# the physics of charge separation preceding LIGHTNING STROKES IN THUNDERCLOUDS <br> ```(NASA-CR-178371) TGE PHYSICS OF CHARGE N88-10452 \\ SEPARATION PRECEDIMG LIGHTNING STROKES IN \\ THONDERCLOODS (Electro Magnetic \\ Applications) 19 p Avail: NTIS HC A03/MF UnClas \\ A01 CSCL 04B G3/47 0103536``` 

A. Kyrala

Electro Magnetic Applications, Inc. P.O. Box 26263 Denver, CO 80226

National Aeronautics and
Space Administration

## Langley Research Center

Hampton, Virginia 23665

# THE PHYSICS OF CHARGE SEPARATION PRECEDING LIGHTNING STROKES IN THUNDERCLOUDS 

by

A. KYRALA -

## 1. Introduction

The present work is divided into three sections after this one of which the first is a quantum-mechanical explanation for the aggregation of charges of like sign, the second is a classical model of the field and electrical forces of charges already aggregated while the third one is stochastic dealing with the development of charge separation by means of the Boltzmann transport equation.

In this context it should be clearly understood that Earnshaw's Theorem [1] implies that a collection of electrical charges cannot be maintained in static equilibrium by electrical forces alone. Thus even after the segregation into positive and negative centers of charge located at different altitudes it would be impossible to maintain such a distribution without the aerodynamical forces and pressure distribution in a containment vortex. By balancing drag against electrostatic forces a deterministic explanation for the vertical locations of maximum charge concentrations of each sign can be attained. The knowledge of this a posteriori distribution with respect to the dependence on electrostatic and aerodynamic forces would then indicate the appropriate input assumptions to be used with the Boltzmann equation.

## 2. A Quantum-Mechanical Explanation for Aggregation of Like Charges

If charges of like sign are to be collected in aggregates of the order of tens of coulombs this clearly cannot be achieved by Coulomb attraction nor is there any other classical mechanism available for such a physical process. There are however two other types of "bonding" in which attractive forces predominate over Coulomb repulsion. These are the nuclear bond and the covalent bond both of which are based on quantum-mechanical exchange interaction which splits each degenerate identical

[^0]energy level of identical particles into two levels and thus can make available a lower energy level for the bound system than for the unbound pair of particles [2]. It is recalled that this argument is the quantum analog of the classical vibration problem of identical coupled oscillators $[3,4]$. Strictly speaking the quantum arguments as formulated for the nuclear or covalent bond applies to identical particles but it can be modified to apply to a two-body problem with a "ball" of collected charge and an individual charge. According to plasma physics the maximum size of such a singlesigned charge ball should not exceed the Debye shielding distance [5] which is the ratio of thermal velocity to plasma frequency. In terms of accumulated charge $Q$ the ball radius would then be $\mathrm{Qe} / 4 \mathrm{kT} \pi$. Given an exchange interaction mechanism for the charge accumulation one would expect that single-signed charge would continue to collect until the field at the ball surface reached a megavolt/meter. At this time a lightning stroke (leader) would be initiated. After the full stroke sequence "relieved" the large charge concentration causing the discharge, recharging by quantum exchange interaction would recur once again accumulating charge up to breakdown field strength. Thus the periodic recharging is also explained by this sequence of events.

## 3. A Posteriori Electrostatic Model

The starting point is the problem of a conducting sphere (Earth) with Charge Q spatially isolated and under the influence of a point charge e located exterior to the sphere [6]. It is supposed that the point charge is maintained at a fixed distance $f_{+}$from the center of the sphere. This is shown in Fig. 1 in which any field point $P$ is at distances $r_{1}, r_{2}, r_{3}$ respectively from the point charge, from its image charge at the inverse point and from the center of the sphere. If the radius of the sphere is a and the distance between point charge and center is $f_{+}$the potential at $P$ is given by

$$
\begin{equation*}
\phi=\frac{e}{r_{1}}-\frac{e a}{f_{+} r_{2}}+\frac{e a}{f_{+} r_{3}}+\frac{Q}{r_{3}} \tag{1}
\end{equation*}
$$

The corresponding surface charge density is

$$
\begin{equation*}
\sigma=\frac{e}{4 \pi a f_{+}}-\frac{e\left(f_{+}^{2}-a^{2}\right)}{4 \pi a r_{1}^{3}}+\frac{Q}{4 \pi a^{2}} \tag{2}
\end{equation*}
$$

