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An Analysis of the Vertical Structure Equation
for Arbitrary Thermal Profiles

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Abstract

The vertical structure equation is a singular Sturm-Liouville problem whose eigenfunctions describe the vertical dependence of the normal modes of the primitive equations linearized about a given thermal profile. The eigenvalues give the equivalent depths of the modes. We study the spectrum of the vertical structure equation and the appropriateness of various upper boundary conditions, both for arbitrary thermal profiles. Our results depend critically upon whether or not the thermal profile is such that the basic state atmosphere is bounded.

In the case of a bounded atmosphere we show that the spectrum is always totally discrete, regardless of details of the thermal profile. For the barotropic equivalent depth, which corresponds to the lowest eigenvalue, we obtain upper and lower bounds which depend only on the surface temperature and the atmosphere height. All eigenfunctions are bounded, but always have unbounded first derivatives. We prove that the commonly invoked upper boundary condition that vertical velocity must vanish as pressure tends to zero, as well as a number of alternative conditions, is well posed.

For unbounded atmospheres, on the other hand, we show that typically there is a continuous spectrum, the boundary condition of vanishing vertical velocity is not well posed, and the eigenfunctions, if any, are unbounded. We point out, however, that owing to the traditional shallowness approximations of the primitive equations, the vertical structure equation has no meaning for unbounded atmospheres. This leads to the conclusion that the vertical structure equation always has a totally discrete spectrum under the assumptions implicit in the primitive equations.

1. Introduction and Main Results

Atmospheric normal modes play a central role in the theory of tidal motions of the atmosphere (Siebert, 1961; Chapman and Lindzen, 1970) and in the theory of planetary-scale traveling waves (Salby, 1984). Normal modes are also important in numerical weather prediction, where they are used to perform objective analysis (Flattery, 1970), to provide basis functions for spectral models (Kasahara, 1977), to initialize models (Daley, 1981), to represent global data (Kasahara and Puri, 1981), and to construct test solutions for models (Dee and da Silva, 1986). Recently normal modes have also been used as the basis of a stability theory for four-dimensional data assimilation methods (Cohn and Dee, 1986).

The vertical structure equation (VSE) is a Sturm-Liouville ordinary differential equation whose eigensolutions describe the vertical structure of the normal modes of the linearized primitive equations. Since the VSE is singular at the top boundary, where pressure vanishes, two fundamental questions are:

(A) What is the nature of the spectrum of the VSE, i.e., when is the spectrum totally discrete and when is there a continuous spectrum?

(B) What boundary condition is appropriate at the top boundary?

The basic state thermal profile appears in a coefficient of the VSE, and various authors have considered these two questions for particular choices of the thermal profile (Taylor, 1936; Dikii, 1965; Wiin-Nielsen, 1971; Staniforth et al., 1985). Our objective in this paper is to answer these questions for arbitrary profiles.

Before summarizing results, we review the background and derivation of the VSE following Daley (1981) and Staniforth et al. (1985). In the usual rotating pressure coordinate system, the adiabatic, hydrostatic primitive equations linearized about a state of rest can be written as

$$\frac{\partial u}{\partial t} - 2\Omega v \sin \phi + \frac{1}{a \cos \phi} \frac{\partial \Phi}{\partial \lambda} = 0 \quad (1.1a)$$

$$\frac{\partial v}{\partial t} + 2\Omega u \sin \phi + \frac{1}{a} \frac{\partial \Phi}{\partial \phi} = 0 \quad (1.1b)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \Phi}{\partial p} \right) + \sigma \omega = 0 \quad (1.1c)$$

$$\frac{\partial \omega}{\partial p} + \frac{1}{a \cos \phi} \left(\frac{\partial u}{\partial \lambda} + \frac{\partial v \cos \phi}{\partial \phi} \right) = 0 \quad (1.1d)$$

Here u , v and w are the perturbation eastward, northward and vertical velocity components in pressure coordinates, and Φ is the perturbation geopotential. The independent variables are time t , pressure p , longitude λ and latitude ϕ . The constants a and Ω are the earth's radius and rotation rate. The thermal structure of the basic state enters through the static stability function σ ,

$$\sigma = \sigma(p) = \frac{RT_0}{p} \left(- \frac{d \log \theta_0}{dp} \right) \quad (1.2a)$$

$$= Rp^{\kappa-1} \left(- \frac{d\theta_0}{dp} \right) \quad (1.2b)$$

$$= \kappa RT_0 p^{-2} - R \frac{dT_0}{dp} p^{-1}, \quad (1.2c)$$

where $T_0 = T_0(p)$ is the basic state temperature profile and $\theta_0 = T_0 p^{-\kappa}$ is the corresponding potential temperature profile; R is the gas constant for dry air and $\kappa = R/c_p$, c_p being the specific heat at constant pressure.

We take the domain of (1.1) to be global, and with pressure ranging from a constant surface value $p = p_s$ up to $p = 0$. The lower boundary condition for (1.1) corresponding to no mass flux through the earth's surface is

$$\frac{\partial}{\partial t} \left(\frac{\partial \Phi}{\partial p} + \frac{p}{RT_0} \sigma \Phi \right) = 0 \quad \text{at } p = p_S . \quad (1.3)$$

We do not yet specify an upper boundary condition. Note, however, that the time rate of change of total energy is given by

$$\frac{\partial}{\partial t} \iiint \frac{1}{2} \left[u^2 + v^2 + \frac{1}{\sigma} \left(\frac{\partial \Phi}{\partial p} \right)^2 + \frac{\Phi^2(p_S)}{RT_0(p_S)} \right] dA dp = \iint \left[\lim_{p \rightarrow 0} \Phi \omega \right] dA , \quad (1.4)$$

where dA is the horizontal area element, the triple integral is over the whole domain, and the double integral is over a spherical shell. Thus a boundary condition

$$\Phi \omega \rightarrow 0 \quad \text{as } p \rightarrow 0 \quad (1.5)$$

would guarantee energy conservation, while the commonly invoked condition

$$\omega \rightarrow 0 \quad \text{as } p \rightarrow 0 \quad (1.6)$$

does not necessarily conserve energy, because one does not know a priori whether $\Phi(t, \lambda, \phi, p)$ is bounded for all time.

Seeking separable solutions

$$\begin{vmatrix} u \\ v \\ \Phi \end{vmatrix} = \begin{vmatrix} \hat{u}(t, \lambda, \phi) \\ \hat{v}(t, \lambda, \phi) \\ \hat{\Phi}(t, \lambda, \phi) \end{vmatrix} X(p) , \quad (1.7)$$

one obtains the vertical structure equation and lower boundary condition (LBC)

$$\frac{d}{dp} \left(\frac{1}{\sigma} \frac{dX}{dp} \right) + \mu X = 0 \quad (1.8)$$

$$\frac{dX}{dp} + \frac{p}{RT_0} \sigma X = 0 \quad \text{at } p = p_S, \quad (1.9)$$

along with the linear global shallow-water system

$$\frac{\partial \hat{u}}{\partial t} - 2\hat{\Omega} \hat{v} \sin \phi + \frac{1}{a \cos \phi} \frac{\partial \hat{\Phi}}{\partial \lambda} = 0, \quad (1.10a)$$

$$\frac{\partial \hat{v}}{\partial t} + 2\hat{\Omega} \hat{u} \sin \phi + \frac{1}{a} \frac{\partial \hat{\Phi}}{\partial \phi} = 0, \quad (1.10b)$$

$$\frac{\partial \hat{\Phi}}{\partial t} + \frac{1}{\mu} \frac{1}{a \cos \phi} \left(\frac{\partial \hat{u}}{\partial \lambda} + \frac{\partial \hat{v} \cos \phi}{\partial \phi} \right) = 0, \quad (1.10c)$$

whose solutions are the Hough harmonics (Hough, 1898; Longuet-Higgins, 1968; Holl, 1970; Kasahara, 1976). Equations (1.8) and (1.10) are coupled through the separation constant μ which appears as an eigenvalue in the VSE (1.8). The corresponding equivalent depth H is $H = 1/\mu g$, g being the acceleration due to gravity. The vertical velocity ω is obtained diagnostically as

$$\omega = - \frac{1}{\sigma} \frac{dX}{dp} \frac{\partial \hat{\Phi}}{\partial t}. \quad (1.11)$$

The candidate boundary conditions (1.5) and (1.6) imply, respectively, that

$$\frac{1}{\sigma} X \frac{dX}{dp} \rightarrow 0 \quad \text{as } p \rightarrow 0, \quad (1.12)$$

or

$$\frac{1}{\sigma} \frac{dX}{dp} \rightarrow 0 \quad \text{as } p \rightarrow 0. \quad (1.13)$$

Staniforth et al. (1985) studied the VSE for the two-parameter family of static stability functions

$$\sigma(p) = Ap^{-\alpha}, \quad (1.14)$$

where $A > 0$ and α are constants, which includes many of the static stability functions considered by previous authors. They showed that for the linear system (1.1) to be valid in the sense of the neglected nonlinear terms being smaller than the retained linear terms, it is necessary that $\alpha \geq 2-\kappa$. On the other hand, for $T_0(p)$ to be finite as $p \rightarrow 0$, they showed that $\alpha \leq 2$. Hence they assumed that

$$2-\kappa \leq \alpha \leq 2. \quad (1.15a,b)$$

In case $\alpha < 2$, they were able to solve the VSE explicitly in terms of Bessel functions. These solutions were used to show that the spectrum is totally discrete, i.e., there is an infinite sequence of eigenvalues μ_n ,

$$\mu_0 < \mu_1 < \mu_2 < \dots \rightarrow \infty, \quad (1.16)$$

for which there is a corresponding sequence of eigenfunctions χ_n satisfying the VSE (1.8), the LBC (1.9), and the upper boundary condition (1.13). For $\alpha = 2$, which includes the much-studied isothermal case, they showed that there is at most one eigenvalue with corresponding eigenfunction satisfying (1.8,1.9,1.13), while the rest of the spectrum is continuous.

The VSE is a singular Sturm-Liouville equation whenever the coefficient $\sigma(p) \rightarrow \infty$ as $p \rightarrow 0$; in particular it is singular for the family of profiles (1.14,1.15). The answer to questions (A) and (B), as well as

other qualitative aspects of solutions, depend solely on the rate at which $\sigma(p) \rightarrow \infty$. More precisely, they depend only on the behavior of $\sigma(p)$, and hence on that of $T_0(p)$, in an arbitrarily small neighborhood of the singular point $p = 0$. For example, we will see that by modifying the isothermal profile near $p = 0$, the continuous spectrum becomes totally discrete.

In analyzing the VSE, one must ensure that any assumptions on the asymptotic behavior of $T_0(p)$ are consistent with the assumptions implicit in the governing nonlinear primitive equations. Recall that the primitive equations are based on certain "shallowness" or "traditional" approximations (Phillips, 1966, 1973), in which gravity g (the magnitude of the resultant of the centrifugal and Newtonian gravitational forces) is taken to be constant and in which the metric factors are replaced by their values near the surface of the earth. These approximations break down for unbounded atmospheres. Therefore we can be assured that an analysis of the linearized system (1.1), and therefore of the VSE, is meaningful only when the basic state atmosphere is bounded.

