

**NASA Contractor Report 178403**

**ICASE REPORT NO. 87-69**

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THE ENERGY CASCADE IN TURBULENCE

(NASA-CR-178403) ON HELICITY FLUCTUATIONS  
AND THE ENERGY CASCADE IN TURBULENCE Final  
Report (NASA) 13 p Avail: NTIS HC  
A03/MF A01

N88-12028

CSCL 20D

G3/34 Unclas  
0108401

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Contract No. NAS1-18107  
November 1987

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**Operated by the Universities Space Research Association**



**National Aeronautics and  
Space Administration**

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**ON HELICITY FLUCTUATIONS AND THE ENERGY  
CASCADE IN TURBULENCE**

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**ABSTRACT**

Recent conjectures concerning the correlation between regions of high local helicity and low dissipation are examined from a rigorous theoretical standpoint based on the Navier-Stokes equations. It is proven that only the solenoidal part of the Lamb vector  $\omega \times u$  (which is directly tied to the nonlocal convection and stretching of vortex lines) contributes to the energy cascade in turbulence. Consequently, it is shown that regions of low dissipation can be associated with either low or high helicity--a result which disproves earlier speculations concerning this direct connection between helicity and the energy cascade. Some brief examples are given along with a discussion of the consistency of these results with the most recent computations of helicity fluctuations in incompressible turbulent flows.

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This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

During the past decade, helical structures have been a subject of increasing interest among researchers in turbulence. Various speculations concerning the relationship between helicity fluctuations and turbulence activity have been put forth beginning with the papers of Levich and his co-workers.<sup>1-5</sup> In particular, claims have been made concerning the direct correlation between regions of high helicity and low dissipation. More recent theoretical investigations and numerical simulations of incompressible turbulent flows have cast grave suspicions on the existence of such a correlation but speculations along these lines continue to persist in the literature. The purpose of this paper is to show conclusively, based on a rigorous analysis of the Navier-Stokes equations, that there is no direct correlation between regions of high helicity and low dissipation in turbulence. Specific examples will be given along with a discussion of the consistency of the results to be derived herein with those obtained previously from direct numerical simulations of the Navier-Stokes equations. Since the results of any given direct numerical simulation of the Navier-Stokes equations, at high Reynolds numbers, are subject to some doubt, it is essential that a corroborating theoretical argument be provided if this issue is to be resolved.

We will consider the incompressible turbulent flow of a homogeneous, viscous fluid governed by the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \quad (1)$$

and continuity equation

$$\nabla \cdot \mathbf{u} = 0. \quad (2)$$

Here,  $\mathbf{u}$  is the velocity vector,  $p$  is the modified pressure (which can include a gravitational body force potential), and  $\nu$  is the kinematic viscosity of the fluid. The acceleration on the left-hand side of Eq. (1) can be written in the alternative form<sup>9</sup>

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \equiv \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} + \frac{1}{2} \nabla |\mathbf{u}|^2 \quad (3)$$

where

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u} \quad (4)$$

is the vorticity vector and  $\boldsymbol{\omega} \times \mathbf{u}$  is usually referred to as the Lamb vector. This gives rise to the alternative form of the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} = -\nabla(p + \frac{1}{2} |\mathbf{u}|^2) + \nu \nabla^2 \mathbf{u}. \quad (5)$$

Since the cascade of energy is a nonlinear velocity effect, it is clear that it must emanate from the Lamb vector  $\boldsymbol{\omega} \times \mathbf{u}$ . Furthermore, since the vector identity

$$(\mathbf{u} \cdot \boldsymbol{\omega})^2 + |\mathbf{u} \times \boldsymbol{\omega}|^2 = |\mathbf{u}|^2 |\boldsymbol{\omega}|^2 \quad (6)$$

holds, it is clear that regions of high helicity density

$$h = \mathbf{u} \cdot \boldsymbol{\omega}$$

are associated with small Lamb vectors  $|\boldsymbol{\omega} \times \mathbf{u}|$  when each are normalized with respect to  $|\mathbf{u}| |\boldsymbol{\omega}|$ . This has motivated several researchers to conjecture that regions of high helicity have an associated low dissipation due

to a reduction in the energy cascade.<sup>1-5</sup> Such conjectures have led to the hypothesis that helical structures are coherent and endure for relatively long times.

