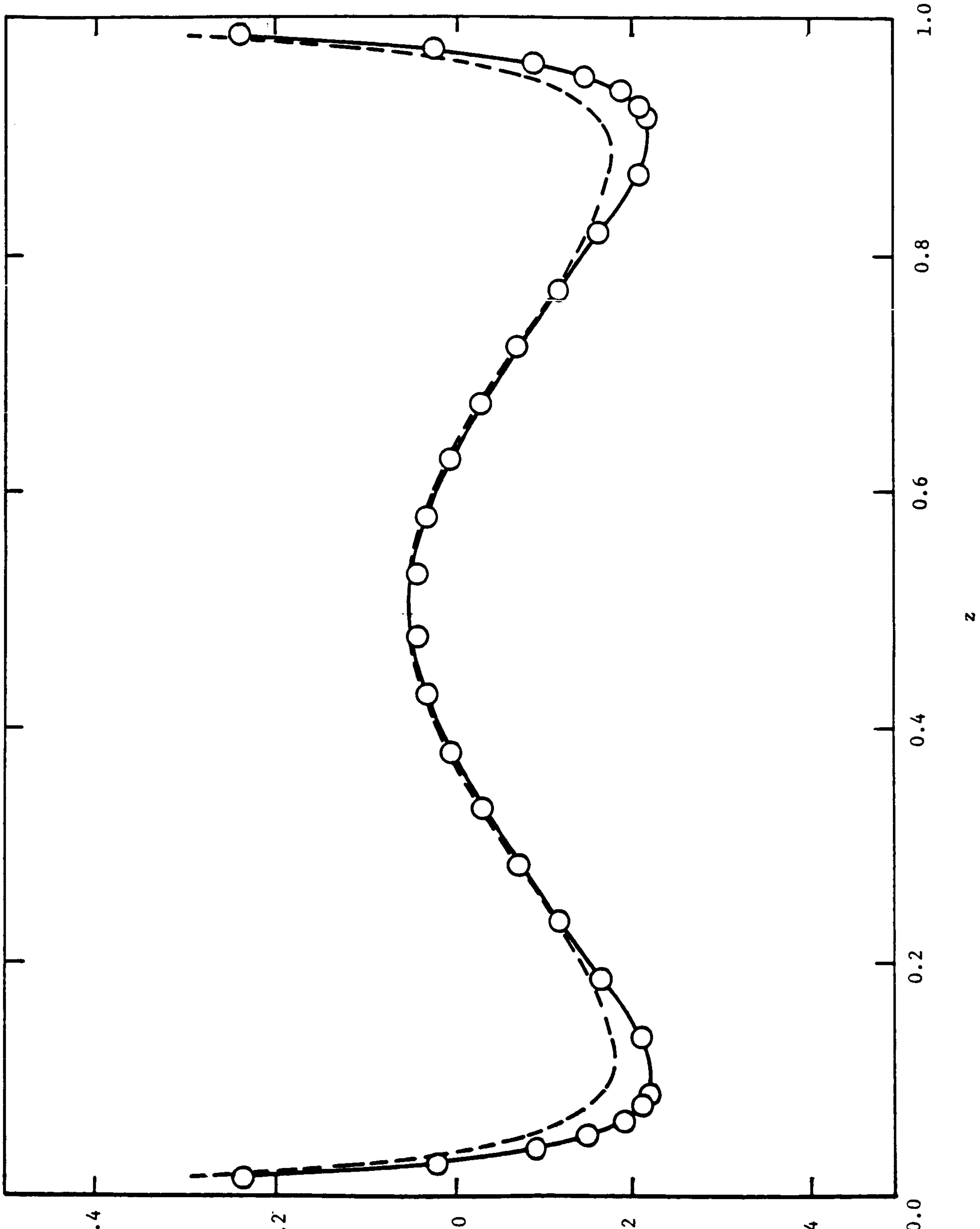


Figure 6(b)