The force on the point charge is given by

$$
\begin{equation*}
F_{+}=\frac{e Q}{f_{+}^{2}}-\frac{e^{2} a}{f_{+}^{3}}-\frac{e^{2} a f_{+}}{\left(f_{+}^{2}-a^{2}\right)^{2}} \tag{3}
\end{equation*}
$$

It is understood that $Q$ and $e$ are both positive in the above formulae but the force on the point charge can change sign for different distances $f$. The charges induced on the sphere must be considered. Thus one has ATTRACTION for LIKE charges provided that

$$
\begin{equation*}
\frac{Q}{e}<\frac{a}{f_{+}}\left[1+\frac{f_{+}^{4}}{\left(f_{+}^{2}-a^{2}\right)^{2}}\right] \tag{4}
\end{equation*}
$$

and the a priori obvious repulsion for the reversed inequality.
In Fig. 2 the same type of analysis applied to a conducting sphere with negative charge $-Q$ and a negative point charge -e at radial distance $f$. yields potential (5), surface charge density (6) and force on -e (7) of

$$
\begin{align*}
& \phi=\frac{e a}{f_{-} r_{2}}-\frac{e}{r_{1}}-\frac{e a}{f_{-} r_{3}}-\frac{Q}{r_{3}}  \tag{5}\\
& \sigma=\frac{e\left(f_{-}^{2}-a^{2}\right)}{4 \pi a r_{1}^{3}}-\frac{e}{4 \pi a f_{-}}-\frac{Q}{4 \pi a^{2}}  \tag{6}\\
& F=\frac{e Q}{f_{-}^{2}}+\frac{e^{2} a}{f_{-}^{3}}-\frac{e^{2} a f}{\left(f_{-}^{2}-a^{2}\right)^{2}} \tag{7}
\end{align*}
$$

In Fig. 3 for a negatively charged ( -Q ) sphere and a positive point charge +e one has for potential (8), surface charge density (9) and force on +e (10)

$$
\begin{equation*}
\phi=\frac{e}{r_{1}}+\frac{e a}{f_{+} r_{3}}-\frac{e a}{f_{+} r_{2}}-\frac{Q}{r_{3}} \tag{8}
\end{equation*}
$$

$$
\begin{align*}
& \sigma=\frac{e}{4 \pi a f_{+}}-\frac{e\left(f_{+}^{2}-a^{2}\right)}{4 \pi a r_{1}^{3}}-\frac{Q}{4 \pi a^{2}}  \tag{9}\\
& F_{+}=-\frac{e Q}{f_{+}^{2}}+\frac{e^{2} a}{f_{+}^{3}}-\frac{e^{2} a f_{-}}{\left(f_{+}^{2}-a^{2}\right)^{2}} \tag{10}
\end{align*}
$$

Finally in Fig. 4 a system with negatively charged sphere (-Q) isolated with two exterior point charges $+e$ at distance $f_{+}$from sphere center and $-e$ at distance $f_{\text {- }}$ from the center is considered. Here five distances from the charges and their inverse points and from the center are utilised as shown in Fig. 4. The potential (11) and surface charge density (12) are given by

$$
\begin{align*}
& \phi=\frac{e}{r_{1}}-\frac{e}{r_{2}}+\frac{e a}{f_{-} r_{3}}-\frac{e a}{f_{+} r_{4}}-\frac{Q}{r_{5}}+\frac{e a}{f_{+} r_{5}}-\frac{e a}{f_{-} r_{5}}  \tag{11}\\
& 4 \pi \sigma=\frac{e}{f_{+}}-\frac{e}{f_{-}}-\frac{e\left(f_{+}^{2}-a^{2}\right)}{a r_{1}{ }^{3}}+\frac{e\left(f^{2}-a^{2}\right)}{a r_{2}{ }^{3}}-\frac{Q}{a^{2}} \tag{12}
\end{align*}
$$