It will now be shown that the speculations made in the literature concerning helicity and dissipation are not supported by the Navier-Stokes equations. In order to accomplish this task, we will decompose the Lamb vector into irrotational and solenoidal parts (i.e., a Helmholtz decomposition) as follows

$$\omega \times u = \nabla \alpha + \nabla \times \beta \quad (7)$$

where  $\alpha$  is scalar field and  $\beta$  is a vector field which can be made solenoidal ( $\nabla \cdot \beta = 0$ ) with no loss of generality. After using this Helmholtz decomposition, Eq. (6) can be written in the form

$$\frac{\partial u}{\partial t} + \nabla \times \beta = -\nabla(p + \frac{1}{2} |u|^2 + \alpha) + \nu \nabla^2 u. \quad (8)$$

The scalar potential  $\alpha$  is absorbed by the pressure and has no effect on the evolution of the velocity field  $u$  which determines the energy cascade. It is thus clear that the energy cascade arises only from the solenoidal part of the Lamb vector  $\nabla \times \beta$ . Consistent with Eq. (6), small magnitudes of the solenoidal part of  $\omega \times u$  need not be associated with large values of  $u \cdot \omega$  (i.e.,  $|\nabla \times \beta|$  can be small, with large  $|\nabla \alpha|$  and correspondingly small  $u \cdot \omega$ ).

For flow in an infinite domain,

$$\beta = \frac{1}{4\pi} \int \frac{\nabla' \times (\omega' \times u')}{|\mathbf{x}' - \mathbf{x}|} d^3 x' \quad (9)$$

where the notation  $\omega' \equiv \omega(\mathbf{x}', t)$  and  $\mathbf{u}' \equiv \mathbf{u}(\mathbf{x}', t)$  is utilized. Eq. (9) follows from a direct application of the vector identity

$$\nabla \times (\nabla \times \beta) = \nabla(\nabla \cdot \beta) - \nabla^2 \beta = -\nabla^2 \beta \quad (10)$$

(where we have made use of the fact that  $\beta$  is solenoidal) along with the construction of the Green's function solution of the resulting Poisson equation.<sup>10</sup> Since,

$$\nabla \times (\omega \times \mathbf{u}) = \mathbf{u} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{u} \quad (11)$$

it follows that

$$\nabla \times \beta = \frac{1}{4\pi} \nabla \times \int \frac{1}{|\mathbf{x}' - \mathbf{x}|} (\mathbf{u}' \cdot \nabla' \omega' - \omega' \cdot \nabla' \mathbf{u}') d^3 \mathbf{x}'. \quad (12)$$

Physically, the vector field  $\mathbf{u} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{u}$  accounts for the convection and stretching of vortex lines in the fluid. Hence, it is obvious that the energy cascade results from the nonlocal convection and stretching of vortex lines. It should be noted that the vorticity stretching term  $\omega \cdot \nabla \mathbf{u}$  gives rise to the usual energy cascade from large to small scales, whereas, the convective vorticity term  $\mathbf{u} \cdot \nabla \omega$  can give rise to a reverse energy cascade from small scales to large scales (this has been rigorously proven for two-dimensional turbulence<sup>11</sup>).