Boundedness of the basic state atmosphere can in fact be expressed as a condition on $T_0(p)$ near $p = 0$, as follows. Since the basic state is hydrostatic, the basic state geopotential Φ_0 is given by

$$\Phi_0(p) - \Phi_0(p_S) + \int_p^{p_S} \frac{RT_0(s)}{s} ds ; \quad (1.17)$$

cf. Staniforth et al. (1985, eq. 2.17). Hence the geopotential extent Φ_E of the domain is

$$\Phi_E = \lim_{p \rightarrow 0} \Phi_0(p) - \Phi_0(p_S) = \lim_{p \rightarrow 0} \int_p^{p_S} \frac{RT_0(s)}{s} ds , \quad (1.18)$$

and the basic state atmosphere is bounded if and only if the limits in (1.18) exist and are finite, i.e.

$$\Phi_E = \int_0^{p_S} \frac{RT_0(s)}{s} ds < \infty . \quad (1.19)$$

In particular, if $T_0(p)$ has a limit as $p \rightarrow 0$ then a necessary condition for the basic state atmosphere to be bounded is that in fact

$$T_0(p) \rightarrow 0 \quad \text{as} \quad p \rightarrow 0 , \quad (1.20)$$

for otherwise the integral in (1.19) would diverge at least logarithmically. In a recent PhD thesis (Sudarshan, 1985), the plausibility of condition (1.20) has also been argued for atmospheres in conductive equilibrium.

Conditions (1.19) and (1.20) will be fundamental in our analysis. We will show in particular that (1.20) guarantees a totally discrete spectrum and that under the stronger condition (1.19), but not necessarily under (1.20), the boundary conditions (1.12) or (1.13), amongst others, are indeed appropriate.

One might argue that neither of conditions (1.19) and (1.20) is physically reasonable. For example, the temperature of the thermospheric plasma varies from about 600K to as much as about 2000K, depending on the amount of solar activity (Gill, 1982, pp. 48-49). Nevertheless, our point of view is that in writing down the vertical structure equation one is addressing the primitive-equation model of the atmosphere, not the atmosphere itself. The shallowness assumptions of the primitive equations then make discussion and analysis of unbounded atmospheres irrelevant. In this paper we do give some results for unbounded atmospheres, partly in

Theorem 1 and partly by way of examples in Sec. 4, but only for the sake of contrast with our results for bounded atmospheres.

The following basic assumptions are made throughout our analysis.

Static stability assumptions. For $0 < p \leq p_S$ we assume that

$$\frac{d\theta_0}{dp} < 0 \quad (1.21)$$

and

$$T_0 > 0 . \quad (1.22)$$

It follows from (1.2) that

$$0 < \sigma(p) < \infty \quad \text{for } 0 < p \leq p_S , \quad (1.23)$$

so that the VSE is regular on any closed interval not containing $p = 0$.

Continuity assumptions. We assume that σ and $d\sigma/dp$ are continuous for $0 < p \leq p_S$. Together with (1.23), this guarantees that all of the classical theory of singular Sturm-Liouville problems applies.

Limit assumptions. We make the technical assumption that the basic state temperature profile $T_0(p)$ is such that both T_0 and $\frac{p}{T_0} \frac{dT_0}{dp}$ have limits as $p \rightarrow 0$. Thus we exclude certain pathological profiles that oscillate infinitely often near $p = 0$. We also assume that $\lim_{p \rightarrow 0} T_0(p)$ is finite.

Entropy assumptions. We assume that

$$\liminf_{p \rightarrow 0} \left(-p^{\frac{1}{2} + \kappa} \frac{d\theta_0}{dp} \right) > 0 . \quad (1.24)$$

Also we assume that

$$1/4 < \kappa < 1/2 ; \quad (1.25)$$

for dry air, $\kappa = 2/7$ for temperatures in the range 250-400K (Batchelor, 1967, p. 45). In Sec. 2 it will be shown that (1.24) implies that the singularity of the VSE is of limit-point type. There it will also be seen that (1.24) holds whenever $\theta_0 \rightarrow \infty$ and $p^{\frac{1}{2}+\kappa} d\theta_0/dp$ has a limit, possibly infinite, as $p \rightarrow 0$. In particular, (1.24) is satisfied for static stability functions of power-law form (1.14) when the linearization validity criterion (1.15a) is met.

Our first main result, proven in Sec. 2, is the following theorem.

Theorem 1. Consider the VSE (1.8), LBC (1.9) and upper boundary condition

$$\int_0^{p_S} X^2(p) dp < \infty, \quad (1.26)$$

and let

$$T_\infty = \lim_{p \rightarrow 0} T_0(p). \quad (1.27)$$

If $T_\infty = 0$ then the spectrum is totally discrete, i.e., there is a countably infinite set of eigenvalues (μ_n) ,

$$\mu_0 < \mu_1 < \mu_2 < \dots \rightarrow \infty, \quad (1.28)$$

and corresponding eigenfunctions (X_n) for which (1.8, 1.9, 1.26) are satisfied. If $T_\infty \neq 0$ then the portion of the spectrum for

$$\mu < \mu_c = \frac{1}{4\kappa RT_\infty} \quad (1.29)$$

is discrete.

The basic state atmosphere must be bounded for consistency with the shallowness approximations in the nonlinear primitive equations. This, in turn, implies that $T_\infty = 0$. Theorem 1 therefore allows us to conclude that under the assumptions implicit in the primitive equations, the VSE always has a totally discrete spectrum.

In general the square-integrability upper boundary condition (1.26) is the only one which is well-posed in the sense of always excluding one of the two linearly independent solutions of the VSE for every μ . The following theorem shows that for bounded basic state atmospheres, however, there are a number of equivalent upper boundary conditions. This theorem is proven in Section 3.

Theorem 2. Consider the VSE (1.8) subject to the LBC (1.9). If the basic state atmosphere is bounded, i.e. if (1.19) holds, then all of the following are equivalent upper boundary conditions:

$$(i) \quad \int_0^{p_S} X^2(p) \, dp < \infty \quad (1.30a)$$

$$(ii) \quad \lim_{p \rightarrow 0} |X(p)| < \infty \quad (1.30b)$$

$$(iii) \quad \lim_{p \rightarrow 0} \frac{1}{\sigma} \frac{dX}{dp} = 0 \quad (1.30c)$$

$$(iv) \quad \lim_{p \rightarrow 0} \frac{1}{\sigma} X \frac{dX}{dp} = 0 \quad (1.30d)$$

$$(v) \quad \lim_{p \rightarrow 0} p^\delta X = 0 \quad \text{if } 0 < \delta \leq 1/2. \quad (1.30e)$$

The equivalence of conditions (i) and (ii) implies that the eigenfunctions for a bounded atmosphere are bounded. It also follows from the theorem that for bounded atmospheres the boundary conditions (1.12, 1.13) implied by $\Phi\omega \rightarrow 0$ (1.5) and $\omega \rightarrow 0$ (1.6), respectively, are in fact equivalent and well posed. Any of conditions (ii)-(v) could be useful in numerical calculations.

In Section 3 we also prove the following theorem, which summarizes our main results for bounded atmospheres.

Theorem 3. Consider the VSE (1.8) for a bounded atmosphere, subject to the LBC (1.9) and any one of the equivalent upper boundary conditions of Theorem 2. Then

- (i) The spectrum is totally discrete.
- (ii) The eigenvalues are all positive and satisfy

$$\frac{1}{\kappa\Phi_E} < \mu_0 < \frac{1}{RT_S}, \quad (1.31a)$$

$$\frac{n}{\kappa\Phi_E - RT_S} < \mu_n \quad \text{for } n \geq 1, \quad (1.31b)$$

where Φ_E is the geopotential extent defined in (1.18) and T_S is the surface temperature. In terms of the equivalent depths $H_n = (g\mu_n)^{-1}$, these inequalities are

$$RT_S < gH_0 < \kappa\Phi_E \quad (1.32a)$$

$$gH_n < \frac{1}{n} (\kappa\Phi_E - RT_S), \quad n \geq 1. \quad (1.32b)$$

(iii) The eigenfunctions $X_n(p)$ form a complete orthogonal basis for the space of functions which are square-integrable over $0 < p < p_S$. Moreover, X_n has precisely n zeros on $0 \leq p \leq p_S$.

(iv) While each eigenfunction X_n remains bounded as $p \rightarrow 0$, the derivative is unbounded. In fact there is a constant $c = c(n) > 0$ such that

$$\left| \frac{dX_n}{dp} \right| > cp^{-1/2}, \quad (1.33)$$

for all p sufficiently small.

Statement (i) of this theorem follows directly from Theorem 1. The bounds in statement (ii) are independent of the details of the thermal profile: they depend only on the surface temperature T_S and geopotential extent Φ_E , in addition to the constants κ , R , and g . For shallow atmospheres, $\kappa\Phi_E$ is not much larger than RT_S . The bound (1.32a) therefore implies that the barotropic equivalent depth H_0 is not sensitive to details of the thermal profile.

Statement (iii) justifies the use of the vertical structure functions X_n as basis functions for spectral expansions. Statement (iv) implies that there is a "boundary layer" near $p = 0$, where each vertical structure function changes rapidly. Numerical methods for both the VSE and the primitive equations themselves should be designed to take this into account.

If one assumes an asymptotic form for $T_0(p)$ near $p = 0$ then the asymptotic form of the eigenfunctions can be obtained. This is done in Sec. 4. As a first example we consider

$$T_0(p) = \sum_{j=0}^{\infty} \tau_j p^{\epsilon_j} \quad \text{near } p = 0, \quad (1.34)$$

where $0 < \epsilon < \kappa$, $\tau_0 = T_\infty = 0$, and $\tau_1 > 0$. Such an atmosphere is bounded, and the equivalence of the boundary conditions in Theorem 2 is verified by displaying the asymptotic behavior of the eigenfunctions.

Second, we consider unbounded atmospheres for which

$$T_0(p) = T_\infty + \sum_{j=1}^{\infty} \tau_j p^j \quad \text{near } p = 0, \quad (1.35)$$

with $T_\infty > 0$, which includes all atmospheres with isothermal tops. In this case we extend the result (1.29) of Theorem 1 by using the asymptotic behavior of the eigenfunctions to show that the spectrum is in fact continuous for $\mu \geq \mu_c$. Hence there is only a finite number of eigenvalues in this case.

For further contrast we consider two families of atmospheres for which $T_0(p)$ is specified throughout the domain but neither (1.34) nor (1.35) is satisfied. For both families $T_0(p) \rightarrow 0$ as $p \rightarrow 0$, but so slowly that the atmosphere is unbounded. We exhibit a closed-form solution which is square-integrable but unbounded, and another which is not square-integrable yet has $\frac{1}{\sigma} \frac{dX}{dp} \rightarrow 0$ as $p \rightarrow 0$. Hence the hypothesis of Theorem 2 that the basic state atmosphere be bounded cannot be weakened.