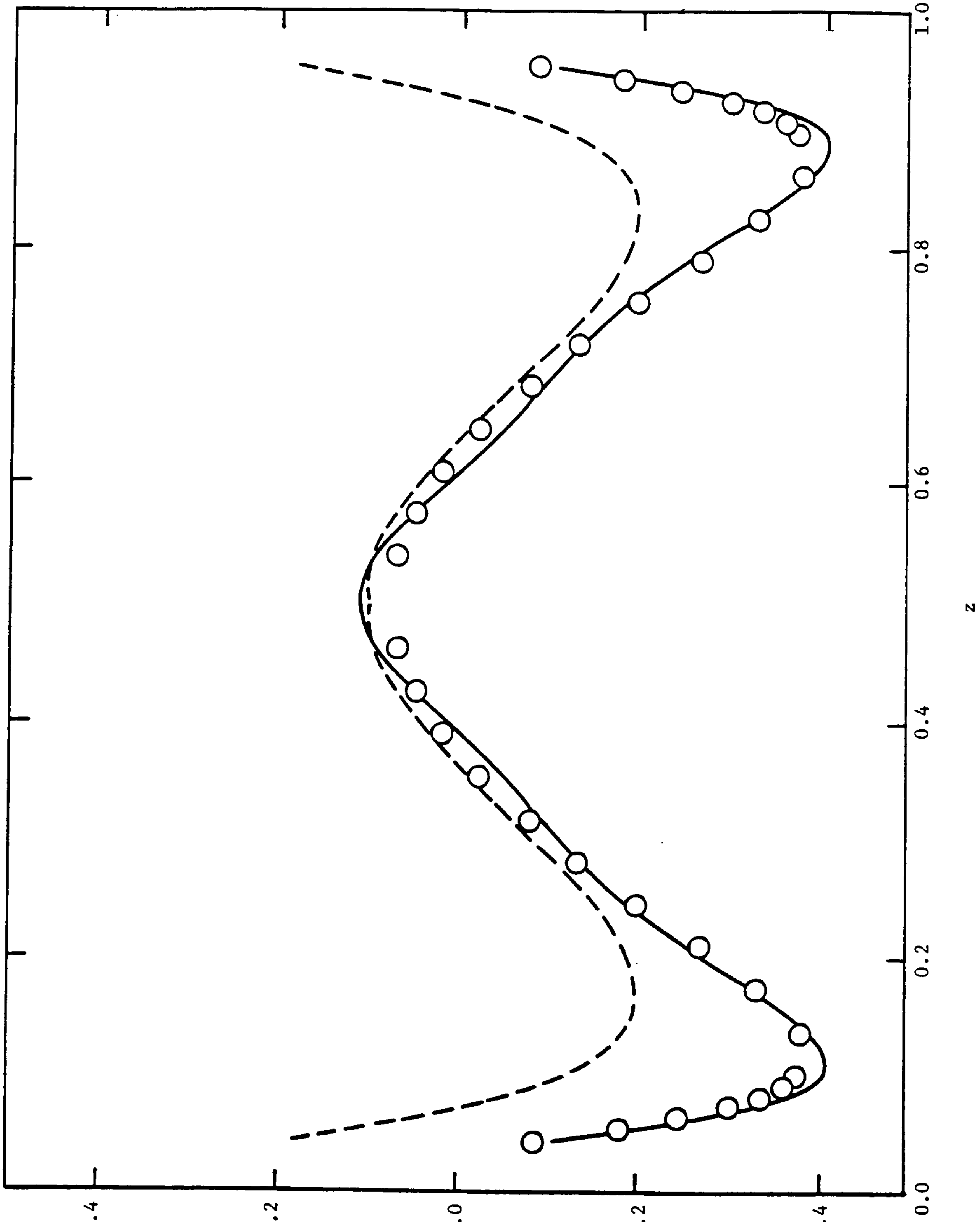


Figure 6(c)