The electrostatic force on the negative charge ball is

$$
\begin{equation*}
F_{-}=\frac{e^{2}}{a^{2}}\left[\left(\frac{a}{s}\right)^{2}-\frac{1}{\left(\frac{f_{2}}{a}\right)\left(\frac{t}{a}-\frac{a}{f_{-}}\right)^{2}}+\frac{1}{\left(\frac{f_{+}}{a}\right)\left(\frac{f}{a}-\frac{a}{f_{+}}\right)^{2}}+\left(\frac{Q}{e}+\frac{a}{f_{-}}-\frac{a}{f_{+}}\right)\left(\frac{a}{f_{-}}\right)^{2}\right] \tag{13}
\end{equation*}
$$

and that on the positive charge ball is

$$
\begin{equation*}
F_{+}=\frac{e^{2}}{a^{2}}\left[-\left(\frac{a}{s}\right)^{2}+\frac{1}{\left(\frac{t_{-}}{a}\right)\left(\frac{f_{+}}{a}-\frac{a}{f_{-}}\right)^{2}}-\frac{1}{\left(\frac{f_{+}}{a}\right)\left(\frac{f_{+}}{a}-\frac{a}{f_{+}}\right)^{2}}-\left(\frac{Q}{e}+\frac{a}{f_{-}}-\frac{a}{f_{+}}\right)\left(\frac{a}{f_{+}}\right)^{2}\right] \tag{14}
\end{equation*}
$$

In a configuration for which both of these forces are zero (i.e., in the absence of drag forces) one would have

$$
\begin{align*}
& F_{-}=0  \tag{15}\\
& F_{+}=0 \tag{16}
\end{align*}
$$

Subtracting one has

$$
\begin{equation*}
F_{-}-F_{+}=0 \tag{17}
\end{equation*}
$$

For $f_{-}$and $f_{+}$approaching infinity the left side of (17) approaches zero while for $f_{-}$and $\mathrm{f}_{+}$approaching a, the left side of (17) approaches infinity so that there must exist a value of $\mathrm{Q} / \mathrm{e}$ for which (17) is satisfied.

Actually null values for the electrostatic forces on the point charges are not appropriate for quasi-stable locations for them since drag and (probably negligible) gravitational forces also exist but a minor modification of the argument can be applied to cases where the point charges enjoy a balance between electrostatic and drag forces experienced in a containing vortex necessary to confer stability (even if transitory) upon the configuration. It is sufficient to equate (13) and (14) to equal and opposite drag forces.

Unfortunately the magnitudes of the drag forces on the positive and negative charge balls depends not only on the (updraft) wind velocity but also on the density of the charge balls. Thus it seems that the best procedure for using this model would be to calculate the electrostatic forces due to Earth and induced charges (including if desired gravitational forces but these are undoubtedly negligible) for a sequence of specified charge magnitudes for the charge balls and then selecting drag forces which would yield null resultant forces at altitudes for the charge balls corresponding to observation of thunderclouds. This could be done by least squares. Presumably the locations most likely to collect charge are the null resultant force altitudes.

In such a numerical analysis it would be easy to endow a thundercloud with one negative and two positive charge balls rather than the above with one positive and one negative charge ball. See Fig. 5. Tables for the electrostatic forces in the twocharge thundercloud are included in Table 1.

## 4. Development of Charge Concentrations and the Collisional Boltzmann Equations

The simplest form for the non-stationary Boltzmann transport equation in which collisions are not neglected is

$$
\begin{equation*}
\partial_{\mathrm{t}} \mathrm{p}+\bar{v} \cdot \nabla p+\bar{a} \cdot \nabla_{v}-p=\gamma p \tag{18}
\end{equation*}
$$

This is formulated for a single species of particle. The probability $p d \tau_{\bar{R}} d \tau_{v}$ that a typical particle be located within $d \tau_{\dot{R}}$ of the vectorial location $\bar{R}$ with a velocity within $d \tau_{v}$ - of vectorial velocity $\bar{v}$ satisfies the normalization condition

$$
\begin{equation*}
\iint p d \tau_{R}^{-} d \tau_{v}=1 \tag{19}
\end{equation*}
$$

$\bar{a}$ is the vectorial acceleration. In terms of the Boltzmannian differential operator

$$
\begin{equation*}
\mathrm{B}_{\mathrm{t}}=\partial_{\mathrm{t}}+\overline{\mathrm{v}} \cdot \nabla_{\overline{\mathrm{R}}}+\overline{\mathrm{a}} \cdot \nabla_{\overline{\mathrm{v}}} \tag{20}
\end{equation*}
$$

(18) may be more concisely written

$$
\begin{equation*}
B_{t p}=\gamma p \tag{21}
\end{equation*}
$$

The collisional frequency $\gamma$ corresponds to the assumption of a single relaxation time. The above formulation applies strictly only for the case of a single species of particle.