We conclude with a brief discussion of our results in Sec. 5.

2. Spectral theory

The object of this section is to prove Theorem 1, which shall be done by application of the classical theory of singular Sturm-Liouville problems. Of fundamental importance is that the VSE is always of limit-point type, which we will first elaborate upon.

From general existence theory it is known that the VSE (1.8), with no boundary conditions imposed, has precisely two linearly independent solutions for any number μ . A Sturm-Liouville differential equation is said to be of limit-point type if for some choice of μ , say $\mu = \mu_0$, one solution is not square-integrable. The classical theory due to Weyl (cf. Coddington and Levinson, 1955, Ch. 9, Theorem 2.1) shows that in fact such an equation has at most one square-integrable solution for any number μ . Thus an appropriate boundary condition for equations of limit-point type is square-integrability, since this condition guarantees uniqueness. The spectrum is then determined by imposing a boundary condition at the regular point, in our case the LBC (1.9).

By choosing $\mu_0 = 0$, to demonstrate that the VSE is of limit-point type it suffices to show that the homogeneous equation

$$\frac{d}{dp} \left(\frac{1}{\sigma} \frac{dX}{dp} \right) = 0, \quad 0 < p < p_S, \quad (2.1)$$

has a solution which is not square-integrable. One solution is $X = \text{constant}$, which of course is square-integrable. The other solution is $X = \psi(p)$,

$$\psi(p) = \int_p^{p_S} \sigma(s) ds. \quad (2.2)$$

We will show that ψ is not square-integrable, i.e.,

$$\int_p^{p_S} \psi^2(s) ds \rightarrow \infty \quad \text{as } p \rightarrow 0, \quad (2.3)$$

by proving the following lemma.

Lemma 2.1. There exists a constant $c_1 > 0$ such that

$$\sigma(p) > c_1 p^{-3/2} \quad \text{for } 0 < p \leq p_S. \quad (2.4)$$

Proof: The static stability assumption (1.21) implies that

$$-p^{\frac{1}{2}+\kappa} \frac{d\theta_0}{dp} > 0 \quad \text{for } 0 < p \leq p_S, \quad (2.5)$$

and by virtue of the entropy assumption (1.24) there is a constant $\delta > 0$ such that in fact

$$-p^{\frac{1}{2}+\kappa} \frac{d\theta_0}{dp} > \delta \quad \text{for } 0 < p \leq p_S. \quad (2.6)$$

Substituting (2.6) into the definition (1.2b) of σ , we then have

$$\sigma(p) > \delta R p^{-3/2} \quad \text{for } 0 < p \leq p_S. \quad (2.7)$$

Lemma 2.2. The VSE is of limit-point type, so that an appropriate upper boundary condition is

$$\int_0^{p_S} X^2(s) ds < \infty. \quad (2.8)$$

Proof: Inserting the bound (2.4) into the definition (2.2) of $\psi(p)$, one obtains by integration

$$\psi(p) > 2c_1(p^{-1/2} - p_S^{-1/2}) . \quad (2.9)$$

Therefore the integral in (2.3) diverges at least logarithmically, and the lemma is proven.

We reiterate that there are no other appropriate boundary conditions without further assumptions on $T_0(p)$. Equivalent boundary conditions for bounded atmospheres are given in Theorem 2.

Lemmas 2.1 and 2.2 depend crucially on the entropy assumption (1.24). Before proceeding with the proof of Theorem 1, we therefore discuss the nature of this assumption. To our knowledge, all thermal profiles considered previously in the literature are such that both

$$\theta_0 \rightarrow \infty \text{ as } p \rightarrow 0 \quad (2.10)$$

and

$$\liminf_{p \rightarrow 0} \left(-p^{\frac{1}{2}+\kappa} \frac{d\theta_0}{dp} \right) = \limsup_{p \rightarrow 0} \left(-p^{\frac{1}{2}+\kappa} \frac{d\theta_0}{dp} \right) . \quad (2.11)$$

In particular, Staniforth et al. (1985) studied the family

$$\sigma(p) = Ap^{-\alpha} , \quad A > 0 , \quad (2.12)$$

which includes most previously-studied profiles, and they showed that linearization validity requires

$$\alpha \geq 2-\kappa . \quad (2.13)$$

Using (2.5) one finds that the potential temperature profile θ_0 corresponding to (2.12) is given by

$$\theta_0(p) - \theta_0(p_S) = \begin{cases} \frac{A}{R} \frac{1}{\alpha - (2 - \kappa)} (p^{-\alpha + (2 - \kappa)} - p_S^{-\alpha + (2 - \kappa)}), & \alpha > 2 - \kappa \\ \frac{A}{R} \log \frac{p_S}{p} & , \alpha = 2 - \kappa \end{cases}, \quad (2.14)$$

while

$$-p^{\frac{1}{2} + \kappa} \frac{d\theta_0}{dp} = \frac{A}{R} p^{\frac{3}{2} - \alpha}. \quad (2.15)$$

Clearly both (2.10) and (2.11) are satisfied.

To complete the discussion of our entropy assumption (1.24), we now assert that it is actually less restrictive than properties (2.10), (2.11).

Claim 2.1. For thermal profiles satisfying (2.10) and (2.11), one has

$$\liminf_{p \rightarrow 0} \left(-p^{\frac{1}{2} + \kappa} \frac{d\theta_0}{dp} \right) = +\infty. \quad (2.16)$$

Hence the entropy assumption (1.24) is satisfied.

Proof: Suppose (2.16) is false. Then (2.11) implies that

$$\limsup_{p \rightarrow 0} \left(-p^{\frac{1}{2} + \kappa} \frac{d\theta_0}{dp} \right) < \infty. \quad (2.17)$$

Hence there is a constant $M > 0$ such that

$$-p^{\frac{1}{2} + \kappa} \frac{d\theta_0}{dp} < M \quad \text{for } 0 < p \leq p_S. \quad (2.18)$$

Integrating from p to p_S , one finds that

$$\theta_0(p) < \theta_0(p_S) + \frac{M}{\frac{1}{2}-\kappa} \left(p_S^{\frac{1}{2}-\kappa} - p^{\frac{1}{2}-\kappa} \right) \quad (2.19)$$

$$< \theta_0(p_S) + \frac{M}{\frac{1}{2}-\kappa} p_S^{\frac{1}{2}-\kappa},$$

where we have used the assumption (1.25b) that $\kappa < 1/2$. Hence $\theta_0(p)$ is bounded, contradicting (2.10), and the claim is established.

So far it has been shown that for any number μ there is at most one square-integrable solution of the VSE: when a solution of the VSE satisfying the upper boundary condition (2.8) exists, it is unique. Theorem 1, whose proof occupies the remainder of this section, concerns the existence of solutions under the additional condition (1.9) at the lower boundary.

Our theorem is simply an application of a classical result (cf. Berkowitz, 1959, Theorem 5.4) in the spectral theory of singular Sturm-Liouville problems, namely that the spectrum is discrete below $\mu = \mu_c$, where

$$\mu_c = \liminf_{p \rightarrow 0} \frac{\sigma(p)}{4\psi^2(p)}, \quad (2.20)$$

the function $\psi(p)$ having been defined in (2.2). In particular, there is a countably infinite set of eigenvalues satisfying (1.28) when $\mu_c = +\infty$. To prove Theorem 1, we need only show that μ_c defined by (2.20) satisfies

$$\mu_c = \begin{cases} +\infty & \text{when } T_\infty = 0 \\ 1/(4\kappa R T_\infty) & \text{when } T_\infty \neq 0 \end{cases} \quad (2.21a)$$

$$(2.21b)$$

To do so, we prove the following two lemmas regarding the asymptotic behavior of $\sigma(p)$ and $\psi(p)$.

Lemma 2.3.

$$\lim_{p \rightarrow 0} p^2 \sigma(p) = \kappa R T_\infty .$$

Proof: Since

$$p^2 \sigma = \kappa R T_0 - R p dT_0/dp , \quad (2.22)$$

by (1.2c), we need only show that $p dT_0/dp$ has a limit and that in fact

$$\lim_{p \rightarrow 0} p \frac{dT_0}{dp} = 0 . \quad (2.23)$$

Except when

$$T_0 \rightarrow 0 \quad \text{and} \quad \left| \frac{p}{T_0} \frac{dT_0}{dp} \right| \rightarrow \infty \quad \text{as} \quad p \rightarrow 0 , \quad (2.24)$$

existence of the limit follows directly from the limit assumptions of Sec. 1, since $p \frac{dT_0}{dp} = (T_0) \left(\frac{p}{T_0} \frac{dT_0}{dp} \right)$. However, the static stability assumption implies that

$$\frac{p}{T_0} \frac{dT_0}{dp} = \kappa + p \frac{d \log \theta_0}{dp} < \kappa \quad (2.25)$$

for $0 < p \leq p_S$, so (2.24) would imply that

$$T_0 \rightarrow 0 \quad \text{and} \quad \frac{p}{T_0} \frac{dT_0}{dp} \rightarrow -\infty \quad \text{as} \quad p \rightarrow 0 . \quad (2.26)$$

This is not possible since $T_0 > 0$ for $0 < p \leq p_S$. Hence $p dT_0/dp$ has a limit.

Now we prove (2.23) by contradiction. Suppose (2.23) does not hold, and define $\tilde{T} = p dT_0/dp$. Then there are constants $\delta > 0$ and $p_* > 0$ such that

$$|\tilde{T}(p)| > \delta \quad \text{for } 0 < p < p_* . \quad (2.27)$$

Hence

$$\left| \int_p^{p_*} dT_0 \right| = \left| \int_p^{p_*} \tilde{T} \frac{dp}{p} \right| > \delta \int_p^{p_*} \frac{dp}{p} , \quad (2.28)$$

i.e.

$$|T_0(p_*) - T_0(p)| > \delta \log \frac{p_*}{p} \rightarrow \infty \quad \text{as } p \rightarrow 0, \quad (2.29)$$

which contradicts our basic assumption that $\lim_{p \rightarrow 0} T_0(p)$ is finite. Therefore (2.23) must be true and the lemma is proven.

Lemma 2.4.

$$\lim_{p \rightarrow 0} p\psi(p) = \kappa RT_\infty .$$

Proof: From the previous lemma and the definition (2.2) of $\psi(p)$ it follows that

$$\psi(p) = \int_p^{p_S} [\kappa RT_\infty + g(s)] s^{-2} ds , \quad (2.30)$$

where

$$g(s) = s^2 \sigma(s) - \kappa RT_\infty \rightarrow 0 \quad \text{as } s \rightarrow 0 . \quad (2.31)$$

From (2.30) it follows that

$$p\psi(p) = \kappa RT_\infty \left(1 - \frac{p}{p_S}\right) + p \int_p^{p_S} g(s) s^{-2} ds , \quad (2.32)$$

so the lemma is proven if we can show that

$$\lim_{p \rightarrow 0} p \int_p^{p_S} g(s) s^{-2} ds = 0. \quad (2.33)$$

This is clearly the case if the integral converges as $p \rightarrow 0$, so assume it does not. Then L'Hospital's rule yields

$$\lim_{p \rightarrow 0} p \int_p^{p_S} g(s) s^{-2} ds = \lim_{p \rightarrow 0} \frac{-g(p)p^{-2}}{-p^{-2}} = \lim_{p \rightarrow 0} g(p) = 0, \quad (2.34)$$

where (2.31) has been used. This completes the proof.