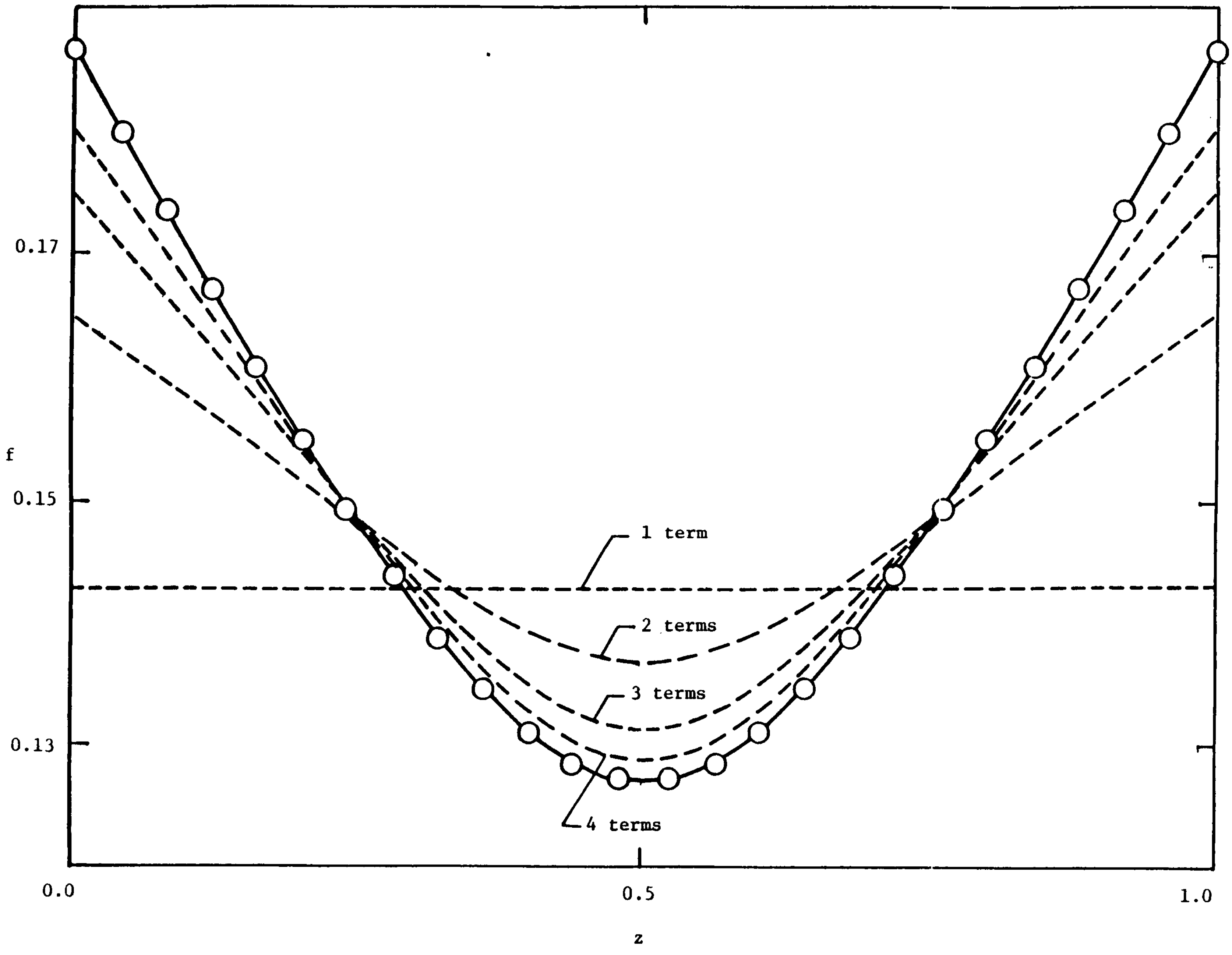


Figure 7(a)