For a plasma containing positive ions, negative ions and neutral particles (21) would be replaced by

$$
\begin{equation*}
B_{t} \bar{p}=\Gamma \bar{p} \tag{22}
\end{equation*}
$$

where the dependent variable has become a probability density vector and the collisional frequency $\Gamma$ has become a collisional frequency matrix. Since this is symmetric there always exists a unitary transformation by means of which the three equations may be uncoupled into separate Boltzmann equations each with an (eigenvalue) effective scalar collisional frequency. Thus it would suffice to deal with an equation of the form (21) with the modification that the collisional frequency term now describes processes involving relaxation times for all three species.

In order to describe condensation or aggregation processes $\gamma$ cannot be a constant, but must be at least a function of position and time. If the collisional frequency term remained positive throughout a region it would mean that the probability of being located in that region would have to increase with time: if it were negative it would have to decrease with time. In any process in which particles were becoming more concentrated in the neighborhood of a point it would be essential that the collisional frequency term be a function of both position and time being positive in a region of decreasing size about the collection point and negative elsewhere so that it could become more likely to be closer to the point and less likely to be away from it.

The stationary collisionless case of (21) is considerably simpler since it becomes

$$
\begin{equation*}
\bar{v} \cdot \nabla_{\bar{R}} p+\bar{a} \cdot \nabla_{\bar{v}} p=0 \tag{23}
\end{equation*}
$$

which is readily seen to be satisfied by any differentiable function whatsoever of $E / k T$ with $E$ total energy and $T$ absolute temperature

$$
\begin{align*}
& p=f\left(\frac{E}{k T}\right), \quad E=\frac{m \bar{v} \cdot \bar{v}}{2}+V  \tag{24}\\
& \nabla_{\bar{R}} p=\frac{\nabla_{\bar{R}} E}{k T} f^{\prime} \\
& \nabla_{\bar{v}} p=\frac{\nabla_{\bar{v}}-E}{k T} f^{\prime}  \tag{25}\\
& \nabla_{\bar{V}} E=m \bar{v} \\
& \nabla_{\bar{R}} E=\nabla_{\bar{R}} V=-\bar{F}
\end{align*}
$$

so that (23) becomes

$$
\begin{equation*}
\frac{f^{\prime}}{\mathrm{kT}} \bar{v} \cdot(-\bar{F}+\mathrm{ma})=0 \tag{26}
\end{equation*}
$$

In the more general case of (18) and for a probability density p with separable velocity dependence so that

$$
\begin{gather*}
p=p_{1}(\bar{R}, t) p_{2}(\bar{v})  \tag{27}\\
\int p_{1}(\bar{R}, t) d \tau_{\bar{R}}=1=\int p_{2}(\bar{v}) d \tau_{\bar{v}}
\end{gather*}
$$

let $q_{1}=\ln p_{1}(\bar{R}, t)$ and $q_{2}=\ln p_{2}(\bar{v})$ and writing (21) in the form

$$
\begin{equation*}
B_{t}(\ln p)=\gamma \tag{28}
\end{equation*}
$$

one has

$$
\begin{equation*}
B_{t}\left(q_{1}+q_{2}\right)=D_{t} q_{1}+\bar{a} \cdot \nabla_{\bar{v}} q_{2}=\gamma \tag{29}
\end{equation*}
$$

which is readily observed to be satisfied by the Ansatz

$$
\begin{align*}
& \mathrm{q}_{2}=-\frac{\mathrm{m}(\overline{\mathrm{v}} \cdot \overline{\mathrm{v}})}{2}  \tag{30}\\
& \overline{\mathrm{a}} \cdot \nabla_{\bar{v}} \mathrm{q}_{2}=-\overline{\mathrm{F}} \cdot \overline{\mathrm{v}}
\end{align*}
$$

provided that

$$
\begin{equation*}
\partial_{t} q_{1}=\gamma \tag{31}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{F}=\nabla_{\bar{R}} q_{1} \tag{32}
\end{equation*}
$$

It then follows that

$$
\begin{equation*}
\nabla_{\bar{R}} \times \bar{F}=\overline{0} \tag{33}
\end{equation*}
$$

and

$$
\begin{align*}
& \Delta \mathrm{q}_{1}=\nabla_{\overline{\mathrm{R}}} \cdot \overline{\mathrm{~F}}  \tag{34}\\
& \Delta \gamma=\nabla_{\overline{\mathrm{R}}} \cdot \partial_{\mathrm{t}} \overline{\mathrm{~F}} \tag{35}
\end{align*}
$$

It is also possible to derive the non-stationary collisional Boltzmann Equation from a Markov chain model [7] and this may be used to set up an equivalent difference equation. Either this or the Method of Characteristics (Appendix A) may be used for numerical solutions. An example of a periodic distribution capable of describing a condensation or aggregation process in positional space is given in Appendix B.