It will now be demonstrated by presenting counterexamples that there is no direct correlation between the turbulence dissipation (which is directly tied to the energy cascade) and local fluctuations in the helicity. Consider the case of two-dimensional turbulence with the velocity field

$$\mathbf{u}(x,y,t) = u_x(x,y,t)\mathbf{i} + u_y(x,y,t)\mathbf{j} \quad (13)$$

and its associated vorticity field

$$\boldsymbol{\omega} = \omega(x,y,t)\mathbf{k}. \quad (14)$$

For such a two-dimensional flow, the helicity density vanishes, i.e.,

$$h = \mathbf{u} \cdot \boldsymbol{\omega} = 0. \quad (15)$$

However, it is well known that two-dimensional turbulence exhibits low dissipation at high Reynolds numbers due to spectral blocking (i.e., energy actually undergoes a reverse cascade from the small scales to the large scales which dramatically reduces the dissipation; see Fjortoft<sup>11</sup>). In a Beltrami flow,

$$\mathbf{u} \times \boldsymbol{\omega} = 0 \quad (16)$$

and, hence, the normalized helicity density  $h_N$  assumes its maximum value of one, i.e.,

$$h_N \equiv \frac{|\mathbf{u} \cdot \boldsymbol{\omega}|}{|\mathbf{u}| |\boldsymbol{\omega}|} = 1 \quad (17)$$

(in general,  $h_N$  can assume the range of values  $0 \leq h_N \leq 1$ ). However, since the energy cascade arises from the Lamb vector  $\boldsymbol{\omega} \times \mathbf{u}$  (which vanishes for a Beltrami flow), it follows that Beltrami flows have low dissipation at high Reynolds numbers. Hence, Beltrami flows and two-dimensional flows each have low dissipation, at high Reynolds numbers, but have helicity densities

that are at the opposite extremes (the former has a small normalized helicity  $h_N = 0$  while the latter has a large normalized helicity  $h_N = 1$ ). It is thus clear that regions of low dissipation can be associated with either high or low helicity density which disproves the earlier claims made concerning the direct correlation between helicity and dissipation. In fact, it can be shown that for a given turbulence dissipation there can be arbitrarily different normalized helicity fluctuations. For example, consider two turbulent flows with velocities  $\mathbf{u}$  and  $\mathbf{u}^*$  that are related as follows

$$\mathbf{u}^* = \mathbf{u} + \mathbf{U}_0(t), \quad (18)$$

where  $\mathbf{U}_0(t)$  can be any random function of time alone. If we take  $\mathbf{U}_0(t)$  to satisfy the constraint

$$|\mathbf{U}_0(t)| \gg |\mathbf{u}(\mathbf{x}, t)|$$

for all  $\mathbf{x}, t$ , it follows that

$$h_N^* \equiv \frac{|\mathbf{u}^* \cdot \boldsymbol{\omega}^*|}{|\mathbf{u}^*| |\boldsymbol{\omega}^*|} = \frac{|(\mathbf{u} + \mathbf{U}_0) \cdot \boldsymbol{\omega}|}{|\mathbf{u} + \mathbf{U}_0| |\boldsymbol{\omega}|} \quad (19)$$

and, hence,

$$h_N^* \approx \frac{|\boldsymbol{\lambda} \cdot \boldsymbol{\omega}|}{|\boldsymbol{\omega}|} = |\cos(\boldsymbol{\lambda}, \boldsymbol{\omega})| \quad (20)$$

where  $\boldsymbol{\lambda} \equiv \mathbf{U}_0/|\mathbf{U}_0|$  is an arbitrary unit vector that is independent of the vorticity vector  $\boldsymbol{\omega}$ . Hence, the normalized helicity  $h_N^*$  associated with the velocity field  $\mathbf{u}^*$  can be varied independently of the corresponding

helicity associated with the velocity field  $\mathbf{u}$ . However, both velocity fields  $\mathbf{u}$  and  $\mathbf{u}^*$  have the same dissipation, i.e.,

$$\nu \overline{\frac{\partial u_k^*}{\partial x_\ell} \frac{\partial u_k^*}{\partial x_\ell}} = \nu \overline{\frac{\partial u_k}{\partial x_\ell} \frac{\partial u_k}{\partial x_\ell}} \quad (21)$$

where an overbar represents an average and the Einstein summation convention applies to repeated indices. It is thus clear that, in general, there is no direct correlation between local fluctuations in the helicity and the dissipation rate of turbulence.