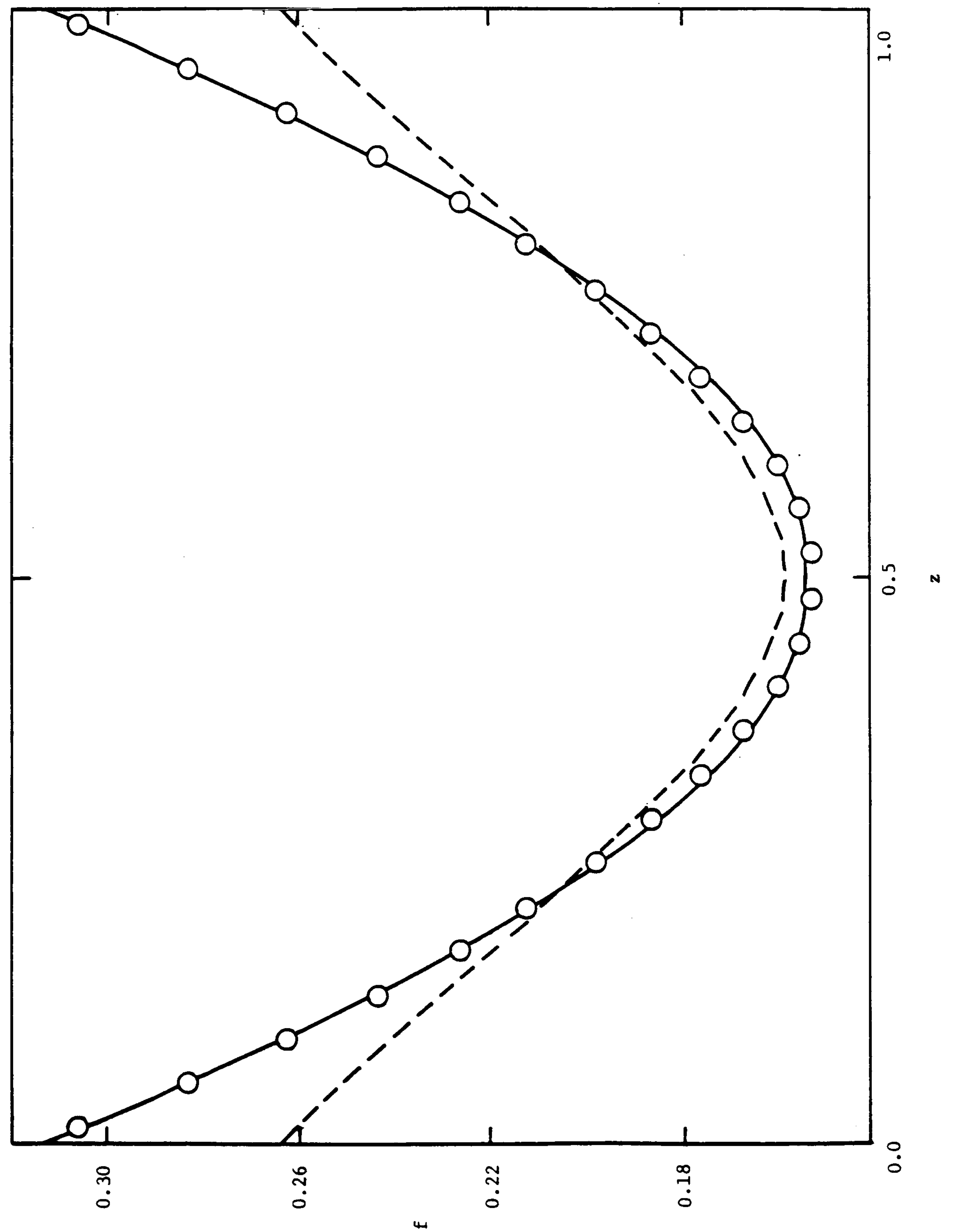


Figure 7(b)