## References

1. Smythe, W.R., "Static and Dynamic Electricity," McGraw Hill, 1939, p. 13.
2. Fluegge, S., "Rechenmethoden der Quantentheorie," Springer, 1952, S.220.
3. Kyrala, A., "Applied Functions of a Complex Variable," Wiley-Interscience, 1972, p. 218.
4. Petrzilka, V., "Zur Theorie zweier gekoppelter Schwingungskreise," ENT, 8, 1930, S.317-324, ENT, 8, 1931, S.122-131.
5. Kyrala, A., "Theoretical Physics," W.B. Saunders, 1967, p. 325.
6. Kottler, F., "Handbuch der Physik," BdXII (Geiger-Scheel), Springer, 1927, S. 431 .
7. Kyrala, A., "Selection Rules, Causality and Unitarity in Statistical and Quantum Physics," Foundation of Physics, Vol. 4, No. 1, p. 31-51.

Table 1

| ＋ | $\begin{aligned} & \text { ALT } \\ & \mathrm{Kms} \end{aligned}$ | F． | FORCES＊ <br> Newtons $F_{+}$ |
| :---: | :---: | :---: | :---: |
| ＋2．000 | ＋1．000 | ＋77．512 | $-77.512$ |
| ＋ |  | ＋5．729 | －5．728 |
| ＋－－cor | ＋2．000 | ＋87．975 | －67．975 |
| ＋4．00 | ＋1．00 | －8．7E\％ | ＋E．7EO |
| ＋4．00 | ＋2．000 | ＋19．402 | －19．402 |
| ＋4．000 | ＋5．00 | ＋89． | －89．-84 |
| ＋5．0） | ＋1．000 | －14．245 | ＋14．240 |
| ＋5．00 | ＋2．000 | ＋0． O 21 | －6．-21 |
| ＋5．0） | ＋－bin | ＋21．507 | －21．507 |
| ＋5． | ＋4． | ＋89．838 | －8¢．8．8 |
| ＋6．00 | ＋1．00 | －16．931 | ＋15．7E2 |
| ＋ 0.000 | ＋2．00 | ＋1．519 | －1．519 |
| ＋s．0y | ＋ | ＋8．716 | －8．716 |
| －o． $\sin$ | －4．0in | ＋22．094 | －2コ．0¢4 |
| ＋6．100 | ＋5． | ＋80．840 | －89．840 |
| ＋7． | ＋1．00 | －18．401 | ＋18．4ici |
| ＋7．09 | ＋2． | －0．800 | ＋0． 800 |
| －7．0n | ＋ 3.00 | ＋4．1こ＝ | －4．1－$=$ |
| ＋7．0ic | ＋4．00） | $+9.442$ | －7．441 |
| －7．000 | ＋5．000 | ＋2コ． | －22．07 |
| ＋7．000 | ＋6．000 | ＋89．904 | －99．904 |
| ＋8．009 | ＋1．000 | －19．416 | ＋19．419 |
| ＋8．00） | ＋2．000 | －2．110 | ＋2．110 |
| ＋8．000 | ＋ | $+1.954$ | －1．954 |
| ＋E．000 | ＋4．000 | ＋4．951 | －4．95 |
| ＋ E .000 | ＋5． 000 | ＋0．781 | －9．7－1 |
| ＋8．09 | ＋6．00） | ＋22．410 | －22．415 |
| ＋8．000 | ＋7．000 | ＋99．957 | －80．0－7 |
| $+9.000$ | ＋1．000 | －20．057 | ＋20．000 |
| ＋ 9.000 | ＋2．000 | －2．929 | ＋2．929 |
| $+9.000$ | ＋ 5.000 | ＋0．750 | －0．700 |
| ＋9．000 | ＋4．000 | ＋2．8こ5 | －2．955 |
| $+7.000$ | ＋5．000 | ＋5．2E8 | －5． 298 |
| － 9.00 | ＋0．009 | ＋9．87\％ | －¢．37＝ |
| ＋ 4.00 | ＋ 7.00 | ＋22．474 | ーここ．474 |
| $+\mathrm{C} .0 \mathrm{O}$ | ＋8．00） | ＋89．75i | －E9．¢5́ |
| $+10.000$ | ＋1．000 | －20．510 | ＋20．511 |
| $+10.0010$ | ＋2．000 | －5．477 | ＋3．477 |
| ＋10．00\％ | ＋3．000 | －9．019 | ＋0．019 |
| $+10.000$ | ＋4．000 | ＋1．06 | －1．－¢ |
| ＋10．000 | ＋5．00\％ | ＋5． 207 | －玉． 20 ¢ |
| ＋10．00\％ | ＋o．000 | ＋5．455 | －5．455 |
| －10．000 | ＋7．000 | ＋9．950 | －9．750 |
| $+10.000$ | ＋8．000 | ＋22．508 | ープ．50 |
| ＋10．000 | ＋9．00） | ＋89．967 | －89．957 |