Finally, comparisons will be made with some recent computational studies of helicity based on direct numerical simulations of the Navier-Stokes equations for homogeneous turbulent flows as well as for turbulent channel flow. The recent computations of Rogers and Moin<sup>7</sup> for turbulent channel flow indicate that peaks in the probability density function for the normalized helicity density are not associated with regions of low dissipation. To be more specific, the pdf for the normalized helicity density conditioned on low dissipation were either fairly flat or slightly peaked at  $h_N = 0$ . This is in stark contrast to the results of Pelz, et al.<sup>4</sup> who claimed that in the interior of the channel, this pdf for  $h_N$  is strongly peaked at  $h_N = 1$  (indicative of a Beltrami flow where  $\mathbf{u} \times \boldsymbol{\omega} = 0$ ). This disparity in results is most likely to have arisen due to inadequate resolution in the computations of Pelz, et al.<sup>4</sup> (the computations of Pelz, et al.<sup>4</sup> were conducted on a  $32^3$  grid whereas those of Rogers and Moin<sup>7</sup> were conducted on a  $128^3$  grid). Similarly, Rogers and Moin<sup>7</sup> found that the pdf for the normalized helicity density (conditioned on low dissipation) obtained from direct numerical simulations of homogeneous turbulent shear flow were fairly flat--another example of the

lack of a direct correlation between high helicity and low dissipation. The recent direct simulations of Kerr<sup>8</sup> (i.e., a  $128^3$  forced pseudo-spectral simulation of turbulence in a periodic box) also indicated that there is no evidence that regions with enhanced normalized helicity are directly involved in retarding the energy cascade process. It is thus clear that the theoretical results presented herein are consistent with the most recent and highly resolved numerical studies of helicity in turbulence.

In conclusion, it has been rigorously demonstrated that there is no direct correlation between regions of high helicity and low dissipation. In fact, regions of low dissipation can be associated with either low or high helicity. Furthermore, it was proven that the theoretical argument which initiated this conjecture was defective in its neglect of the fact that low magnitudes of the solenoidal part of the Lamb vector (which is the only contributor to the energy cascade process) are not necessarily associated with high values of helicity. It is therefore highly doubtful that helicity fluctuations can play an important role in the correlation of turbulence activity that is directly tied to the energy cascade process.

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## Report Documentation Page

1. Report No. NASA CR-178403 ICASE Report No. 87-69		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle ON HELICITY FLUCTUATIONS AND THE ENERGY CASCADE IN TURBULENCE				5. Report Date November 1987	
				6. Performing Organization Code	
7. Author(s) Charles G. Speziale				8. Performing Organization Report No. 87-69	
				10. Work Unit No. 505-90-21-01	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225				11. Contract or Grant No. NAS1-18107	
				13. Type of Report and Period Covered Contractor Report	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225				14. Sponsoring Agency Code	
15. Supplementary Notes Langley Technical Monitor: Richard W. Barnwell Submitted to Physical Review Letters Final Report					
16. Abstract Recent conjectures concerning the correlation between regions of high local helicity and low dissipation are examined from a rigorous theoretical standpoint based on the Navier-Stokes equations. It is proven that only the solenoidal part of the Lamb vector $\omega \times u$ (which is directly tied to the nonlocal convection and stretching of vortex lines) contributes to the energy cascade in turbulence. Consequently, it is shown that regions of low dissipation can be associated with either low or high helicity--a result which disproves earlier speculations concerning this direct connection between helicity and the energy cascade. Some brief examples are given along with a discussion of the consistency of these results with the most recent computations of helicity fluctuations in incompressible turbulent flows.					
17. Key Words (Suggested by Author(s)) turbulence, helicity, coherent structures, energy cascade			18. Distribution Statement 34 - Fluid Mechanics and Heat Transfer  Unclassified - unlimited		
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified		21. No. of pages 11	22. Price A02	