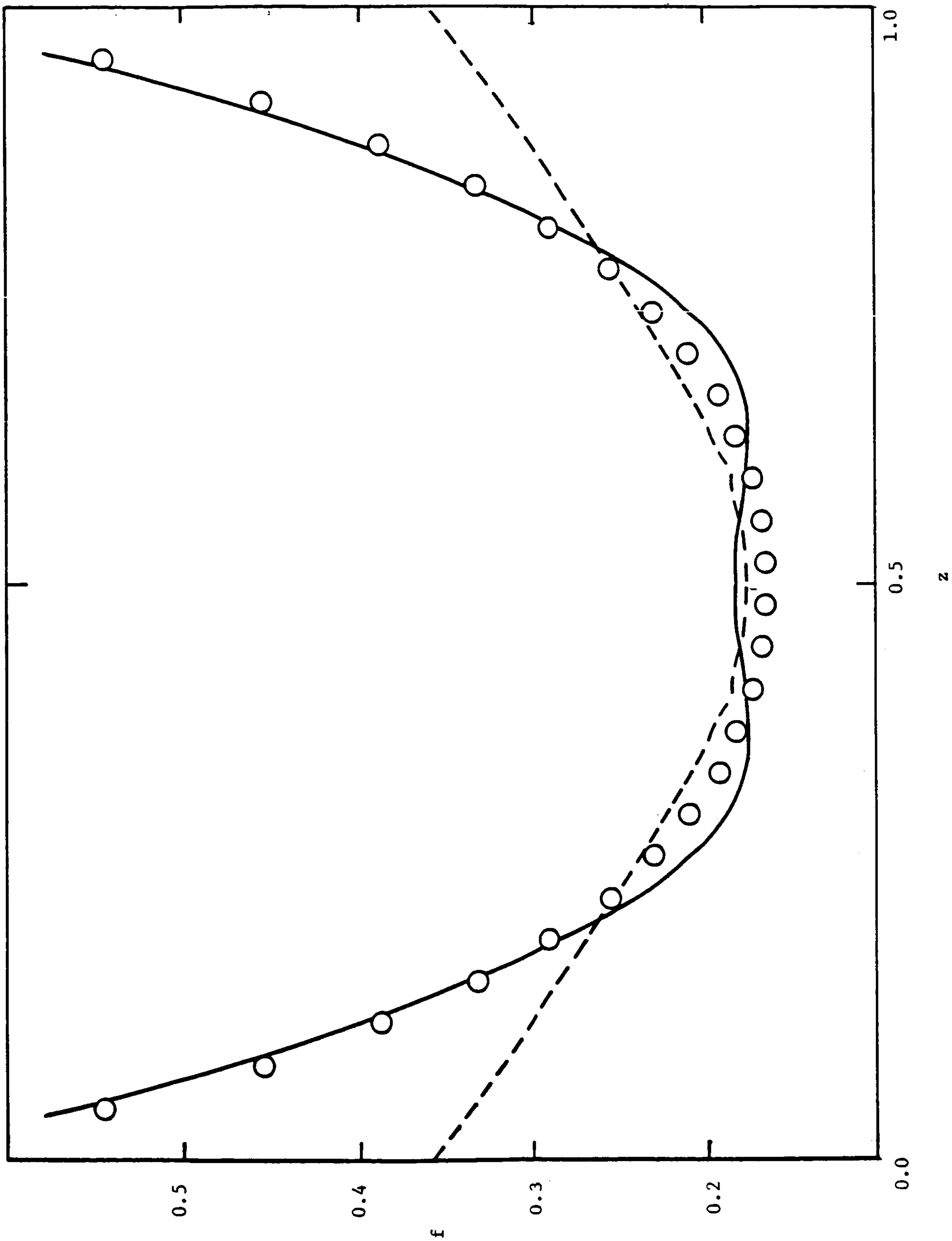
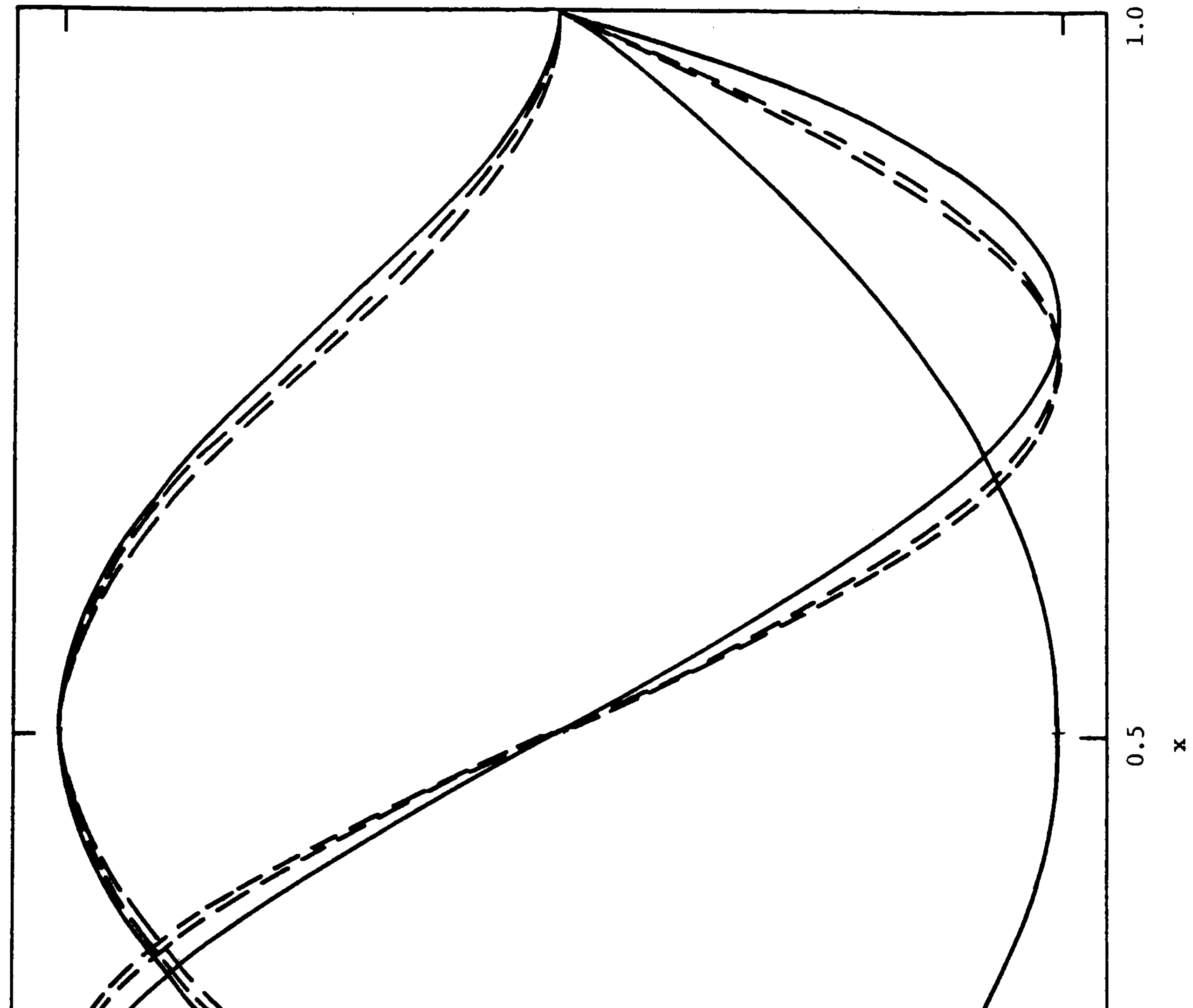