The proof of Theorem 1 in case $T_\infty \neq 0$ is now immediate, upon rewriting (2.20) as

$$\mu_c = \liminf_{p \rightarrow 0} \frac{p^2 \sigma(p)}{4[p\psi(p)]^2}. \quad (2.35)$$

Lemmas 2.3 and 2.4 imply that both the numerator and denominator here have limits, namely κRT_∞ and $4(\kappa RT_\infty)^2$, so that the result (2.21b) follows.

Finally we treat the case $T_\infty = 0$ and complete the proof of Theorem 1 by verifying (2.21a). To do so we prove the following lemma.

Lemma 2.5. For $0 < p \leq p_S$,

$$(i) \quad \int_p^{p_S} \sigma(s) ds < \frac{RT_0(p)}{p}$$

$$(ii) \quad \frac{\sigma(p)}{\psi^2(p)} > \frac{1}{RT_0} \left(-p \frac{d \log \theta_0}{dp} \right).$$

Proof: Using (1.2b) and (2.2), we prove statement (i) by integrating by parts:

$$\begin{aligned}
 \psi(p) &= \int_p^{p_S} \sigma(s) \, ds = -R \int_p^{p_S} p^{\kappa-1} \frac{d\theta_0}{dp} \, dp \\
 &= -R \left[p^{\kappa-1} \theta_0 \Big|_p^{p_S} + (1-\kappa) \int_p^{p_S} s^{\kappa-2} \theta_0(s) \, ds \right] \\
 &< R p^{\kappa-1} \theta_0 \\
 &= RT_0/p, \tag{2.36}
 \end{aligned}$$

where the inequality was obtained since $\kappa < 1$ and $\theta_0 > 0$. From the definition (1.2a) of σ we then obtain

$$\frac{\sigma}{\psi^2} > \frac{p^2}{R^2 T_0^2} \frac{RT_0}{p} \left(-\frac{d \log \theta_0}{dp} \right) = \frac{1}{RT_0} \left(-p \frac{d \log \theta_0}{dp} \right), \tag{2.37}$$

thus proving statement (ii).

From part (ii) of this lemma it is clear that the proof of Theorem 1 is complete if we can show that

$$\liminf_{p \rightarrow 0} \frac{1}{RT_0} \left(-p \frac{d \log \theta_0}{dp} \right) = +\infty. \tag{2.38}$$

Since $T_0 \rightarrow 0$ as $p \rightarrow 0$, this is certainly true in case

$$\liminf_{p \rightarrow 0} \left(-p \frac{d \log \theta_0}{dp} \right) > 0. \tag{2.39}$$

The only other possibility is that

$$\liminf_{p \rightarrow 0} \left(-p \frac{d \log \theta_0}{dp} \right) = 0, \quad (2.40)$$

since $-d \log \theta_0 / dp > 0$ for $0 < p \leq p_s$ by static stability. Now

$$p \frac{d \log \theta_0}{dp} = p \frac{d \log T_0}{dp} - \kappa. \quad (2.41)$$

But $\lim_{p \rightarrow 0} (p d \log T_0 / dp)$ exists by the limit assumption, so (2.40) implies

that in fact

$$\lim_{p \rightarrow 0} \left(-p \frac{d \log \theta_0}{dp} \right) = 0. \quad (2.42)$$

Hence, for each $\epsilon > 0$ there exists a $p_* > 0$ such that

$$-p \frac{d \log \theta_0}{dp} < \epsilon \quad \text{for } 0 < p < p_*. \quad (2.43)$$

Integrating from p to p_* , one obtains

$$\log \frac{\theta_0(p)}{\theta_0(p_*)} < \log \left(\frac{p}{p_*} \right)^{-\epsilon}, \quad (2.44)$$

and therefore

$$T_0 < \text{const.} \times p^{\kappa - \epsilon}. \quad (2.45)$$

Now

$$\frac{1}{RT_0} \left(-p \frac{d \log \theta_0}{dp} \right) = \frac{1}{RT_0^2} \left(-p^{1+\kappa} \frac{d \theta_0}{dp} \right), \quad (2.46)$$

so equation (2.45) implies that

$$\frac{1}{RT_0} \left(-p \frac{d \log \theta_0}{dp} \right) > \text{const.} \times p^\beta \left(-p^{\frac{1}{2}+\kappa} \frac{d\theta_0}{dp} \right), \quad (2.47)$$

where

$$\beta = -2\left(\kappa - \frac{1}{4} - \epsilon\right). \quad (2.48)$$

Since $\kappa > 1/4$ by (1.25a), we may choose ϵ so that $\beta < 0$. The result (2.38) then follows from (2.47) by virtue of the entropy assumption (1.24), thereby completing the proof of Theorem 1.

3. Bounded atmospheres

In this section we prove Theorems 2 and 3. The first lemma expresses boundedness of the basic state atmosphere as a condition on the static stability function σ .

Lemma 3.1. Suppose the basic state atmosphere is bounded, i.e., Φ_E defined by (1.18) is finite. Then

$$(i) \quad T_0(p) \rightarrow 0 \quad \text{as } p \rightarrow 0 \quad (3.1)$$

and

$$(ii) \quad \int_0^{p_S} s \sigma(s) ds = \kappa \Phi_E - RT_S < \infty. \quad (3.2)$$

Proof: By the definition of (1.17) of $\Phi_0(p)$ and from our basic assumption that $\lim_{p \rightarrow 0} T_0(p)$ exists, it follows that the limit must in fact be zero, for otherwise the integral in (1.17) would diverge at least logarithmically.

To prove part (ii), write (1.17) as

$$\begin{aligned} \Phi_0(p) &= \Phi_0(p_S) + R \int_p^{p_S} \theta_0(s) s^{\kappa-1} ds \\ &= \Phi_0(p_S) + \frac{R}{\kappa} \left[T_0(s) \Big|_p^{p_S} - \int_p^{p_S} s^{\kappa} \frac{d\theta_0(s)}{ds} ds \right], \end{aligned} \quad (3.3)$$

or, using (1.2b),

$$\Phi_0(p) = \Phi_0(p_S) + \frac{1}{\kappa} RT_S - \frac{1}{\kappa} RT_0(p) + \frac{1}{\kappa} \int_p^{p_S} s \sigma(s) ds. \quad (3.4)$$

From (3.1) one then obtains

$$\lim_{p \rightarrow 0} \int_p^{p_S} s \sigma(s) ds = \kappa \lim_{p \rightarrow 0} \Phi_0(p) - \kappa \Phi_0(p_S) - RT_S < \infty, \quad (3.5)$$

completing the proof.

The following two lemmas are fundamental and will be used extensively. The first of these simply states several versions and immediate consequences of the variation-of-constants formula for the VSE.

Lemma 3.2. Suppose that $X(p)$ satisfies the VSE (1.8). Let $0 < p \leq p_S$ and $0 < p_* \leq p_S$. Then

$$(i) \quad X(p) = X(p_*) + a\psi(p) - \mu \int_p^{p_*} [\psi(p) - \psi(s)] X(s) ds \quad (3.6)$$

where

$$\psi(p) = \int_p^{p_*} \sigma(s) ds \quad (3.7)$$

$$a = -\frac{1}{\sigma} \frac{dX}{dp} + \mu \int_p^{p_*} X(s) ds = \text{const.} \quad (3.8)$$

Here $\psi = \psi(p)$ depends only on p_* , while a depends on p_* but not on p .

(ii) If X satisfies the LBC (1.9) then $X(p_S) \neq 0$ and

$$\tilde{X}(p) = X(p) / X(p_S) \quad (3.9)$$

satisfies

$$\tilde{X}(p) = 1 + \frac{p_S}{RT_S} \psi(p) - \mu \int_p^{p_S} [\psi(p) - \psi(s)] \tilde{X}(s) ds. \quad (3.10)$$

(iii) If $\frac{1}{\sigma} \frac{dX}{dp} \rightarrow 0$ as $p \rightarrow 0$ then

$$\frac{1}{\sigma} \frac{dX}{dp} = -\mu \int_0^p X(s) ds \quad (3.11)$$

and

$$X(p) = X(p_*) + \mu \psi(p) \int_0^{p_*} X(s) ds - \mu \int_p^{p_*} [\psi(p) - \psi(s)] X(s) ds \quad (3.12a)$$

$$= X(p_*) + \mu \int_p^{p_*} \left[\int_0^s X(p') dp' \right] \sigma(s) ds. \quad (3.12b)$$

(iv) If $\frac{1}{\sigma} \frac{dX}{dp} \rightarrow 0$ as $p \rightarrow 0$ and if X satisfies the LBC (1.9), then

$$\frac{p_S}{RT_S} = \mu \int_0^{p_S} \bar{X}(s) ds, \quad (3.13)$$

where \bar{X} is defined in (3.9).

Proof: By differentiating (3.6), dividing by $\sigma(p)$, and differentiating the result, one obtains the VSE. Thus (3.6) defines a solution of the VSE. The same argument can be applied in reverse by virtue of the continuity assumptions, so that the VSE is in fact equivalent to (3.6) and part (i) is proven.

In part (ii), if $X(p_S) = 0$ then the LBC would imply $dX(p_S)/dp = 0$, hence the VSE would imply $X(p) = 0$. To verify (3.10), evaluate (3.8) at $p = p_S$ and use the LBC to find

$$a = \frac{p_S}{RT_S} X(p_S) - \mu \int_{p_*}^{p_S} X(s) ds. \quad (3.14)$$

Now substitute (3.14) into (3.6), choose $p_* = p_S$, and divide by $X(p_S)$ to obtain (3.10).

To verify (3.11) in part (iii), let $p \rightarrow 0$ in (3.8), so that

$$a = \mu \int_0^{p_*} X(s) ds, \quad (3.15)$$

then substitute (3.15) into (3.8) to obtain (3.11). Equation (3.12a) follows by substituting (3.15) into (3.6). Equation (3.12b) follows most easily by multiplying (3.11) by $\sigma(p)$ and integrating the result from p to p_* .

Part (iv) is verified by equating (3.14) and (3.15), and then dividing by $X(p_S)$.

Lemma 3.3. Suppose that the basic state atmosphere is bounded, and that X satisfies the VSE (1.8) and LBC (1.9). Then

- (i) X has finitely many zeros.
- (ii) If $\mu \leq 0$ then in fact X has no zeros.
- (iii) Both X and $\frac{1}{\sigma} \frac{dX}{dp}$ have limits as $p \rightarrow 0$.