－$F$ ．and $F_{+}$forces are equal in magnitude before drag forces are introduced

Table 1－cont＇d

| ＋ | $\begin{aligned} & \text { ALT } \\ & \text { Kms } \end{aligned}$ | F． | $F_{+}$ |
| :---: | :---: | :---: | :---: |
| ＋11．000 | $+1.000$ | －20．840 | ＋20．840 |
| ＋11．000 | ＋2．000 | －5．864 | ＋5．365 |
| ＋11．000 | ＋5．000 | －0．522 | $+0.525$ |
| $+11.000$ | ＋4．00\％ | ＋0．941 | －0．941 |
| ＋11．000 | ＋5．000 | ＋2．060 | －2．056 |
| $+11.000$ | ＋6．000 | ＋5．39 | －こ．39 |
| ＋11．000 | ＋7．000 | ＋5．547 | －5．547 |
| ＋11．000 | ＋8．000 | ＋9．996 | －9．996 |
| ＋11．000 | ＋9．000 | ＋22．529 | －22．529 |
| $+11.000$ | $+10.000$ | ＋99．975 | －89．975 |
| $+12.000$ | ＋1．000 | －21．088 | ＋21．089 |
| ＋12．00\％ | ＋2．000 | －4．149 | －4．149 |
| $+12.000$ | $+3.000$ | －0．976 | ＋0．876 |
| ＋12．000 | $+4.000$ | ＋0．462 | －0．462 |
| $+12.000$ | ＋5．000 | ＋1．357 | －1．357 |
| $+12.000$ | ＋6．000 | ＋2．261 | －2．260 |
| $+12.000$ | $+7.000$ | ＋こ．496 | － 5.496 |
| ＋12．00\％ | ＋8．00\％ | ＋5．602 | －5．602 |
| $+12.000$ | ＋9．000 | ＋10．624 | －10．624 |
| $+12.000$ | $+16.000$ | ＋22．54 | －22．54\％ |
| ＋12．000 | ＋11．000 | ＋89．980 | －89．980 |
| ＋12．000 | ＋1．000 | －21．280 | ＋21．291 |
| ＋12．000 | ＋2．000 | －4．304 | ＋4．864 |
| ＋13．000 | $+5.000$ | －1．155 | $+1.130$ |
| ＋15．000 | ＋4．0030 | $+6.127$ | －0．127 |
| ＋1E．006 | ＋5．000 | $+6.894$ | －0．80 |
| $+12.000$ | ＋6．000 | $+1.570$ | －1．570 |
| $+15.900$ | ＋7．000 | ＋2． 574 | －2．375 |
| $+15.000$ | ＋6．090 | $+. .559$ | －3．5ड7 |
| ＋15．000 | ＋9．006 | ＋5． 57 | －5．637 |
| ＋12．000 | ＋10．000 | $+10.043$ | －10．04 -20 |
| ＋13．000 | ＋11．000 | $+22.552$ | －22．5E2 |
| ＋1．2．900 | ＋12．000 | ＋8¢．984 | －85．983 |
| ＋14．000 | $+1.000$ | －21．4．32 | ＋21．4ここ |
| ＋14．000 | ＋2．000 | －4． 5.1 | ＋4．5 51 |
| ＋14．009 | ＋ 2.000 | －1．－52 | ＋1．552 |
| ＋14．000 | ＋4．000 | －0．117 | ＋0．117 |
| $+17.000$ | －5．000 | ＋0．570 | －0．570 |
| ＋14．000 | ＋6．000 | ＋1．110 | －1．115 |
| ＋14．000 | ＋7．000 | $+1.690$ | －1． 690 |
| ＋14．000 | ＋9．000 | ＋2．442 | －2．442 |
| ＋14．000 | ＋4．000 | ＋ 5.598 | －-578 |
| ＋14．000 | ＋10．000 | ＋5．600 | －5． 500 |
| ＋14．000 | ＋11．000 | ＋10．050 | －10．050 |
| ＋14．000 | ＋12．000 | ＋22．55¢ | $-22.559$ |
| ＋14．009 | ＋13．000 | ＋89．986 | －89．986 |