Figure 7(c)



1.0
0.0
-1.0
u_j
0.0
0.5
1.0
x

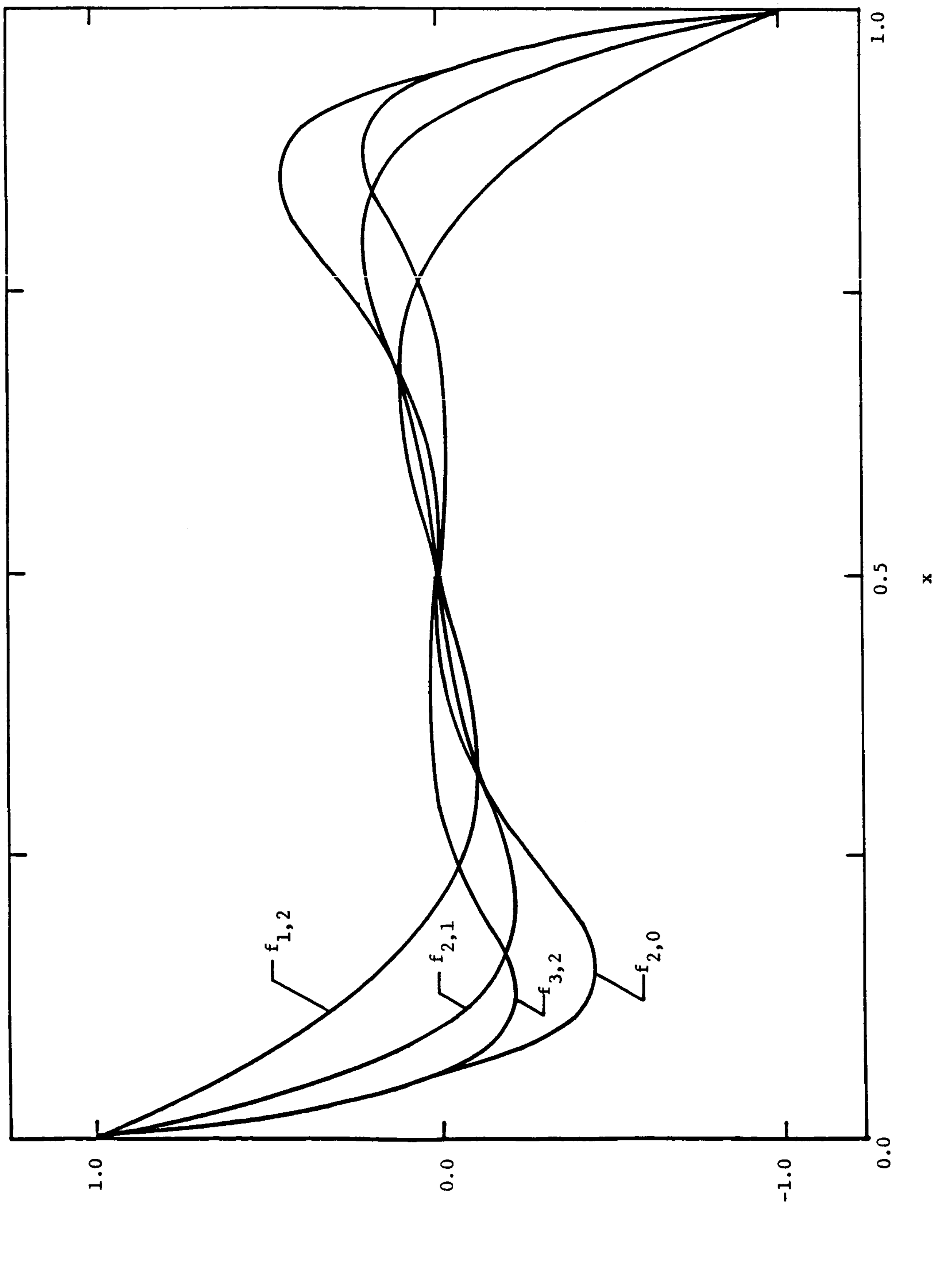


Figure 9

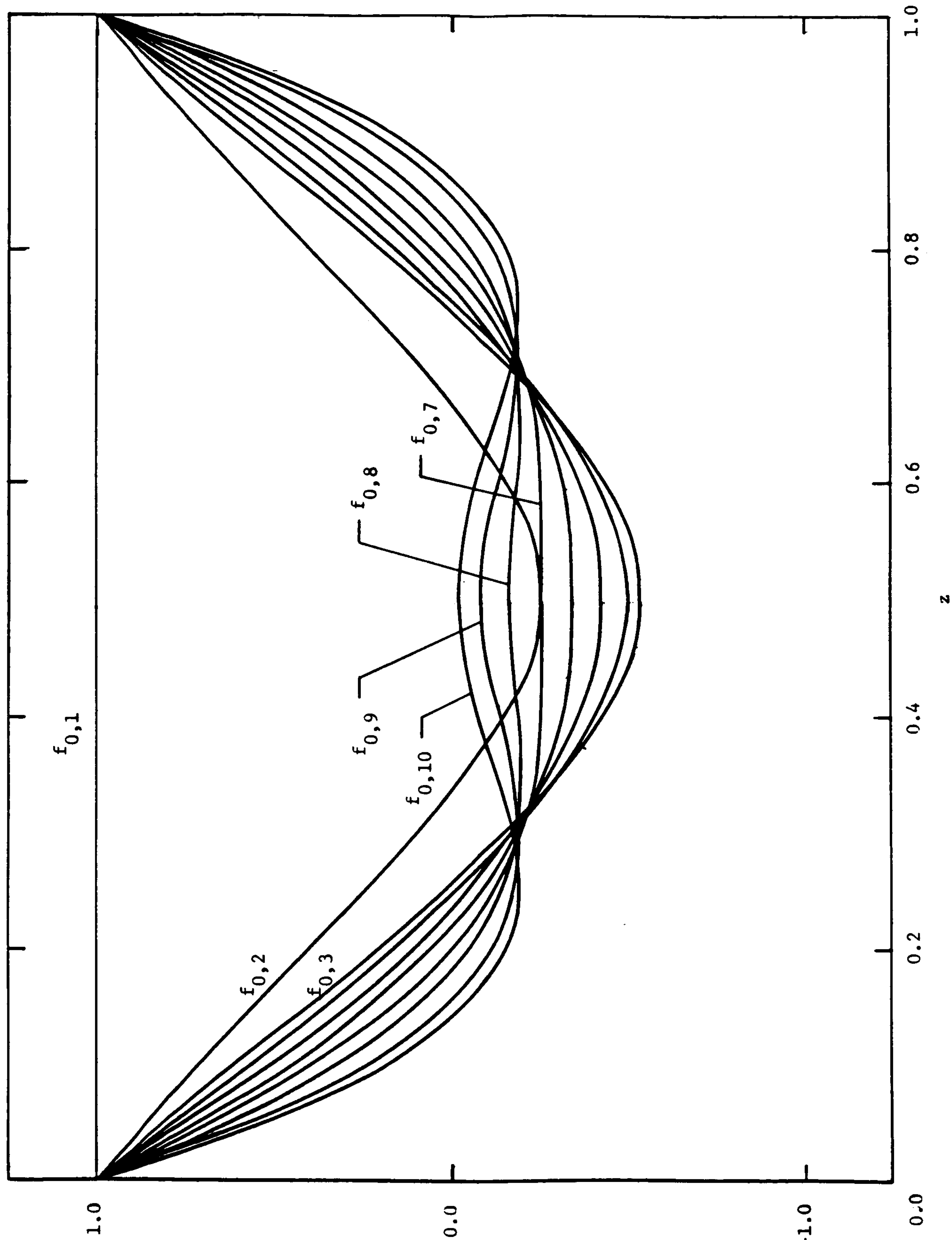


Figure 10

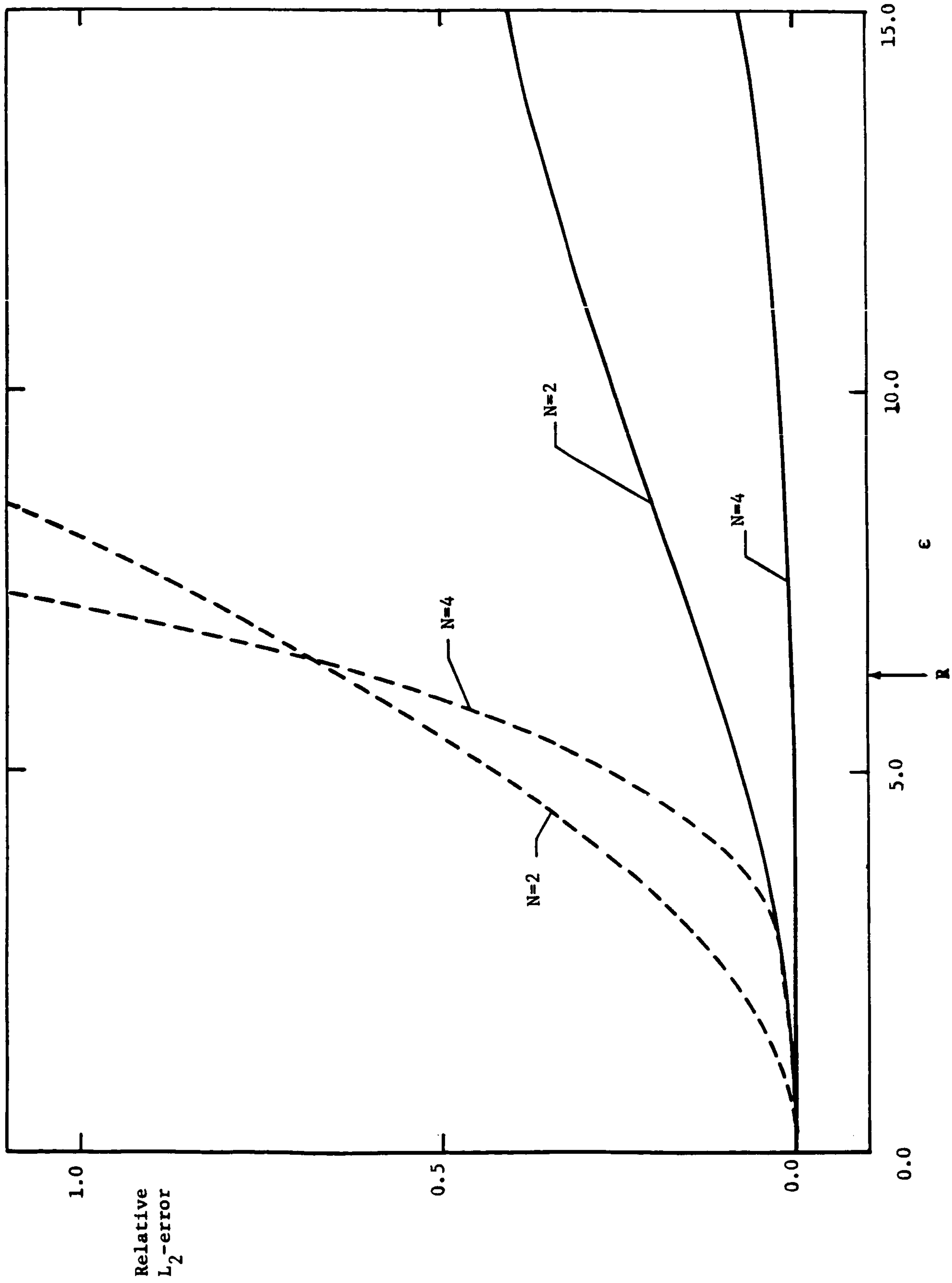


Figure 11

Standard Bibliographic Page

1. Report No. NASA CR-178354 ICASE Report No. 87-55	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A HYBRID PERTURBATION GALERKIN TECHNIQUE WITH APPLICATIONS TO SLENDER BODY THEORY		5. Report Date August 1987
		6. Performing Organization Code
7. Author(s) James F. Geer and Carl M. Andersen		8. Performing Organization Report No. 87-55
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No. 505-90-21-01
		11. Contract or Grant No. NAS1-18107
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Richard W. Barnwell

Submitted to SIAM J. Appl. Math.

Final Report

16. Abstract

A two-step hybrid perturbation-Galerkin method to solve a variety of applied mathematics problems which involve a small parameter is presented. The method consists of: (1) the use of a regular or singular perturbation method to determine the asymptotic expansion of the solution in terms of the small parameter; (2) construction of an approximate solution in the form of a sum of the perturbation coefficient functions multiplied by (unknown) amplitudes (gauge functions); and (3) the use of the classical Bubnov-Galerkin method to determine these amplitudes. This hybrid method has the potential of overcoming some of the drawbacks of the perturbation method and the Bubnov-Galerkin method when they are applied by themselves, while combining some of the good features of both. The proposed method is applied to some singular perturbation problems in slender body theory. The results obtained from the hybrid method are compared with approximate solutions obtained by other methods, and the degree of applicability of the hybrid method to broader problem areas is discussed.

17. Key Words (Suggested by Authors(s)) perturbations, slender body theory, Galerkin method, reduced basis method	18. Distribution Statement 64 - Numerical Analysis Unclassified - unlimited

19. Security Classif.(of this report) Unclassified	20. Security Classif.(of this page) Unclassified	21. No. of Pages 57	22. Price A04