Proof: The zeros of any solution of a singular Sturm-Liouville equation can accumulate only at the singular point. First we show that if X has any zeros at all, then $\mu > 0$, thereby proving statement (ii). Denote by p_1 the location of the first zero to the left of p_S , so \bar{X} defined in (3.9) is such that $\bar{X}(p_1) = 0$ and $\bar{X}(p) > 0$ for $p_1 < p < p_S$. Setting $p = p_1$ in (3.10) then gives

$$0 = 1 + \frac{p_S}{RT_S} \psi(p_1) - \mu \int_{p_1}^{p_S} [\psi(p_1) - \psi(s)] \bar{X}(s) ds, \quad (3.16)$$

from which it follows that $\mu > 0$ because $\psi(p_1) > 0$ and the integrand is positive.

To prove (i), suppose $\mu > 0$. Let p_i and p_j be two successive zeros of $X(p)$, with $0 < p_i < p_j < p_S$. Choosing $p_* = p_j$ in (3.6) gives

$$X(p) = a \psi(p) - \mu \int_p^{p_j} [\psi(p) - \psi(s)] X(s) ds, \quad (3.17)$$

where

$$\psi(p) = \int_p^{p_j} \sigma(p') dp'. \quad (3.18)$$

Now either $X(p) > 0$ or $X(p) < 0$ for $p_i < p < p_j$. In the former case, since $\mu > 0$ we find from (3.17) that

$$0 < X(p) < a \psi(p) \quad \text{for } p_i < p < p_j, \quad (3.19)$$

hence $a > 0$, while in the latter,

$$0 < -X(p) < -a \psi(p) \quad \text{for } p_i < p < p_j, \quad (3.20)$$

and $a < 0$. Combining, we have

$$0 < |X(p)| < |a| \psi(p) \quad \text{for } p_i < p < p_j, \quad (3.21)$$

and $a \neq 0$.

Setting $p = p_i$ in (3.17) gives

$$0 = a \psi(p_i) - \mu \int_{p_i}^{p_j} [\psi(p_i) - \psi(s)] X(s) ds, \quad (3.22)$$

so that

$$\begin{aligned}
 |a|\psi(p_i) - \mu \int_{p_i}^{p_j} [\psi(p_i) - \psi(s)] |X(s)| ds \\
 < \mu \int_{p_i}^{p_j} [\psi(p_i) - \psi(s)] |a| \psi(s) ds \\
 < \mu \int_{p_i}^{p_j} \psi(p_i) |a| \psi(s) ds
 \end{aligned} \tag{3.23}$$

where (3.21) has been used. Therefore

$$\begin{aligned}
 1 &< \mu \int_{p_i}^{p_j} \psi(s) ds \\
 &= \mu \int_{p_i}^{p_j} \left[\int_s^{p_j} \sigma(p') dp' \right] ds \\
 &= \mu \int_{p_i}^{p_j} (s-p_i) \sigma(s) ds \\
 &< \mu \int_{p_i}^{p_j} s \sigma(s) ds .
 \end{aligned} \tag{3.24}$$

Summing over all pairs of successive zeros of X , it follows that the total number n of zeros for $0 < p \leq p_s$ satisfies

$$n-1 < \mu \int_0^{p_S} s \sigma(s) ds . \quad (3.25)$$

The integral is finite by Lemma 3.1, hence part (i) is proven.

Finally we prove part (iii). Our continuity assumptions on σ guarantee that every solution X of the VSE is continuous for $0 < p \leq p_S$. Since X also has finitely many zeros it follows that $\lim_{p \rightarrow 0} \int_p^{p_*} X(s) ds$ exists for every $p_* > 0$. Therefore (3.8) implies that $\frac{1}{\sigma} \frac{dX}{dp}$ has a limit as $p \rightarrow 0$. Now choose p_* such that $X(p)$ does not change sign for $0 < p < p_*$. Then it follows from (3.8) that dX/dp has at most one zero for $0 < p < p_*$. Therefore X is monotone in some neighborhood of $p = 0$, and hence X has a limit as $p \rightarrow 0$. This completes the proof of Lemma 3.3.

The most difficult part of the proof of Theorem 2 is showing that statement (iii) of the theorem implies statement (ii) of the theorem. We therefore begin by proving a lemma which gives several consequences of statement (iii).

Lemma 3.4. Suppose the basic state atmosphere is bounded, and that X satisfies the VSE (1.8), the LBC (1.9), and $\frac{1}{\sigma} \frac{dX}{dp} \rightarrow 0$ as $p \rightarrow 0$. Then

$$(i) \quad \mu > 0 \quad (3.26)$$

$$(ii) \quad pX(p) \rightarrow 0 \text{ as } p \rightarrow 0 \quad (3.27)$$

(iii) If X has any zeros, then

$$1 < \mu \int_0^{p_1} s \sigma(s) ds , \quad (3.28)$$

where p_1 denotes the first zero of $X(p)$ to the right of $p = 0$.

(iv) Given any $\epsilon > 0$, there is a $p_0 > 0$ such that

$$\mu \int_0^p s \sigma(s) ds \leq \epsilon \quad \text{if } 0 < p \leq p_0. \quad (3.29)$$

If $\epsilon < 1$ and $0 < p \leq p_0$, then

$$\int_0^p |X(s)| ds < \frac{1}{1-\epsilon} p |X(p)|. \quad (3.30)$$

Proof: To prove part (i), suppose that $\mu \leq 0$. We then have $\bar{X}(p) = X(p)/X(p_S) > 0$ for $0 < p \leq p_S$ by Lemma 3.3(ii), hence the right-hand side of (3.13) is nonpositive. This is impossible since the left-hand side is positive. Therefore $\mu > 0$.

To prove part (ii), observe that since X has finitely many zeros by Lemma 3.3(i), there is a $p_* > 0$ such that $X(p) \neq 0$ for $0 < p < p_*$. Since $\mu > 0$, for this choice of p_* in (3.12a) we find that for $0 < p < p_*$,

$$|X(p)| < |X(p_*)| + \mu \int_p^{p_*} \sigma(s) ds \int_0^{p_*} |X(s)| ds \quad (3.31)$$

$$= |X(p_*)| + |a| \int_p^{p_*} \sigma(s) ds$$

$$< |X(p_*)| + |a| \int_p^{p_S} \sigma(s) ds$$

$$< |X(p_*)| + |a| \frac{RT_0(p)}{p}, \quad (3.32)$$

where we have used (3.8) and Lemma 2.5(i). Therefore

$$0 < p|X(p)| < p|X(p_*)| + |a|RT_0(p) \rightarrow 0 \text{ as } p \rightarrow 0, \quad (3.33)$$

by (3.1), and part (ii) of the lemma is established.

Now let p_1 be as defined in statement (iii). Then setting $p_* = p_1$ in (3.31) yields

$$|X(p)| < \mu \int_p^{p_1} \sigma(s) ds \int_0^{p_1} |X(s)| ds, \quad (3.34)$$

and integrating (3.34) from zero to p_1 leads to

$$\begin{aligned} 1 &< \mu \int_0^{p_1} \left[\int_p^{p_1} \sigma(s) ds \right] dp \\ &= \mu \left\{ p \int_p^{p_1} \sigma(s) ds \Big|_0^{p_1} + \int_0^{p_1} p \sigma(p) dp \right\} \\ &= \mu \int_0^{p_1} p \sigma(p) dp - \mu \lim_{p \rightarrow 0} p \int_p^{p_1} \sigma(s) ds, \end{aligned} \quad (3.35)$$

provided the indicated limit exists. In fact the limit is zero since, by Lemma 2.5(i),

$$0 < p \int_0^{p_1} \sigma(s) ds < p \int_p^{p_1} \sigma(s) ds < RT_0(p) \rightarrow 0 \text{ as } p \rightarrow 0. \quad (3.36)$$

This proves statement (iii).

The first part of statement (iv) follows immediately from Lemma 3.1(ii). If $\epsilon < 1$ then, in particular,

$$\mu \int_0^{p_0} s \sigma(s) ds < 1. \quad (3.37)$$

If X has any zeros, then (3.28) and (3.37) together imply that $p_0 < p_1$. Hence $X(p) \neq 0$ for $0 < p \leq p_0$. But from (3.11) we have

$$\begin{aligned} \frac{d}{dp} (pX) &= X + (p\sigma) \left(\frac{1}{\sigma} \frac{dX}{dp} \right) \\ &= X + (p\sigma) \left[-\mu \int_0^p X(p') dp' \right], \end{aligned} \quad (3.38)$$

and integrating from zero to p and using (3.27) gives

$$pX(p) = \int_0^p X(s) ds - \mu \int_0^p s \sigma(s) \left[\int_0^s X(p') dp' \right] ds. \quad (3.39)$$

Since $X(p)$ does not change sign for $0 < p \leq p_0$ we therefore have on this interval

$$\begin{aligned} p|X(p)| &= \int_0^p |X(s)| ds - \mu \int_0^p s \sigma(s) \left[\int_0^s |X(p')| dp' \right] ds \\ &> \int_0^p |X(s)| ds - \mu \int_0^p s \sigma(s) \left[\int_0^p |X(p')| dp' \right] ds \\ &= \left[1 - \mu \int_0^p s \sigma(s) ds \right] \int_0^p |X(s)| ds \end{aligned}$$

$$\geq (1-\epsilon) \int_0^p |X(s)| ds, \quad (3.40)$$

where the last inequality follows from (3.29). This completes the proof of Lemma 3.4.

We are now ready to prove Theorem 2. Referring to the statements in the theorem numbered (i) through (v), we shall prove the theorem by showing that

$$(iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)$$

$$(i) \Leftrightarrow (iv)$$

$$(i) \Leftrightarrow (v)$$

Proof that (iii) \Rightarrow (ii): Let ϵ and p_0 be as stated in Lemma 4(iv), and let $0 < p \leq p_* \leq p_0$. From (3.12b) and (3.30) it then follows that

$$|X(p)| \leq |X(p_*)| + \frac{\mu}{1-\epsilon} \int_p^{p_*} s \sigma(s) |X(s)| ds. \quad (3.41)$$

By virtue of the Gronwall inequality (cf. Coddington and Levinson, 1955, p. 37, or Hale, 1969, pp. 36-37), it follows from (3.41) that in fact

$$|X(p)| \leq |X(p_*)| \left\{ 1 + \frac{\mu}{1-\epsilon} \int_p^{p_*} s \sigma(s) \exp \left[\frac{\mu}{1-\epsilon} \int_p^s p' \sigma(p') dp' \right] ds \right\}. \quad (3.42)$$

The integrals here may be bounded in terms of ϵ , using (3.29), as follows:

$$|X(p)| \leq |X(p_*)| \left\{ 1 + \frac{\mu}{1-\epsilon} \int_p^{p_*} s \sigma(s) \exp\left(\frac{\epsilon}{1-\epsilon}\right) ds \right\}$$

$$\leq |X(p_*)| \left\{ 1 + \frac{\epsilon}{1-\epsilon} \exp\left(\frac{\epsilon}{1-\epsilon}\right) \right\} < \infty . \quad (3.43)$$

But $X(p)$ has a limit as $p \rightarrow 0$ by Lemma 3.3(iii), hence (3.43) implies that the limit is finite.