Table 1-cont'd

| + | ALT Kms | F. | FORCES Newtons $F_{+}$ |
| :---: | :---: | :---: | :---: |
| $+15.000$ | $+1.000$ | -21.554 | +21.554 |
| +15.000 | $+2.000$ | -4.60.5 | +4.064 |
| +15.000 | +3.000 | -1.494 | +1.484 |
| +15.006 | +4.009 | -0.301 | +6.30 |
| +15.000 | +5.000 | +0.505 | -0.355 |
| $+15.000$ | +6.000 | +0.800 | -0.800 |
| $+15.000$ | +7.000 | +1.242 | -1.242 |
| +15.000 | +8.000 | +1.764 | -1.764 |
| $+15.000$ | +9.000 | +2.487 | -2.486 |
| $+15.60$ | +10.000 | +5.626 | -3.626 |
| +15.000 | +11.000 | +5.678 | -5.678 |
| +15.000 | +12.000 | $+10.070$ | -10.070 |
| +15.000 | +15.000 | +22.580 | -22.580 |
| +15.000 | +14.000 | +90.125 | -90.125 |
| $+16.000$ | +1.000 | -21.655 | +21.65: |
| +16.000 | +2.000 | -4.770 | +4.771 |
| $+10.000$ | $+3.000$ | -1. 0.05 | +1.605 |
| +10.600 | +4.000 | -0.444 | +0.445 |
| +10.000 | +5.000 | +0.159 | -0.158 |
| $+16.000$ | +6.000 | +0.571 | -0.571 |
| +16.000 | +7.000 | $+6.952$ | -0.922 |
| $+16.000$ | +8.000 | +1.320 | -1.520 |
| $+16.000$ | +9.000 | +1.812 | -1.811 |
| +16.000 | $+10.000$ | +2.516 | -2.510 |
| +16.000 | +11.000 | +3.645 | -3.644 |
| +16.000 | +12.000 | +5.689 | -5. 699 |
| $+16.000$ | $+15.000$ | +10.077 | -10.076 |
| +16.000 | +14.000 | +22.584 | -22.584 |
| $+16.000$ | +15.000 | +89.989 | -89.989 |
| +17.000 | +1.000 | $-21.735$ | +21.735 |
| $+17.000$ | +2.000 | -4.858 | +4.850 |
| +17.000 | +3.000 | -1.702 | $+1.705$ |
| +17.000 | +4.000 | -0.558 | +0. 558 |
| +17.000 | +5.000 | +0.022 | -0.021 |
| $+17.000$ | +6.000 | +0. 399 | -0.399 |
| $+17.000$ | +7.000 | +0.707 | -0.707 |
| +17.000 | +8.000 | $+1.01=$ | -1.01\% |
| +17.000 | +9.000 | +1. 5.71 | -1.371 |
| +17.000 | +10.000 | -1.644 | -1.544 |
| $+17.600$ | +11.000 | +2. 5.7 | -2. 5.7 |
| $+17.000$ | +12.000 | +5.658 | -5.659 |
| $+17.000$ | +15.000 | +5.697 | -5.697 |
| $-17.000$ | $+14.000$ | +10.082 | -10.081 |
| +17.000 | +15.090 | +2こ.570 | -22.507 |