Proof that (ii) \Rightarrow (i): Every solution of the VSE is continuous, and therefore bounded, for $p_* \leq p \leq p_S$ with $p_* > 0$ arbitrary. Statement (ii) implies that in fact this is true for $0 \leq p \leq p_S$. Therefore X is square-integrable.

Proof that (i) \Rightarrow (iii): Lemma 3.3(iii) guarantees that

$$W = \lim_{p \rightarrow 0} \left| \frac{1}{\sigma} \frac{dX}{dp} \right| \quad (3.44)$$

exists. We will show that (i) \Rightarrow (iii) by proving the contrapositive.

Suppose that $W \neq 0$, so that there is a $p_* > 0$ such that

$$\left| \frac{1}{\sigma} \frac{dX}{dp} \right| > \frac{1}{2} W > 0 \quad \text{for } 0 < p < p_* . \quad (3.45)$$

From Lemma 2.1 it then follows that

$$\left| \frac{dX}{dp} \right| > \frac{1}{2} W c_1 p^{-3/2} \quad \text{for } 0 < p < p_* . \quad (3.46)$$

Upon integrating from p to p_* , this yields

$$|X(p) - X(p_*)| > W c_1 (p^{-1/2} - p_*^{-1/2}) \quad \text{for } 0 < p < p_* . \quad (3.47)$$

It follows that

$$\begin{aligned}
\int_p^{p_S} X^2(s) ds &\geq \int_p^{p_*} X^2(s) ds \\
&\geq \int_p^{p_*} |X(s) - X(p_*)|^2 ds - \int_p^{p_*} X^2(p_*) ds \\
&> W^2 c_1^2 \int_p^{p_*} (s^{-1/2} - p_*^{-1/2})^2 ds - (p_* - p) X^2(p_*) \\
&\rightarrow \infty \text{ as } p \rightarrow 0.
\end{aligned} \tag{3.48}$$

Proof that (i) \Rightarrow (iv): We have already shown that (i) implies both (ii) and (iii). But (ii) and (iii) together imply (iv).

Proof that (iv) \Rightarrow (i): Statement (iv) implies that $\lim_{p \rightarrow 0} X(p) = 0$ or $\lim_{p \rightarrow 0} \frac{1}{\sigma} \frac{dX}{dp} = 0$, since Lemma 3.3(iii) states that both of these limits exist for every solution of the VSE. In the first case, (i) follows trivially. In the latter case (i) also follows since we already showed that (iii) \Rightarrow (i).

Proof that (i) \Rightarrow (v): This is trivial since (i) \Rightarrow (ii) \Rightarrow (v).

Proof that (v) \Rightarrow (i): We will show that (v) \Rightarrow (iii), by proving the contrapositive. As shown previously, the negation of (iii) implies (3.47). Multiplying (3.47) by p^δ gives

$$|p^\delta X(p) - p^\delta X(p_*)| > W c_1 [p^{\delta-1/2} - p^\delta p_*^{-1/2}] , \tag{3.49}$$

and therefore, letting $p \rightarrow 0$, we have

$$\liminf_{p \rightarrow 0} p^\delta |X(p)| \geq W c_1 > 0 , \tag{3.50}$$

for $0 < \delta \leq 1/2$. This completes the proof of Theorem 2.

Proof of Theorem 3: Part (i) follows directly from Theorem 1 and Lemma 3.1(i). We do not prove part (iii), but only point out that for limit-point Sturm-Liouville problems with totally discrete spectra, the facts about completeness, orthogonality, and number of zeros of the eigenfunctions are still the same as for regular Sturm-Liouville problems; cf. Coddington and Levinson (1955, Chapter 9, Theorem 3.1 and Problem 1).

We now prove part (ii). Positivity of the eigenvalues was already stated and proven as Lemma 3.4(i). To obtain the eigenvalue bounds we use the fact that the eigenfunction X_n corresponding to eigenvalue μ_n has precisely n zeros. In particular, $\tilde{X}_0(p) = X_0(p)/X_0(p_S) > 0$ for $0 < p \leq p_S$. Since $\mu_0 > 0$, it then follows from (3.11) that $d\tilde{X}_0/dp < 0$ for $0 < p \leq p_S$, so in fact $\tilde{X}_0(p) > \tilde{X}_0(p_S) = 1$ for $0 < p < p_S$. From (3.13) we then have

$$\frac{p_S}{RT_S} > \mu_0 p_S, \quad (3.51)$$

proving the upper bound in (1.31a).

To obtain the lower bound, use (3.13) and Lemma 3.4(ii) to write

$$\begin{aligned} \frac{1}{RT_S} - \frac{\mu_0}{p_S} \int_0^{p_S} \tilde{X}_0(s) \, ds \\ - \frac{\mu_0}{p_S} \left[s\tilde{X}_0(s) \Big|_0^{p_S} - \int_0^{p_S} s \frac{d\tilde{X}_0(s)}{ds} \, ds \right] \\ = \mu_0 - \frac{\mu_0}{p_S} \int_0^{p_S} [s\sigma(s)] \left[\frac{1}{\sigma(s)} \frac{d\tilde{X}_0(s)}{ds} \right] \, ds. \end{aligned} \quad (3.52)$$

Now from (3.11) and (3.13) we have for $0 < p < p_S$

$$\begin{aligned}
 -\frac{1}{\sigma} \frac{d\tilde{X}_0}{dp} &= \mu_0 \int_0^p \tilde{X}_0(s) ds \\
 &< \mu_0 \int_0^{p_S} \tilde{X}_0(s) ds \\
 &= \frac{p_S}{RT_S} .
 \end{aligned} \tag{3.53}$$

Introducing (3.53) into (3.52) and using Lemma 3.1(ii), we get

$$\begin{aligned}
 \frac{1}{RT_S} &< \mu_0 + \frac{\mu_0}{RT_S} \int_0^{p_S} s \sigma(s) ds \\
 &= \mu_0 \frac{\kappa \Phi_E}{RT_S} ,
 \end{aligned} \tag{3.54}$$

which completes the proof of (1.31a).

The bound (1.31b) for $n \geq 1$ is actually a refinement of (3.25). If we denote the first zero of X_n by p_1 and the last by p_n , we have from (3.24) that in fact

$$n-1 < \mu_n \int_{p_1}^{p_n} s \sigma(s) ds . \tag{3.55}$$

Adding inequalities (3.28) and (3.55) then gives

$$\begin{aligned}
n &< \mu_n \int_0^{p_n} s \sigma(s) ds \\
&< \mu_n \int_0^{p_S} s \sigma(s) ds
\end{aligned} \tag{3.56}$$

and (1.31b) follows from Lemma 3.1(ii).

Inequalities (1.32a,b) follow directly from (1.31a,b). We remark that, while the bounds in (1.32a) usually provide a good estimate of the barotropic equivalent depth as pointed out in the Introduction, the bounds in (1.32b) become progressively worse as n increases. This is clear from the derivation of (3.24). Better bounds can be obtained if one has some information about $T_0(p)$.

It remains to prove statement (iv). If p_1 denotes the first zero of X_n , it follows from (3.11) that $X_n(p)$ is monotone for $0 < p < p_1$, hence

$$|X_n(p)| \geq |X_n(p_*)| > 0 \tag{3.57}$$

for $0 < p \leq p_* < p_1$. Again using (3.11) we then have

$$\begin{aligned}
\left| \frac{dX_n(p)}{dp} \right| &= \mu_n \sigma(p) \int_0^p |X_n(s)| ds \\
&> \mu_n |X_n(p_*)| p \sigma(p) \\
&> \mu_n |X_n(p_*)| c_1 p^{-1/2}
\end{aligned} \tag{3.58}$$

for $0 < p \leq p_* < p_1$, where we have used Lemma 2.1. This completes the proof.

4. Examples

In this section we give four examples that illustrate some of our theoretical results. In the first two examples, we specify the asymptotic form of $T_0(p)$ near $p = 0$ and we apply the Frobenius (power series) method (e.g., Coddington and Levinson, 1955, Section 4.8) to determine the asymptotic behavior of all solutions of the VSE. These asymptotic results depend neither on $T_0(p)$ away from $p = 0$ nor on the lower boundary condition. In Example 1 the atmosphere is bounded and the asymptotic results allow one to verify directly the equivalent boundary conditions of Theorem 2. In Example 2 we have $\lim_{p \rightarrow 0} T_0(p) \neq 0$ and we show that the spectrum is continuous for $\mu \geq \mu_c$, a result which complements Theorem 1.

In the third and fourth examples we specify $T_0(p)$ throughout the domain, and we obtain closed-form solutions of the VSE satisfying the lower boundary condition. In both examples $T_0(p) \rightarrow 0$ as $p \rightarrow 0$, but slowly enough that the atmosphere is unbounded. These two examples are used as counter-examples showing that Theorem 2 is not true when its stated hypothesis of a bounded atmosphere is replaced by the somewhat weaker hypothesis that $T_0(p) \rightarrow 0$ as $p \rightarrow 0$. In particular, Example 3 shows that it is possible for non-square-integrable solutions of the VSE to have $\frac{1}{\sigma} \frac{dX}{dp} \rightarrow 0$ as $p \rightarrow 0$, while Example 4 shows that it is possible for square-integrable solutions (eigenfunctions) to be unbounded.

Example 1. Suppose that near $p = 0$, $T_0(p)$ is an analytic function of p^ϵ ,

$$T_0(p) = \sum_{j=0}^{\infty} \tau_j p^{\epsilon j}, \quad (4.1)$$

where we assume that

$$0 < \epsilon < \kappa, \quad \tau_0 = 0, \quad \tau_1 > 0. \quad (4.2)$$

It follows that our basic assumptions are all satisfied near $p = 0$, and that

$$T_{\infty} = \lim_{p \rightarrow 0} T_0(p) = 0. \quad (4.3)$$

Hence the spectrum is totally discrete.

From (4.1) and the definition (1.2c) of σ , it follows that near $p = 0$, σ has the form

$$\sigma(p) = R p^{\epsilon-2} \sum_{j=0}^{\infty} [\kappa - \epsilon(j+1)] r_{j+1} p^{\epsilon j}. \quad (4.4)$$

By taking $r_j = 0$ for $j \geq 2$, this family reduces to the family (1.14, 1.15) considered by Staniforth et al. (1985), not including the limiting cases $\alpha = 2 - \kappa$ and $\alpha = 2$.

Using (4.4) we write

$$\frac{1}{\sigma(p)} = p^{2-\epsilon} B(p^{\epsilon}) \quad (4.5a)$$

where

$$B(q) = \sum_{j=0}^{\infty} b_j q^j; \quad (4.5b)$$

the coefficients b_j can be calculated from the coefficients in (4.4). Now substituting (4.5a) into the VSE (1.8), and making the coordinate transformation

$$q = p^{\epsilon}, \quad (4.6)$$

the VSE becomes

$$q^2 \frac{d^2 X}{dq^2} + qC(q) \frac{dX}{dq} + D(q)X = 0, \quad (4.7a)$$

where

$$C(q) = q \frac{d}{dq} (\log q^{1/\epsilon} B(q)) \quad (4.7b)$$

$$D(q) = \frac{\mu}{\epsilon^2} \frac{q}{B(q)} . \quad (4.7c)$$

The Frobenius method applies since the coefficients C and D are analytic functions of q . Therefore the solutions of the VSE have the form

$$X(q) = q^\beta \left[1 + \sum_{j=1}^{\infty} a_j q^j \right] \quad (4.8)$$

in a neighborhood of $p = 0$, where the series converge uniformly and can be differentiated term-by-term. The exponent β and coefficients a_j are obtained by substituting (4.8) into (4.7a), using the power-series expansions for $C(q)$ and $D(q)$, and then equating coefficients of like powers of q . Using (4.6) one finds that the regular solution $X^{(1)}(p) = X^{(1)}(p; \mu)$ and the singular solution $X^{(2)}(p) = X^{(2)}(p; \mu)$ are given near $p = 0$ by

$$X^{(1)}(p) = 1 - \mu R r_0 \frac{\kappa - \epsilon}{\epsilon} p^\epsilon + \sum_{j=2}^{\infty} a_j^{(1)} p^{\epsilon j} , \quad (4.9a)$$

$$X^{(2)}(p) = p^{\epsilon-1} \left[1 + \sum_{j=1}^{\infty} a_j^{(2)} p^{\epsilon j} \right] , \quad (4.9b)$$

where the coefficients $a_j^{(1)}$ and $a_j^{(2)}$ can be expressed in terms of the r_j in (4.1).

It is a simple matter to verify that $X^{(1)}$ satisfies each of the boundary conditions of Theorem 2, whereas $X^{(2)}$ satisfies none of them. In particular the n^{th} eigenfunction X_n is given by

$$X_n(p) = X^{(1)}(p; \mu_n) , \quad (4.10)$$

μ_n being the n^{th} eigenvalue.

Example 2. Suppose that near $p = 0$, $T_0(p)$ is an analytic function of p ,

$$T_0(p) = \sum_{j=0}^{\infty} \tau_j p^j, \quad \tau_0 > 0. \quad (4.11)$$

If $\tau_j = 0$ for $j \geq 1$, this reduces to the familiar atmosphere with an isothermal top. Our basic assumptions are all satisfied near $p = 0$, and

$$T_{\infty} = \lim_{p \rightarrow 0} T_0(p) = \tau_0 \neq 0. \quad (4.12)$$

Thus the atmosphere is unbounded and from Theorem 1 it follows that the portion of the spectrum for

$$\mu < \mu_c = \frac{1}{4\kappa R T_{\infty}} \quad (4.13)$$

is discrete. We will show that the spectrum is continuous for $\mu \geq \mu_c$.

From (4.1) and the definition (1.2c) of σ , we have

$$\sigma(p) = R p^{-2} \sum_{j=0}^{\infty} (\kappa - j) \tau_j p^j. \quad (4.14)$$

Therefore

$$\frac{1}{\sigma(p)} = p^2 B(p), \quad (4.15a)$$

where B is the analytic function

$$B(p) = \sum_{j=0}^{\infty} b_j p^j \quad (4.15b)$$

whose coefficients can be expressed in terms of the τ_j . The VSE can then be written as

$$p^2 \frac{d^2 X}{dp^2} + p C(p) \frac{dX}{dp} + D(p) X = 0, \quad (4.16a)$$

where

$$C(p) = \frac{1}{pB} \frac{dp^2 B}{dp} \quad (4.16b)$$

and

$$D(p) = \frac{\mu}{B} \quad (4.16c)$$

are analytic. Application of the Frobenius method results in the following three cases.

If $\mu < \mu_c$, we find that near $p = 0$ the two linearly independent solutions are of the form

$$X^{(1)}(p) = X^{(1)}(p; \mu) = p^{\beta_+} \left[1 + \sum_{j=1}^{\infty} a_j^{(1)} p^j \right] \quad (4.17a)$$

$$X^{(2)}(p) = X^{(2)}(p; \mu) = p^{\beta_-} \left[1 + \sum_{j=1}^{\infty} a_j^{(2)} p^j \right] \quad (4.17b)$$

where

$$\beta_{\pm} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{\mu}{\mu_c}} \quad (4.17c)$$

Clearly $X^{(1)}$ is square-integrable and $X^{(2)}$ is not. For any eigenvalue $\mu_n < \mu_c$ it follows that the corresponding eigenfunction is

$$X_n(p) = X^{(1)}(p; \mu_n) \quad (4.18)$$

The actual number of eigenvalues $\mu < \mu_c$, if any, depends upon the thermal structure all the way to $p = p_S$, as well as upon the lower boundary condition. This number is always finite.

If $\mu = \mu_c$, we find instead that

$$X^{(1)}(p) = p^{-1/2} \left[1 + \sum_{j=1}^{\infty} a_j^{(1)} p^j \right] \quad (4.19a)$$

$$X^{(2)}(p) = p^{-1/2}(\log p) \left[1 + \sum_{j=1}^{\infty} a_j^{(2)} p^j \right] . \quad (4.19b)$$

It is clear that $X^{(1)}$ is not square-integrable. Since $|X^{(2)}(p)| > |X^{(1)}(p)|$ for all p sufficiently small, it follows that $X^{(2)}$ is not square-integrable either, nor is any linear combination of $X^{(1)}$ and $X^{(2)}$.

If $\mu > \mu_c$, two linearly independent solutions are

$$X^{(1)}(p) = p^{-1/2}[\cos(\gamma \log p)] \left[1 + \sum_{j=1}^{\infty} a_j^{(1)} p^j \right] \quad (4.20a)$$

and

$$X^{(2)}(p) = p^{-1/2}[\sin(\gamma \log p)] \left[1 + \sum_{j=1}^{\infty} a_j^{(2)} p^j \right] \quad (4.20b)$$

where

$$\gamma = \frac{1}{2} \sqrt{\frac{\mu}{\mu_c} - 1} . \quad (4.20c)$$

Observe that $X^{(1)}(p)$ is not square-integrable, for

$$\int_p s^{-1} \cos^2(\gamma \log s) ds = -\frac{1}{2} \log p - \frac{1}{4\gamma} \sin(2\gamma \log p) , \quad (4.21)$$

which is unbounded as $p \rightarrow 0$. Similarly it follows that $X^{(2)}$ is not square-integrable, nor is any linear combination of $X^{(1)}$ and $X^{(2)}$.

Thus there are no square-integrable solutions of the VSE for $\mu \geq \mu_c$. This means that the spectrum is continuous for $\mu \geq \mu_c$: for every such μ there is a unique solution of the VSE satisfying the lower boundary condition (1.9), but the solution is not square-integrable. However, one can still expand arbitrary square-integrable functions in terms of the totality of these solutions for $\mu \geq \mu_c$ and the eigenfunctions (if any) for $\mu < \mu_c$, in a manner analogous to the Fourier integral; cf. Coddington

and Levinson (1955, Chapter 9, Theorem 3.1), and Staniforth et al. (1985, p. 346). Thus if one is interested in spectral expansions, then no upper boundary condition should be imposed when $\mu \geq \mu_c$. We reiterate that continuous spectra cannot arise for bounded atmospheres. The continuous spectrum in this example is spurious in the sense that the primitive equations on which the VSE is based do not apply to unbounded atmospheres, due to the traditional shallowness approximations.

Since the atmosphere is unbounded in this example, Theorem 2 does not apply. In particular, for all $\mu > 0$ it follows from our three sets of asymptotic formulas that both $X^{(1)}$ and $X^{(2)}$ satisfy

$$\frac{1}{\sigma} \frac{dX}{dp} \rightarrow 0 \quad \text{as } p \rightarrow 0. \quad (4.22)$$

Thus the boundary condition (4.22) is generally not well posed, because it is not strong enough to eliminate the non-square-integrable solutions when $\mu < \mu_c$.

The boundary condition (4.22) is important because it is derived from the commonly invoked boundary condition (1.6) that $\omega \rightarrow 0$ as $p \rightarrow 0$ in the primitive equations. Theorem 2 implies that (4.22) is well-posed for bounded atmospheres, while Example 2 shows that it is generally not well-posed when $T_\infty \neq 0$. In the next example $T_\infty = 0$ but the atmosphere is unbounded. We will see that the boundary condition (4.22) is still not well-posed.

Example 3. Let $\sigma_S > 0$ be arbitrary, and let $\epsilon > 0$ be an arbitrary parameter. Consider the VSE problem for

$$\sigma(p) = \frac{RT_S}{p^2} \left(1 - \epsilon \log \frac{p}{p_S}\right)^{-2} \left(\frac{\sigma_S p_S^2}{RT_S} - \epsilon \log \frac{p}{p_S}\right). \quad (4.23)$$

From (1.2b) it follows that the temperature profile $T_0(p)$ corresponding to any $\sigma(p)$ is given by

$$T_0(p) = p^\kappa \left[T_S p_S^{-\kappa} + R^{-1} \int_p^{p_S} s^{1-\kappa} \sigma(s) ds \right]. \quad (4.24)$$

Upon substituting (4.23) into (4.24) it is easily verified that our basic assumptions in Sec. 1 are all satisfied.

It can also be established that

$$T_\infty = \lim_{p \rightarrow 0} T_0(p) = 0, \quad (4.25)$$

as follows. For σ given by (4.23) it is clear that the integral in (4.24) diverges as $p \rightarrow 0$. Therefore L'Hospital's rule applies to give asymptotically

$$\begin{aligned} T_0(p) &\sim \frac{1}{R} \left[\frac{-p^{1-\kappa} \sigma(p)}{-\kappa p^{-\kappa-1}} \right] \quad \text{as } p \rightarrow 0 \\ &= \frac{1}{\kappa R} p^2 \sigma(p) \\ &\sim T_S (-\kappa \epsilon \log \frac{p}{p_S})^{-1}, \end{aligned} \quad (4.26)$$

from which (4.25) follows immediately. By Theorem 1 the spectrum of the VSE is therefore totally discrete. Nevertheless the basic state atmosphere is unbounded because from (4.26) we have that

$$\int_p \frac{RT_0(s)}{s} ds \sim \frac{RT_S}{\kappa \epsilon} \int_p \frac{ds}{-s \log \frac{s}{p_S}}$$

$$= \frac{RT_S}{\kappa \epsilon} \log(-\log \frac{p}{p_S}) , \quad (4.27)$$

so from (1.18) it follows that $\Phi_0(p) \rightarrow \infty$ as $p \rightarrow 0$.

By differentiation it can be verified that the function

$$X(p) = \left(\frac{p}{p_S} \right)^{-1} \left(1 - \epsilon \log \frac{p}{p_S} \right)^{-\nu/\epsilon} , \quad (4.28)$$

where

$$\nu = 1 - \frac{\sigma_S p_S^2}{RT_S} , \quad (4.29)$$

satisfies the LBC and VSE with

$$\mu = \frac{\nu - \epsilon}{RT_S} . \quad (4.30)$$

On the other hand, this X is not an eigenfunction because it is clearly not square-integrable. However, one finds that

$$\frac{1}{\sigma} \frac{dX}{dp} = - \left(1 - \epsilon \log \frac{p}{p_S} \right)^{-(\nu-\epsilon)/\epsilon} . \quad (4.31)$$

Therefore (4.22) holds whenever the parameters are such that $\epsilon < \nu$. Once again we see that the boundary condition (4.22) is too weak to exclude solutions of the VSE which are not square-integrable. This example also shows that Theorem 2 does not hold under the hypothesis that $T_0 \rightarrow 0$ as $p \rightarrow 0$, which was shown by Lemma 2.1 to be weaker than the stated hypothesis that the atmosphere be bounded.