Table 1 －cont＇d

| ＋ | ALT <br> Kms | F． | FORCES <br> Newtons $F_{+}$ |
| :---: | :---: | :---: | :---: |
| ＋17．00 | ＋16．000 | ＋85．990 | －89．990 |
| $+18.00$ | ＋1．00） | －21．80こ | ＋21．804 |
| ＋18．00 | ＋2．0n | －4．950 | －4．751 |
| ＋18．000 | ＋－－ 0 | －1．782 | ＋1．アヨこ |
| ＋18．00 | ＋4．000 | －0．649 | ＋0．650 |
| ＋10．00 | ＋5．000 | －0．086 | ＋0．087 |
| ＋18．00 | ＋5．090 | ＋0．207 | －0．200 |
| ＋18．00 | $+7.000$ | ＋0．5 59 | －0．538 |
| ＋18．000 | ＋6．000 | ＋0．791 | －0．791 |
| ＋18．00 | ＋ヲ． | ＋1．060 | －1．006 |
| $+18.000$ | ＋10．000 | ＋1．405 | －1．405 |
| ＋18．0 | ＋11．90 | ＋1．906 | －1．30i |
| ＋18．009 | ＋12．00 | ＋2．552 | －2． 551 |
| ＋18．00 | ＋15．000 | ＋5．067 | －三． 0 － 7 |
| ＋18．000 | ＋14．000 | ＋5．705 | －5．703 |
| ＋1E．000 | $+15.000$ | $+10.080$ | －10．080 |
| $+13.000$ | －10．000 | ＋3コ．5フコ | －22．571 |
| $+18.00$ | ＋17．000 | ＋89．991 | －89．701 |
| ＋19．090 | ＋1．00 | －21．85： | ＋21．8่2 |
| ＋19．009 | ＋2．000 | －4．991 | ＋4．972 |
| $+19.000$ | －5．090 | －1．849 | ＋1．849 |
| $+19.000$ | ＋4．000 | －0．724 | ＋0．725 |
| ＋19．00 | ＋5．00） | －0．174 | $+0.174$ |
| ＋19．00） | ＋6．00） | ＋0．162 | －0．102 |
| ＋19．009 | ＋7．00 | －0．409 | －0．409 |
| ＋19．00 | ＋8．90 | ＋0．623 | －0．525 |
| $+19.000$ | ＋9．00） | ＋0．847 | －0．846 |
| ＋19．009 | $+10.000$ | ＋1．103 | －1．102 |
| $+19.000$ | ＋11．00） | ＋1．429 | －1．429 |
| $+19.000$ | ＋12．000 | ＋1．88こ | －1．882 |
| ＋19．009 | $+15.0 \mathrm{Og}$ | ＋2．56 | －2．502 |
| －10．000 | ＋14．000 | ＋3．675 | －5．675 |
| $+15.000$ | ＋15．009 | ＋5．700 | －5．700 |
| $+19.000$ | ＋16．000 | ＋10．0日 | －10．08こ |
| ＋17．000 | ＋17．009 | ＋22．575 | －22．573 |
| ＋19．000 | ＋18．00） | ＋89．992 | －89．902 |
| ＋20．000 | ＋1．00 | －21．911 | ＋21．911 |
| ＋20．000 | ＋2．000 | －5．043 | ＋5．044 |
| ＋20．000 | ＋3．0\％ | －1．905 | ＋1．905 |
| ＋20．000 | ＋4．000 | －0．786 | ＋0．787 |
| ＋20．000 | $+5.000$ | －0．245 | ＋0．245 |
| $+20.000$ | ＋6．000 | ＋0．078 | －0．078 |
| $+20.000$ | $+7.000$ | ＋0．307 | －0．307 |
| $+20.000$ | ＋8．000 | ＋0．498 | －0．498 |
| －20．000 | $+9.000$ | ＋0．68こ | －0．035 |

Table 1-cont'd

| + | ALT <br> Kms | $F_{\text {L }}$ | FORCES Newtons $F_{+}$ |
| :---: | :---: | :---: | :---: |
| $+20.000$ | $+10.000$ | +0.885 | -0.884 |
| +20.000 | +11.000 | +1.12日 | -1.129 |
| +20.000 | +12.000 | +1.447 | -1.447 |
| +20.000 | +15.000 | +1.895 | -1.895 |
| +20.000 | +14.000 | +2.571 | -2.571 |
| +20.000 | +15.000 | +3.679 | -3.679 |
| +20.000 | +16.000 | +5.710 | $-5.710$ |
| $+20.000$ | $+17.000$ | $+10.085$ | -10.095 |
| +20.000 | +18.000 | +22.574 | -22.574 |
| + 20.000 | +19.000 | +89.992 | -85.992 |



Figure 1
Figure 2


Figure 3


## Appendix A

The equations of the characteristics for (21) are given by

$$
\begin{align*}
& d \bar{R}=\bar{v} d t, d \bar{v}=\bar{a} d t,  \tag{A1}\\
& d p=\gamma d t \quad\left(\partial_{\mathrm{t}} \gamma \neq 0, \nabla_{\bar{R}} \gamma \neq \overline{0}\right)
\end{align*}
$$

## Appendix B

$$
p=\frac{\tan ^{2} \omega t}{\sqrt{\pi}} e^{-x^{2} \tan ^{4} \omega t} \quad \text { for } t \neq n \pi
$$




[^0]:    - Physics Dept. Arizona State University Tempe, Arizona 85287