Notice also that if $\epsilon \geq \nu$, this example shows that it is possible for a function to satisfy the LBC and the VSE with $\mu \leq 0$.

We remark that the parameters σ_S and ϵ in this example can be chosen in such a way that $T_0(p)$ is arbitrarily close to being constant on

any interval $p_* \leq p \leq p_S$ with $p_* > 0$. To see this, choose σ_S so that $dT_0/dp = 0$ at $p = p_S$, i.e.,

$$\sigma_S = \kappa R T_S p_S^{-2}. \quad (4.32)$$

Then (4.23) reads

$$\begin{aligned} \sigma(p) &= \kappa R T_S p^{-2} (1 - \epsilon \log \frac{p}{p_S})^{-2} (1 - \frac{\epsilon}{\kappa} \log \frac{p}{p_S}) \\ &= \kappa R T_S p^{-2} [1 + O(\epsilon)]. \end{aligned} \quad (4.33)$$

Therefore $\sigma(p)$ can be made arbitrarily close to the isothermal static stability function $\kappa R T_S p^{-2}$ over any interval away from $p = 0$, by choosing ϵ sufficiently small. For such a choice of parameters, it follows that $T_0(p) \approx T_S$ from the surface up to a point p_* arbitrarily close to $p = 0$, after which (4.26) applies and $T_0(p)$ drops to zero.

It is well known, and also follows from Example 2, that the VSE for isothermal atmospheres has a continuous spectrum. Example 3 shows that by modifying the isothermal atmosphere in a neighborhood of $p = 0$, a totally discrete spectrum can be obtained.

This example was created by choosing $\frac{1}{\sigma} \frac{dX}{dp}$ (judiciously) to be of the form (4.31), then differentiating to obtain $-\mu X$, then differentiating once more to obtain $-\mu dX/dp$. The LBC then determines μ , and σ is recovered upon dividing dX/dp by $\frac{1}{\sigma} \frac{dX}{dp}$. The following example is constructed in the same way, but from a different choice of $\frac{1}{\sigma} \frac{dX}{dp}$. As in Example 3, we will have $T_\infty = 0$ but an unbounded atmosphere. The example will exhibit an eigenfunction which is unbounded. This is in contrast with Theorem 2, which states that for bounded atmospheres all eigenfunctions are bounded.

Example 4. Let $\sigma_S > 0$ be arbitrary, and let a parameter δ satisfy

$$0 < \delta < \min\left(\frac{\sigma_S p_S^2}{RT_S}, 1\right). \quad (4.34)$$

Define for given σ_S and δ the positive parameter ϵ ,

$$\epsilon = \left(\frac{1-\delta}{\delta}\right) \left(\frac{\sigma_S^2 p_S}{RT_S} - \delta\right), \quad (4.35)$$

and consider the VSE problem for

$$\sigma(p) = \frac{RT_S}{p^2} \left(1 - \epsilon \log \frac{p}{p_S}\right)^{-2} \left[\frac{\sigma_S^2 p_S}{RT_S} - \left(\frac{\sigma_S^2 p_S}{RT_S} - \delta\right) \log \frac{p}{p_S} \right]. \quad (4.36)$$

As in Example 3 it can be verified that the basic assumptions are satisfied, that $T_0(p) \rightarrow 0$ as $p \rightarrow 0$, and that the atmosphere is unbounded. In particular the spectrum is totally discrete. One can make the temperature profile corresponding to (4.36) nearly isothermal as in Example 3, by the choice (4.32) of σ_S and by choosing δ sufficiently close to κ . For the choice $\delta = 1/2$, (4.36) is identical to the static stability function (4.23) of Example 3 with $\epsilon = \frac{\sigma_S^2 p_S}{RT_S} - \frac{1}{2}$.

By differentiation it can be verified that the function

$$X(p) = \left(1 - \epsilon \log \frac{p}{p_S}\right)^{(\delta-\epsilon)/\epsilon} \left[\delta - \left(\frac{\sigma_S^2 p_S}{RT_S} - \delta\right) \log \frac{p}{p_S} \right] \quad (4.37)$$

satisfies the LBC and VSE with

$$\mu = \frac{1-\delta}{RT_S}. \quad (4.38)$$

Clearly we have asymptotically

$$X(p) \sim \frac{1}{\epsilon} \left(\frac{\sigma_S^2 p_S}{RT_S} - \delta\right) \left(1 - \epsilon \log \frac{p}{p_S}\right)^{\delta/\epsilon}, \quad (4.39)$$

so $X(p) \rightarrow \infty$ as $p \rightarrow 0$. However, this X is square-integrable because

$$\int_0^1 (-\log x)^{2\delta/\epsilon} dx = \Gamma\left(\frac{2\delta}{\epsilon} + 1\right), \quad (4.40)$$

where Γ is the gamma-function, which is finite. Thus X is an unbounded eigenfunction. In fact, since X has no zeros, μ given by (4.38) is the lowest eigenvalue.

5. Concluding remarks

In this article we have presented an analysis of the vertical structure equation for quite arbitrary thermal profiles, using classical techniques for singular Sturm-Liouville problems. We have argued that the vertical structure equation is assuredly meaningful only for bounded atmospheres because the primitive equations from which it is derived apply only to bounded, in fact shallow atmospheres, owing to the traditional shallowness approximations. For bounded atmospheres we have shown that the spectrum of the vertical structure equation is always totally discrete, that the barotropic equivalent depth H_0 corresponding to the lowest eigenvalue satisfies the inequality $RT_s/g < H_0 < \kappa\Phi_g/g$ independently of the thermal profile, that the eigenfunctions are all bounded but have unbounded first derivatives, and that any one of a number of upper boundary conditions, including conditions which follow from requiring mass or energy conservation in the linearized primitive equations, are equivalent and well posed. These results are in stark contrast to the situation for unbounded atmospheres, for which we have shown that typically the spectrum is partly continuous and the eigenfunctions, if any, are square-integrable but unbounded.

The primitive equations, with the traditional shallowness approximations, are only a model of the atmosphere. Our results pertain to this model and not necessarily to any real atmosphere. Under the model assumptions there is a vertical structure equation and its spectrum is totally discrete. A primitive equation system without shallowness approximations would be a more realistic model of the atmosphere. It will be interesting to determine whether such a model has a vertical structure equation and, if so, the extent to which results on the spectrum of and boundary conditions for the classical vertical structure equation carry over. These issues will be studied in a forthcoming paper. Work on numerical treatment of the vertical structure equation and the primitive

equations themselves, in light of the equivalent boundary conditions of Theorem 2, is also underway.

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REFERENCES

- Batchelor, G. K., 1967. An Introduction to Fluid Dynamics. Cambridge University Press, London, 615 pp.
- Chapman, S., and R. Lindzen, 1970. Atmospheric Tides. Gordon and Breach, New York, 200 pp.
- Coddington, E. A., and N. Levinson, 1955. Theory of Ordinary Differential Equations. McGraw-Hill, New York, 429 pp.
- Cohn, S. E., and D. P. Dee, 1986. Observability of discretized partial differential equations. SIAM J. Numer. Anal., to appear.
- Daley, R., 1981. Normal mode initialization. Rev. Geophys. Space Phys., 19, pp. 450-468.
- Dee, D. P., and A. M. da Silva, 1986. Using Hough harmonics to validate and assess nonlinear shallow-water models. Mon. Wea. Rev., 114, pp. 2191-2196.
- Dikii, I. A., 1965. The terrestrial atmosphere as an oscillating system. Izv. Atmosph. Oceanic Phys., 1, pp. 275-286.
- Flattery, T. W., 1970. Spectral models for global analysis and forecasting. Proc. Sixth AWS Technical Exchange Conf., U. S. Naval Academy, Air Weather Service Tech. Rep. 242, pp. 42-53.
- Gill, A. E., 1982. Atmosphere-Ocean Dynamics. Academic Press, New York, 662 pp.
- Hale, J. K., 1969. Ordinary Differential Equations. Wiley-Interscience, New York, 332 pp.
- Holl, P., 1970. The completeness of the orthogonal system of the Hough functions. B. Haurwitz, trans. (1979), Dept. of Atmos. Sci. Report, Colorado State University, 13 pp. Translation of "Die Vollständigkeit des Orthogonalsystems der Hough-Functionen, Nachrichten der Wissenschaften in Göttingen, II. Mathematisch-Physikalische Klasse, 7, pp. 159-168.

- Hough, S. S., 1898. On the application of harmonic analysis to the dynamical theory of the tides — Part II. On the general integration of Laplace's dynamical equations. Phil. Trans. Roy. Soc. London, A191, pp. 139-185.
- Kasahara, A., 1976. Normal modes of ultralong waves in the atmosphere. Mon. Wea. Rev., 104, pp. 669-690.
- Kasahara, A., 1977. Numerical integration of the global barotropic primitive equations with Hough harmonic expansions. J. Atmos. Sci., 34, pp. 687-701.
- Kasahara, A., and K. Puri, 1981. Spectral representation of three-dimensional global data in normal mode functions. Mon. Wea. Rev., 109, pp. 37-51.
- Longuet-Higgins, M. S., 1968. The eigenfunctions of Laplace's tidal equations over a sphere. Phil. Trans. Roy. Soc. London, A262, pp. 511-601.
- Phillips, N. A., 1966. The equations of motion for a shallow rotating atmosphere and the "traditional approximation." J. Atmos. Sci., 23, pp. 626-628.
- Phillips, N. A., 1973. Principles of large scale numerical weather prediction. In: Dynamic Meteorology, P. Morel, ed., Reidel, Dordrecht, 622 pp.
- Salby, M. L., 1984. Survey of planetary-scale traveling waves: The state of theory and observations. Rev. Geophys. Space Phys., 22, pp. 209-236.
- Siebert, M., 1961. Atmospheric tides. In: Advances in Geophysics, 7, H. E. Landsberg and J. van Mieghem, eds., Academic Press, New York, pp. 105-187.

- Staniforth, A., M. Béland, and J. Côté, 1985. An analysis of the vertical structure equation in sigma coordinates. Atmosphere-Ocean, 23, pp. 323-358.
- Sudarshan, E., 1985. Studies in Atmospheric Physics: Normal Mode Oscillations in the Earth's Atmosphere. Ph.D. Dissertation, Sri Venkateswara University, Tirupathi, India, 163 pp.
- Taylor, G. I., 1936. The oscillations of the atmosphere. Proc. Roy. Soc. London, A156, pp. 318-326.
- Wiin-Nielsen, A., 1971. On the motion of various vertical modes of transient, very long waves. Part II. The spherical case. Tellus, 23, pp. 207-217.